INTERSECTION REPRESENTATION OF DIGRAPHS IN TREES WITH FEW LEAVES

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Abstract. The *leafage* of a digraph is the minimum number of leaves in a host tree in which it has a subtree intersection representation. We discuss bounds on the leafage in terms of other parameters (including Ferrers dimension), obtaining a string of sharp inequalities.

1. INTRODUCTION

An intersection representation of a digraph D assigns an ordered pair (S_v, T_v) to each vertex $v \in V(D)$ such that $uv \in E(D)$ if and only if $S_u \cap T_v \neq \emptyset$. We call S_v and T_v the source set and sink set of v. This model was first described by Beineke and Zamfirescu [1] under the name connection digraph. An essentially equivalent model in terms of bipartite graphs was introduced by Harary, Kabell, and McMorris [7].

When each set in an intersection representation is a subtree of a fixed host tree, we have a subtree representation. Every *n*-vertex digraph has a subtree representation in a star with *n* leaves. Not every digraph has a subtree representation in a path; those that do are the interval digraphs, which are characterized in [15,16]. We define the *leafage* l(D) of a digraph *D* to be the minimum number of leaves in a host tree in which *D* has a subtree representation. Thus leafage is a measure of distance from an interval digraph, and the subtree representations in stars show that $l(D) \leq n(D)$. An analogous parameter

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for chordal (undirected) graphs is studied in [8]. Further results about adjacency matrices of interval digraphs appear in [9,10,15,16,17].

We obtain lower bounds on leafage using the idea of Ferrers dimension. The successors of a vertex v are $\{u \in V(D): vu \in E(D)\}$; the predecessors are $\{u \in V(D): uv \in E(D)\}$. A digraph is a Ferrers digraph [14] if its successor sets are linearly ordered by inclusion, which is equivalent to the adjacency matrix A(D) having no 2 by 2 permutation submatrix. Viewing a digraph as a relation $D \subseteq V(D) \times V(D)$, the Ferrers dimension f(D) of D is the minimum number of Ferrers digraphs on V(D) whose intersection is D (introduced in [2]). Since the complement in $V \times V$ of a Ferrers digraph is also a Ferrers digraph, this also equals the minimum number of Ferrers digraphs whose union is \overline{D} .

Interval digraphs all have Ferrers dimension at most 2; a digraph D is an interval digraph if and only if \overline{D} is the union of two disjoint Ferrers digraphs [15]. This generalizes to a lower bound on l(D) using Ferrers digraphs. Let $f^*(D)$ denote the minimum number of pairwise disjoint Ferrers digraphs whose union is \overline{D} . These are Ferrers digraphs whose intersection is D and whose pairwise unions are $V(D) \times V(D)$. Having imposed an extra condition on the minimization, we have $f^*(D) \ge f(D)$; we prove that $l(D) \ge f^*(D)$.

On the upper side, we study the related *catch leafage* $l^*(D)$ of a digraph D. This is the minimum number of leaves in a host tree in which D has a subtree representation such that each sink subtree is a single vertex. (Such representations, particularly when the host tree is a path, are studied in [12,13,15].) This condition restricts the allowable representations, so $l^*(D) \ge l(D)$. We prove that $l^*(D) \le w(P(D))$, where w(P(D)) is the width of the inclusion poset P(D) on the sets whose incidence vectors are the columns of the adjacency matrix A(D). We also give a sufficient condition for equality in this bound.

We thus obtain the chain of inequalities

$$f(D) \le f^*(D) \le l(D) \le l^*(D) \le w(P(D)) \le n(D).$$

We present examples to show that each inequality is best possible. We also present examples to show that each bound is arbitrarily weak, as any one of these parameters can be at most 3 when the next parameter is arbitrarily large.

The upper bound w(P(D)) is easily computable, but the lower bounds are not. Cogis [2] and Doignon, Ducamp, and Falmagne [4] proved an easily testable characterization of the digraphs with Ferrers dimension at most 2, but Yannakakis [18] proved that recognition of Ferrers dimension 3 is NP-complete. Müller [11] found a polynomial-time recognition algorithm for interval digraphs (leafage 2). Other than this, we do not know the complexity of recognizing digraphs with bounded values for any of $\{f^*(D), l(D), l^*(D)\}$.

