A New Upper Bound on the Cyclic Chromatic Number†

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Abstract: A cyclic coloring of a plane graph is a vertex coloring such that vertices incident with the same face have distinct colors. The minimum number of colors in a cyclic coloring of a graph is its cyclic chromatic number χ^c. Let Δ^* be the maximum face degree of a graph. There exist plane graphs with $\chi^c = \lfloor \frac{3}{2}\,\Delta^* \rfloor$. Ore and Plummer [5] proved that $\chi^c \leq 2\,\Delta^*$, which bound was improved to $\lfloor \frac{9}{5}\,\Delta^* \rfloor$ by Borodin, Sanders, and Zhao [1], and to $\lceil \frac{5}{3}\,\Delta^* \rceil$ by Sanders and Zhao [7].

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We introduce a new parameter k^*, which is the maximum number of vertices that two faces of a graph can have in common, and prove that $\chi^c \leq \max\{\Delta^* + 3k^* + 2, \Delta^* + 14, 3k^* + 6, 18\}$, and if $\Delta^* \geq 4$ and $k^* \geq 4$, then $\chi^c \leq \Delta^* + 3k^* + 2$. J Graph Theory 54: 58–72, 2007

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1. INTRODUCTION

Throughout this article, G is a connected plane graph with vertex set V_G, edge set E_G, and face set F_G. In what follows, G can have multiple edges but no loops, while a *simple* graph has no multiple edges. The *degree* of a vertex v, denoted by $d_G(v)$, is the number of edges incident with v. The *degree* of a face f, denoted by $d_G(f)$, is the number of vertices incident with f. We use Δ_G and Δ_G^* to denote the *maximum vertex degree* and *maximum face degree* of G, respectively.

For a cycle C we denote the sets of vertices of G lying strictly inside C and strictly outside C by $Int_G(C)$ and $Ext_G(C)$, respectively. We say C is a *separating cycle* if both $Int_G(C)$ and $Ext_G(C)$ are not empty.

A *cyclic coloring* of a plane graph is a vertex coloring such that two different vertices incident with the same face receive distinct colors. The minimum number of colors needed for a cyclic coloring, the *cyclic chromatic number*, is denoted by χ_G^c. This concept was introduced by Ore and Plummer [5].

In the remainder the subscript G will often be omitted when it is clear what graph we are dealing with. And instead of, say, "an edge incident with a face" or "a face incident with a vertex," we will sometimes write "an edge of a face" or "a face of a vertex."

It is obvious that a cyclic coloring of a 2-connected plane graph requires at least Δ^* colors. Note that the following plane graph has $\chi^c = \lfloor \frac{3}{2}\Delta^* \rfloor$: Take disjoint triangles $x_1x_2x_3$, $y_1y_2y_3$ and join each x_i with y_i by a path all internal vertices of which have degree 2, where one path has length $\lceil \frac{1}{2}\Delta^* \rceil - 1$, while the other two have length $\lfloor \frac{1}{2}\Delta^* \rfloor - 1$. It is conjectured (see Jensen and Toft [4], page 37) that any plane graph G has $\chi^c \leq \lfloor \frac{3}{2}\Delta^* \rfloor$. Clearly, this bound, if true, would be best possible. Ore and Plummer [5] proved that $\chi^c \leq 2\Delta^*$, which bound was improved to $\lfloor \frac{9}{5}\Delta^* \rfloor$ by Borodin, Sanders, and Zhao [1], and to $\lceil \frac{5}{3}\Delta^* \rceil$ by Sanders and Zhao [7].

In this article we prove a bound for the cyclic chromatic number that depends on Δ^* and the following easily computable parameter of the graph. For a face f in a plane graph G, let $V_G(f)$ be the set of vertices of f. Let k_G^* (or just k^*) be the maximum number of vertices that two faces of G can have in common:

$$k_G^* = \max\{|V_G(f_1) \cap V_G(f_2)| \mid f_1, f_2 \in F_G,\ f_1 \neq f_2\}.$$

Our main result is the following.

Theorem 1.1. *Every connected plane graph G has*

$$\chi_G^c \leq \max\{\Delta_G^* + 3k_G^* + 2, \Delta_G^* + 14, 3k_G^* + 6, 18\}.$$

Observe that for graphs with small enough k^* the bound of Theorem 1.1 is better than any general bound depending on Δ^* only. No serious attempt has been made by the authors to make the additive constants in Theorem 1.1 as small as possible. It seems very likely that our proof method plus some extra detail analysis of special cases can provide smaller values for these constants. However, we do not see how to improve the constant 3 in front of k^*. We suggest the following conjecture, which if true is best possible.

Conjecture 1.2. *Every plane graph G with Δ^*_G and k^*_G large enough has a cyclic coloring with $\Delta^*_G + k^*_G$ colors.*

In particular this conjecture implies $\chi^c_G \le \lfloor \frac{3}{2}\, \Delta^*_G \rfloor$ if Δ^*_G is large enough.

