Topological Properties of Incomplete WK-Recursive Networks

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Abstract

The WK-recursive networks, which were originally proposed by Vecchia and Sanges, have suffered from the rigorous restriction on the number of nodes. Like the other incomplete networks, the incomplete WK-recursive networks is proposed to relieve this restriction. In this paper, it is first shown that the structures of the incomplete WK-recursive networks are conveniently represented with multistage graphs. This representation can provide a uniform look at the incomplete WK-recursive networks. By its aid, we (1) compute the connectivities of the incomplete WK-recursive networks, (2) show that they are hamiltonian if their connectivities are greater than one, and (3) propose a sufficient and necessary condition for a hamiltonian path in an incomplete WK-recursive network with connectivity 1.

1 Introduction

In the recent decade, a number of networks have been proposed in the literature [1, 5, 15, 17, 18, 19]. For these networks, many nice topological properties have been derived and many efficient algorithms have been developed. However, a major defect of these networks is that they are not truly expansible. A network is *expansible* if no changes with respect to node configuration and link connections are necessary when it is expanded.

We have emphasized two topological advantages, i.e., expansibility and equal degree, with the consideration of easy implementation and low cost. Recently, the WK-recursive networks [22] owning these two properties have been proposed. They offer high degree of regularity, scalability and symmetry which very well conform to a modular design and implementation of distributed systems involving a large number of computing elements. A VLSI implementation of a 16-node WK-recursive network had been realized at the Hybrid Computing Research Center [22]. Later this prototype network had been further extended to 64 nodes [23]. Some variants of the WK-recursive networks have been proposed recently [7, 8].

Although the WK-recursive networks own many nice properties (see [4, 6, 9-11, 22, 23]), there is a rigorous restriction on the number of their nodes. As we will see in the next section, the number of nodes contained in a WK-recursive network is restricted to d^t, where d is the degree and t is the level. Thus, as d=4, extra $3 \cdot 4^7$=49152 nodes are required to expand from a 7-level WK-recursive network to a 8-level WK-recursive network. Almost all of the networks mentioned earlier in this section suffered from the same problem. Therefore, some incomplete structures [12, 13, 14, 16] have been proposed as a solution to this problem.

In this paper, we define the incomplete WK-recursive networks that require the number of nodes to be a multiple of d, where d is the size of the basic building block. Since each basic building block of the WK-recursive networks contains d nodes, the incomplete WK-recursive networks can be expanded or contracted in arbitrary units of basic building blocks. We then compute the connectivities and hamiltonicity of the incomplete WK-recursive networks.

In the next section, the WK-recursive networks are reviewed and the incomplete WK-recursive networks are formally defined. The connectivities and hamiltonicity are discussed, respectivity, in Sections 3 and 4. Finally, this paper is concluded with some remarks in Section 5.

2 WK-Recursive Networks and Incomplete WK-Recursive Networks

The WK-recursive networks can be constructed recursively by grouping basic building blocks. Any complete graph can serve as a basic building block. For convenience, we use K(d, t) to denote a WK-recursive network of level t whose basic building blocks are each a d-node complete graph, where d>1 and $t \geq 1$. K(d, 1), which is the basic building block, is a d-node complete graph, and K(d, t) for $t \geq 2$ is composed of d K(d, t-1)'s which are connected as a complete graph. Each node of K(d, t) has degree d and can be uniquely identified by a sequence of t digits. We define K(d, t) formally as follows.

Definition 2.1. The node set of K(d, t) is denoted by $\{a_{t-1}a_{t-2}...a_1a_0 \mid a_i \in \{0, 1, ..., d\text{-}1\}$ for $0 \leq i \leq t\text{-}1\}$. Node adjacency is defined as follows: $a_{t-1}a_{t-2}...a_1a_0$ is adjacent to (1) $a_{t-1}a_{t-2}...a_1b$, where $0 \leq b \leq d$-1 and $b \neq a_0$, and (2) $a_{t-1}a_{t-2}$

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$\ldots a_{i+1}a_{i-1}(a_i)^i$ if $a_i \neq a_{i-1}$ and $a_{i-1}=a_{i-2}= \ldots =a_1=a_0$, where $(a_i)^i$ represents i consecutive a_i's. The links of (1) are named *substituting links* and assigned label 0. The links of (2) are named *flipping links* and assigned label i. The flipping links with label i are referred to as i-flipping links. Besides, there are *open links* whose one end node is a^t, where $0 \leq a \leq d-1$, and the other end node is unspecified. The open links are labeled t.

Since each node is incident with d-1 substituting links and one flipping link (or open link), K(d, t) has degree d. The structures of K(4, 1) and K(4, 3) are illustrated in Figure 1. Intuitively, the substituting links are those within basic building blocks, the i-flipping links each connect two embedded K(d, i)'s, and the open links are left for future expansion. For example, let us consider the incident links of node 311 in Figure 1. The one to node 133 is a 2-flipping link, and the others are substituting links.

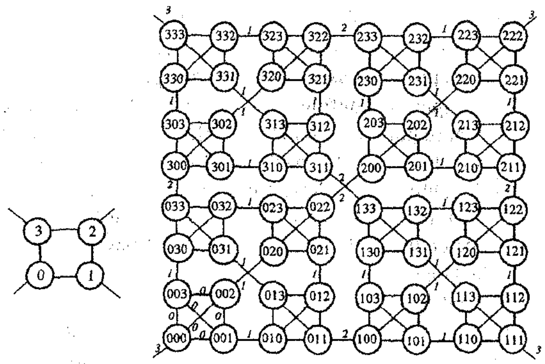

Figure 1. The structures of K(4, 1) and K(4, 3).

Definition 2.2. Define $c_{t-1}c_{t-2}\ldots c_r \cdot$K($d$, r) to be the induced subgraph of K(d, t) by $\{c_{t-1}c_{t-2}\ldots c_r a_{r-1}\ldots a_1 a_0 \mid a_i \in \{0, 1, \ldots, d-1\}$ for $0 \leq i \leq r-1\}$, where $1 \leq r \leq t-1$ and c_{t-1}, c_{t-2}, ..., c_r are all integers from $\{0, 1, \ldots, d-1\}$.

In Figure 1, for example, 20$\cdot$K(4, 1) is the subgraph induced by {200, 201, 202, 203}.

Definition 2.3. Node $a_{t-1}a_{t-2}\ldots a_1a_0$ is a *k-frontier*, where $1 \leq k \leq t$, if $a_{k-1}= \ldots =a_1=a_0$.

Note that by Definition 2.3 a k-frontier is automatically an l-frontier, where $1 \leq l < k$. Both end nodes of a k-flipping link are k-frontiers. An embedded K(d, r) contains one (r+1)-frontier and d-1 r-frontiers.

