Very weak zero one law for random graphs with order and random binary functions

Saharon Shelah*

Institute of Mathematics, The Hebrew University Department of Mathematics, Rutgers University Department of Mathematics, University of WI, Madison

> done: August 1993, October 31, 1993 last corrections introduced June 5, 1996 printed: July 4, 2018

^{*} The research partially supported by the United States-Israel Binational Science Foundation and NSF under grant #144-EF67; Publication no 548.

Very weak zero one law..., Sh 548

0 Introduction

Let $G_{<}(n, p)$ denote the usual random graph G(n, p) on a totally ordered set of *n* vertices. (We naturally think of the vertex set as $1, \ldots, n$ with the usual <). We will fix $p = \frac{1}{2}$ for definiteness. Let $L^{<}$ denote the first order language with predicates equality (x = y), adjacency $(x \sim y)$ and less than (x < y). For any sentence A in $L^{<}$ let $f(n) = f_A(n)$ denote the probability that the random $G_{<}(n, p)$ has property A. It is known Compton, Henson and Shelah [CHSh245] that there are A for which f(n) does not converge. Here we show what is called a *very weak zero-one law* (from [Sh 463]):

Theorem 0.1 For every A in language $L^{<}$

$$\lim_{n \to \infty} (f_A(n+1) - f_A(n)) = 0$$

Note, as an extreme example, that this implies the nonexistence of a sentence A holding with probability 1 - o(1) when n is even and with probability o(1) when n is odd (as in Kaufman, Shelah [KfSh201]).

In §2 we give the proof, based on a circuit complexity result. In §3 we prove that result, which is very close to the now classic theorem that parity cannot be given by an AC^0 circuit. In §4 we give a very weak zero-one law for random two-place functions. The proof is very similar, the random function theorem being perhaps of more interest to logicians, the random graph theorem to discrete mathematicians.

The reader should thank Joel Spencer who totally rewrote the paper (using the computer science jargon rather than the logicians one), and with some revisions up to the restatement in the proof of 2.1 but with 3.1, this is the version presented here. We thank the referee for comments on the exposition, and we thank Tomasz Luczak and Joel Spencer for reminding me this problem on $G_{\leq}(n, p)$ in summer 93.

On a work continuing this of Boppana and Spencer see 3.2(5).

1 The Proof.

Let G be a fixed graph on the ordered set $1, \ldots, 2n+1$. For a property A and for i = n, n+1 let $g(i) = g_{G,A}(i)$ denote the probability that $G \upharpoonright_S$ satisfies A where S is chosen uniformly from all subsets of $1, \ldots, 2n+1$ of size precisely *i*. We shall show Very weak zero one law..., Sh 548

Theorem 1.1 g(n+1) - g(n) = o(1)

More precisely, given A and $\epsilon > 0$ there exists n_0 so that for any G as above with $n \ge n_0$ we have $|g(n+1) - g(n)| < \epsilon$.

We first show that Thm.0.1 follows from Thm.1.1. The idea is that a random $G_{\leq}(i,p)$ on i = n or n + 1 vertices is created by first taking a random $G_{\leq}(2n+1,p)$ and then restricting to a random set S of size i. Thus (fixing A) f(n), f(n+1) are the averages of $g_G(n), g_G(n+1)$ over all G. By Thm.1.1 we have $g_{G,A}(n) - g_{G,A}(n+1) = o(1)$ for all G and therefore their averages are only o(1) apart.

Now we show Thm.1.1. Fix G, A as above. Let P(S) be the Boolean value of the statement that $G|_S$ satisfies A. For $1 \leq x \leq 2n + 1$ let z_x denote the Boolean value of " $x \in S$ " so that P(S) is a Boolean function of z_1, \ldots, z_{2n+1} . We claim this function has a particularly simple form. Any A can be built up from primitives $x = y, x < y, x \sim y$ by \wedge, \neg and, critically, \exists_x . As G is fixed the primitives have values true or false. Let \wedge, \neg be themselves. Consider $\exists_x W(x)$ where for each $1 \leq x \leq 2n + 1$ we let W(x) on $G|_S$ is given by $W^*(x)$. Then $\exists_x W(x)$ has the interpretation $\exists_{x \in S} W(x)$ which is expressed as $\bigvee_{x=1}^{2n+1} (z_x \wedge W^*(x))$. For convenience we can be redundant and replace $\forall_x W(x)$ by $\bigwedge_{x=1}^{2n+1} (z_x \Rightarrow W^*(x))$. For example $\forall_x \exists_y x \sim y$ becomes