2. SUBTREE REPRESENTATIONS AND LEAFAGE

We use $u \to v$ to denote the successor relation; $u \to v$ means "uv is an edge". A branch point of a tree is a vertex of degree at least 3. We show first that leafage is well-defined.

THEOREM 1. If D is a digraph with n vertices, then D has a subtree representation in a star with at most n leaves.

Proof: In a star H with n leaves, assign distinct leaves as sink sets for the n vertices. For each $v \in V(D)$, let S_v be the star induced by the center of H and the leaves corresponding to the successors of v. Then $u \to v$ if and only if $T_v \subseteq S_u$, and hence this is a representation.

The bound $l(D) \leq n(D)$ is sharp, as it holds with equality for the digraph D_n in Theorem 2. Our tool for proving lower bounds on l(D) is a property of subtrees of a tree. If T_i, T_j, T_k are subtrees of a tree, then we say that T_k is between T_i and T_j if $T_i \cap T_j = \emptyset$ and the unique path from T_i to T_j contains a vertex of T_k (possibly at the start or end). A collection of pairwise disjoint subtrees having the property that none is between two others is an asteroidal collection of subtrees.

LEMMA 1. If T_1, \ldots, T_n is an asteroidal collection of subtrees of a tree T, then T has at least n leaves.

Proof: We may assume that the path from any leaf of T to the nearest branch point contains a vertex of some T_i ; otherwise, we could delete the vertices before the branch point to reduce the number of leaves without changing the hypotheses. For each leaf v of T, we assign to v the first subtree encountered on the path from v to its nearest branch point. If T has fewer than n leaves, then some subtree T_k in our list is not assigned to any leaf. Let x be a vertex of T_k , and let P be a maximal path containing x. The endpoints of P are leaves of the tree, and T_k is between the subtrees assigned to those leaves.

LEMMA 2. If v, w have a common successor u that is not a successor of z in a digraph D, then S_z is not between S_v and S_w in any subtree representation of D. Similarly, if v, w have a common predecessor u that is not a predecessor of z in D, then T_z is not between T_v and T_w in any subtree representation of D.

Proof: If S_z is between S_v and S_w , then $S_v \cap S_w = \emptyset$, and T_u must contain the unique path from S_v to S_w in the host. This contradicts $S_z \cap T_u = \emptyset$, since S_z has a vertex on this path. The proof of the other statement is similar.

Subtrees of a tree satisfy the *Helly property*; the members of a pairwise intersecting family of (sub)trees have a common vertex (see, for example, [6, p. 92]).

LEMMA 3. If in a subtree representation of D the source subtrees are pairwise intersecting and the sink subtrees are pairwise intersecting, then A(D) has a row of 1's or a column of 0's, and similarly A(D) has a column of 1's or a row of 0's.

Proof: In such a representation, the source subtrees have a common vertex, and the sink subtrees have a common vertex. Let s, t denote these vertices, respectively. If s = t, then A(D) is all 1's and the claim holds. If $s \neq t$, let x be the vertex of $\bigcup S_i$ that is closest to t on the unique s, t-path in T. Suppose $x \in S_k$. If A(D) has no row of 1's, then $x \neq t$ and some sink subtree T_j fails to contain x. However, $t \in T_j$, and hence T_j intersects no source subtree, forcing a column of 0's in A(D). The other claim follows by considering the vertex of $\bigcup T_i$ that is closest to s on the s, t-path in T.

Because permuting rows or columns is simply a relabeling of source or sink sets, leafage can be viewed as a property of a 0,1-matrix (the adjacency matrix A(D)) rather than a property of a digraph. We next show that asteroidal collections are forced by complements of permutation matrices.

THEOREM 2. For $n \ge 3$, let D_n be the digraph such that $A(D_n) = J - I$, where J is the matrix of all 1's and I is the identity matrix. In every subtree representation of D_n , either the source subtrees have a common vertex and the sink subtrees form an asteroidal collection, or the sink subtrees have a common vertex and the source subtrees form an asteroidal collection.