It is easy to check that if a plane graph G is 3-connected, then $k^*_G = 2$. So Theorem 1.1 has the following corollary.

Corollary 1.3. *Every 3-connected plane graph G has $\chi^c_G \le \max\{\Delta^*_G + 14, 18\}$.*

This upper bound on χ^c for 3-connected graphs of the form $\Delta^*_G + O(1)$ is actually close to best possible: the k-wheel W_k on $k+1$ vertices is a 3-connected plane graph with $\chi^c = \Delta^* + 1$. The first bound of this order, $\chi^c \le \Delta^* + 9$, was obtained by Plummer and Toft [6]. They also conjectured that any 3-connected plane graph has $\chi^c \le \Delta^* + 2$. The best known bounds for the cyclic chromatic number of 3-connected plane graphs seem to be $\chi^c \le \Delta^* + 1$ for $\Delta^* \ge 60$ by Enomoto, Horňák, and Jendrol' [2], and $\chi^c \le \Delta^* + 2$ for $\Delta^* \ge 24$ by Horňák and Jendrol' [3].

In the next section we give some further definitions and prove an auxiliary structural result. The proof of Theorem 1.1 itself can be found in Section 3.

2. DEFINITIONS AND STRUCTURAL RESULT

Throughout this section let $\beta \ge 4$ be an integer and G a simple 2-connected plane graph.

By a *triangle* we mean a face of degree three; an *S-face* ("small face") is a face of degree between 4 and $\beta - 1$, while a *B-face* ("big face") is a face of degree at least β. A *BB-edge* is an edge incident with two B-faces; *BS-edges* ("S" for small) and *BT-edges* ("T" for triangle) are defined analogously.

A *d-vertex* is a vertex of degree d. A *BBB-vertex* is a 3-vertex incident with three B-faces. A vertex is called *onerous* if it is either a 3-vertex incident with a triangle and two B-faces, or a 4-vertex incident with two non-adjacent triangles and two B-faces. A triangle is *onerous* if it is incident with three onerous vertices.

We next classify the vertices and edges of G incident with B-faces. An edge is called *separating* if it is a BB-edge, and *irregular* if it is a BS- or BT-edge. A vertex is *separating* if it is an onerous 4-vertex, or a 2-vertex incident with two B-faces; otherwise a vertex is *irregular*. Observe that if $G \ne C_n$, then every B-face of G has at least one irregular element (vertex or edge).

To describe the boundary of a B-face f, we define a *separating path* of f to be a single onerous 4-vertex of f, or a maximal path $P = v_1e_1v_2e_2 \cdots v_{\ell-1}e_{\ell-1}v_\ell$, $\ell \ge 2$, on the boundary of f such that every edge e_i and every internal vertex $v_2, \ldots, v_{\ell-1}$ is separating. By this definition each separating path joins two B-faces in G.

The boundary of a B-face f is called an *irregular path* of f if Q is maximal with the property that every edge e_i and every internal vertex $v_2, \ldots, v_{\ell-1}$ is irregular. If $\ell = 1$, then Q is just one irregular vertex incident with two separating edges of f. It is easy to see that each edge of f belongs to a unique irregular or separating path of f, and each end vertex of an irregular path Q is an irregular vertex or an onerous 4-vertex. Note that if a B-face f has at least one separating element on its boundary, then each irregular path of f divides two separating paths of f.

An irregular path Q is called *onerous* if Q is a single BBB-vertex, or Q contains an edge of an onerous triangle adjacent to f. From the definitions above it follows that each onerous irregular path has at most one edge. A separating path P of f is called *onerous* if it is bounded by two onerous irregular paths (by edges of two onerous triangles if P is formed by one onerous 4-vertex).

We say that a B-face f with at least one separating vertex or edge on its boundary has *dimension* $\dim(f) = m \geq 1$ if f is incident with exactly m separating paths (and m irregular paths). We set $\dim(f) = 0$ if f has no irregular vertex or edge (and hence $G = C_n$). A B-face f is *admissible* if it is incident with at least one onerous vertex or separating 2-vertex. An admissible B-face f of dimension 5 is called *critical* if it has at least 4 onerous irregular paths and each irregular path of f has at most one edge.

We are now ready to give the main structural result.