Now, we begin to introduce the incomplete WK-recursive networks. The incomplete WK-recursive networks are subgraphs of the WK-recursive networks. For convenience, we use IK(d, t) to denote an incomplete WK-recursive network with N nodes, where $d^{t-1}<N<d^t$ is a multiple of d. The restriction to N is because K(d, 1) remains the basic building block for IK(d, t). The structure of IK(d, t) with N nodes can be uniquely determined by the associated coefficient vector, as defined below.

Definition 2.4. The *coefficient vector* associated with an N-node IK(d, t) is a (t-1)-tuple (b_{t-1}, b_{t-2}, ..., b_1) satisfying $N=b_{t-1}d^{t-1}+b_{t-2}d^{t-2}+ \ldots +b_1 d$, where $1 \leq b_{t-1} \leq d-1$ and $0 \leq b_i \leq d-1$ for $1 \leq i \leq t-2$.

Let $V(b_{t-1}, b_{t-2}, \ldots, b_1)$ denote the node set of IK(d, t) with coefficient vector (b_{t-1}, b_{t-2}, ..., b_1) and $V(i \cdot$K(d, t-1)) denote the node set of $i\cdot$K(d, t-1), where $0 \leq i \leq d-1$. The set $V(b_{t-1}, b_{t-2}, \ldots, b_1)$ can be defined recursively as follows.

$$V(b_{t-1}, b_{t-2}, \ldots, b_1) = V(0\cdot K(d, t-1))+V(1\cdot K(d, t-1))+ \ldots +V((b_{t-1}-1)\cdot K(d, t-1))+V(b_{t-1}\cdot(b_{t-2}, b_{t-3}, \ldots, b_1)),$$

where + denotes a union operation and $b_{t-1}\cdot(b_{t-2}, b_{t-3}, \ldots, b_1)$ represents an IK(d, t-1) with coefficient vector (b_{t-2}, b_{t-3}, ..., b_1) that is contained in $b_{t-1}\cdot$K(d, t-1) provided $b_{t-2} \neq 0$. If $b_{t-2}= b_{t-3} = \ldots =b_r=0$ and $b_{r-1} \neq 0$, where $1<r \leq t-2$, then $b_{t-1}\cdot(b_{t-2}, b_{t-3}, \ldots, b_1)$ represents an IK(d, r) with coefficient vector (b_{r-1}, b_{r-2}, ..., b_1) that is contained in $b_{t-1}0^{t-r-1}\cdot$K(d, r).

For example, the coefficient vector of IK(5, 6) with 8225 nodes is (2, 3, 0, 4, 0) and its node set can be expressed as follows.

$$\begin{aligned}
&V(2, 3, 0, 4, 0)\\
&= V(0\cdot K(5, 5))+V(1\cdot K(5, 5))+V(2\cdot(3, 0, 4, 0))\\
&= V(0\cdot K(5, 5))+V(1\cdot K(5, 5))+V(20\cdot K(5, 4))+V(21\cdot K(5, 4)) +V(22\cdot K(5, 4))+V(23\cdot(0, 4, 0))\\
&= V(0\cdot K(5, 5))+V(1\cdot K(5, 5))+V(20\cdot K(5, 4))+V(21\cdot K(5, 4))+V(22\cdot K(5, 4))+V(230\cdot(4, 0))\\
&= V(0\cdot K(5, 5))+V(1\cdot K(5, 5))+V(20\cdot K(5, 4))+V(21\cdot K(5, 4))+V(22\cdot K(5,4))+V(2300\cdot K(5, 2))+V(2301\cdot K(5, 2))+V(2302\cdot K(5, 2))+V(2303\cdot K(5, 2))
\end{aligned}$$

The structure of IK(d, t) with coefficient vector (b_{t-1}, b_{t-2}, ..., b_1) is defined as follows.

Definition 2.5. IK(d, t) with coefficient vector (b_{t-1}, b_{t-2}, ..., b_1) is the induced subgraph of K(d, t) by $V(b_{t-1}, b_{t-2}, \ldots, b_1)$.

See Figure 2 where the structure of IK(4, 3) with coefficient vector (3, 2) is shown.

3 Connectivity

The *connectivity* of a connected network is defined as the minimum number of nodes whose removal can result in the network disconnected. Connectivity is usually adopted as a measure for fault tolerance in networks because Menger's theorem [3] states that the number of node-disjoint paths between two nodes of a network is at least its connectivity. Since IK(d, t) is a subgraph of K(d, t), the connectivity of the former is not greater than the connectivity of the latter. The connectivity of K(d, t) is known to be d-1 [4]. In this section, the connectivity of IK(d, t) is computed. First, some necessary definitions and lemmas are introduced.

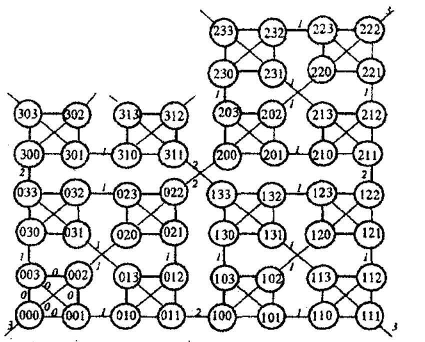

Figure 2. The structures of IK(4, 3) with coefficient vector (3, 2).

According to Definition 2.5, IK(d, t) with coefficient vector (b_{t-1}, b_{t-2}, ..., b_1) contains b_{t-1} embedded K(d, t-1)'s, b_{t-2} embedded K(d, t-2)'s, ..., and b_1 embedded K(d, 1)'s. For $1 \le i \le t-1$, the b_i embedded K(d, i)'s are $b_{t-1}b_{t-2}...b_{i+1}0 \cdot$K($d$, i), $b_{t-1}b_{t-2}...b_{i+1}1 \cdot$K($d$, i), ..., and $b_{t-1}b_{t-2}...b_{i+1}(b_i-1) \cdot$K($d$, i). Let G_i represent the induced subgraph of IK(d, t) with coefficient vector (b_{t-1}, b_{t-2}, ..., b_1) by $V(b_{t-1}b_{t-2}...b_{i+1}0 \cdot K(d, i)) + V(b_{t-1}b_{t-2}...b_{i+1}1 \cdot K(d, i)) + ... + V(b_{t-1}b_{t-2}...b_{i+1}(b_i-1) \cdot K(d, i))$, and R_n^m, where $1 \le n \le m \le t-1$, the connectivity of $G_m+G_{m-1}+ ... +G_n$. Then, R_1^{t-1} is the connectivity of IK(d, t) with coefficient vector (b_{t-1}, b_{t-2}, ..., b_1). In Figure 2, for example, we have $R_2^2=2$, $R_1^1=1$, and the connectivity of the IK(4, 3) is $R_1^2=2$.