Proof: Let the vertices of D_n be $\{1, \ldots, n\}$; we have $i \to j$ if and only if $i \neq j$. Consider a subtree representation of D_n . By Lemma 3, the source subtrees and sink subtrees cannot both be pairwise intersecting; we may assume by symmetry that there is a disjoint pair of source subtrees.

When i, j, k are distinct vertices, we have $i \to k, j \to k$, and $k \neq k$. Thus Lemma 2 implies that no source subtree can be between two other source subtrees. With betweenness forbidden, S_i and S_k cannot intersect if $S_i \cap S_j = \emptyset$. We conclude that if some pair of source subtrees is disjoint, then the source subtrees are pairwise disjoint, and none is between two others. Hence they form an asteroidal collection.

With the source subtrees pairwise disjoint, consider the sink subtrees. For any distinct vertices i, j, k, we must have T_j containing the path from S_i to S_k and T_i containing the path from S_j to S_k . Hence $T_i \cap T_j \neq \emptyset$, and the sink subtrees are pairwise intersecting. The Helly property then implies that the sink subtrees have a common vertex.

Together, Lemma 1 and Theorem 2 imply that $l(D_n) = n$.

3. LEAFAGE AND DISJOINT FERRERS DIMENSION

We next prove our main lower bound on leafage. We use $N_D^+(u)$ to denote the successor set and $N_D^-(u)$ to denote the predecessor set of a vertex u in a digraph D.

THEOREM 3. If D is a digraph, then $l(D) \ge f^*(D)$.

Proof: Suppose that l(D) = k, and let $\{(S_v, T_v): v \in V(D)\}$ be a representation of D in a host tree with k leaves. When k = 2, the result follows from the characterization of interval digraphs in [15]. For $k \ge 3$, we construct k pairwise disjoint Ferrers digraphs whose union is \overline{D} . With the host tree T embedded in the plane, let the leaves be x_1, \ldots, x_n in clockwise order around the tree. Let P_i denote the path in T from x_i to x_{i+1} , indexed cyclically.

For each leaf x_i of the host tree T, we construct an associated Ferrers digraph D(i). The edges of \overline{D} consist of those pairs uv such that $S_u \cap T_v = \emptyset$, meaning that the unique shortest path from S_u to T_v has length at least 1. Let D(i) consist of those edges uv in \overline{D} such that the first edge on the path from S_u to T_v lies on P_i , with S_u between x_i and T_v (see Fig. 1). If S_u has no vertex on P_i , then u has no successors in D(i). If the last vertex of $S_{u'}$ on P_i is closer to x_{i+1} than the last vertex of S_u on P_i , then $N_{D(i)}^+(u') \subseteq N_{D(i)}^+(u)$, by construction. Hence the D(i)'s are Ferrers digraphs.

The paths P_i together cover each edge of the host tree exactly once in each direction. Since each edge is covered in each direction, $\bigcup_i D(i) = \overline{D}$. Since each edge is covered only once, and when $S_u \cap T_v = \emptyset$ there is a unique first edge on the path from S_u to T_v , the subgraphs $\{D(i)\}$ are pairwise disjoint.



Fig. 1. Ferrers digraphs from subtree representation

This provides another proof that the leafage of the digraph D_n is n. Since each pair of ones on the diagonal of $A(\overline{D}_n)$ induce a 2 by 2 permutation submatrix, no pair of them can be covered by a single Ferrers digraph contained in \overline{D}_n .

Although the inequalities $f(D) \leq f^*(D) \leq l(D) \leq n(D)$ are best possible, with equality throughout when $D = D_n$, the gaps can be arbitrarily large. For an interval digraph, $f(D) = f^*(D) = l(D) = 2$. By the characterization of interval digraphs in [15], $f^*(D) = 2$ implies l(D) = 2. Nevertheless, there exist digraphs D with $f^*(D) = 3$ and l(D) = n(D).

THEOREM 4. Leafage is not bounded by any function of f^* when $f^* \ge 3$. In particular, let E_n be the *n*-vertex digraph with $A(E_n) = {I \choose Y^T 0}$, where *I* denotes the n-1 by n-1 identity matrix and Y denotes a column vector of n-1 ones. If $n \ge 3$, then $l(E_n) = n$, but $f^*(E_n) = f(E_n) = 3$.