$$\wedge_x \left[z_x \Rightarrow \vee_{y \sim x} z_x \right]$$

Thus P(S) can be built up from z_1, \ldots, z_{2n+1} be means of the standard \neg, \land, \lor and \land, \lor over (at most) 2n + 1 inputs. That is (see §3) P(S) can be expressed by an AC^0 circuit over z_1, \ldots, z_{2n+1} (of course with the number of levels bounded by the length d_A of the sequence A (can get less) and the number of nodes bounded by $d_A n^{d_A}$). Now g(i), for i = n, n + 1, is the probability P holds when a randomly chosen set of precisely i of the z's are set to True. From Thm.2.1 below g(n + 1) - g(n) = o(1) giving Thm. 1.1 and hence Thm. 0.1.

2 AC^0 Functions

We consider Boolean functions of z_1, \ldots, z_m . (In our application m = 2n+1.) The functions $z_i, \neg z_i$, called literals, are the level 0 functions. A level i + 1 function is the \land or \lor of polynomially many level i functions. An AC^0 function is a level d function for any constant d. By standard technical means we can express any AC^0 function in a "levelled" form so that the level i + 1 functions used are either all \land s of level i functions or all \lor s of level ifunctions and the choice alternates with i (at most doubling the number of levels). It is a classic result of circuit complexity that parity is not an AC^0 function. Let C be an AC^0 function. For $0 \le i \le m$ let $f(i) = f_C(i)$ denote the probability C holds when precisely i of the z_j are set to True and these i are chosen randomly.

Theorem 2.1 f(n+1) - f(n) = o(1)

Called a restriction ρ balanced if $|\{i : \rho(i) = 0\}| = |\{i : \rho(i) = 1\}|$. Now more fully the theorem says

(*) for every ε , d, t there is $n_{\varepsilon,d,t}$ satisfying: if $n \ge n_{\varepsilon,d,t}$ and C is an AC^0 Boolean circuit of z_1, \ldots, z_{2n+1} of level $\le d$ with $\le n^t$ nodes then $|f_C(n+1) - f_C(n)| < \varepsilon$.

This statement is proved by induction on d.

We choose the following

- (i) $c_0 = (\ell n 4)t > 0$
- (ii) $\varepsilon = \frac{1}{2}, \varepsilon_{\ell} = \frac{1}{2^{1+\ell}}$
- (iii) k is such that $\varepsilon \cdot k \ge t$
- (iv) we choose k_{ℓ} inductively on $\ell \leq k$ such that k_{ℓ} large enough.
- (v) c_1 a large enough real
- (vi) n_0 is large enough

For a node x of the circuit C let \mathcal{Y}_x be the set of nodes which fans into it; (without loss of generality in the level 1 we have only OR).

First we assume d > 2. Note

 \otimes_1 drawing as below a balanced restriction ρ with domain with $\leq n$ elements, with probability $\geq 1 - \varepsilon/3$ we have: in $C^1 = C \upharpoonright_{\rho}$, every node of the level 1 (i.e. for which \mathcal{Y}_x is a set of atoms) satisfies $|\mathcal{Y}_x| \leq c_0(\ln n)$. [Why? Choose randomly a set \mathbf{u}_0 of [n/2] pairwise disjoint pairs of numbers among $\{1, \ldots, 2n+1\}$, and then for each $\{i, j\} \in \mathbf{u}$ decide with probability half that $\rho(i) = 0$, $\rho(j) = 1$ and with probability half that $\rho(i) = 1$, $\rho(j) = 0$ (independently for disjoint pairs). This certainly gives a balanced ρ .