For easy reference, we refer to $b_{t-1}b_{t-2}... b_{i+1}r \cdot$K($d$, i) as the (r+1)th K(d, i) within G_i in the subsequent discussion, where $0 \le r \le b_i-1$. Besides, a coefficient vector (b_{t-1}, b_{t-2}, ..., b_1) is written as (b_{t-1}, b_{t-2}, ..., b_i, $*$), provided $b_1=b_2= ... =b_{i-1}=0$ and $b_i \neq 0$. For example, (2, 3, 0, 4, 0) is written as (2, 3, 0, 4, $*$), and (2, 3, 4) is written as (2, 3, 4, $*$).

Lemma 3.1. For IK(d, t) with coefficient vector (b_{t-1}, b_{t-2}, ..., b_i, $*$), $R_m^m=b_m-1$ if $b_m \ge 2$, where $1 \le i \le m \le t-1$.

Proof. G_m can be regarded as a b_m-node complete graph with each node being a K(d, m). The connectivity of K(d, m) is known to be d-1. Since at least b_m-1 ($<d$-1) nodes have to be removed in order to disconnect a b_m-node complete graph, the connectivity of G_m is b_m-1. Hence, $R_m^m=b_m-1$. Q.E.D.

Lemma 3.2. For IK(d, t) with coefficient vector (b_{t-1}, b_{t-2}, ..., b_i, $*$), $R_{m-1}^m=\min\{b_{m-1}, b_m\}$ if $b_{m-1} \ge 1$ and $b_m \ge 1$, where $1 \le i < m \le t-1$.

Proof. We first assume $b_m < b_{m-1}$. For $0 \le j \le b_m-1$, there is an m-flipping link between $b_{t-1}b_{t-2}... b_{m+1}j \cdot$K($d$, m) and $b_{t-1}b_{t-2}...b_m j \cdot$K($d$, m-1). There are several possibilities (and their combinations) to disconnect G_m+G_{m-1}. To isolate one or more nodes from a K(d, m-1) or a K(d, m) requires removing at least d-1 nodes. To isolate one or more K(d, m)'s from G_m requires removing at least b_m nodes. To isolate one or more K(d, m-1)'s from G_{m-1} requires removing at least b_{m-1}-1 nodes. To separate G_{m-1} from G_m requires removing at least b_m nodes. Hence, the connectivity of G_m+G_{m-1} is b_m.

With similar arguments, the connectivity of G_m+G_{m-1} can be proved to be b_m if $b_m=b_{m-1}$, and b_{m-1} if $b_m > b_{m-1}$. This completes the proof. Q.E.D.

Lemma 3.3. For IK(d, t) with coefficient vector (b_{t-1}, b_{t-2}, ..., b_i, $*$), $R_{m-1}^{m+1}=b_m+1$ if $b_{m+1} > b_m$ and $b_m < b_{m-1}$, and $\min\{b_{m+1}, b_m, b_{m-1}\}$ else, where $b_{m+1} \ge 1$, $b_m \ge 0$, $b_{m-1} \ge 1$, and $1 \le i < m \le t-2$.

Proof. There are five cases: (1) $b_{m+1} \le b_m \le b_{m-1}$; (2) $b_{m+1} \ge b_m \ge b_{m-1}$; (3) $b_{m+1} < b_m$ and $b_m > b_{m-1}$; (4) $b_{m+1} > b_m$, $b_m < b_{m-1}$, and $b_m \neq 0$; (5) $b_m=0$, to be considered.

Case 1. $b_{m+1} \le b_m \le b_{m-1}$.

Note that for $i < j \le t-1$, there are $\min\{b_j, b_{j-1}\}$ j-flipping links connecting G_j and G_{j-1}. By Lemma 3.2, $R_m^{m+1}=b_{m+1}$ and $R_{m-1}^m=b_m$. Since no link exists between G_{m+1} and G_{m-1} (see Figure 3(a) where the links among each G_j are omitted), the connectivity of $G_{m+1}+G_m+G_{m-1}$ is b_{m+1}.

Case 2. $b_{m+1} \ge b_m \ge b_{m-1}$.

By Lemma 3.2, $R_m^{m+1}=b_m$ and $R_{m-1}^m=b_{m-1}$. The connectivity of $G_{m+1}+G_m+G_{m-1}$ is b_{m-1} with the arguments similar to Case 1.

Case 3. $b_{m+1} < b_m$ and $b_m > b_{m-1}$.

By Lemma 3.2, $R_m^{m+1}=b_{m+1}$ and $R_{m-1}^m=b_{m-1}$. The connectivity of $G_{m+1}+G_m+G_{m-1}$ is $\min\{b_{m+1}, b_{m-1}\}$ with the arguments similar to Case 1.

Case 4. $b_{m+1} > b_m$, $b_m < b_{m-1}$, and $b_m \neq 0$.

By Lemma 3.2, $R_m^{m+1}=b_m$ and $R_{m-1}^m=b_m$. There are b_m (m+1)-flipping links, i.e., $(b_{t-1}b_{t-2}... b_{m+2}0(b_{m+1})^{m+1}, b_{t-1}b_{t-2}...b_{m+2}b_{m+1}0^{m+1})$, $(b_{t-1}b_{t-2}...b_{m+2}1(b_{m+1})^{m+1}, b_{t-1}b_{t-2}...b_{m+2}b_{m+1}1^{m+1})$, ..., and $(b_{t-1}b_{t-2}...b_{m+2}(b_m-1)(b_{m+1})^{m+1}, b_{t-1}b_{t-2}...b_{m+2}b_{m+1}(b_m-1)^{m+1})$, between G_{m+1} and G_m (see Figure 3(b)). Besides, there is exactly one (m+1)-flipping link between G_{m+1} and G_{m-1} as explained as follows. There is an (m+1)-flipping link, i.e., $(b_{t-1}b_{t-2}...b_{m+2}b_m(b_{m+1})^{m+1}, b_{t-1}b_{t-2}...b_{m+2}b_{m+1}(b_m)^{m+1})$, connecting $b_{t-1}b_{t-2}...b_{m+2}b_m \cdot$K($d$, m+1) and $b_{t-1}b_{t-2}...b_{m+2}b_{m+1}b_m b_m \cdot$K($d$, m-1) which belong to G_{m+1} and G_{m-1}, respectively. For $j > b_m$, the link $(b_{t-1}b_{t-2}...b_{m+2}j(b_{m+1})^{m+1}, b_{t-1}b_{t-2}...b_{m+2}b_{m+1}j^{m+1})$ does not exist because $b_{t-1}b_{t-2}\ldots b_{m+2}b_{m+1}j^{m+1}$ is not a node in the IK(d, t). Hence, the connectivity of $G_{m+1}+G_m+G_{m-1}$ is b_m+1.

