TRADING CROSSINGS FOR HANDLES AND CROSSCAPS

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ABSTRACT. Let $c_k = cr_k(G)$ denote the minimum number of edge crossings when a graph G is drawn on an orientable surface of genus k. The (orientable) *crossing sequence* $c_0, c_1, c_2, \ldots$ encodes the trade-off between adding handles and decreasing crossings.

We focus on sequences of the type $c_0 > c_1 > c_2 = 0$; equivalently, we study the planar and toroidal crossing number of doubly-toroidal graphs. For every $\epsilon > 0$ we construct graphs whose orientable crossing sequence satisfies $c_1/c_0 > 5/6 - \epsilon$. In other words, we construct graphs where the addition of one handle can save roughly 1/6th of the crossings, but the addition of a second handle can save 5 times more crossings.

We similarly define the *non-orientable crossing sequence* $\tilde{c}_0, \tilde{c}_1, \tilde{c}_2, \ldots$ for drawings on non-orientable surfaces. We show that for every $\tilde{c}_0 > \tilde{c}_1 > 0$ there exists a graph with non-orientable crossing sequence $\tilde{c}_0, \tilde{c}_1, 0$. We conjecture that every strictly-decreasing sequence of non-negative integers can be both an orientable crossing sequence and a non-orientable crossing sequence (with different graphs).

1. INTRODUCTION

One of the most important classes of graphs are the *planar* graphs: those which can be drawn on the plane (or on the surface of a sphere) so that edges don't cross. Not every graph is planar. Traditionally, there are two ways to draw such non-planar graphs. The first is to draw them on the plane allowing edge crossings. The goal is then to minimize the number of crossings. The second is to draw the graph on a sphere with either handles or crosscaps attached so that edges do not cross. The goal is then to minimize the number of handles or crosscaps.

In this paper we combine the above concepts. Namely, we investigate the crossing number of a graph drawn on a surface. Let $c_k = cr_k(G)$ denote the minimum number of crossings in a drawing of a graph G on the orientable surface with k handles, S_k. We call $c_0, c_1, c_2, \ldots$ the *(orientable) crossing sequence* of G. By studying crossing sequences we can study the "trade-off" between crossings and handles when drawing a graph. Similarly, we define the *non-orientable crossing sequence* $\tilde{c}_0, \tilde{c}_1, \tilde{c}_2, \ldots$ for graphs embedded in non-orientable surfaces with crosscaps and examine the trade-off between crossings and crosscaps.

What do crossing sequences look like? It is easy to see that they are strictly decreasing (we can always use a handle or crosscap to eliminate a single crossing) and eventually zero (when k achieves the orientable or non-orientable genus of the graph). Moreover, it is easy to construct examples where the difference between consecutive numbers in the crossing sequence is arbitrarily large. In the orientable case, take a graph with large planar crossing number but with a drawing where all crossings involve a fixed edge; in the non-orientable case take a projective planar

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graph with a large face-width (see [8] for information about the face-width of an embedding).

Crossing sequences were first introduced by Širáň [13]. He tried to characterize orientable crossing sequences. A sequence $c_0, c_1, \ldots$ is *convex* if $c_i - c_{i+1} \leq c_{i-1} - c_i$ for all i. Širáň showed [13] that any decreasing convex sequence of non-negative integers was the orientable crossing sequence of some graph. He conjectured that all orientable crossing sequences are convex.

A counterexample to Širáň's conjecture would be a non-convex crossing sequence. For example, is there a crossing sequence where $c_1 - c_2 > c_0 - c_1$? Loosely speaking, if adding the second handle saves more edges than adding the first handle, why not add the second handle first? Despite its non-intuitive nature, we will show that it is possible to achieve this inequality with the crossing sequence of a graph.

More strongly, we believe the following.

Conjecture 1.1. *Any strictly decreasing sequence of non-negative integers is the crossing sequence of some graph. It is also the non-orientable crossing sequence of a (different) graph.*

To study the effect on the number of crossings of adding successive handles we focus on the simplest special case: sequences $c_0 > c_1 > c_2 = 0$. In other words, we examine the planar crossing number c_0 and the toroidal crossing number c_1 of graphs which embed in the double torus without crossings. Our main result is the following (where $\binom{n}{k}$ denotes a binomial coefficient).