Now if x is a node of C of the level 1, the probability that ρ does not decide the truth value which the node compute is $\leq (\frac{1}{4})^{|\mathcal{Y}_x|}$. Note: after drawing **u**, if \mathcal{Y}_x contains a pair from **u** the probability is zero, we only increase compared to drawing just a restriction. So the probability that for some x of the level 1 of C, $|\mathcal{Y}_x| \geq (\ln 4)t(\ln n) + 1$ and the truth value is not computed, is $\leq |C| \times (\frac{1}{4})^{(\ell n 4)t(\ell n n)+1} \leq 1/2$, so there is ρ_0 for which for any such x the truth value is computed.]

Next, we say that a restriction ρ' extends a restriction ρ if $\rho'(i) \neq \rho(i) \Rightarrow \rho(i) = *$. Now

 \otimes_2 Choosing randomly a restriction ρ_1 as below we have: ρ_1 is a balanced restriction extending ρ such that $|\{i : i \in \{1, \ldots, 2n+1\}, \rho(i) = *\}| \geq 2[n^{\varepsilon}] + 1$ and with probability $\geq 1 - \varepsilon/3$ for every node y of C of the level 1 we have, $|\mathcal{Y}_y| \leq k$.

[Why? We draw a set \mathbf{u}_1 of $(2n+1-|\operatorname{dom}(\rho_0)|-(2[n^{\varepsilon_0}]+1))/2$ pairs from $\{i : \rho_0(i) = *\}$ pairwise disjoint and for each $\{i, j\} \in \mathbf{u}$, decide with probability $\frac{1}{2}$ that $\rho_1(i) = 0$, $\rho_1(j) = 1$ and with probability half that $\rho_1(i) = 1$, $\rho_1(j) = 0$.

For each node $y \in C^1$ of the level 1 the probability that "the number of $y' \in \mathcal{Y}_y$ not assigned a truth value by ρ_1 is $\geq k + 1$ " is at most $\begin{pmatrix} |\mathcal{Y}_y| \\ k+1 \end{pmatrix} \times (-1)^{k+1} \leq (-1)^{k+1} = -\varepsilon_0(k+1) \leq -t$

 $\begin{pmatrix} \frac{1}{2n^{\varepsilon_0}+1} \end{pmatrix}^{k+1} \leq (c_0 \ln n)^{k+1} \cdot n^{-\varepsilon_0(k+1)} < n^{-t}. \\ \text{We now choose by induction on } \ell \leq k \text{ a restriction } \rho_{2,\ell} \text{ such that}$

$$\otimes_3$$
 (a) $\rho_{2,\ell_0} = \rho_1, \, \rho_{2,\ell} \subseteq \rho_{2,\ell+1}, \, 2n+1-(2[n^{\varepsilon_\ell}]+1) = |\mathrm{dom}\rho_{2,\ell}|$

(b) every $y \in C$ of the level 2 there is a set $w_{y,\ell}$ of $\leq k_{\ell}$ atoms such that: if $z \in \mathcal{Y}_y$, then $|\mathcal{Y}_z \setminus w_{y,\ell}| \leq k - \ell$.

Now for $C \upharpoonright \rho_{2,k}$ we can invert AND and OR (multiplying the size by a constant $\leq c_1$) decreasing d by one thus carrying the induction step.

For $\ell = 0$ let $\rho_{2,0} = \rho_1$. For $\ell + 1$, for each $y \in C$ of level 2 let $\Xi = \{\nu : \nu \text{ a restriction with domain } w_{y,\ell}\}$ let

 $\mathcal{Y}_{y}^{\nu} = \{ z \in \mathcal{Y}_{y} : \text{ the truth value at } z \text{ is still not computed under } \rho_{2,\ell} \cup \nu \},\$

and try to choose by induction on i an atom $z_{y,\ell,\nu,i} \in \mathcal{Y}_y^{\nu} \setminus \{z_{y,\ell,\nu,j} : j < i\}$, such that $\operatorname{dom}(z_{y,\ell,\nu,i})$ is disjoint to $\bigcup_{j < i} \operatorname{dom}(z_{y,\ell,\nu,j}) \setminus w_{y,\ell}$. Let it be defined if $i < i_{y,\ell}$.