Case 5. $b_m=0$.

This case is a degenerated case of Case 4 (see Figure 3(c)). The connectivity of $G_{m+1}+G_m+G_{m-1}$ is 1. Q.E.D.

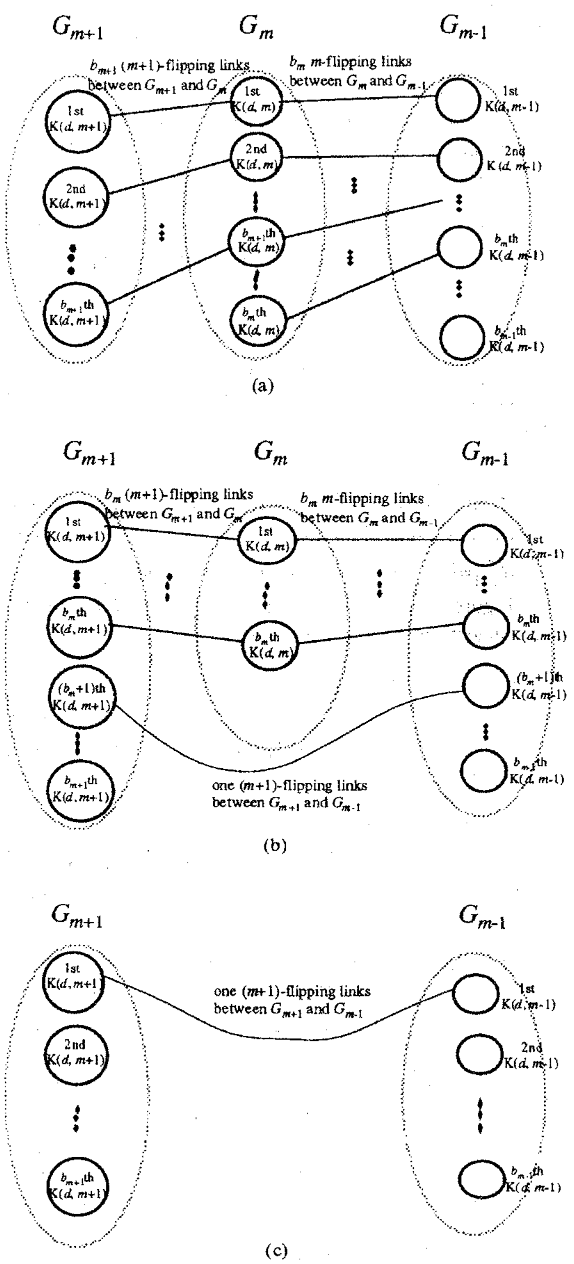

Figure 3. The proof of Lemma 3.3. (a) Case 1. (b) Case 4. (c) Case 5.

Lemma 3.4. For IK(d, t) with coefficient vector $(b_{t-1}, \ldots, b_m, r, \ldots, r, b_n, \ldots, b_i, *)$ (i.e., $b_{m-1}=b_{m-2}=\ldots=b_{n+1}=r$), where $1\le i\le n<m\le t-1$, $m\ge n+2$, $b_m>r$, $b_n>r$, and $0\le r\le d-2$, $R_n^m=r+1$. Moreover, there exists exactly one m-flipping link between G_m and G_n, and no other link exists between G_x and G_y, where $m\le x\le t-1$ and $i\le y\le n$.

Proof. By Lemma 3.2, $R_{m-1}^m=R_{m-2}^{m-1}=\ldots=R_n^{n+1}=r$. It is not difficult to see that for $n+1\le k\le m-2$ and $n+2\le l\le m-1$, no link exists between G_m and G_k and between G_n and G_l. With the arguments similar to Case 4 in the proof of Lemma 3.3, there is exactly one m-flipping link, i.e., $(b_{t-1}b_{t-2}\ldots b_{m+1}r(b_m)^m,\ b_{t-1}b_{t-2}\ldots b_{m+1}b_m r^m)$, between G_m and G_n. Hence, the connectivity of $G_m+G_{m-1}+\ldots+G_n$ is $r+1$.

Then we proceed to show that no other link exists between G_x and G_y. We first assume $x\neq m$ and $y\neq n$. If a (x-flipping) link exists between G_x and G_y, its two end nodes should be $b_{t-1}b_{t-2}\ldots b_{x+1}\alpha(b_x)^x$ and $b_{t-1}b_{t-2}\ldots b_{x+1}b_x(\alpha)^x$ for some $\alpha>r$. However, since $x>m$ and $\alpha>r$, $b_{t-1}b_{t-2}\ldots b_{x+1}b_x(\alpha)^x$ is not a node in the IK(d, t). Similarly, it can be proved that no link exists between G_x and G_y if $x=m$ and $i\le y<n$ or $m<x\le t-1$ and $y=n$. Q.E.D.

Theorem 3.1. For IK(d, t) with coefficient vector $(b_{t-1}, b_{t-2}, \ldots, b_i, *)$, $1\le i\le t-1$, its connectivity, i.e., R_i^{t-1} can be determined as follows.

(1) If $i=t-1$, then $R_i^{t-1}=b_{t-1}-1$.

(2) Otherwise, letting $k=\min\{b_{t-1}, b_{t-2}, \ldots, b_i\}$, $R_i^{t-1}=k$ if $b_{t-1}=k$ or $b_i=k$, and $k+1$ else.

Proof. We prove this theorem by induction on i.

Induction basis. Lemmas 3.1, 3.2, and 3.3 show the validity of the theorem for $i=t-1$, $t-2$, and $t-3$, respectively.

Induction hypothesis. Assume the theorem is valid for $i=m+1$, where $1\le m\le t-4$. Let $k'=\min\{b_{t-1}, b_{t-2}, \ldots, b_{m+1}\}$.

Induction step. We now discuss the case of $i=m$. Three cases are considered according to the value of k'.

Case 1. $k'=b_{t-1}$.