Proof: Because the last three rows and columns of $A(E_n)$ form a row permutation of $A(D_3)$, we have $f^*(E_n) \ge f(E_n) \ge 3$. For equality, partition the zeros of $A(\overline{E}_n)$ into three sets; those in the upper right of the submatrix I, those in the lower left of the submatrix I, and the 0 in the lower right corner. These sets yield Ferrers digraphs, so $f^*(E_n) \le 3$.

To show that $l(E_n) = n$, we name the vertices by the row and column indices of the matrix and let $\{(S_i, T_i): 1 \leq i \leq n\}$ be a subtree representation of E_n in the host tree T. By Lemma 1, it suffices to show that the source subtrees or the sink subtrees form an asteroidal collection in T.

We have $S_n \cap T_n = \emptyset$; let P be the unique path from S_n to T_n in T. For each k < n, we have $S_k \cap T_n \neq \emptyset$, $S_n \cap T_k \neq \emptyset$, and $S_k \cap T_k \neq \emptyset$. Consider also i < n. If P contains a vertex x of $S_i \cap T_i$, then the nonadjacency of i and k implies that x separates S_k and T_k . This contradicts $S_k \cap T_k \neq \emptyset$, so S_i cannot intersect T_i in P. We conclude that $S_i \cap T_i$ is contained in the component of T - E(P) containing T_n or in the component of T - E(P)containing S_n . By symmetry, we may assume the latter. Since $n \to i$, we now have $P \subset T_i$. Applying this argument for all vertices other than n yields that all $S_i \cap T_i$ lie in the same component of T - E(P), since there are no edges except loops among these vertices. Thus $P \subset T_i$ and $P \cap S_i = \emptyset$ for all i < n.

Now consider disjointness and betweenness of the source subtrees. Since $i \to i, n \to i$, and $j \not\to i$, Lemma 2 forbids S_j between S_i and S_n for i, j < n. Since P separates S_n from the others, this implies that the source subtrees are pairwise disjoint. Furthermore, if S_j is between S_i and S_k for i, j, k < n, then the union of the paths from S_n to the trees S_i and S_k must intersect S_j , which puts S_j between S_n and one of $\{S_i, S_k\}$. Hence the source subtrees are pairwise disjoint, and none is between two others. They form an asteroidal collection, and Lemma 1 applies.

Every n by n (adjacency) matrix with leafage n is a minimal forbidden submatrix for leafage less than n. We next present another such family. Given the adjacency matrix A(D) of a digraph D, let H(D) be the graph with vertices corresponding to the zeros of A(D) and edges corresponding to the pairs of zeros contains in a 2 by 2 permutation submatrix. Cogis [2] and Doignon-Ducamp-Falmagne [4] proved that D has Ferrers dimension 2 if and only if H(D) is bipartite; here we need only the obvious necessity of the condition.

THEOREM 5. Let C_n be the digraph consisting of a directed cycle of length n plus a loop at each vertex. Then $l(C_n) = n$, but $f(C_n) = f^*(C_n) = 3$.

Proof: Assume that the cycle is $1 \to 2 \to \cdots \to n \to 1$. Partition the zeros of $A(C_n)$ into three sets: those in the last row, those in the first n-1 rows below the diagonal, and the remainder. These sets form Ferrers digraphs, so $f^*(C_n) \leq 3$. To prove that $f(C_n) > 2$, we observe that the positions

$$\{(i, i + \lceil n/2 \rceil): 1 \le i \le \lfloor n/2 \rfloor\} \cup \{(i, i + 1 - \lceil n/2 \rceil): \lceil n/2 \rceil \le i \le n\}$$

form an odd cycle in $H(C_n)$.

We use induction on n to prove that $l(C_n) = n$. The claim holds for n = 3 because $A(C_3)$ is a permutation of $A(D_3)$. For n > 3, let **T** be the host tree for an optimal representation of C_n . Suppose first that $S_{i-1} \cap S_i \neq \emptyset$ for some *i* (all indexing is circular). The subtree T_i must intersect both of these, so by the Helly property S_{i-1}, T_i, S_i have a common vertex x in **T**. No other source subtree intersects T_i , and no other sink subtree intersects S_{i-1} and S_i ; hence no other assigned subtree contains x. Every two consecutive subtrees in the list $T_{i+1}, S_{i+1}, T_{i+2}, \ldots, S_{i-2}, T_{i-1}$ intersect; hence their union is connected and contained in one component of $\mathbf{T} - x$. The remaining components of $\mathbf{T} - x$ can be deleted without changing the intersection digraph, so we may assume that x is a leaf.