Now $\rho_{2,\ell+1}$ will for each $\nu \in \Xi$ decide that ν make the truth value computed in y true, or will leave only $\leq (k_{\ell+1} - k_{\ell})/2^{k_{\ell}}$ of the atoms in $\bigcup_{i} \operatorname{dom} z_{y,\ell,\nu,i} \setminus w_{y,\ell}$ undetermined (this is done as in the previous two stages).

But now by $\otimes_1 + \otimes_2$, $C \upharpoonright_{\rho_{2,k}}$ can be considered having d-1 levels (because, as said above we can invert the AND and OR in level 1 and 2).

We have translate our problem to one with $[n^{\varepsilon_k}]$, d-1, $\varepsilon_k(t+\varepsilon_1)$, $\frac{\varepsilon}{3}$ instead n, d, t, ε (the $t+\varepsilon$ is just for $n^{t+\varepsilon} > c_1 n^t$).

Also note: ε , c_1 does not depend on n. So we can use the induction hypothesis. We still have to check the case $d \leq 2$, we still are assuming level 1 consist of cases of OR, and for almost all random ρ_1 (as in \otimes_1) for every xof level 1 we have $|\mathcal{Y}_x| \leq \mathbf{c}_0 \ln n$ (so again changing n).

So as above we can add this assumption. Choose randomly a complete restriction ρ^0 with $|\{i : \rho^0(i) = 1\}| = n$, and let ρ^1 be gotten from ρ^0 by changing one zero to 1, so $|\{i : \rho^1(i) = 1\}| = n + 1$.

Now the probability that $C \upharpoonright_{\rho^0} = 0$ but $C \upharpoonright_{\rho^1} = 1$ is small: it require that for some node x of level 1 is made false in $C \upharpoonright_{\rho^0}$ while there is no such xfor $C \upharpoonright_{\rho^1}$, but if x(*) is such for $C \upharpoonright_{\rho^0}$ it is made true then with probability $\geq 1 - \frac{|\mathcal{Y}_x|}{2n+1} \geq 1 - \frac{c_0 \ln n}{n}$ the z_i changed is not in $\mathcal{Y}_{x(*)}$. Contradiction, thus finishing the proof.

3 Two Place Functions

Here we consider the random structure $([n], F_n)$ where $F_n(x, y)$ is a random function from $[n] \times [n]$ to [n]. (We no longer have order. A typical sentence would be $\forall_x \exists_y F(x, y) = x$.): Again for any sentence A we define $f(n) = f_A(n)$ to be the probability A holds in the space of structures on [n] with uniform distribution. Again it is known [CHSh245] that convergence fails, there are A for which f(n) does not converge. Again our result is a very weak zero one law.

Theorem 3.1 For every A

$$\lim_{n \to \infty} f_A(n+1) - f_A(n) = 0$$

Again let m = 2n+1. Let $F^*(x, y, z)$ be a *three*-place function from $[m] \times [m] \times [m]$ [m] to [m]. For $S \subset [m]$ of cardinality i = n or n + 1 we define F_S^* , a partial function from $[S] \times [S]$ to [S] by setting $F_S^*(x, y) = F^*(x, y, z)$ where z is the minimal value for which $F^*(x, y, z) \in S$. If there is no such z then $F_S^*(x, y)$ is not defined. This occurs with probability $(\frac{m-i}{m})^m$ for any particular x, y so the probability F_S^* is not always defined is at most $i^2(\frac{m-i}{m})^m = o(1)$.