In this case $R_{m+1}^{t-1}=b_{t-1}$. Since $b_{t-1}\le b_{t-2}$, no link emits from G_{t-1} to G_j, where $m\le j\le t-3$. Consequently, removing b_{t-1} nodes will seperate G_{t-1} from $G_{t-2}+G_{t-3}+\ldots+G_m$. We first assume $b_{m+1}\le b_m$. By Lemma 3.2 we have $R_m^{m+1}=b_{m+1}$. There are three possibilities (or their combinations) to disconnect $G_{t-1}+G_{t-2}+\ldots+G_m$. One is to disconnect G_m which requires removing at least b_{m+1} nodes. Another is to seperate G_m from $G_{t-1}+G_{t-2}+\ldots+G_{m+1}$ which requires removing at least b_{m+1} nodes. The other is to disconnect $G_{t-1}+G_{t-2}+\ldots+G_{m+1}$ which requires removing at least b_{t-1} nodes. Hence, $R_m^{t-1}=\min\{b_{t-1}, b_{m+1}\}=b_{t-1}$. Note that $b_{t-1}=\min\{b_{t-1}, b_{t-2}, \ldots, b_{m+1}, b_m\}$.

On the other hand, if $b_{m+1}>b_m$, $R_m^{m+1}=b_m$ by Lemma 3.2, and no link emits from G_m to G_l, where $m+2\le l\le t-1$.

Similarly, R_m^{t-1} can be determined as $\min\{b_{t-1}, b_m\}$. Note that since $b_{t-1}=\min\{b_{t-1}, b_{t-2}, ..., b_{m+1}\}$, $\min\{b_{t-1}, b_m\}=\min\{b_{t-1}, b_{t-2}, ..., b_{m+1}, b_m\}$.

Case 2. $k'=b_{m+1}$.

In this case $R_{m+1}^{t-1}=b_{m+1}$. If $b_{t-1}=b_{t-2}= ... =b_{m+1}$, the discussion is the same as Case 1 because $b_{t-1}=k'$. Otherwise, let $j=\min\{l \mid m+2\le l\le t-1$ and $b_l>b_{m+1}\}$. If $b_{m+1}<b_m$, by Lemma 3.4 there is a j-flipping link between G_m and G_j. Also note that no link exists between G_m and G_s for $s\ne j$ and $s\ne m+1$. If $b_{m+1}\ge b_m$, no link exists between G_m and G_s for $s\ne m+1$. With the arguments similar to Case 1, it can be proved that $R_m^{t-1}=b_{m+1}+1$ and $b_{m+1}=\min\{b_{t-1}, b_{t-2}, ..., b_{m+1}, b_m\}$ if $b_{m+1}<b_m$, and $R_m^{t-1}=b_m$ and $b_m=\min\{b_{t-1}, b_{t-2}, ..., b_{m+1}, b_m\}$ else.

Case 3. $k'\ne b_{t-1}$ and $k'\ne b_{m+1}$.

We assume $k'=b_r$, where $m+1<r<t-1$. In this case $R_{m+1}^{t-1}=b_r+1$. Let $j=\max\{l \mid m<l<r$ and $b_l>b_r\}$. By Lemma 3.4, no link exists between G_m and G_s, where $j<s\le t-1$. With the arguments similar to Case 1, it can be proved that (1) if $b_{m+1}\le b_m$, $R_m^{t-1}=b_r+1$ and $b_r=\min\{b_{t-1}, b_{t-2}, ..., b_r, ..., b_{m+1}, b_m\}$; (2) if $b_{m+1}>b_m$ and $b_m<b_r+1$, $R_m^{t-1}=b_m$ and $b_m=\min\{b_{t-1}, b_{t-2}, ..., b_r, ..., b_{m+1}, b_m\}$; (3) if $b_{m+1}>b_m$ and $b_m\ge b_r+1$, $R_m^{t-1}=b_r+1$ and $b_r=\min\{b_{t-1}, b_{t-2}, ..., b_r, ..., b_{m+1}, b_m\}$.

Q.E.D.

We have the following corollary immediately.

Corollary 3.1. For IK(d, t) with coefficient vector ($b_{t-1}, b_{t-2}, ..., b_i, *$), letting $k=\min\{b_m, b_{m-1}, ..., b_n\}$, where $1\le i\le n<m\le t-1$, $b_m\ne 0$, and $b_n\ne 0$, $R_n^m=k$ if $b_m=k$ or $b_n=k$, and $k+1$ else.

For $i\le n<m\le t-1$, an m-flipping link between G_m and G_n is called a *jumping* m-flipping link if $m-n>1$. Note that by Lemma 3.4 the flipping links of an IK(d, t) with coefficient vector ($b_{t-1}, b_{t-2}, ..., b_i, *$) can be determined from its coefficient vector. We take IK(6, 10) with coefficient vector (4, 3, 4, 2, 1, 1, 3, $*$) as an illustrative example. There are two jumping flipping links. One is between the 4th K(6, 9) within G_9 and the 4th K(6, 7) within G_7, and the other is between the 2nd K(6, 6) within G_6 and the 2nd K(6, 3) within G_3. An easy way to determine jumping flipping links is that for any local minimal value, say b_r, in the sequence $b_{t-1}, b_{t-2}, ..., b_i$, there exists a jumping (m-flipping) link between G_m and G_n, where $i<r<t-1$ and $m=\min\{l \mid r<l\le t-1$ and $b_l>b_r\}$ and $n=\max\{l \mid i\le l<r$ and $b_l>b_r\}$, if m and n exist. This link connects the (b_r+1)th K(d, m) and the (b_r+1)th K(d, n). All non-jumping flipping links exist between G_m and G_{m-1}, where $i<m\le t-1$. More specifically, $\min\{b_m, b_{m-1}\}$ m-flipping links connect the jth K(d, m) within G_m and the jth K(d, m-1) within G_{m-1} for all $1\le j\le\min\{b_m, b_{m-1}\}$.

4 Hamiltonicity

A cycle (path) in a network is called a *hamiltonian cycle* (*path*) if it contains every node of the network exactly once. A network is *hamiltonian* if it contains a hamiltonian cycle. A hamiltonian network can embed a ring with unit expansion and unit dilation. In this section, we show that IK(d, t) with connectivity greater than one is hamiltonian. Moreover, we propose a sufficient and necessary condition for a hamiltonian path in an IK(d, t) with connectivity one. Chen and Duh [4] have shown that K(2, t) contains a hamiltonian path, and K(d, t) contains a hamiltonian cycle for $d\ge 3$. Moreover, they have shown the following result.

Lemma 4.1. [4] There is one hamiltonian path between any two t-frontiers in K(d, t).

Since IK(2, t) has a linear structure, it contains a hamiltonian path. In this section, we concentrate our attention on the hamiltonicity of IK(d, t) for $d\ge 3$. First we adapt Lemma 4.1 to IK(d, t).