Theorem 1.2. *For every $m > 0$, there exists a graph which embeds in the double torus without crossings, has planar crossing number $4\binom{3m}{2}$, and has toroidal crossing number $3\binom{3m}{2} + 3\binom{m}{2}$.*

Corollary 1.3. *For every $\epsilon > 0$, there exists doubly-toroidal graph whose orientable crossing sequence $c_0, c_1, c_2 = 0$, has the property $(c_1 - c_2)/(c_0 - c_1) > 5 - \epsilon$. Equivalently, adding the first handle saves at most $1/6^{th}$ of the crossings.*

Analogously, we study the effect on the number of crossings of adding successive crosscaps by focusing on the simplest special case: $\tilde{c}_0 > \tilde{c}_1 > \tilde{c}_2 = 0$. Thus we examine the planar crossing number $\tilde{c}_0$ and the projective-plane crossing number $\tilde{c}_1$ of graphs which embed in the Klein bottle without crossings. In this non-orientable case we have the following stronger result.

Theorem 1.4. *For every $\tilde{c}_0 > \tilde{c}_1$ there exists a graph which embeds on Klein's bottle without crossings, has planar crossing number $\tilde{c}_0$, and projective-plane crossing number $\tilde{c}_1$.*

In Section 2 we introduce weights on edges and define the weighted crossing number. The purpose of the weights is to control, even prohibit, the crossings of some edges in a drawing achieving a crossing number. In Section 3 we use the weights to fix a large portion (called a "patch") of any drawing that achieves a crossing number. Again, the purpose is to restrict the types of embedding so that the crossing number on a surface is more easily determined. In Section 4 we give the (weighted patched) graphs demonstrating our theorem for non-orientable surfaces. In Section 5 we give the (weighted patched) graphs demonstrating our theorem for orientable surfaces. We conclude in Section 6 with some directions for future research.

2. Weights

In this section we introduce weights on edges and the concept of the weighted crossing number. Loosely speaking by assigning a large enough (possibly infinite) weight to an edge we can guarantee it is not involved in a crossing in a drawing which achieves the crossing number.

A *weight* on an edge is a positive integer (or ∞) assigned to that edge. A *weighted graph* is a graph together with a weight on each edge. Let $wt(e)$ denote the weight of edge e. Suppose that G is a weighted graph drawn on some surface. The *weighted crossing number* of the drawing is the sum $wt(e) \cdot wt(e')$ over all pairs e, e' of crossing edges. The (weighted) *crossing number* of a weighted graph G is the minimum weighted crossing number over all drawings of G. The weighted crossing number corresponds to the usual crossing number if all of the weights are 1.

Proposition 2.1. *Let G be a cubic weighted graph. Then there exists a simple unweighted graph G' with the same crossing sequence as G.*

Proof. We will first modify G to a non-simple graph G'' with the same crossing sequence. Let e be an edge of G with weight k. Form G'' by replacing e with k parallel edges $e_1, e_2, \ldots, e_k$, each with weight 1. (If the reader prefers finite graphs, we replace an edge with weight ∞ with a sufficiently large number of parallel edges.)

We claim that G'' has the same crossing sequence as G. Consider a drawing of G''. Among the new edges $e_1, \ldots, e_k$ pick the one with the fewest number of crossings, say e_1. We redraw e_i alongside e_1 without increasing the total number of crossings. Once all k edges are redrawn in parallel alongside e_1, we can replace them with a single edge of weight k. Thus we have constructed a drawing of G with the same weighted crossing number.

We form the desired G' by repeating this process for each edge of weight exceeding one. We then subdivide any parallel edges to make G' simple as claimed. □

3. Patches

Patches are a common method in topological graph theory to fix certain portions of an embedding. A *patched graph* is a graph G and a collection $P_1, \cdots, P_k$ of edge-disjoint closed trails in G (so vertices may be repeated but not edges). An embedding of G is a *patch embedding* if each P_i appears as a face boundary. Notice that we have not assigned a direction to the P_i; in an oriented surface we accept either P_i or P_i^{-1} as the face boundary. In some instances it is helpful to talk about the theory of oriented patches (see [2], [5] or [15]), but it is not needed here.

