Various Results on the Toughness of Graphs

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Received March 1997; accepted 3 August 1998

Abstract: Let G be a graph and let $t \geq 0$ be a real number. Then, G is t-tough if $t\omega(G - S) \leq |S|$ for all $S \subseteq V(G)$ with $\omega(G - S) > 1$, where $\omega(G - S)$ denotes the number of components of $G - S$. The toughness of G, denoted by $\tau(G)$, is the maximum value of t for which G is t-tough [taking $\tau(K_n) = \infty$ for all $n \geq 1$]. G is minimally t-tough if $\tau(G) = t$ and $\tau(H) < t$ for every proper spanning subgraph H of G. We discuss how the toughness of (spanning) subgraphs of G and related graphs depends on $\tau(G)$, we give some sufficient degree conditions implying that $\tau(G) \geq t$, and we study which subdivisions of 2-connected graphs have minimally 2-tough squares. Networks 33: 233–238, 1999

Keywords: t-tough graph; toughness (of subgraphs); minimally t-tough graph; square of a graph; degree sum condition

1. INTRODUCTION

In 1973, Chvátal [8] introduced a new graph invariant called toughness, and he investigated its relation to the existence of Hamilton cycles. Although many results concerning the relation between toughness and cycle structure have been obtained since, the main conjectures raised in [8] are still open, and little progress has been made toward (dis)proving these conjectures.

More recently, toughness has also been considered as a vulnerability measure for networks, since it measures how extensively a graph breaks up into components when a set of vertices is removed. Unfortunately, this measure is difficult to calculate: As shown by Bauer et al. [3], it is NP-hard to determine the toughness of a graph.

Despite these negative observations, many new results involving the toughness of a graph have appeared, and new and interesting ideas have been added to the field, in an attempt to solve the main problems and to get more of a grip on the concept of toughness. The results presented here should be considered in the same vein.

2. PRELIMINARIES

We start this section with some terminology and notation. We refer to [4] and [12] for any undefined terminology and notation and consider finite undirected graphs without loops and multiple edges only.

Let G be a graph and let $t \geq 0$ be a real number. Then, G is *t-tough* if $t\omega(G - S) \leq |S|$ for all $S \subseteq V(G)$ with $\omega(G - S) > 1$, where $\omega(G - S)$ denotes the number of

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AMS Subject Classifications (1991): 05C75, 05C35

 CCC 0028-3045/99/030233-06

components of $G - S$. The *toughness* of G, denoted by $\tau(G)$, is the maximum value of t for which G is t-tough [taking $\tau(K_n) = \infty$ for all $n \geq 1$]. A set $S \subseteq V(G)$ with the property that $\tau(G) = |S|/[\omega(G - S)]$ is called a *toughness-determining set* of G. A *Hamilton cycle* (*Hamilton path*) of G is a cycle (path) of G containing every vertex of G. A graph G is called *Hamiltonian* if it contains a Hamilton cycle and it is called *Hamiltonian-connected* if between any two vertices of G there exists a Hamilton path of G.

The concept of toughness was introduced in [8], where relations between toughness and the existence of Hamilton cycles or *k-factors* (k-regular spanning subgraphs) are studied, and several conjectures are stated. The following conjecture is still open:

Conjecture 1. *There exists a t_0 such that every t_0-tough graph is Hamiltonian.*

Little progress has been made toward proving Conjecture 1, although in [10], it is shown that every 2-tough graph on at least three vertices contains a 2-factor and that there exist $(2 - \epsilon)$-tough graphs not containing a 2-factor (hence, non-Hamiltonian) for arbitrarily small $\epsilon > 0$.

For many years, it was expected that if Conjecture 1 were true then t_0 would be equal to 2, but, recently, in [2], that idea was refuted by constructing $(\frac{9}{4} - \epsilon)$-tough non-Hamiltonian graphs for arbitrary $\epsilon > 0$.

The results in Section 5 were obtained before Conjecture 1 was refuted for $t_0 = 2$ and were a previous attempt to find a counterexample to this conjecture within the class of (spanning subgraphs of) squares of 2-connected graphs. The results in the other sections are based on an attempt to get more of a grip on the concept of toughness. In Section 3, we express the toughness of spanning subgraphs of a graph G and related graphs in terms of the toughness of G. In the main result of Section 4, a certain value of the toughness is guaranteed by a degree sum condition.