We generate a random three-place F^* and then consider F_S^* with S a random set of size i = n or n+1. Conditioning on F_S^* being always defined it then has the distribution of a random two-place function on i points. Thus $\Pr[A]$ over $[n], F_n$ is within o(1) of $\Pr[A]$ when $F_n = F_S^*$ is chosen in this manner. Thus, as in §2, it suffices to show for any F^* and A that, letting g(i) denote the probability F_S^* satisfies A with S a uniformly chosen i-set, g(n+1) - g(n) = o(1). Again fix F^* and A and let z_x be the Boolean value of $x \in S$ for $1 \leq x \leq 2n+1$. In A replace the ternary relation F(a,b) = c by $\wedge_{y < d} \neg z_{F(a,b,y)} \& z_{F(a,b,d)}$. (For d = 1 this is simply True.) As in §2 replace $\exists_x P(x)$ by $\bigvee_x (z_x \wedge P^*(x))$ where $P^*(x)$ has been inductively defined as the replacement of P(x). Then the statement that F_S^* satisfies A becomes a Boolean function of the z_1, \ldots, z_m , as before it is an AC^0 function, and by §2 we have g(n+1) - g(n) = o(1).

*

The following discussion is directed mainly for logicians but may be of interest for CS-oriented readers as well.

Discussion 3.2 (1) Note that the results of [Sh463] cannot be gotten in this way as the proof here use high symmetry. The problem there was: let $\bar{p} = \langle p_i : i \in \mathbb{N} \rangle$ be a sequence of probabilities such that $\sum_i p_i < \infty$. Let $G(n, \bar{p})$ be the random graph with set of nodes $[n] = \{1, \ldots, n\}$ and the edges drawn independently, and for $i \neq j$ the probability of $\{i, j\}$ being an edge is $p_{|i-j|}$.

The very weak 0-1 law was proved for this context in [Sh463] (earlier on this context (probability depending on distance) was introduced and investigated in Luczak and Shelah [LuSh435]). Now drawing $G(2n + 1, \bar{p})$ and then restricting ourselves to a random $S \subseteq \{1, \ldots, 2n + 1\}$ with n, and with n + 1 elements, fail as $G(2n + 1, \bar{p}) \upharpoonright_S$ does not have the same distribution as $G(|S|, \bar{p})$.

(2) We may want to phrase the result generally;

- One way: just say that M_n, M_{n+1} can be gotten as above : draw a model on $[2n + 1] = \{1, \ldots, 2n + 1\}$ (i.e. with this universe), then choose randomly subsets P_n^{ℓ} with $n + \ell$ elements and restrict yourself to it.
- (3) Two random linear order satisfies the very weak 0-1 zero law (mean: take two random functions from [n] to $[0,1]_{\mathbb{R}}$). The proof should be clear.
- (4) All this is for fixed probabilities; we then can allow probabilities depending on n e.g. we may consider G_<(n, p_n) is the model with set of elements {1,...,n}, the order relation and we draw edges with edge probability p_n depending on n. This call for estimating two number (for φ first order sentence):

$$\alpha_n = |\operatorname{Prob}(G_{<}(n, p_{n+1}) \models \varphi) - \operatorname{Prob}(G_{<}(n, p_n) \models \varphi)|$$
$$\beta_n = |\operatorname{Prob}(G_{<}(n, p_{n+1}) \models \varphi) - \operatorname{Prob}(G_{<}(n+1, p_{n+1} \models \varphi))|$$

As for β_n the question is how much does the proof here depend on having the probability $\frac{1}{2}$. Direct inspection on the proof show it does not at all (this just influence on determining the specific Boolean function with 2n+1 variables) so we know that β_n converge to zero.

As for α_n , clearly the question is how fast p_n change.

(5) As said in [Sh463] we can also consider lim(Prob_{n+h(n)}(M_{n+h(n)} ⊨ ψ) − Prob_n(M_n ⊨ ψ)) = 0, i.e. characterize the function h for which this holds but this was not dealt with there. Hopefully there is a threshold phenomena. Probably this family of problems will appeal to mathematicians with an analytic background.

Another problem, closer to my heart, is to understand the model theory: in some sense first order formulas cannot express too much, but can we find a more direct statement fulfilling this?