Theorem 2.1. *Let $\beta \geq 8$ be an integer and G a 2-connected plane graph. Then G has at least one of the following configurations:*

a. *two adjacent triangles;*
b. *a vertex of degree at most 4 incident with at most one B-face;*
c. *an admissible B-face of dimension at most 4 incident with at most 5 irregular edges;*
d. *two B-faces f_1, f_2 joined by an onerous separating path $P_{12} = v_1 e_1 \cdots e_{\ell-1} v_\ell$, where f_1 is critical, $\dim(f_2) \leq 6$, and f_2 has at most 4 irregular edges that are not incident with v_1, v_ℓ.*

Proof. We first show that it suffices to prove Theorem 2.1 for plane graphs without onerous 4-vertices. Let G be an arbitrary 2-connected plane graph. We form a new graph G_1 by replacing each onerous 4-vertex v in G incident with triangles vxy and vzt by a pair of onerous 3-vertices v_1, v_2, where v_1 is adjacent to v_2, x, y, while v_2 is adjacent to v_1, z, t. By this definition, G_1 is 2-connected and has the same set of triangles, B-faces, S-faces, and irregular edges as G. Moreover, for every B-face f we have $\dim_G(f) = \dim_{G_1}(f)$. Observe that every onerous element (vertex, triangle, irregular, or separating path) of G corresponds to an onerous element (or a pair of onerous elements) of the same type in G_1. It follows that if some claim of Theorem 2.1 holds for G_1 then it is also valid for G.

So assume that $\beta \geq 8$ is an integer and G is a counterexample to Theorem 2.1 without onerous 4-vertices. We next establish the following properties of G:

1. G has no adjacent triangles;
2. $\delta_G \geq 2$;
3. every vertex of degree at most 4 is incident with at least two B-faces;
4. every 2-vertex is separating;
5. every 3-vertex is either an onerous vertex, a BBB-vertex, or is incident with two B-faces and one S-face;
6. G has no onerous 4-vertex;

6$'$. every 4-vertex is incident with at most one triangle;

7. every d-vertex, $d \geq 5$, is incident with at most $\lfloor d/2 \rfloor$ triangles;
8. an admissible B-face of dimension at most 4 has at least 6 irregular edges;
9. every two irregular paths of a B-face are vertex-disjoint;
10. if a critical B-face f_1 is joined through an onerous separating path $P_{12} = v_1 e_1 \cdots e_{\ell-1} v_\ell$ with another B-face f_2, then $\dim(f_2) \geq 7$ or f_2 has at least 5 irregular edges that are not incident with v_1, v_ℓ.

Claims (1), (3), (6), (8), (10) are directly implied by the assumptions made and the fact that G fails to satisfy any of (a)–(d) in Theorem 2.1; (2) follows from the 2-connectedness of G; (4) and (5) are consequences of (3); while (6$'$) follows from (1), (3), and (6). Claims (7) and (9) follow from (1) and (6), respectively.

Euler's Formula $|V_G| - |E_G| + |F_G| = 2$ for G can be rewritten as

$$\sum_{x \in V_G \cup F_G} (d(x) - 4) = \sum_{x \in V_G \cup F_G} \mu_1(x) = -8,$$

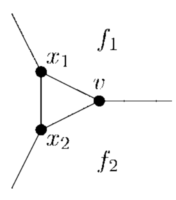
where $\mu_1(x) = d(x) - 4$ is called the initial *charge* of an element (vertex or face) x. By (2), only triangles and vertices of degree 2 and 3 have negative initial charge.

We next redistribute initial charges according to the following rules:

(**R1**) A 2-vertex receives 1 from each incident B-face.

(**R2**) Let v be a 3-vertex incident only with B- and S-faces. Then v receives $1/3$ from each incident B-face if v is a BBB-vertex, and $1/2$ from each incident B-face if v is incident with exactly two B-faces.

(**R3**) Let v be an onerous 3-vertex incident with a triangle vx_1x_2 and B-faces $f_1 = vx_1 \ldots$ and $f_2 = vx_2 \ldots$ (see sketch below).



If $d(x_1) = 3$ and $d(x_2) > 3$, then v receives $1/2$ from f_1 and $5/6$ from f_2. If $d(x_1) = d(x_2) = 3$ or $d(x_i) > 3$, $i = 1, 2$, then v receives $2/3$ from both f_1 and f_2.

(R4) Let v be a 4-vertex incident with a triangle T and (non-triangular) faces f_1, f_2, and f_3 in a cyclic order. Then v receives $1/6$ from both f_1 and f_3 if f_1 and f_3 are B-faces, or v receives $1/6$ from f_1 and f_2 if f_3 is an S-face and (hence) f_1 and f_2 are B-faces.

(R5) A triangle receives $1/3$ from each incident vertex.

(R6) Let v be a vertex of degree at least 5 incident with a triangle T_1, a B-face f and a triangle T_2 in a cyclic order. Then v gives $1/3$ to f.

Denote the resulting charge of an element $x \in V_G \cup F_G$ after applying rules (R1)–(R6) by $\mu_2(x)$. Because we always move charge from one element to another,

$$\sum_{x \in V_G \cup F_G} \mu_2(x) = \sum_{x \in V_G \cup F_G} \mu_1(x) = -8.$$

We next check that all vertices and most faces of G have a non-negative charge μ_2. First consider vertices.