3. TOUGHNESS OF SUBGRAPHS AND RELATED GRAPHS

3.1. Toughness of Spanning Subgraphs

In this section, we will prove that every noncomplete graph with toughness greater than one contains a spanning subgraph with toughness exactly one. In fact, we will prove a stronger result from which the above follows as a special case. One could wonder (as we did) whether for all noncomplete graphs G with $\tau(G)$ not an integer there exists a spanning subgraph H with $\tau(H) = \lfloor \tau(G) \rfloor$. We were not able to prove or disprove this. At the Ph.D. defense of the third author, we were informed by Brouwer [5] that the toughness τ of a graph can drop considerably below the closest integer $\lfloor \tau \rfloor$ if one deletes an arbitrary edge. As an example, he considered the so-called Higman–Sims graph HS (see, e.g., [6], p. 391), a 22-regular distance-regular triangle-free graph on 100 vertices such that each pair of nonadjacent vertices has precisely six common neighbors. The complement $\overline{HS}$ of this graph is a 77-regular claw-free graph on 100 vertices with $\alpha = 2$ and $\tau = \kappa/2 = \delta/2 = 38.5$. For an arbitrary edge $e = uv \in E(\overline{HS})$, $\overline{HS} - e$ has a cut set of 92 vertices, leaving only u, v, and a K_6 consisting of the six common neighbors of u and v in HS; hence, $\tau(\overline{HS} - e) \leq \frac{92}{3} < 31$.

In the next proof, we use the following easy observation: Let S be a cut set of a graph G. Then, $|S| \geq 2\tau(G)$ [8].

Theorem 2. *Let G be a graph ($\neq K_1, K_2, \ldots, K_{\lfloor (4i+7)/3 \rfloor}$) with $\tau(G) > i$ for some positive integer i. Then, there exists a spanning subgraph H of G with $(2i + 1)/3 \leq \tau(H) \leq i$.*

Proof. Suppose there does not exist a spanning subgraph H of G with $(2i + 1)/3 \leq \tau(H) \leq i$. Let H_1 and H_2 be spanning subgraphs of G such that $H_2 = H_1 - e$ for some $e \in E(H_1)$ and such that $\tau(H_1) > i$ and $\tau(H_2) < (2i + 1)/3$. Then, there exists a cut set S of H_2 for which $\tau(H_2) = |S|/[\omega(H_2 - S)] < (2i + 1)/3$, and, thus, $\omega(H_2 - S) > (3|S|)/(2i + 1)$. Since $\omega(H_2 - S)$ and $3|S|$ are integers, we get $\omega(H_2 - S) \geq (3|S| + 1)/(2i + 1)$. First, suppose that S is a cut set of H_1. Then, $\omega(H_1 - S) \geq \omega(H_2 - S) - 1 \geq [(3|S| + 1)/(2i + 1)] - 1$. On the other hand, since $\tau(H_1) > i$, $\omega(H_1 - S) < |S|/i$; hence, $\omega(H_1 - S) \leq (|S| - 1)/i$. Thus, we get $(3|S| + 1 - 2i - 1)/(2i + 1) \leq |S| - 1/i$. For $i = 1$, this clearly leads to a contradiction. So let $i > 1$. Then, equivalently, $3i|S| - 2i^2 \leq 2i|S| + |S| - 2i - 1$; hence, $(i - 1)|S| \leq 2i^2 - 2i - 1$. So, the following holds for $|S|$: $|S| \leq (2i^2 - 2i - 1)/(i - 1) = 2i - 1/(i - 1) < 2i$. But, on the other hand, $|S| \geq 2\tau(H_1) > 2i$, a contradiction. This leaves us with the case that S is not a cut set of H_1. This means that e (in H_1) connects the only two components of $H_2 - S$. Since S is a toughness-determining set of H_2, $\omega(H_2 - S) = 2$, and since $\tau(H_2) < (2i + 1)/3$, $|S| < (4i + 2)/3$; hence, $|S| \leq (4i + 1)/3$. If $S \cup \{u\}$ is a cut set of H_1 for some $u \in V(H_1)$, then $|S \cup \{u\}| \geq 2\tau(H_1) > 2i$, so $|S| \geq 2i$. This contradicts $|S| \leq (4i + 1)/3$ since $i \geq 1$. This implies that the two components of $H_2 - S$ are K_1. But then $|V(G)| = |S| + 2 \leq [(4i + 1)/3] + 2 = (4i + 7)/3$; thus, $|V(G)| \leq \lfloor (4i + 7)/3 \rfloor$. This implies that G is not complete. But, then, $\tau(G) \leq |S|/2 \leq (|V(G)| - 2)/2 \leq (4i + 1)/6 < i$, our final contradiction. ∎