Lemma 4.2. There are two hamiltonian paths, one between 0^t and 1^t and the other between $0(b_{t-1})^{t-1}$ and $1(b_{t-1})^{t-1}$, in IK(d, t) with coefficient vector (b_{t-1}, $*$), where $b_{t-1}\ge 2$.

Proof. A hamiltonian path between 0^t and 1^t can be constructed as follows:

$$0^t \to_{(H,t-1)} 0(b_{t-1}-1)^{t-1} \to (b_{t-1}-1)0^{t-1} \to_{(H,t-1)} (b_{t-1}-1)(b_{t-1}-2)^{t-1} \to (b_{t-1}-2)(b_{t-1}-1)^{t-1} \to_{(H,t-1)} (b_{t-1}-2)(b_{t-1}-3)^{t-1} \to (b_{t-1}-3)(b_{t-1}-2)^{t-1} \to_{(H,t-1)} \cdots \to_{(H,t-1)} 21^{t-1} \to 12^{t-1} \to_{(H,t-1)} 1^t,$$

where $\to$ indicates a flipping link and $\to_{(H,t-1)}$ indicates a hamiltonian path in a K(d, t-1). A hamiltonian path between $0(b_{t-1})^{t-1}$ and $1(b_{t-1})^{t-1}$ can be obtained by substituting $0(b_{t-1})^{t-1}$ and $1(b_{t-1})^{t-1}$, respectively, for 0^t and 1^t in the construction above. The correctness is assured by Lemma 4.1.

Q.E.D.

Lemma 4.3. There are two node-disjoint paths, one between $0(b_{t-1})^{t-1}$ and 0^t and the other between $1(b_{t-1})^{t-1}$ and 1^t, in IK(d, t) with coefficient vector (b_{t-1}, $*$), where $b_{t-1}\ge 2$, such that they contain every node of the IK(d, t) exactly once.

Proof. There are b_{t-1} K(d, t-1)'s, i.e., 0·K(d, t-1), 1·K(d, t-1), 2·K(d, t-1), ..., and (b_{t-1}-1)· K(d, t-1), contained in the IK(d, t). Clearly, $0(b_{t-1})^{t-1}$, 0^t, $1(b_{t-1})^{t-1}$, and 1^t are all (t-1)-frontiers. We construct two node-disjoint paths according to the following two cases.

Case 1. $b_{t-1}=2$.

By Lemma 4.1, there is one hamiltonian path between $0(b_{t-1})^{t-1}$ and 0^t in $0{\cdot}K(d, t-1)$. Likewise, there is one hamiltonian path between $1(b_{t-1})^{t-1}$ and 1^t in $1{\cdot}K(d, t-1)$. These two paths are node-disjoint, and they contain every node of the IK(d, t) exactly once.

Case 2. $b_{t-1}>2$.

Note that $0{\cdot}K(d, t-1)$ is composed of $00{\cdot}K(d, t-2)$, $01{\cdot}K(d, t-2)$, $02{\cdot}K(d, t-2)$, ..., and $0(d-1){\cdot}K(d, t-2)$, and $1{\cdot}K(d, t-1)$ is composed of $10{\cdot}K(d, t-2)$, $11{\cdot}K(d, t-2)$, $12{\cdot}K(d, t-2)$, ..., and $1(d-1){\cdot}K(d, t-2)$. A path between $0(b_{t-1})^{t-1}$ and 0^t is constructed as follows:

$$0(b_{t-1})^{t-1} \rightarrow_{(H,t-2)} 0b_{t-1}(b_{t-1}+1)^{t-2} \rightarrow 0(b_{t-1}+1)(b_{t-1})^{t-2} \rightarrow_{(H,t-2)} 0(b_{t-1}+1)(b_{t-1}+2)^{t-2} \rightarrow 0(b_{t-1}+2)(b_{t-1}+1)^{t-2} \rightarrow_{(H,t-2)} \cdots \rightarrow_{(H,t-2)} 0(d-1)0^{t-2} \rightarrow 00(d-1)^{t-2} \rightarrow_{(H,t-2)} 0^t,$$

where $\rightarrow$ indicates a flipping link and $\rightarrow_{(H,t-2)}$ indicates a hamiltonian path in a K(d, t-2). The hamiltonicity is assured by Lemma 4.1. Actually this path contains every node of $0b_{t-1}{\cdot}K(d, t-2)$, $0(b_{t-1}+1){\cdot}K(d, t-2)$, $0(b_{t-1}+2){\cdot}K(d, t-2)$, ..., $0(d-1){\cdot}K(d, t-2)$, and $00{\cdot}K(d, t-2)$ exactly once.

On the other hand, a path between $1(b_{t-1})^{t-1}$ and 1^t is constructed as the concatenation of the following four paths:

(1) $1(b_{t-1})^{t-1} \rightarrow_{(H,t-2)} 1b_{t-1}(b_{t-1}+1)^{t-2} \rightarrow 1(b_{t-1}+1)(b_{t-1})^{t-2} \rightarrow_{(H,t-2)} 1(b_{t-1}+1)(b_{t-1}+2)^{t-2} \rightarrow 1(b_{t-1}+2)(b_{t-1}+1)^{t-2} \rightarrow_{(H,t-2)} \cdots \rightarrow_{(H,t-2)} 1(d-1)0^{t-2} \rightarrow 10(d-1)^{t-2} \rightarrow_{(H,t-2)} 10^{t-1} \rightarrow 01^{t-1}$;

(2) $01^{t-1} \rightarrow_{(H,t-2)} 012^{t-2} \rightarrow 021^{t-2} \rightarrow_{(H,t-2)} 023^{t-2} \rightarrow 032^{t-2} \rightarrow_{(H,t-2)} \cdots \rightarrow_{(H,t-2)} 0(b_{t-1}-2)(b_{t-1}-1)^{t-2} \rightarrow 0(b_{t-1}-1)(b_{t-1}-2)^{t-2} \rightarrow_{(H,t-2)} 0(b_{t-1}-1)^{t-1} \rightarrow (b_{t-1}-1)0^{t-1}$;

(3) $(b_{t-1}-1)0^{t-1} \rightarrow_{(H,t-1)} (b_{t-1}-1)(b_{t-1}-2)^{t-1} \rightarrow (b_{t-1}-2)(b_{t-1}-1)^{t-1} \rightarrow_{(H,t-1)} (b_{t-1}-2)(b_{t-1}-3)^{t-1} \rightarrow_{(H,t-1)} \cdots \rightarrow_{(H,t-1)} 32^{t-1} \rightarrow 23^{t-1} \rightarrow_{(H,t-1)} 21^{t-1} \rightarrow 12^{t-1}$;