Lemma 2.2. *Every $v \in V_G$ has $\mu_2(v) \geq 0$.*

Proof. If $d(v) \leq 4$, then by (2)–(6′) and (R1)–(R5), we have $\mu_2(v) = 0$. If v is a 5-vertex, then by (7) and (R5)–(R6), v gives $1/3$ to at most two triangles and at most one B-face. Therefore, in this case we have $\mu_2(v) \geq 1 - 2 \times 1/3 - 1/3 = 0$. Finally, if $d(v) \geq 6$, then v sends at most $1/3$ to each incident face by (R5)–(R6). Hence, $\mu_2(v) \geq d(v) - 4 - d(v) \times 1/3 = 2\,(d(v) - 6)/3 \geq 0$. ■

We now start looking at the faces. If T is a triangle, then by (R5), $\mu_2(T) = -1 + 3 \times 1/3 = 0$. Note that an S-face never sends or receives charge by any rule (R1)–(R6). Therefore, for any such face f we have $\mu_2(f) = \mu_1(f) \geq 0$. This implies the following property.

Lemma 2.3. *If $f \in F_G$ is a triangle or an S-face, then $\mu_2(f) \geq 0$.*

So we are left with B-faces. By $c_f(v)$ denotes the amount of charge that a B-face f gives to one of its vertices v by rules (R1)–(R4) (it may happen that $c_f(v) = 0$), and set $c_f(v) = -1/3$ if f receives $1/3$ from v by (R6). We say that a B-face f *saves* charge $sc_f(v) = 1 - c_f(v)$ on its vertex v. It follows from (R1)–(R4) and (R6) that $sc_f(v) = 0$ if and only if $d(v) = 2$ (i.e., v is a separating vertex), and $sc_f(v) \geq 1/6$ otherwise (and then v is an irregular vertex). Furthermore, $sc_f(v) \geq 5/6$ if $d(v) \geq 4$, and $sc_f(v) \geq 1$ if $d(v) \geq 5$. If $Q = v_1 e_1 v_2 \cdots e_{\ell-1} v_\ell$ is an irregular path of f then we say that f *saves* charge $sc_f(Q) = \sum_{i=1}^{\ell} sc_f(v_i)$ on Q. Note that by (9), any two irregular paths of f are vertex disjoint, so if v is an irregular vertex of f, then

$sc_f(v)$ is counted in exactly one $sc_f(Q)$. Because of (6) this implies

$$\mu_2(f) = \sum_{i=1}^{m} sc_f(Q_i) - 4, (*)$$

where $m = \dim(f)$ and $Q_1, \dots, Q_m$ are the irregular paths of f. In particular, $\mu_2(f) \geq 0$ if and only if f saves the total of at least 4 on its irregular paths.

The next claim determines the amount of charge that a B-face can save on its irregular path.

Proposition 2.4. *Let $Q = v_1 e_1 v_2 \cdots e_{\ell-1} v_\ell$, $\ell \geq 1$, be an irregular path of a B-face f.*

a. *If Q is onerous, then $sc_f(Q) = 2/3$.*
b. *If Q is not onerous, then $sc_f(Q) \geq 1$.*
c. *If $2 \leq i \leq \ell - 1$, then $sc_f(v_i) \geq 5/6$.*
d. *If $\ell = 3$, then $sc_f(Q) \geq 3/2$.*
e. *If $\ell \geq 4$, then $sc_f(Q) \geq (5\,\ell - 8)/6$.*

Proof.

a. This part follows from (R2) and (R3).
b. Suppose Q is not onerous. First assume $Q = \{v_1\}$, that is, $\ell = 1$. Let u, w be the neighbors of v_1 on the boundary of f. Since Q is an irregular path, both $v_1 u$ and $v_1 w$ are separating. From this we see that $c_f(v_1) = 0$, hence $sc_f(Q) = sc_f(v_1) = 1$.

   So assume $\ell \geq 2$ and consider the edge $v_1 v_2$ of Q. If $v_1 v_2$ is a BS-edge, then by (R2) and (R4) we have $sc_f(v_1) \geq 1/2$, $sc_f(v_2) \geq 1/2$, and hence $sc_f(Q) \geq 1$. So we are left with the case when $v_1 v_2$ is a BT-edge and $sc_f(v_1) < 1/2$. The last inequality, in particular, implies $d(v_1) = 3$. Since Q is not an onerous irregular path, applying (R3) to v_1 shows that $d(v_2) > 3$. Finally we get $sc_f(Q) \geq sc_f(v_1) + sc_f(v_2) \geq 1/6 + 5/6 = 1$.
c. Since $v_{i-1} v_i$, $v_i v_{i+1}$ are irregular edges, hence non-BB-edges, it follows that v_i is incident with at least two non-B-faces in G. Taking into account (3), this implies that $d(v_i) \geq 4$.