Let P be the path in **T** from x to the nearest branch point. By symmetry, we may assume that S_{i-1} contains as much of P as S_i . If S_i does not contain all of P, then T_{i+1} intersects S_{i-1} , which is forbidden. Hence $P \subseteq S_i \cap S_{i-1}$, and no sink subtree other than T_i intersects P. If another source subtree extends onto P, then deleting its edges on P does not change the intersection digraph. We can now delete T_i and replace S_{i-1}, S_i by a single source subtree with edge set $(E(S_{i-1}) \cup E(S_i)) - E(P)$ to obtain a representation of C_{n-1} with $l(C_n) - 1$ leaves. By the induction hypothesis, this yields $l(C_n) \ge n$.

Hence we may assume that $S_{i-1} \cap S_i = \emptyset$ for all *i*, and by symmetry also $T_{i-1} \cap T_i = \emptyset$ for all *i*. In this case, let P_i be the portion of S_i that is the unique T_i, T_{i+1} -path, and let Q_i be the portion of T_i that is the unique S_{i-1}, S_i -path. Note that $Q_i \cap P_i$ and $P_i \cap Q_{i+1}$ are single vertices. The union of all these paths is thus a closed walk in which no consecutive edges are the same. Such a walk contains a cycle, which is impossible in a host tree. Hence this case does not arise.

We have presented examples with fixed $f^*(D)$ and large l(D). Also $f^*(D)$ may be arbitrarily large when f(D) = 2. We construct a two-parameter family of adjacency matrices. The matrix $M_{k,m}$ is a km by km matrix consisting of k rows and k columns of mby m blocks. The diagonal blocks are the identity matrix, the blocks below the diagonal consist entirely of 0's, and the blocks above the diagonal consist entirely of 1's. The zeros can be covered by two Ferrers digraphs, each consisting of all the subdiagonal blocks and half of each diagonal block; hence $f(M_{k,m}) = 2$. We will prove that $f^*(M_{k,m}) = c + 1$ when $k = 1 + {c \choose 2}$ and m is sufficiently large. (In this discussion we use the notation $M_{k,m}$ for both the digraph and its adjacency matrix.)

Let I_n denote the *n*-vertex digraph whose adjacency matrix is the identity. A partition of \overline{I}_n into *c* Ferrers digraphs can be viewed as a special *c*-coloring of the 0's in the *n* by *n* identity matrix I_n . We say that colors *A*, *B* are a *crossed pair* if *A*, *B* appear together in some row and appear together in some column.

LEMMA 4. If $n \ge 3c!/2$, then every partition of \bar{I}_n into c Ferrers digraphs has a crossed pair of colors.

Proof: The proof is by induction on c. For c = 2, a 2-coloring of the 0's in the 3 by 3 identity matrix cannot have all rows or all columns monochromatic without having a 2 by 2 permutation matrix with 0's in one color. For c > 2, let n = 3c!/2 and r = 3(c-1)!/2. Consider a partition of \bar{I}_n into c Ferrers digraphs, and suppose that the corresponding coloring has no crossed pair.

Since each row of the identity matrix has n-1 0's, the pigeonhole principle implies that each row has at least $\lceil (3(c-1)!/2)(c/c) - 1/c \rceil = r$ 0's in some color. By symmetry, we may assume there are 0's of color A in the first r columns of row r+1 (see Fig. 2). Let D be the subdigraph induced by the first r vertices, with K the corresponding submatrix. By the induction hypothesis, every partition of \overline{D} into c-1 Ferrers digraphs yields a coloring of the 0's in K with a crossed pair of colors. Hence we may assume that all c colors (including A) appear in K.