The lower bound on $\tau(H)$ in Theorem 2 cannot be raised. Let G be a $K_{(4i+8)/3}$, where we assume that $(4i + 8)/3$ is an integer. Then, for every proper spanning subgraph H of G, $\tau(H) \leq \tau(G - e) = \{[((4i + 8)/3] - 2)/2\} = (2i + 1)/3$. We do not know whether there exist noncomplete examples showing this. The complement of the Higman–Sims graph mentioned before shows that the toughness of G can drop considerably below $\lfloor \tau(G) \rfloor$ if one deletes an edge.

By taking $i = 1$ in Theorem 2, we get the following result as a special case:

Corollary 3. *Let G be a graph ($\neq K_1, K_2, K_3$) with $\tau(G) > 1$. Then, there exists a spanning subgraph H of G with $\tau(H) = 1$.*

3.2. Toughness of Subgraph-related Graphs

The next two lemmas give a relation between the toughness of a graph G and the toughness of the join of a small complete graph and a component of $G - S$, where S is a toughness-determining set of G. These lemmas might turn out to be useful in inductive proofs of structural results on graphs with a certain toughness, because they guarantee the same toughness for usually smaller graphs, each containing one of the components of $G - S$.

By $G \vee H$, we denote the *join* of two disjoint graphs G and H, that is, the graph obtained from G and H by joining every vertex of G to every vertex of H.

First, we give a result for arbitrary toughness-determining sets:

Lemma 4. *Let k be a positive integer, and let G be a graph with $\tau(G) \geq k$; S, an arbitrary toughness-determining set of G; and H_i, an arbitrary component of $G - S$. Then, $\tau(K_k \vee H_i) \geq k$.*

Proof. Suppose that $\tau(K_k \vee H_i) < k$. Then, there exists a subset $S_i' \subset V(K_k \vee H_i)$ such that $|S_i'| < k\omega((K_k \vee H_i) - S_i')$. Let $V(K_k) = \{v_1, \ldots, v_k\}$. Define $S_i = S_i' \setminus \{v_1, \ldots, v_k\}$. Then, $|S_i| = |S_i'| - k$ and $\omega((K_k \vee H_i) - S_i') = \omega(H_i - S_i)$. Now,

$$\tau(G) \leq \frac{|S \cup S_i|}{\omega(G - (S \cup S_i))} = \frac{|S| + |S_i|}{\omega(G - S) + \omega(H_i - S_i) - 1} < \frac{|S| + |S_i|}{\omega(G - S) + \frac{|S_i|}{k}} \leq \frac{|S|}{\omega(G - S)},$$

a contradiction. ■

To show that the lower bound on $\tau(K_k \vee H_i)$ in Lemma 4 cannot be increased, consider, for example, the graph $G_{p,k}$ ($p \geq 2$, $k \geq 1$) obtained by joining a K_{pk} to p disjoint copies of a K_{k+2} with one edge e deleted. Then, $\tau(G_{p,k}) = (pk)/p = [(p + 1)k]/(p + 1) = \ldots = (2pk)/(2p) = k$. Let $S = V(K_{pk})$. Then, $H_i = K_{k+2} - e$ and $\tau(K_k \vee H_i) = \tau(K_k \vee (K_{k+2} - e)) = (k + k)/2 = k$.

Although Lemma 4 is sharp, it can be improved in the following manner if we only look at maximum toughness-determining sets. Here, it is convenient to adopt the notation that $K_{k-1} \vee H_i = H_i$ if $k = 1$:

Lemma 5. *Let k be a positive integer, and let G be a graph with $\tau(G) \geq k$; S, a maximum toughness-determining set of G; and H_i, an arbitrary component of $G - S$. Then, $\tau(K_{k-1} \vee H_i) \geq k$.*