   If $d(v_i) = 4$, then by (R4) we have $c_f(v_i) = 1/6$ or $c_f(v_i) = 0$, hence $sc_f(v_i) \geq 5/6$. If $d(v_i) \geq 5$, then from (R6) we get that $c_f(v_i) = -1/3$ or $c_f(v_i) = 0$, hence $sc_f(v_i) \geq 1$.
d. If both $v_1 v_2$ and $v_2 v_3$ are BS-edges, then by (R2) and (R4) we have $sc_f(v_1) \geq 1/2$, $sc_f(v_2) = 1$, and $sc_f(v_3) \geq 1/2$, which implies that $sc_f(Q) \geq 1/2 + 1 + 1/2 > 3/2$. If $v_1 v_2$ is a BS-edge while $v_2 v_3$ is a BT-edge, then it follows from (R2), (R4), and (c) that $sc_f(v_1) \geq 1/2$, $sc_f(v_2) \geq 5/6$, and $sc_f(v_3) \geq 1/6$. Thus, $sc_f(Q) \geq 3/2$. Finally, assume that both $v_{i-1} v_i$ and $v_i v_{i+1}$ are

BT-edges. In this case (3) and (6) show that $d(v_i) \geq 5$, so by (R6) we have $sc_f(v_2) = 4/3$. This gives $sc_f(Q) \geq 1/6 + 4/3 + 1/6 > 3/2$.

e. Applying (c) yields $sc_f(Q) \geq (\ell - 2) \times 5/6 + 2 \times 1/6 = (5\,\ell - 8)/6$. ∎

We are now ready to describe the faces of G with a negative charge μ_2.

Lemma 2.5. *Let $f \in F_G$ be a face with $\mu_2(f) < 0$. Then f is a critical B-face and one of the following statements holds:*

a. $\mu_2(f) = -2/3$*, and f has five onerous irregular paths;*
b. $\mu_2(f) \geq -1/3$*, and f has exactly four onerous irregular paths.*

Proof. By Lemma 2.3, f is a B-face. Assume that f is not admissible. Then according to (R2), (R4), and (R6), f gives at most $1/2$ to each incident vertex. This implies $\mu_2(f) \geq d(f) - 4 - d(f)/2 = (d(f) - 8)/2 \geq 0$, a contradiction.

Denote the number of vertices in the longest irregular path of f by ℓ. If $\dim(f) = 1$ or f has no separating edge, then $\ell \geq 7$ by (8). Using $(*)$ and Proposition 2.4(c) gives $\mu_2(f) = sc_f(Q) - 4 \geq (5 \cdot 7 - 8)/6 - 4 = 1/2 > 0$.

Let $\dim(f) = m \geq 2$, and let $Q_1, \ldots, Q_m$ be the irregular paths of f. W.l.o.g., we can assume that Q_1 has ℓ vertices. First consider the case $m = 2$. Claim (8) shows that $\ell \geq 4$. If $\ell \geq 6$, then by $(*)$ and Proposition 2.4(a), (b), (e), we have $\mu_2(f) = sc_f(Q_1) + sc_f(Q_2) - 4 \geq (5 \cdot 6 - 8)/6 + 2/3 - 4 = 1/3 > 0$. If $\ell = 5$, then Q_2 has at least three vertices due to (8). Applying $(*)$ and Proposition 2.4(d), (e) yields $\mu_2(f) \geq (5 \cdot 5 - 8)/6 + 3/2 - 4 = 1/3 > 0$. Finally, if $\ell = 4$, then both Q_1 and Q_2 have four vertices by (8), and hence $\mu_2(f) \geq 2 \times (5 \cdot 4 - 8)/6 - 4 = 0$.

Suppose $m = 3$. It follows from (8) that $\ell \geq 3$, and if $\ell = 3$, then each irregular path of f has three vertices. If this is the case, then $(*)$ and Proposition 2.4(d) imply that $\mu_2(f) = 3 \times 3/2 - 4 = 1/2 > 0$. If $\ell = 4$, then claim (8) shows that either Q_2 or Q_3 has at least three vertices. Using $(*)$ and Proposition 2.4, we obtain $\mu_2(f) \geq (5 \cdot 4 - 8)/6 + 3/2 + 2/3 - 4 = 1/6 > 0$. If $\ell \geq 5$, then from $(*)$ and Proposition 2.4(a), (b), (e) we get $\mu_2(f) \geq (5 \cdot 5 - 8)/6 + 2 \times 2/3 - 4 = 1/6 > 0$.

Let $m = 4$. Again from (8) we obtain $\ell \geq 3$. If $\ell \geq 4$, then $\mu_2(x) \geq (5 \cdot 4 - 8)/6 + 3 \times 2/3 - 4 = 0$ due to $(*)$ and Proposition 2.4. If $\ell = 3$, then, by (8), f has at least two irregular paths with three vertices. Thus, $\mu_2(f) \geq 2 \times 3/2 + 2 \times 2/3 - 4 = 1/3 > 0$.

If $m \geq 6$, then $\mu_2(f) \geq 6 \times 2/3 - 4 = 0$ by $(*)$ and Proposition 2.4(a), (b).