Patches are also known under many different names. Archdeacon referred to identified disk spaces [1], Širáň called them relative graphs [14], while Mohar and Robertson [10] used the word patches.

We also want to consider drawings with edge crossings in a patched graph. Specifically, we assign weight ∞ to all edges in a patch cycle and consider the minimum weighted crossing number over all patch embeddings in a given surface. Therefore edges in a patch boundary cycle are not allowed to cross any other edge, nor can the patch face contain any other portion of the graph. Define the orientable and non-orientable crossing sequence for patched graphs similar to that for unpatched graphs.

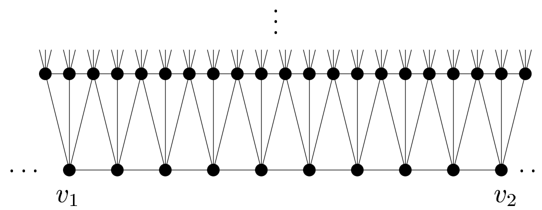

FIGURE 1. A patch extension

For our purposes we need to consider only cubic patched graphs. Notice that in a cubic patched graph the patch trails are all vertex-disjoint simple cycles. We also allow weights on non-patch edges of a patched graph.

Let d be a positive integer and let $C = (v_1, \ldots, v_n)$ be a simple cycle. A *d-dense patch extension on C* is a 3-connected planar graph P with the following properties:

(1) the outer face of P is bounded by the cycle C' formed by replacing each edge of C with a path on $2d + 2$ edges,
(2) every internal face of P is a triangle, and
(3) any path in P joining two non-adjacent vertices of C' and internally disjoint from C' is of length at least $2nd + 2n + 6d + 1$.

Observe that it is easy to construct a d-dense patch extension for any length simple cycle. See for example Figure 1 which illustrates a patch extension along an edge $v_1 v_2$.

The goal of this section is to prove the following.

Proposition 3.1. *Let G be a weighted patched graph. Then there exists a weighted graph G' with the same crossing sequence.*

Proof. We prove the proposition for orientable sequences only. The proof in the non-orientable case is an easy modification. Let d be the largest subscript such that c_d is non-zero (that is, $d + 1$ is the genus of the patched graph). We form the weighted graph G' from G by identifying each patch cycle C_i with the outer face of a d-dense patch extension P_i. Each edge in the patch extension is assigned infinite weight. The weights of edges which are not in patch cycles are left unchanged.

We need to show that for each patch drawing of G with weighted crossing number k, there is a drawing of G' with the same crossing number. But this is obvious: we can place the patch P_i inside the face bounded by C_i.

Now suppose that G' is drawn on S_g $(g \le d)$ with weighted crossings $k > 0$. We need to demonstrate a drawing of the patched G on the same surface with the same weighted crossing number. Observe that edges in a patch extension P receive infinite weight, and therefore are not involved in any crossings in the drawing of G'. Now consider the restriction of the drawing that only considers the patch extension P, capping off any unused handles or crosscaps. This restriction is an embedding of P. Let $C = (v_1, \ldots, v_n)$ be the cycle on the boundary of P.

Claim: *Either*

(1) *No face has more than one v_i, or*
(2) *There is a face F so that $(v_1, \ldots, v_n)$ appear on the boundary of F in that order.*

In the second case we can replace the given embedding of P with the planar embedding of the patch. This may lower the genus of the embedding, but will not increase the crossing number. In the first case each of the n edges connecting P to the remainder of G' lie on a different handle. (Recall that G is cubic and hence our patches are disjoint.) It is now easy to re-embed using the planar embedding of P. Finally, in either case we can fill the embedding of P to get an embedding of the patched graph G in the original surface.