Proof. Suppose that $\tau(K_{k-1} \vee H_i) < k$. Then, as in the proof of the previous lemma, there now exists a nonempty set $S_i \subset V(H_i)$ such that $(|S_i| - 1)/k + 1 < \omega(H_i - S_i)$. Hence, $\omega(H_i - S_i) \geq 1 + |S_i|/k$. So,

$$\tau(G) < \frac{|S \cup S_i|}{\omega(G - (S \cup S_i))} = \frac{|S| + |S_i|}{\omega(G - S) + \omega(H_i - S_i) - 1} \leq \frac{|S| + |S_i|}{\omega(G - S) + \frac{|S_i|}{k}} \leq \frac{|S|}{\omega(G - S)},$$

a contradiction. ■

To show that the lower bound on $\tau(K_{k-1} \vee H_i)$ in Lemma 5 cannot be increased, consider, for example, the graph $H_{p,k}$ ($p \geq 2$, $k \geq 1$) obtained by joining a K_{pk} to p disjoint copies of a C_4 joined to a K_{k-1} having $k \geq 2$. Then, $\tau(H_{p,k}) = k$. Thus, $S = V(K_{pk})$ is a maximum toughness-determining set of G. Since $H_i = K_{k-1} \vee C_4$, we conclude that $\tau(K_{k-1} \vee H_i) = k$.

4. SUFFICIENT CONDITIONS FOR *T*-TOUGHNESS

In this section, we present some sufficient conditions guaranteeing a graph to be t-tough.

Chvátal and Erdös [9] proved that a graph G on at least three vertices with $\alpha(G) \leq \kappa(G)$ is Hamiltonian and, hence, 1-tough. It is very easy to extend this result to t-toughness.

Lemma 6. *Let G be a graph. If $t\alpha(G) \leq \kappa(G)$, then G is t-tough.*

Proof. Suppose that G is not t-tough. Then, there exists a cut set S of G with $\omega(G - S) > |S|/t$. Clearly, $\alpha(G) \geq \omega(G - S)$, so $\alpha(G) > |S|/t$, which implies that $t\alpha(G) > |S| \geq \kappa(G)$. ■

The connectivity condition in Lemma 6 cannot be relaxed and still guarantee t-toughness, as can easily be seen from the graph $K_p \vee \bar{K}_q$ with $p > q$: $\alpha = q$, $\kappa = p$ and $\tau = p/q = \kappa/\alpha$.

In a similar way, one might try to extend degree conditions for Hamiltonicity (hence, 1-toughness) to results for t-toughness, for example, the following result of Ore [14]:

Theorem 7. *Let G be a graph on $n \geq 3$ vertices such that for every two nonadjacent vertices u and v of G, $d(u) + d(v) \geq n$. Then, G is Hamiltonian.*

The degree condition in this theorem can be changed as follows to obtain a degree condition which implies t-toughness. In the following results, we use $\sigma_k(G)$ to denote the minimum degree sum of the vertices of an independent set of k vertices of the graph G [if such exists; otherwise, we define $\sigma_k = k(|V(G)| - 1)$]:

Theorem 8. *Suppose that k and $kt \geq 1$ are integers. If G is a kt-connected graph on n vertices with $\sigma_{k+1} \geq (k + 1)(nt/(t + 1))$, then G is t-tough.*

Proof. Suppose that $G \neq K_n$ and G is not t-tough. Then, for some $S \subseteq V(G)$: $\omega(G - S) > |S|/t$. If $\omega(G - S) < k + 1$, then $|S|/t < \omega(G - S) \leq k$, implying that $\kappa(G) \leq |S| < kt$, a contradiction. So, $\omega(G - S) \geq k + 1$. Let $v_1, v_2, \ldots, v_{k+1}$ denote $k + 1$ vertices from distinct components of $G - S$. Then, $(k + 1)(nt)/(t + 1) \leq \sigma_{k+1} \leq \sum_{i=1}^{k+1} d(v_i) \leq (k + 1)|S| + n - |S| - \omega(G - S) < k|S| + n - |S|/t$. Hence, $(k + 1)tn < k(t + 1)|S| + n(t + 1) - [|S|(t + 1)]/t$ and $(kt - 1)n < [kt(t + 1)]/t\,|S| - [(t + 1)/t]|S| = (kt - 1)[(t + 1)/t]|S|$. If $kt = 1$, we obtain a contradiction. Hence, assume that $kt > 1$. Then, $|S| > (tn)/(t + 1)$, hence, $\omega(G - S) > |S|/t > n/(t + 1)$. But, now, $n \geq |S| + \omega(G - S) > (tn)/(t + 1) + n/(t + 1) = n$, a contradiction. ∎

Note that if G is kt-connected with $k \geq \alpha(G)$, then we know that G is t-tough (by Lemma 6). In the case when $k < \alpha(G)$, G has an independent set of $k + 1$ vertices and the degree sum condition makes sense.