Finally, we come to the conclusion that $m = 5$. If $\ell \geq 3$, then from $(*)$ and Proposition 2.4 we get $\mu_2(f) \geq 3/2 + 4 \times 2/3 - 4 = 1/6 > 0$. Hence each irregular path of f has at most one edge. If f has at most three onerous irregular paths, then $\mu_2(f) \geq 3 \times 2/3 + 2 \cdot 1 - 4 = 0$ by $(*)$ and Proposition 2.4(a), (b). So either f has five onerous irregular paths and then $\mu_2(f) = 5 \times 2/3 - 4 = -2/3$ by Proposition 2.4(a), or f has exactly four onerous irregular paths and $\mu_2(f) \geq 4 \times 2/3 + 1 - 4 = -1/3$ due to Proposition 2.4(a), (b). Clearly, in the first case we have the situation of claim (a), while the second implies (b). ∎

From now on, for a critical B-face we say that it is either of type (a) or of type (b), according to Lemma 2.5. We see that a critical face of type (a) has five onerous separating paths, while a critical face of type (b) has three onerous separating paths. From (10) we know that every onerous separating path of a critical face f joins f with another B-face having specific properties. At this point we introduce another rule of charge distribution:

(**R7**) Let f_1 be a critical B-face joined through an onerous separating path with another B-face f_2. Then f_2 gives $1/6$ to f_1.

Denote the resultant charge of an element (vertex or face) x after applying rules (R1)–(R7) by $\mu_3(x)$. Clearly, $\sum_{x \in V_G \cup F_G} \mu_3(x) = -8$. The final contradiction in proving Theorem 2.1 now follows from the following lemma.

Lemma 2.6. *Every $x \in V_G \cup F_G$ has $\mu_3(x) \geq 0$.*

Proof. Since (R7) deals only with specific B-faces described in (10), it follows from the Lemmas 2.2, 2.3, and 2.5 that if $x \in V_G \cup F_G$ is not such a face then $\mu_3(x) = \mu_2(x) \geq 0$.

If f is a critical face of type (a), then Lemma 2.5(a) implies $\mu_2(f) = -2/3$, and f is incident with five onerous separating paths. Applying (R7) gives $\mu_3(f) = -2/3 + 5 \times 1/6 = 1/6 > 0$. If f is a critical face of type (b), then Lemma 2.5 (b) shows that $\mu_2(f) \geq -1/3$, and f is incident with three onerous separating paths. In this case, $\mu_3(f) \geq -1/3 + 3 \times 1/6 = 1/6 > 0$.

Suppose f is a B-face which gives charge to at least one critical face f_1 by (R7). Let $P_1 = v_1 e_1 \cdots e_{\ell-1} v_\ell$ be an onerous separating path between f and f_1. It follows from (10) that if $\dim(f) \leq 6$, then f has at least five irregular edges that are not incident with v_1, v_ℓ. Since P_1 is bounded by two onerous irregular paths Q_1, Q_2 of f and each Q_i has at most one edge, $\dim(f) = m \geq 3$. If $m \geq 8$, then, using $(*)$, (R7), and Proposition 2.4(a), (b), we obtain $\mu_3(f) \geq m \times 2/3 - 4 - m \times 1/6 = (m-8)/2 \geq 0$.

So assume that $3 \leq m \leq 7$. First, we provide a lower bound on $\mu_2(f)$. If $m = 7$, then $\mu_2(f) \geq 7 \times 2/3 - 4 = 2/3$, due to $(*)$ and Proposition 2.4(a), (b). If $m \leq 6$, then by (10) there are at least five edges in the irregular paths of f other than Q_1 and Q_2. Direct calculations similar to those in proving Lemma 2.5 combined with $(*)$ and Proposition 2.4 show that $\mu_2(f) \geq 1$ if $m = 3$, $\mu_2(f) \geq 5/6$ if $m \in \{4, 6\}$, and $\mu_2(f) \geq 2/3$ if $m = 5$. This implies $\mu_3(f) \geq 5/6 - 4 \times 1/6 = 1/6 > 0$ if $m \leq 4$. Furthermore, in the case $5 \leq m \leq 7$ we still have $\mu_3(f) \geq 2/3 - 4 \times 1/6 = 0$ provided that f makes at most four donations of $1/6$ by (R7). Since $m \leq 7$, it suffices to prove that it is impossible for f to give charge to three consecutive adjacent B-faces by (R7).

Suppose there are three consecutive onerous separating paths P_1, P_2, P_3 on the boundary of f joining f with critical faces f_1, f_2, f_3, respectively. By the definition of an onerous separating path, the faces f, f_1, f_2 either have a BBB-vertex in common or are adjacent to a common onerous triangle, and the same is true for f, f_2, f_3.