To establish the claim, suppose that there is a face containing v_i and v_j with boundary walk W. If W contains a sub-path internally disjoint from C' which joins two non-adjacent vertices of C', then

$$|W| \geq 2nd + 2n + 6d + 1. \tag{1}$$

Let f_p be the number of faces in the planar embedding and let f_g be the number of faces in the S_g-embedding. Then $f_g = f_p - 2g \geq f_p - 2d$. Let e denote the number of edges in P. Observe that in the planar embedding all faces are triangles except for one $n(2d+2)$-gon, and hence $3(f_p - 1) + n(2d+2) = 2e$. Therefore,

$$\begin{aligned} 2e - 3f_g &\leq 2e - 3(f_p - 2d) \\ &= n(2d+2) + 6d - 3. \end{aligned}$$

It follows that every face of the S_g-embedding of P has length at most $2nd+2n+6d$, contradicting Equation (1). (Indeed, if one face had length at least $2nd+2n+6d+1$ then $2e \geq 3(f_g - 1) + 2nd + 2n + 6d + 1$.) It follows that there is no walk between non-adjacent vertices of C' in W that is internally disjint from C'.

In P, v_i and v_j (vertices of the original cycle C) divide C' into two v_iv_j-sub-walks W_1 and W_2. It follows that W contains the vertices of W_i in the same order as they appear in C'. There are two possibilities: either W contains the vertices of C' in the same order as they appear in $W_iW_i^{-1}$ for some i, or W contains the vertices $W_1W_2^{-1}$. In the latter case $(v_1, \ldots v_n)$ appear in order as claimed, so it only remains to rule out the former case.

Suppose that W contains the vertices of $W_1W_1^{-1}$ in that order. Recall that every edge of C has been replaced by a path with $2d+2$ edges and that $d \geq g$. Hence, among the vertices of W_1 we have $2g+3$ consecutive vertices $u_0, u_1, \ldots u_{2g+2}$. Inside of the face W we draw $g+1$ chords connecting the two occurrences of u_i, where $i = 1, 3, \ldots, 2g+1$. These chords correspond to simple cycles in the surface, and hence they are homologically non-null. Because the surface is of genus g, in any collection of $g+1$ disjoint cycles there are at least two cycles (chords in our case) whose removal disconnects the surface. However, by the choice of the u_i's, we have that removal of two vertices disconnects the graph P, a contradiction.

We have established the claim and so the proposition follows. □

4. Non-orientable Crossing Sequences

We begin with the non-orientable case because the construction is slightly easier to describe.

Theorem 4.1. *Let $\tilde{c}_0 > \tilde{c}_1 > \tilde{c}_2 = 0$. Then there exists a weighted patched graph G with non-orientable crossing sequence $\tilde{c}_0, \tilde{c}_1, \tilde{c}_2$.*

Proof. We begin by showing that $n, n-1, 0$ is the non-orientable crossing sequence of a patched graph. The graph is shown in Figure 2. This graph comprises a Hamiltonian cubic graph together with a patch (shown as shading) on a fixed

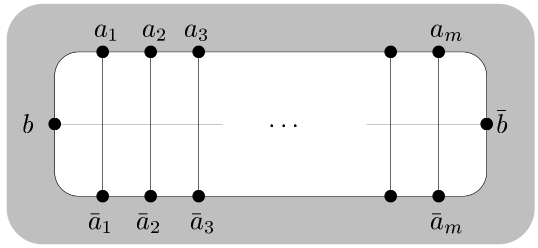

FIGURE 2. The patched graph for non-orientable sequences

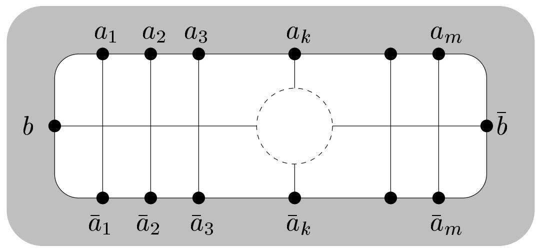

FIGURE 3. Embedding the patched graph in the projective plane

Hamiltonian cycle $(b, a_1, \ldots, a_m, \bar{b}, \bar{a}_m, \ldots, \bar{a}_1, b)$. We need to show the two claims about the non-orientable crossing numbers.

We first show that the planar crossing number is as claimed. There is a unique embedding of this patched graph in the plane with a simple cycle on the boundary of the only unpatched face. All other edges must lie in this face. Two edges cross if and only if their edge ends alternate in the boundary of this cycle. Hence the crossing pairs of edges are exactly $b\bar{b}$ and $a_i\bar{a}_i$, $i = 1, \ldots, m$. There are exactly n edge crossings.