From Theorem 8, we can simply derive an extension of Theorem 7 to t-toughness. To show that it is an extension, we need the following lemma:

Lemma 9. *Let G be a graph on n vertices such that $\sigma_2 \geq 2nt/(t + 1)$, where $1 \leq t \leq n - 1$. Then, G is $\lceil t \rceil$-connected.*

Proof. Suppose that G is not $\lceil t \rceil$-connected. Then, there exists a cut set $S \subseteq V(G)$ with $|S| < \lceil t \rceil$. Since $|S|$ is an integer, this implies that $|S| \leq \lceil t \rceil - 1 < t$. Since S is a cut set, $G - S$ consists of at least two components. Choose a vertex u in one component and a vertex v in another component. Then, $\sigma_2 \leq d(u) + d(v) \leq 2|S| + n - 2 - |S| = n + |S| - 2 < n + t - 2$, so $2nt/(t + 1) < n + t - 2$ or, equivalently, $(n - t)(t - 1) < -2$, which is absurd since $1 \leq t \leq n - 1$. ∎

Corollary 10. *Let G be a graph on n vertices such that $\sigma_2 \geq 2nt/(t + 1)$, where $1 \leq t \leq n - 1$. Then, G is t-tough.*

Corollary 10 generalizes a result of Bauer et al. [3] in which the minimum-degree condition $\delta(G) \geq nt/(t + 1)$ is shown to be sufficient for t-toughness.

5. SPANNING SUBGRAPHS OF SQUARES

One possible approach to disproving the 2-tough-conjecture (Conjecture 1 with $t_0 = 2$) was to try to disprove the equivalent conjecture that all 2-tough graphs are Hamiltonian-connected (see [1]), that is, by finding a graph which is 2-tough but not Hamiltonian-connected. To avoid the difficulty of checking whether the considered graphs are 2-tough (which is an NP-hard problem [3]), one could restrict the search for a counterexample to a class of graphs every member of which is known to be 2-tough. An example of such a class of graphs is the class consisting of the squares of all 2-connected graphs. The *square* G^2 of a graph G is the graph obtained from G by joining all vertices at distance 2 in G. By an elementary result of Chvátal [8], the square of a k-connected graph is k-tough. Unfortunately, the square of a 2-connected graph is also Hamiltonian-connected, by an observation in [7] based on the beautiful deep result due to Fleischner [11] that all squares of 2-connected graphs are Hamiltonian. So, all attempts to find a counterexample within this class of squares of 2-connected graphs will fail, that is, if we restrict ourselves to graphs within this class. But, by the ease of showing that squares of 2-connected graphs are 2-tough, and the difficulty of showing that these squares are Hamiltonian-connected, one could be tempted to think that there exist spanning subgraphs of these squares that are 2-tough, but not Hamiltonian-connected. This led us to define and study minimally 2-tough graphs or, more generally, minimally t-tough graphs. A graph G is called *minimally t-tough* if $\tau(G) = t$ and there does not exist a proper spanning subgraph H of G with $\tau(H) = t$. By the above motivation, we concentrate on squares of 2-connected graphs and consider the problem which of these graphs are (not) minimally 2-tough. We need one more definition. Let G and H be graphs, and s, an integer with $s \geq 0$. G is an *s-subdivision* of H if G can be obtained from H by replacing every edge uv of H by a path between u and v with at least s internal vertices. We call G an s-subdivision if it is an s-subdivision of some graph H. Clearly, a graph which is an s-subdivision ($s \geq 1$) is also an $(s - 1)$-subdivision, and an s-subdivision of a 2-connected graph is 2-connected.

As mentioned in [13] during the Workshop on the Hamiltonicity of 2-Tough Graphs held at Enschede, The Netherlands, in November 1995 (sponsored by EIDMA: the Euler Institute for Discrete Mathematics and Its Applications), "sufficiently" subdivided 2-connected 3-regular graphs have minimally 2-tough squares. In this section, we will further specify what "sufficiently" subdivided means.

To prove that the square of a 4-subdivision of a 2-connected 3-regular graph is minimally 2-tough, we first prove that if H is a 2-connected 3-subdivision then $\tau(H^2) = 2$.