This means that there exist separating paths P_{12}, P_{23} joining f_2 with f_1, f_3, respectively. Since f_2 is critical, it has a sequence of at least three consecutive onerous separating paths on its boundary. In particular, at least one of P_{12}, P_{23} must be onerous. However, since each of f_i is critical and has dimension 5, this contradicts (10). ■

This completes the proof of Theorem 2.1. ■

3. PROOF OF THEOREM 1.1

Throughout this section we fix $\beta = 8$. For a plane graph G we set

$$M_G^* = \max\{\Delta_G^* + 3\,k_G^* + 2,\, \Delta_G^* + 14,\, 3\,k_G^* + 6,\, 18\}.$$

Suppose G is a counterexample to Theorem 1.1 with the fewest edges. Note that if a plane graph H satisfies $\Delta_H^* \le \max\{\Delta_G^*, 4\}$ and $k_H^* \le \max\{k_G^*, 4\}$, then $M_H^* \le M_G^*$.

We first prove some structural properties of G and then apply Theorem 2.1 to show that G cannot exist.

Lemma 3.1. *G has no multiple edges.*

Proof. Suppose G has edges e_1, e_2, both joining vertices v_1 and v_2. If the cycle $C = v_1e_1v_2e_2v_1$ is not separating, then removing e_2 gives a graph H with $\Delta_H^* = \Delta_G^*$ and $k_H^* = k_G^*$, so $M_H^* = M_G^*$. By minimality of G, H has a cyclic coloring with at most M_G^* colors, which is also a cyclic coloring of G with at most M_G^* colors, a contradiction.

If C is separating, then denote the subgraphs of G induced by $C \cup Int(C)$ and $C \cup Ext(C)$ by G_1 and G_2, respectively. It is straightforward that $\Delta_{G_i}^* \le \Delta_G^*$ and $k_{G_i}^* \le k_G^*$, $i = 1, 2$. By minimality of G, both G_1 and G_2 can be colored with at most M_G^* colors. Since G_1 and G_2 have only two vertices in common and each face of G is also a face in G_1 or in G_2, we can combine the colorings of G_1 and G_2 to obtain a cyclic coloring of G using at most M_G^* colors. ■

Lemma 3.2. *G is 2-connected.*

Proof. Suppose G has a cut vertex z, and let G_1, G_2 be two subgraphs obtained by separating G along z. Again we can color both G_1 and G_2 with at most M_G^* colors. Also, G_1 has a face f_1 and G_2 has a face f_2 whose boundaries together form the boundary of a face f in G. Since $M_G^* > \Delta_G^* + 1 \ge |V_G(f)| + |\{z\}| = |V_G(f_1)| + |V_G(f_2)|$, we can use different colors for all vertices of f_1 and f_2, and use the same color for z. Combining the colorings of G_1 and G_2 gives a coloring of G. ■

By Lemmas 3.1 and 3.2, G is a simple 2-connected graph. Hence G must have one of the configurations described in Theorem 2.1.

Lemma 3.3. *G has no adjacent triangles.*

Proof. Suppose G has adjacent triangles $T_1 = uvx$, $T_2 = uvy$. Remove the edge uv from G. The resultant graph H has fewer edges than G and has only one face $f = uxvy$ which is not in G. Since $d_H(f) = 4$, f has at most four vertices in common with any other face and hence $\Delta^*_H \le \max\{\Delta^*_G, 4\}$ and $k^*_H \le \max\{k^*_G, 4\}$. Therefore, H has a cyclic coloring using at most $M^*_H \le M^*_G$ colors, which is also a cyclic coloring of G. ■

A *cyclic neighbor* of a vertex v is a vertex $u \ne v$ such that there is a face incident with both u and v. The *cyclic degree* $d^c_G(v)$ of a vertex v in G is the number of cyclic neighbors of v.

Proposition 3.4. *G cannot have a vertex of degree at most* 4 *and cyclic degree at most $M^*_G - 1$.*

Proof. Suppose v is such a vertex with degree $d \le 4$. Denote the neighbors of v in a cyclic order by $u_1, u_2, \dots, u_d$. Form the plane graph H by removing the vertex v and adding edges $u_1u_2, u_2u_3, \dots, u_{d-1}u_d, u_du_1$. By this definition, H has fewer vertices than G and the new face formed by the edges u_iu_{i+1} has degree at most 4, so $\Delta^*_H \le \max\{\Delta^*_G, 4\}$, $k^*_H \le \max\{k^*_G, 4\}$. Hence, H has a cyclic coloring using at most M^*_G colors. This also gives a cyclic coloring of G with at most M^*_G colors in which v is not colored yet. Since $d^c_G(v) \le M^*_G - 1$, there is at least one color not appearing on the cyclic neighbors of v. Hence the coloring can be extended to a cyclic coloring of G with at most M^*_G colors, a contradiction. ■

Lemma 3.5. *G cannot have a vertex of degree at most* 4 *incident with at most one B-face.*

Proof. The cyclic degree of a vertex v is at most the sum of the degrees of the faces incident with v subtracted by $2\,d_G(v)$. Indeed, v itself is counted in each of these face degrees, and each neighbor of v is counted in at least two of such degrees. Since a non-B-face has degree at most 7, while a B-face has degree at most Δ^*_G, it follows that any vertex v with $d_G(v) \le 4$ and that is incident with at most one B-face has $d^c_G(v) \le \Delta^*_G + 3 \cdot 7 - 2 \cdot 4 = \Delta^*_G + 13 \le M^*_G - 1$, contradicting Proposition 3.4. ■

At this point we know that G must have one of the structures (c), (d) in Theorem 2.1. In order to show that these options also lead to a contradiction, we do some further analysis of the structure of B-faces and separating paths of G.