We next show the graph embeds on the Klein bottle (the non-orientable surface of genus 2). This is clear since $G - \{b\bar{b}\}$ is planar. Hence we can add a single anti-handle (the equivalent of two crosscaps) to accommodate the edge $b\bar{b}$.

We need only show the claim for the projective crossing number. We can assume that all non-infinity weight edges lie in a single face which contains a crosscap. Notice that the crossing number is at most $n-1$ since we can exclude a single crossing in the planar embedding by adding a crosscap. (See Figure 3, the dotted circle represents a crosscap).

Suppose that we have a drawing achieving the projective crossing number. Then there must exist a k such that $b\bar{b}$ does not cross $a_k\bar{a}_k$ (if not, there would exist at least n crossings). Consider the unique projective embedding without crossings of the graph formed by deleting all edges $a_i\bar{a}_i$ where $i \neq k$. This embedding must be induced by the drawing achieving the crossing number. But in this embedding a_i and $\bar{a}_i$ lie on different faces. Hence the edge $a_i\bar{a}_i$ must cross another edge in the drawing. Since there are $n-1$ such edges, there are at least $n-1$ crossings.

To achieve an arbitrary sequence $\tilde{c}_0 > \tilde{c}_1 > \tilde{c}_2 = 0$ we vary the construction slightly. Namely, we pick one of the edges $a_k\bar{a}_k$ and assign it weight $\tilde{c}_1 - \tilde{c}_0$. The

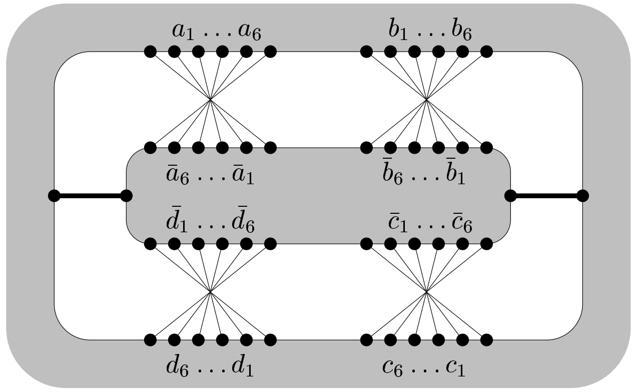

FIGURE 4. The patched graph G_m for orientable sequences ($m = 2$)

total number of vertical edges is $m = \tilde{c}_2 - \tilde{c}_1 + 1$. Again, it is easy to see the graph has the desired drawings on the plane and on Klein's bottle. A drawing on the projective plane can eliminate at most one unweighted crossing. This is minimized by eliminating the crossing with edge $a_k\bar{a}_k$. □

5. Orientable Crossing Sequences

In this section we give graphs with an interesting orientable crossing sequence. These graphs will embed on the double-torus but not on the torus. The toroidal crossing number will be almost 5/6th of the planar crossing number. Thus the addition of a second handle saves a greater number of edge crossings than did the addition of the first handle. By Propositions 2.1 and 3.1 it suffices to show the result for weighted patched graphs.

The weighted patched graph G_m, $m = 2$, is shown in Figure 4. It has two patched faces (outside the large rectangle and inside the small rectangle), and two edges with infinite weights (shown in bold). There are $12m$ other edges of weight 1. These $12m$ edges are broken into 4 groups of size $3m$, each group on vertices a_i, b_i, c_i, d_i respectively. The graph G_m, $m \neq 2$ is defined similarly, differing only in the number $3m$ of edges in each of the 4 groups. In particular, any two edges from the same group cross in the drawing shown.

Theorem 5.1. *The orientable crossing sequence of G_m is*

$$c_0 = 4\binom{3m}{2}, \quad c_1 = 3\binom{3m}{2} + 3\binom{m}{2}, \quad c_2 = 0.$$

Proof. Let H_m denote the patched subgraph consisting of the two patch faces and the two infinite weight edges in G_m. Observe that H_m has exactly two embeddings in an unoriented plane. In one embedding the faces are as shown in Figure 5; in the other the inside patch face can be reflected about a horizontal line containing the infinite-weight edges so that a_i and $\bar{a}_i$ lie on different unpatched faces. We call these the matched and unmatched embeddings respectively.