Let *i* be the index of a row in *K* in which color *A* appears. If another color appears in row *i* of *K*, then it crosses *A* in the full matrix. Thus we may assume that row *i* of *K* has only color *A*. Now, to avoid the forbidden submatrix in color *A*, position i, r + 1 must have some other color *B*. Now colors *A* and *B* appear in a row together, so they cannot appear in a column together. This contradicts the observation that every color, including B, appears in K.

$$K \begin{bmatrix} 1 & & & \\ & 1 & & \\ & A & A & A & A & A \\ & & & 1 & & \\ & & & 1 & & \\ & & & 1 & & \\ & & & 1 & & \\ & & & 1 & & \\ & & & 1 & & \\ & & & 1 & & \\ & & & A & A & A & A & 1 \end{bmatrix}$$

Fig. 2. Coloring 0's in an identity matrix.

The bound 3c!/2 in Lemma 4 is not best possible. For c = 2, 3, 4, the bound is 3, 9, 36, but the actual minimum values forcing the desired behavior are 3, 4, 6. We are content with the bound arising from the short argument in Lemma 4 because our aim is to show that $f^*(M_{k,m})$ grows arbitrarily large.

THEOREM 6. If $k \ge 1 + \binom{c}{2}$ and $m \ge 3c!/2$, then $f^*(M_{k,m}) > c$.

Proof: Suppose $\overline{M}_{k,m}$ has a partition into c pairwise-disjoint Ferrers digraphs. By Lemma 4, in each copy of I_m in the block structure of $M_{k,m}$, the corresponding coloring has a crossed pair of colors. Since there are more than $\binom{c}{2}$ diagonal blocks, by the pigeonhole principle some pair of colors A, B is crossed twice.

Let r, s be the indices of the diagonal blocks where A, B are crossed, with r < s. Let j be the column within diagonal block r where A, B both appear, and let i be the row within diagonal block s where A, B both appear. Position i, j of block s, r is now forced to have both color A and color B to avoid the forbidden substructure for the Ferrers digraphs given by colors A and B. This is impossible.

It is worth noting that $f^*(M_{k,m}) \leq c$ for all m when $k \leq \binom{c}{2}$. This is illustrated by the block coloring in Fig. 3.

A B	1	1	1	1	1	1	1	1	1
A	$A \backslash C$	1	1	1	1	1	1	1	1
B	C	$B \backslash C$	1	1	1	1	1	1	1
A	A	A	$A \backslash D$	1	1	1	1	1	1
B	B	B	D	$B \backslash D$	1	1	1	1	1
C	C	C	D	D	$C \backslash D$	1	1	1	1
A	A	A	A	A	A	$A \backslash E$	1	1	1
B	B	B	B	B	B	E	$B \backslash E$	1	1
C	C	C	C	C	C	E	E	$C \backslash E$	1
$\setminus D$	D	D	D	D	D	E	E	E	$D \setminus E /$

Fig. 3. A 5-coloring of the 0's in the blocks of $M_{10,l}$.

We previously gave examples with f(D) = 3, $f^*(D) = 3$, and l(D) large. We next prove that the family $M_{k,m}$ includes examples with f(D) = 2, $f^*(D) = 3$, and l(D) large.

THEOREM 7. For $m \ge 3$, $M_{2,m}$ is a 2*m*-vertex digraph with $f(M_{2,m}) = 2$, $f^*(M_{2,m}) = 3$, and $l(M_{2,m}) = m$.

Proof: With four blocks of order m, $M_{2,m} = \begin{pmatrix} I & 1 \\ 0 & I \end{pmatrix}$. The value $f(M_{2,m}) = 2$ was obtained before Lemma 4. The lower bound on f^* comes from Theorem 6 (with c = 2), and the upper bound comes from the coloring illustrated in Fig. 3 (with c = 3).

To prove that $l(M_{2,m}) \leq m$, we construct a representation with m leaves. Let the host tree be the union of m paths q, s_i, t_i with q as a common endpoint. Let S_i and T_{i+m} be the entire *i*th path, for $1 \leq i \leq m$. Let S_{i+m} be the single vertex s_i , and let T_i be the single vertex t_i .