Lemma 11. *Let H be a 3-subdivision of a 2-connected graph and let $G = H^2$. Then, $\tau(G) = 2$.*

Proof. From [8] we know that the square of every 2-connected graph is 2-tough, which means that $\tau(G) \geq 2$. Suppose that $H = C_n$. Then, every vertex in G has degree 4; thus, $\tau(G) \leq 2$. So suppose that $H \neq C_n$. Now, label the vertices of G with degree exceeding 2 in H as $v_1, v_2, \ldots, v_k$. Construct S such that it contains all the vertices v_i, and for every v_i, $d_H(v_i) - 1$ vertices which are neighbors of v_i in H. Here, $d_H(v_i)$ denotes the degree of v_i in H. Then, $|S| = \sum_{i=1}^{k} d_H(v_i)$. By deleting S from G, there is a contribution of one to $\omega(G - S)$ for every two vertices v_i, v_j of G which are connected by a path containing only internal vertices of degree 2 in H. So, $\omega(G - S) = \frac{1}{2}\sum_{i=1}^{k} d_H(v_i)$ and $\tau(G) \leq |S|/[\omega(G - S)] = 2$. ■

Theorem 12. *Let H be a 4-subdivision of a 2-connected 3-regular graph and let $G = H^2$. Then, G is minimally 2-tough.*

Proof. From Lemma 11, we know that $\tau(G) = 2$, so it is sufficient to prove that $\tau(G - e) < 2$ for an arbitrary edge $e \in E(G)$. This is clear if e is incident to a vertex of degree 4 in G. Construct S as is Lemma 11, and consider the three remaining possibilities for e: First, assume that e is incident to two vertices $w_1 \in S$ and $w_2 \in S$ of degree 5. Denote the common neighbor of w_1 and w_2 with degree 5 by w_3, and define $S' = (S \cup \{w_3\})\backslash\{w_1, w_2\}$. Then, $|S'| = |S| - 1$ and $\omega((G - e) - S') = \omega(G - S)$. So, $\tau(G - e) \leq |S'|/[\omega((G - e) - S')] < |S|/[\omega(G - S)] = 2$. Next, assume that e is incident to two vertices $w_1 \in S$ and $w_2 \notin S$ of degree 5. Define $S' = S\backslash\{w_1\}$. As in the previous case, $\tau(G - e) < 2$. Finally, assume that e is incident to a vertex w_1 of degree 6 and a vertex w_2 of degree 5. Denote by P the path in H containing e between w_1 and the next vertex x of degree 3 in H. Note that we may choose S in such a way that the predecessor of x on P is not in S. Define $S' = (S \cup N_G(w_1) \cup N_G(w_2))\backslash\{w_1, w_2\}$. Then, $\tau(G - e) \leq |S'|/[\omega((G - e) - S')] = (|S| + 3)/[\omega(G - S) + 2] < 2$. ■

We will now prove that if two neighbors u and v with $d(u) + d(v) \geq 6$ exist in a 2-connected graph H then H^2 is not minimally 2-tough. To prove that, we need the following lemma. A graph G is *minimally 2-connected* if $\kappa(G) = 2$ and $\kappa(H) < 2$ for every proper spanning subgraph of G:

Lemma 13. *Let $G = H^2$ be a minimally 2-tough graph. Then, H is minimally 2-connected and triangle-free.*

Proof. Suppose that H is not minimally 2-connected. Then, there exists a proper spanning subgraph F of H which is 2-connected. But, then, F^2 is a proper spanning subgraph of G and F^2 is 2-tough, a contradiction. Next, suppose that H has a triangle $uvwu$. Because H is minimally 2-connected, w is a cut vertex of $H - uv$. Denote the components of $(H - uv) - w$ as H_1 (the component which contains u) and H_2 (the component which contains v). Notice that there are no other components of $(H - uv) - w$; otherwise, $H - w$ would be disconnected. By similar arguments, v is a cut vertex of $H - uw$ but not of H. Because H_1 is connected, there are no edges between w and H_1 other than uw; otherwise, $H - uw$ would not have v as a cut vertex. Analogously, there are no edges between H_2 and w other than vw. Now, H_1 can only contain u and H_2 can only contain v; otherwise, u or v would be a cut vertex of H. So, $H = K_3$, a contradiction, since, then, $\tau(G) \neq 2$. ■