Our proof proceeds by a sequence of claims.

Claim 1: The planar crossing number of G_m is $4\binom{3m}{2}$. The upper bound on this crossing number is established by the drawing in Figure 4. To show the lower bound

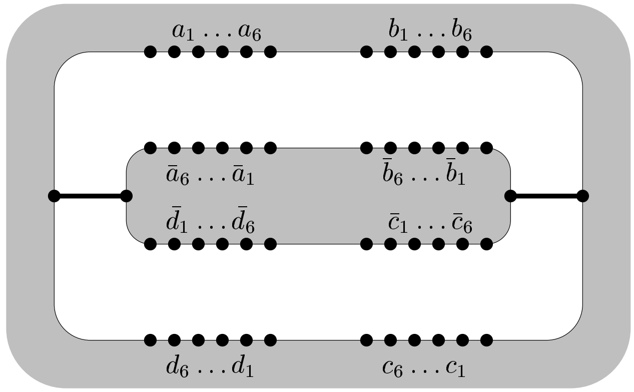

FIGURE 5. The matched embedding of H_m.

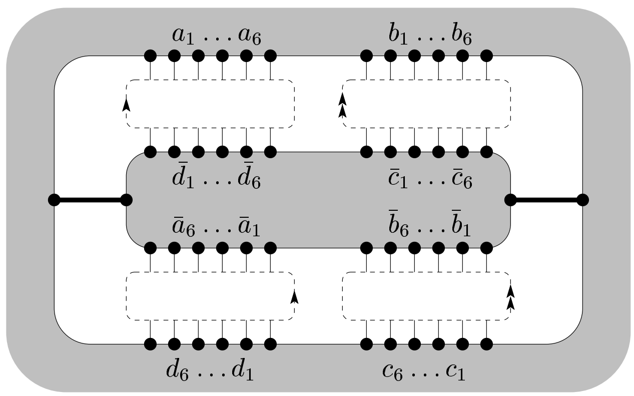

FIGURE 6. An embedding of G_m in the double torus

we first consider the embedding of H_m. If H_m has the unmatched embedding, then there is no face with a_1 and $\bar{a}_1$ on the same face. It follows that the edge $a_1\bar{a}_1$ crosses a second edge of infinite weight or crosses a patch. No such drawing has a finite crossing number.

It follows that H_m has the matched embedding. For each $x \in \{a, b, c, d\}$ and each i there is a unique unpatched face of H_m containing x_i and $\bar{x}_i$. It follows that in any drawing achieving the minimal crossing number has the edge $x_i\bar{x}_i$ in this face. Two edges cross in this face if and only if they lie in the same group. A simple count gives the desired lower bound.

Claim 2: G_m embeds on the double torus The proof is established in Figure 6. In this figure there are two handles indicated by a dotted ellipse labelled with one or two arrows to indicate the orientation. Identifying the arrows in the directions indicated gives the desired embedding.

Claim 3: The toroidal crossing number of G_m where H_m has the unmatched embedding is at least $18m^2$. To avoid crossing an infinite weight edge the boundary

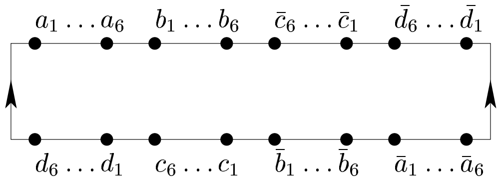

FIGURE 7. A cylindrical face of H_m with the unmatched embedding

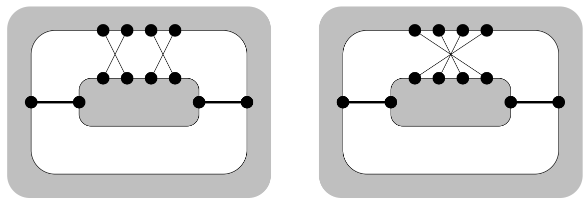

FIGURE 8. Two non-toroidal patched graphs

cycles of the two unpatched faces in the planar embedding must bound the same face in the toroidal embedding restricted to H_m. Hence this face is a cylinder bounded by two cycles as shown in Figure 7.