It remains to prove that $l(M_{2,m}) \geq m$. Consider a subtree representation with source subtrees S_1, \ldots, S_{2m} and sink subtrees T_1, \ldots, T_{2m} for the vertices indexed by the rows and columns of $M_{2,m}$ in order. If T_{m+1}, \ldots, T_{2m} have no common point, then some T_i, T_j among these are disjoint. Since T_i and T_j must intersect each of S_1, \ldots, S_m , those subtrees contain the T_i, T_j -path and hence have a common point. Similarly, if S_1, \ldots, S_m have no common point, then T_{m+1}, \ldots, T_{2m} must. By symmetry, we may assume that S_1, \ldots, S_m have a common point q.

We now show that T_1, \ldots, T_m is an asteroidal collection of subtrees. If $T_i \cap T_j \neq \emptyset$ with $i, j \leq m$, then the entire path from q to the closest vertex of $T_i \cap T_j$ belongs to at least one of $\{S_i, S_j\}$, which contradicts the requirement that each of $\{S_i, S_j\}$ intersects exactly one of $\{T_i, T_j\}$. If T_j is between T_i and T_k , then let P be the path between T_i and T_k , and let r be the vertex of P closest to q. Depending on the location of r relative to T_j on P, the q, T_k -path in S_k or the q, T_i -path in S_i intersects T_j , contradicting their disjointness from T_j . Thus T_1, \ldots, T_m is an asteroidal collection, and Lemma 1 implies that the host has at least m leaves.

4. CATCH LEAFAGE

If D has a subtree representation in which every sink subtree is a single vertex, then we say this is a *catch representation*, and D is a *catch-tree digraph*. In discussing catch representations, we say "sink point" instead of "sink subtree" to make the usage clear. If D has a catch-tree representation in which the host is a path, then D is a *catch-interval digraph*. The corresponding classes in which the source sets are single vertices are merely those whose adjacency matrices are the transposes of the digraphs in the classes defined above. Catch-interval digraphs are characterized in [12] under the name "interval catch digraphs" and in [15] under the name "interval-point digraphs".

The catch leafage $l^*(D)$ is the minimum number of leaves in a host tree in which D has a catch-tree representation; the catch-interval digraphs are the digraphs with catch-leafage 2. In the proof of Theorem 1, we gave every *n*-vertex digraph a catch representation in a star with *n* leaves, so catch leafage is well-defined. Since every catch-tree representation is a subtree representation, we have $n \ge l^*(D) \ge l(D)$. We may make several simplifying assumptions about the form of optimal catch-tree representations. In a catch-tree representation, sink subtrees can occupy the same vertex of the host if and only if the corresponding columns of the matrix are identical. We may split such a vertex of the host (without increasing the number of leaves), including the source subtrees to cover both. Thus we may assume that in catch representations each vertex is occupied by at most one sink point. Also, if a vertex of degree at most two in the host tree is not assigned as a sink point, then an edge incident to it can be contracted.

Recall that the predecessor set for v is $N^{-}(v) = \{u: u \to v\}$; this is the set whose incidence vector is the column of the adjacency matrix corresponding to v. Because the source sets occupy single vertices, a catch-tree representation can be described by listing, for each vertex of the host tree, the non-empty collection of source sets containing it. This will be a catch-tree representation if and only if 1) among these sets appear the predecessor sets, and 2) the set of vertices assigned to each source set forms a subtree of the host.

Therefore, our analysis of catch leafage focuses on the columns of the adjacency matrix as incidence vectors for the predecessor sets. We define an associated partial order. Let P(D), the *incidence poset* of the digraph D, be the collection of predecessor sets in D, ordered by inclusion. For simplicity, we will use the same notation V_j to refer to a predecessor set or the column of the adjacency matrix that is its incidence vector.

The width w(P) of a poset P is the maximum size of its antichains (collections of pairwise incomparable elements). Dilworth's Theorem [3] says that the elements of a finite poset P can be partitioned into w(P) disjoint chains.

THEOREM 8. The inequality $l^*(D) \leq w(P(D))$ holds for every digraph D with $w(P(D)) \geq 2$.

Proof: Let k = w(P(D)), and let C_1, \ldots, C_k be a partition of P(D) into k disjoint chains. Let the host tree T be a subdivision of a star with k leaves. That is, T consists of a central point of degree k from which k paths emerge. Assign the central vertex the set of all predecessors, and assign to each emerging path the sets on one of the chains C_i , in decreasing order. The predecessor sets all appear at vertices, and the occurrences of each predecessor form a subtree, so this is a catch-tree representation.