* * *

Another way to present the first problem for our case is: close (or at least narrow) the analytic gap between [CHSh245] and the present paper. After this work, Boppana and Spencer [BS], continuing the present paper and [CHSh245], address the problem and completely solve it. More specifically they proved the following.

For every sentence A there exists a number t so that $m(n) = O(n \ln^{-t} n)$ implies

$$\lim_{n \to \infty} f_A(n + m(n)) - f_A(n) = 0.$$

And

For every number t there exists a sentence A and a function $m(n) = O(n \ln^{-t} n)$ so that $f_A(n + m(n)) - f_A(n)$ does not approach zero.

Together we could say: a function m(n) has the property that for all A and all $m'(n) \leq m(n)$ we have $f_A(n + m'(n)) - f_A(n) \to 0$ if and only if $m(n) = o(n \ln^{-t} n)$ for all t.

For improving the bound from this side they have used Hastad switching lemma [Hastad] (see [AS], §11.2, Lemma 2.1).

- (6) If we use logic stronger than first order , it cannot be too strong (on monadic logic see [KfSh201]), but we may allow quantification over subsets of size k_n , e.g. $\log(n)$ there are two issues:
 - (A) when for both n and n + 1 we quantify over subsets of size k_n , we should just increase M by having the set $[n]^{k_n}$ as a set of extra elements, so in (*), P is chosen as a random subset of $\{1, 2, 3, \ldots, 2n, 2n + 1\}$ with n or n + 1 elements but the model has about $(2n + 1)^{k_n}$ elements; this require stronger theorem, still true (up to very near to exponentiation)
 - (B) if $k_n \neq k_{n+1}$ we need to show it does not matter, we may choose to round $k_n = \log_2(n)$ so only for rare *n* the value change so we weaken a little the theorem *or* we may look at sentences for which this does not matter.

Maybe more naturally, together with choosing randomly \mathcal{M}_n we choose a number \underline{k}_n , and the probability of $\underline{k}_n = k_n + i$ if $i \in [-k_n/2, k_n/2]$ being $1/k_n$.

And we ask for " $p_n^{\varphi} =: \operatorname{Prob}(\mathcal{M}_n \models \varphi)$ where the monadic quantifier is interpreted as varying on set with $\leq \underline{k}_n$ elements) for sentence φ (the point of the distribution of \underline{k}_n is just that for n, n+1 they differ a little). E.g. if for a random graph on n (probability 0.5) we ask on the property "the size of maximal clique of size at most $[\log_2 n]^2$ is even" it satisfies the very weak zero one law

Of course we know much more on this, still it shows that this old result (more exactly - a weakened version) can be put in our framework.

References

- [A] M. Ajtai, $A \Sigma_1^1$ formula on finite structures, Annals of Pure and Applied Logic **24**(1983):1–49.
- [AS] N. Alon and J. Spencer, The Probabilistic Method, John Wiley (New York), 1992.
- [BS] R.B. Boppana and J.H. Spencer (1995), Smoothness laws for random ordered graphs, to appear in Logic and Random Structures: DIMACS Workshop, November 5–7, 1995, DIMACS series in discrete mathematics and theoretical computer science, American Mathematical Society, Providence, Rhode Island.
- [Hastad] J. Hastad, Almost optimal lower bounds for small depth circuits, in S. Micali, ed. Advances in Computer Research, Vol. 5, JAI Press, Greenwich CT, 143-170
- [KfSh201] M. Kaufman and S. Shelah, On random models of finite power and monadic logic, J. Symb. Logic **54**(1985):285–293.
- [CHSh245] K. Compton, C.S. Henson and S. Shelah, Non-convergence, Undecidability and Intractability in asymptotic problems, Annals of Pure and Applied Logic 36(1987):207–224.
- [LuSh435] T. Luczak and S. Shelah, *Convergence in homogeneous random graphs*, Random Structures and Algorithm, **6** (1965):371–391.

Very weak zero one law..., $\mathbf{Sh}~\mathbf{548}$

[Sh463] S. Shelah, On the very weak 0 - 1 law for random graphs with order, J. of Logic and Computation, **6** (1966):137–159.