Consider an arbitrary set of three vertices x_i, y_j, z_k where $i+j+k \equiv 0 (mod\ 3m)$ and x, y, z are distinct elements from the set $\{a, b, c, d\}$. Observe that the embedding of $H_m \cup \{x_i\bar{x}_i, y_j\bar{y}_j, z_k\bar{z}_k\}$ is non-planar. Hence there must exist a crossing among these three new edges in an optimal drawing of G_m. There are $4(3m)^2$ such triples. These triples count each crossing at most twice: if say $x_i\bar{x}_i$ and $y_j\bar{y}_j$ cross, then the third edge is either $z_k\bar{z}_k$ or $w_k\bar{w}_k$ where $i+j+k \equiv 0 \quad (mod\ 3m)$ and x, y, z, w are all distinct. Hence this drawing of G_m has at least $4(3m)^2/2 = 18m^2$ crossings as claimed.

The reader may note that considering these edges as points and these triples as blocks yields a group-divisible design with 4 groups of size $3m$ each, block size 3, and $\lambda = 2$. See [7] for details.

Claim 4: The toroidal crossing number of G_m where H_m has the matched embedding is $3\binom{3m}{2} + 3\binom{m}{2}$. We first note that one of the two unpatched faces in the planar embedding of H_m is also a face in the restricted embedding of G_m. Without loss of generality say it is the face with $d_i\bar{d}_i$ and $c_j\bar{c}_j$. It follows that there are $2\binom{3m}{2}$ crossings in that face.

We now turn our attention to crossings in the other face. First, suppose that there exist edges $a_i\bar{a}_i$, $a_j\bar{a}_j$ which do not cross and edges $b_k\bar{b}_k$, $b_l\bar{b}_l$ which do not cross. Then the restriction of G_m to $H_m \cup \{a_i\bar{a}_i, a_j\bar{a}_j, b_k\bar{b}_k, b_l\bar{b}_l\}$ has no crossings. However, this patched embedding is non-toroidal, a contradiction (see the left side of Figure 8). We conclude that without loss of generality each pair of edges $b_k\bar{b}_k, b_l\bar{b}_l$ cross in an optimal drawing of G_m. Note that this gives a total of $3\binom{3m}{2}$ crossings among edges of the form $x_i\bar{x}_i$ where $x \in \{b, c, d\}$.

We finally turn our attention to crossings on edges $a_i\bar{a}_i$. First suppose that there were four pairwise non-crossing edges $a_i\bar{a}_i, a_j\bar{a}_j, a_k\bar{a}_k, a_l\bar{a}_l$. Then the drawing

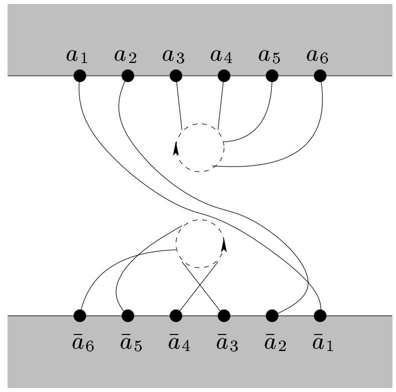

FIGURE 9. Part of the torodial drawing of G_m with crossing number $3\binom{3m}{2} + 3\binom{m}{2}$

of G_m restricted to $H_m \cup \{a_i\bar{a}_i, a_j\bar{a}_j, a_k\bar{a}_k, a_l\bar{a}_l\}$ has no crossings. But this patched embedding is non-toroidal, a contradiction (see the right side of Figure 8).

Now form a graph T_m as follows. The vertex set of T_m are the $3m$ edges $a_i\bar{a}_i$. Join two of these vertices i, j with an edge of T_m if the corresponding edges $a_i\bar{a}_i, a_j\bar{a}_j$ do not cross in the drawing of G_m. Note that to minimize the number of crossings in G_m is to maximize the number of edges in T_m. Also note that since there does not exist four pairwise non-crossing edges in G_m, there is no induced K_4 in T_m. By Turán's Theorem [16] the maximum number of edges in a K_4-free graph on $3m$ vertices is $\binom{3m}{2} - 3\binom{m}{2}$. Hence there are at least $3\binom{m}{2}$ crossings involving edges $a_i\bar{a}_i$ and the claim follows. (See [12] for a similar application of Turán's Theorem to crossing numbers.)