Theorem 14. *Let H be a 2-connected graph with $d_H(u) + d_H(v) \geq 6$ for some, u, $v \in V(H)$ with $uv \in E(H)$, and let $G = H^2$. Then, G is not minimally 2-tough.*

Proof. Suppose that G is minimally 2-tough. Then, $\tau(G - e) < 2$ for every $e \in E(G)$, so this also holds if we let $e = uv$. Hence, there exists a vertex cut $S \subset V(G - e)$ such that $\omega((G - e) - S) > |S|/2$. Since ω is an integer, $\omega((G - e) - S) \geq \lceil(|S| + 1)/2\rceil = \lfloor|S|/2\rfloor + 1$. First, suppose that S is a vertex cut of G. Since G is 2-tough, we have $\omega(G - S) \leq \lfloor|S|/2\rfloor$, which implies that $\omega((G - e) - S) = \lfloor|S|/2\rfloor + 1$ and $\omega(G - S) = \lfloor|S|/2\rfloor$. Two vertices of two distinct components of $G - S$ have no common neighbors in S in H; otherwise, in G, there would be an edge between these components. Since H is 2-connected, every component other than the component containing e has at least two distinct neighbors in S in H. Now, consider the number of neighbors in S in H of the component of $G - S$ containing e. Since $d_H(u) + d_H(v) \geq 6$ by assumption and since u and v cannot have neighbors in H in the components of $(G - e) - S$, except by the edge e, $|N_H(u) \cap S| + |N_H(v) \cap S| \geq 4$. If $|N_H(u) \cap N_H(v) \cap S| \neq 0$, then H contains a triangle. This is impossible by Lemma 13. Hence, the component of $G - S$ containing e has at least four distinct neighbors in S in H. We conclude that $|S| \geq 2(\lfloor|S|/2\rfloor - 1) + 4 = 2(\lfloor|S|/2\rfloor + 1) \geq 2\{[(|S| - 1)/2] + 1\} = |S| + 1$, a contradiction.

Next suppose that S is not a vertex cut of G. Then, $2 = \omega((G - e) - S) > |S|/2$, implying that $|S| \leq 3$. But for the same reasons as before, u and v have together at least four distinct neighbors in S in H, a contradiction. ■

So, squares of 4-subdivisions of 3-regular 2-connected graphs are minimally 2-tough whereas squares of 2-con-

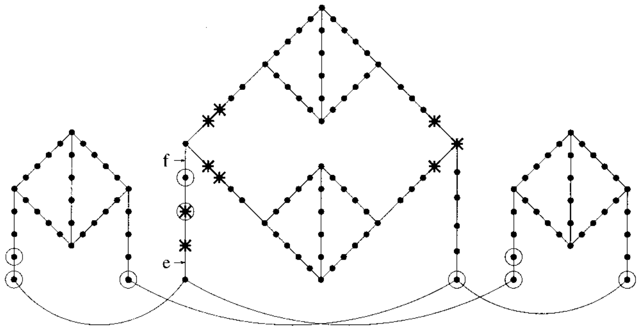

Fig. 1. The square of this graph is minimally 2-tough.

nected graphs with at least one vertex of degree exceeding 3 are 2-tough, but not minimally 2-tough. The same is true for squares of 2-connected graphs which have two adjacent vertices with degree sum 6 or more. What can we say about squares of s-subdivisions with $s \leq 3$? There are examples of squares of 3-subdivisions which are minimally 2-tough and examples of squares of 3-subdivisions which are not minimally 2-tough. If we take the square of the 3-subdivision drawn in Figure 1, the resulting graph G is minimally 2-tough.

To prove this, it is sufficient to show that deletion of the marked edge e or f decreases the toughness. All other edges are incident to a vertex of degree 4 or can be proved to decrease the toughness in either the same way or as in the proof of Theorem 12. $\tau(G - e) \leq \frac{9}{5} < 2$ due to the vertex cut marked with circles in the figure and $\tau(G - f) \leq \frac{9}{5} < 2$ due to the vertex cut marked with stars in the figure.

The example that we give of a 3-subdivision whose square is not minimally 2-tough is less spectacular. Let H be the graph obtained from K_4 by replacing each edge by a path with exactly three internal vertices and let $G = H^2$. Clearly, G is 2-tough. But $G - e$ is also 2-tough, where e is an edge of H incident to a vertex of degree 3 in H (we omit further details).

The above examples illustrate the difficulties in obtaining a full classification of all minimal 2-tough squares of 3-subdivisions.

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