(4) $12^{t-1} \rightarrow_{(H,t-2)} 123^{t-2} \rightarrow 132^{t-2} \rightarrow_{(H,t-2)} 134^{t-2} \rightarrow 143^{t-2} \rightarrow_{(H,t-2)} \cdots \rightarrow_{(H,t-2)} 1(b_{t-1}-2)(b_{t-1}-1)^{t-2} \rightarrow 1(b_{t-1}-1)(b_{t-1}-2)^{t-2} \rightarrow_{(H,t-2)} 1(b_{t-1}-1)1^{t-2} \rightarrow 11(b_{t-1}-1)^{t-2} \rightarrow_{(H,t-2)} 1^t$,

where the hamiltonicity is assured by Lemma 4.1. Path (1) contains all nodes of $1b_{t-1}{\cdot}K(d, t-2)$, $1(b_{t-1}+1){\cdot}K(d, t-2)$, ..., $1(d-1){\cdot}K(d, t-2)$, and $10{\cdot}K(d, t-2)$. Path (2) contains all nodes of $01{\cdot}K(d, t-2)$, $02{\cdot}K(d, t-2)$, ..., and $0(b_{t-1}-1){\cdot}K(d, t-2)$. Path (3) contains all nodes of $(b_{t-1}-1){\cdot}K(d, t-1)$, $(b_{t-1}-2){\cdot}K(d, t-1)$, ..., and $2{\cdot}K(d, t-1)$. Path (4) contains all nodes of $12{\cdot}K(d, t-2)$, $13{\cdot}K(d, t-2)$, ..., $1(b_{t-1}-1){\cdot}K(d, t-2)$, and $11{\cdot}K(d, t-2)$. All nodes appear in these paths exactly once. It is not difficult to check that the two paths we have constructed between $0(b_{t-1})^{t-1}$ and 0^t and between $1(b_{t-1})^{t-1}$ and 1^t are node-disjoint, and they contain every node of the IK(d, t) exactly once. To illustrate the construction, Figure 4 shows two node-disjoint paths, one between 033 and 000 and the other between 133 and 111, in IK(4, 3) with coefficient vector (3, $*$). Q.E.D.

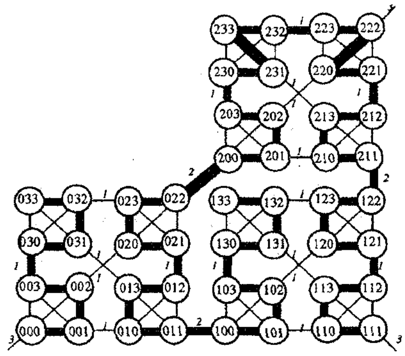

Figure 4. Two node-disjoint paths, one between 033 and 000 and the other between 133 and 111 in IK(4, 3) with coefficient vector (3,$*$).

A necessary condition for a hamiltinian graph is that its connectivity must be greater than 1. In the following, we show that the latter is also a sufficient condition for a hamiltonian IK(d, t).

Theorem 4.1. An IK(d, t), where $d \geq 3$, is hamiltonian if its connectivity is greater than 1.

Proof. Suppose $(b_{t-1}, b_{t-2}, ..., b_i, *)$ is the coefficient vector of the IK(d, t) and $R_i^{t-1}>1$. If $i=t-1$, by Theorem 3.1 we have $b_{t-1} \geq 3$. The IK(d, t) is composed of b_{t-1} K(d, t-1)'s that are connected as a b_{t-1}-node complete graph. By the aid of Lemma 4.1, it is not difficult to see that there exists a hamiltonian cycle in the IK(d, t). So, in the rest of the proof, we assume $1 \leq i < t-1$. By Theorem 3.1 we have $b_{t-1} \geq 2$ and $b_i \geq 2$. By Lemma 4.2, there exists a path between $0(b_{t-1})^{t-1}$ and $1(b_{t-1})^{t-1}$ which contains every node of G_{t-1} exactly once, and there exists a path between $b_{t-1}b_{t-2}...b_{i+1}0^{i+1}$ and $b_{t-1}b_{t-2}...b_{i+1}1^{i+1}$ which contains every node of G_i exactly once. Since $R_i^{t-1}>1$, we have $b_m \geq 1$ for all $i<m<t-1$. A hamiltonian cycle in the IK(d, t) is constructed according to the following two cases.

Case 1. $b_m \geq 2$ for all $i<m<t-1$.

Lemma 4.3 assures that for $i<m<t-1$, there exist two node-disjoint paths in G_m, one between $U_{m,0}=b_{t-1}b_{t-2}...b_{m+1}0(b_m)^m$ and $V_{m,0}=b_{t-1}b_{t-2}...b_{m+1}0^{m+1}$ and the other between $U_{m,1}=b_{t-1}b_{t-2}...b_{m+1}1(b_m)^m$ and $V_{m,1}=b_{t-1}b_{t-2}...b_{m+1}1^{m+1}$, such that they contain every node of G_m exactly once. A hamiltonian cycle in the IK(d, t) is thus formed as shown in Figure 5(a), where

(a) $(0(b_{t-1})^{t-1}, V_{t-2,0})$ and $(1(b_{t-1})^{t-1}, V_{t-2,1})$ define two $(t-1)$-flipping links;

(b) $(U_{i+1,0}, b_{t-1}b_{t-2}...b_{i+1}0^{i+1})$ and $(U_{i+1,1}, b_{t-1}b_{t-2}...b_{i+1}1^{i+1})$ define two $(i+1)$-flipping links;

(c) $(U_{m,0}, V_{m-1,0})$ and $(U_{m,1}, V_{m-1,1})$ define two m-flipping links, where $i+1<m<t-1$.

Case 2. $b_m=1$ for one or more m's between i and $t-1$.

We assume $b_m=1$ for exactly one m. The extension to multiple m's is very straightforward. According to Lemma 4.1, there exists a path between $X=b_{t-1}b_{t-2}...b_{m+1}01^m$ and $Y=b_{t-1}b_{t-2}...b_{m+1}0^{m+1}$ which contains every node of G_m exactly once. As shown in Figure 5(b), there is an $(m+1)$-flipping link between $U_{m+1,0}$ and Y, an m-flipping link between X and $V_{m-1,0}$, and a jumping $(m+1)$-flipping link between $U_{m+1,1}$ and $V_{m-1,1}$. A hamiltonian cycle in the IK(d, t) can be constructed similar to Case 1.

Q.E.D.