A toroidal drawing of G_m with crossing number $3\binom{3m}{2} + 3\binom{m}{2}$ can be obtained with a small modification of the drawing G_m in Figure 4: Place near the $a_i\bar{a}_i$ group of edges a single handle that carries two-thirds $(= 2m)$ of the edges, as shown in Figure 9. The crossings that occur in this drawing are the $\binom{3m}{2}$ crossings in each of the groupings $b_i\bar{b}_i$, $c_i\bar{c}_i$ and $d_i\bar{d}_i$, plus the $3\binom{m}{2}$ crossings that occur in the $a_i\bar{a}_i$ group near the handle.

After having established Claims 1-4, the proof of Proposition 5.1 is complete. □

6. Conclusion

We conclude with a discussion of some directions for future research.

Initially, the authors felt that a non-convex crossing sequence was not possible. However, we quickly discovered a counter-example with crossing sequence $3, 2, 0$. This was subsequently generalized to the following example. Consider the weighted patched graph illustrated in Figure 10. The thick edges have weights $a \geq b \geq c$, and the optimal planar drawing is the one given. One can easily show that an optimal toroidal drawing is achieved by attaching the ends of a handle to dashed regions 1 and 2 to carry the edge crossing the edge weighted a. The double-toroidal embedding is achieved by adding a second handle to dashed region 3 and the first handle. Hence, the crossing sequence is $a + b + c, b + c, 0$. In other words, one can obtain any crossing sequence $c_0, c_1, c_2 = 0$, where

$$\frac{c_1 - c_2}{c_0 - c_1} \leq 2.$$

For some time, the authors felt that this upper bound on the ratio might be the best possible. However, our Theorem 5.1 now leads us to believe it can be arbitrarily large (see Conjecture 1.1).

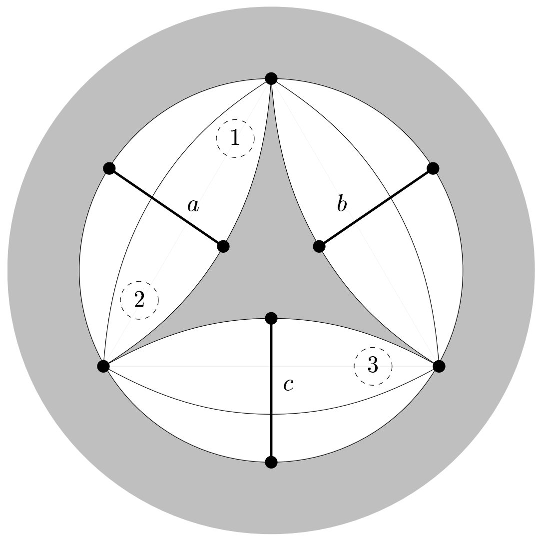

FIGURE 10. First non-convex example

In [4], Archdeacon and Bonnington discuss crossings that arise when two graphs are jointly embedded on the same surface – the only crossings permissible were crossings involving edges from different graphs. If one relaxes this restriction, the notion of jointly embedding graphs can give rise to "joint crossing sequences". Let $cr_k(G, H)$ denote the minimum number of crossings possible in a drawing of G and H simultaneously on the same surface S_k. Now, it is clear that $cr_{2k}(G, G) \leq 2cr_k(G)$. Indeed, to obtain a drawing that achieves this upper bound, take two optimal drawings of G on two disjoint copies of S_k, and "glue" the two surfaces to a single S_{2k}.

Do there exist graphs G for which $cr_{2k}(G, G) < 2cr_k(G)$? (In other words, is there a better way of drawing two copies of G on S_{2k} other than the method described?) Our main theorem implies that (at least for some graphs and $k = 1$) the answer is in the affirmative. Indeed, let G be one of the double-toroidal graphs from Theorem 5.1. Take a double-toroidal embedding of G and insert into one of the faces of this embedding a (planar) drawing of a second copy of G. Therefore, clearly we have $cr_2(G, G) \leq cr_0(G)$. However, by Theorem 5.1, $cr_2(G, G) < 2cr_1(G)$, as required. (In the nonorientable case, one can obtain an even larger difference between $cr_2(G, G)$ and $2cr_1(G)$.)

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