Fulkerson [5] observed that Dilworth's Theorem is equivalent to the König-Egerváry Theorem on matchings in bipartite graphs. Thus bipartite matching or other algorithms can be used to compute w(P(D)). Nevertheless, this is only a bound on $l^*(D)$, and this bound also can be arbitrarily bad. The digraph D consisting of a directed path plus a loop at each vertex has catch leafage 2 but w(P(D)) = n - 1, so w(P(D)) is not bounded by any function of $l^*(D)$.

Note that w(P(D)) = 1 when D is a Ferrers digraph. Thus Theorem 8 requires $w(P(D)) \ge 2$, and we see that the break between w(P(D)) and n(D) can be large.

We now have the chain of inequalities

 $f(D) \le f^*(D) \le l(D) \le l^*(D) \le w(P(D) \le n(D).$

One may have equality throughout (achieved by D_n). To prove that there can be arbitrarily bad breaks between any pair, it suffices to produce examples where l(D) is bounded and $l^*(D)$ is large. To do this, we prove a sufficient condition for $l^*(D) = w(P(D))$. **THEOREM 9.** If D is a digraph such that P(D) has a unique maximal element and $w(P(D)) \ge 2$, then $l^*(D) = w(P(D))$.

Proof: Let V_0 be the unique maximal element, and let $A = V_1, \ldots, V_k$ denote a maximum antichain in P(D). Let q_i denote the vertex of the host assigned to V_i in an optimal catch representation. Iteratively delete leaves of the host tree that are not in $\{q_i\}$ until all remaining leaves belong to $\{q_i\}$. If the number of leaves (other than q_0) is less than k, then some set V_i in A is assigned to a non-leaf q_i . Every path from q_0 to another remaining vertex can be extended to reach a remaining leaf. In particular, the path from q_0 to q_i belongs to a path from q_0 to a leaf assigned q_j . Since $V_j \subseteq V_0$ and each predecessor is assigned to the vertices of a tree, this entire path including q_i belongs to the source subtrees for V_j . This yields $V_j \subseteq V_i$, contradicting the choice of A as an antichain.

THEOREM 10. Catch leafage is not bounded by any function of leafage. If F_n denotes the *n*-vertex digraph whose adjacency matrix is $\binom{I}{Y^T} \binom{Y}{1}$, where *I* denotes the n-1 by n-1 identity matrix and Y denotes a column vector of n-1 ones, then $f^*(F_n) = l(F_n) = 2$, but $l^*(F_n) = n-1$.

Proof: The upper left and lower right zeros in the portion I of the adjacency matrix yield two disjoint Ferrers digraphs whose union is \overline{F}_n . As proved in [15], this is equivalent to leafage 2. On the other hand, the predecessor set of the last vertex contains all the other predecessor sets, so $l^*(F_n) = w(P(F_n)) = n - 1$.

The sufficient condition in Theorem 9 does not characterize equality in $l^*(D) \leq w(P(D))$. For the digraph C_n consisting of a directed cycle plus loops, we have seen that $l(C_n) = n$. Also the columns of $A(C_n)$ form an antichain, so $l(C_n) = l^*(C_n) = w(P(C_n)) = n$.

This example shows also that leafage and catch leafage can drop arbitrarily much when a single vertex is deleted. Deleting one vertex from a cycle with loops leaves a path with loops. The former has leafage and catch leafage n; the latter has leafage and catch leafage 2.

Our proof of $l^*(D) \leq w(P(D))$ shows that every digraph has a catch representation in a host tree having only one branch point, and if P(D) has a unique maximum this can be achieved in a host tree with the minimum number of leaves. This is not true of all digraphs. The digraph D with adjacency matrix below contains C_4 and thus has catch leafage at least 4. However, every catch representation of D in a host tree with four leaves has two branch points. We thus close by mentioning two further optimization problems for digraphs with catch leafage k: Among catch representations in trees with k leaves, what is the minimum number of branch points, and what is the minimum number of vertices?

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

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