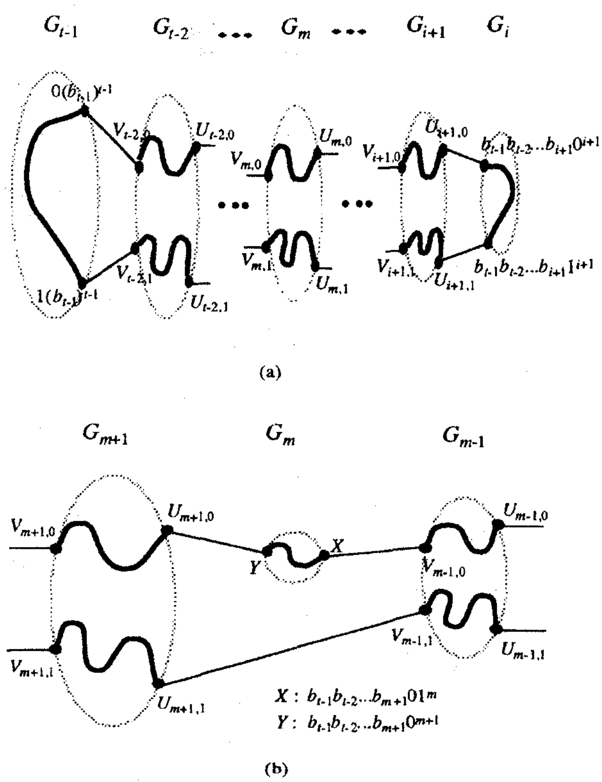

Figure 5. The proof of Theorem 4.1. (a) Case 1. (b) Case 2.

Theorem 4.1 guarantees a hamiltonian cycle in IK(d, t) with $d\geq 3$ if its connectivity is greater than 1. For IK(d, t) with connectivity 1, there is no hamiltonian cycle, and there is not necessarily a hamiltonian path. For example, no hamiltonian path exists in IK(4, 4) with coefficient vector (1, 2, 1, *). In what follows, we identify the class of IK(d, t)'s with connectivity 1 which contain a hamiltonian path.

For IK(d, t) with coefficient vector $(b_{t-1}, b_{t-2}, ..., b_i, *)$ and $R_i^{t-1}=1$, we can partition it into blocks. G_k is a *block* if $b_k\neq 0$ and ($R_k^{k+1}=1$ or $b_{k+1}=0$) and ($R_{k-1}^{k}=1$ or $b_{k-1}=0$). $G_m+G_{m-1}+ ... +G_n$, where $b_m\neq 0$, $b_n\neq 0$ and $m>n$, is a *block* if $R_n^m>1$ and ($R_n^{m+1}=1$ or $b_{m+1}=0$) and ($R_{n+1}^{m}=1$ or $b_{n+1}=0$). The partition can be easily done by examining the coefficient vector. As an illustrative example let us consider IK(6, 10) with coefficient vector (1, 2, 1, 2, 0, 1, *). By Lemma 3.4, there are two jumping flipping links. One is between the 2nd K(6, 8) within G_8 and the 2nd K(6, 6) within G_6, and the other is between the first K(6, 6) within G_6 and the first K(6, 4) within G_4. Clearly G_9 and G_4 are two blocks because $R_8^9=1$ and $b_5=0$, respectively. $G_8+G_7+G_6$ is another block because $R_6^8=2$, $R_6^9=1$, and $b_5=0$. Hence IK(6, 10) with coefficient vector (1, 2, 1, 2, 0, 1, *) can be partitioned into $\{G_9, G_8+G_7+G_6, G_4\}$. Intuitively, if each G_j ($4\leq j\leq 9$) with $b_j\neq 0$ is regarded as a node, then the flipping links between G_9 and G_8 and between G_6 and G_4 are two bridges [2], and each block is either a single node or a maximal biconnected component in the resulting graph. The following two lemmas have proven in [20].

Lemma 4.4.[20] An IK(d, t) with connectivity 1 contains a hamiltonian path if it consists of one or two blocks.

Lemma 4.5.[20] Consider an IK(d, t) with coefficient vector $(b_{t-1}, b_{t-2}, ..., b_i, *)$ and $R_i^{t-1}=1$ that contains three or more blocks. There is a hamiltonian path in the IK(d, t) if and only if for each block, say $G_m+G_{m-1}+ ... +G_n$, in the IK(d, t), no $b_{r+1}, b_r, b_{r-1}, ..., b_s, b_{s-1}$ exist such that $b_{r+1}\in\{0, 1\}$, $b_r=b_{r-1}= ... =b_s=2$, $b_{s-1}\in\{0, 1\}$, and $r-s+1$ is odd, where $m\neq t-1$, $n\neq i$, and $n\leq s\leq r\leq m$.

Combining Lemmas 4.4 and 4.5, we have a necessary and sufficient condition for a hamiltonian path in an IK(d, t) with connectivity 1.

Theorem 4.2. For IK(d, t) with coefficient vector $(b_{t-1}, b_{t-2}, ..., b_i, *)$ and $R_i^{t-1}=1$, it contains a hamiltonian path if and only if either of the following two conditions holds:

(1) it contains one or two blocks;

(2) for each block, say $G_m+G_{m-1}+ ... +G_n$, in the IK(d, t), no $b_{r+1}, b_r, b_{r-1}, ..., b_s, b_{s-1}$ exist such that $b_{r+1}\in\{0, 1\}$, $b_r=b_{r-1}= ... =b_s=2$, $b_{s-1}\in\{0, 1\}$, and $r-s+1$ is odd, where $m\neq t-1$, $n\neq i$, and $n\leq s\leq r\leq m$.

5 Concluding Remarks

Deriving topological properties for incomplete networks is far more difficult than for complete networks. The reason is that complete networks of different sizes preserve great topological similarity, whereas incomplete networks may have a significant difference in their topologies. For example, K(d, t) looks very similar to K(d, t-1), whereas two IK(d, t)'s with different coefficient vectors may look very unlike in their topologies. Many of topological properties of the incomplete star networks [13], [16] are still unknown, although they have been well solved for the star networks [1]. Most of the results obtained for the incomplete star networks are restricted to a special case: $N=c \cdot k!$, where N is the number of nodes.

In this paper, we have shown it very convenient to represent the structure of an incomplete WK-recursive networks by a "multistage-like" graph $G_{t-1}+G_{t-2}+ \dots +G_{i}$. This representation provides a uniform look at the incomplete WK-recursive networks, and thus facilitates the derivation of many properties. By the aid of this representation, we have computed the connectivities and hamiltonicity. Moreover, we have suggested a tight upper bound on the diameters. The methods adopted in this paper are different from Chen and Duh's for the WK-recursive networks [4]. Readers who are interested in the incomplete WK-recursive networks are refered to [20] and [21] for more results. Precisely, using the prune-and-search technique a linear-time algorithm for computing the diameters can be found in [20], and a distributed shortest-path routing algorithm can be found in [21].

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