Property 3.6. *A separating path of a B-face has at most k^*_G vertices.*

Proof. Indeed, any such path lies on the boundary of two different B-faces. ■

Proposition 3.7. *Let v be a* 2*-vertex or an onerous vertex incident with a B-face f_1 of dimension m. If the face f_1 has at most t irregular edges on its boundary, then $d^c_G(v) \le \Delta^*_G + (m-1)\,k^*_G + t - m - 1$.*

Proof. Suppose v is incident with a B-face f_1 of dimension m and f_1 has at most t irregular edges. By Lemma 3.5, v is also incident with another B-face f_2. First observe that every cyclic neighbor of v is incident with either f_1 or f_2. This is clear if v is a 2-vertex. If v is onerous, then v is incident with one or two triangles. However, it follows from the definition of an onerous vertex that the vertices of these triangles are also incident with either f_1 or f_2.

By the above, v is a separating vertex or an onerous 3-vertex and hence belongs to a separating path P_1 joining f_1 with f_2. Let $P_2, \ldots, P_m$ be the other separating paths of f_1. Denote the number of irregular paths of f_1 consisting of a single vertex by m_1. Since f_1 has dimension m, there are exactly $m_2 = m - m_1$ irregular paths of f_1 having at least one edge. Clearly, each end vertex of an irregular path is also an end vertex of some separating path. So if an irregular path consists of a single vertex x, then x is an end vertex of two separating paths of f_1. Hence f_1 has m_1 vertices that are covered twice by separating paths. On the other hand, every irregular path of f_1 with $r \geq 1$ edges has $r - 1$ internal vertices that are not covered by separating paths. As f_1 has m_2 such irregular paths formed by at most t irregular edges, the total number of vertices of f_1 not covered by separating paths can be at most $t - m_2$.

These arguments, combined with Property 3.6 and the fact that every vertex of P_1 is incident with f_2, yield

$$\begin{aligned} d_G^c(v) &\leq d_G(f_2) - 1 + |V_G(P_2)| + \cdots + |V_G(P_m)| - m_1 + t - m_2 \\ &\leq \Delta_G^* - 1 + (m-1)\,k_G^* + t - m. \end{aligned}$$

■

Lemma 3.8. *G cannot have an admissible B-face of dimension at most* 4 *incident with at most* 5 *irregular edges.*

Proof. Suppose f is such a face. Since f is admissible, it has a vertex v which is either a separating 2-vertex or an onerous vertex. Using $t = 5, m \leq 4$, and $k_G^* \geq 2$ in Proposition 3.7, we deduce that $d_G^c(v) \leq \Delta_G^* + 3k_G^* < M_G^* - 1$, a contradiction with Proposition 3.4. ■

Proposition 3.9. *A critical B-face cannot have two adjacent BBB-vertices on its boundary.*

Proof. Let f be such a face, and let v_1, v_2 be adjacent BBB-vertices on its boundary. Then $e = v_1v_2$ is a BB-edge and $P = v_1\, e\, v_2$ is an onerous separating path of f. An easy analysis as in the proof of Proposition 3.7 and Lemma 3.8 shows that f is incident with an onerous or separating vertex v such that $d_G^c(v) \leq \Delta_G^* - 1 + |V_G(P)| + 3k_G^* + 5 - 5 = \Delta_G^* + 3k_G^* + 1 \leq M_G^* - 1$. Again we obtain a contradiction with Proposition 3.4. ■

Using Theorem 2.1 and the previous claims in this section, we conclude that G has B-faces f_1 and f_2 as described in Theorem 2.1(d). In particular, f_1 is a critical

B-face joined with f_2 through an onerous separating path $P_{12} = v_1 e_1 \cdots e_{\ell-1} v_\ell$. The definition of an onerous separating path shows that there is a unique B-face $f_3 \notin \{f_1, f_2\}$ incident with v_1 if v_1 is a BBB-vertex, or with the onerous triangle incident with v_1 if v_1 is an onerous vertex. Similarly, at the other end of the path P_{12} we can find a unique B-face f_4.

By the definition of an onerous irregular path there exists a separating path P_{13} which joins f_1 with f_3 and starts at the vertex a_{13} which can be v_1 or a vertex of an onerous triangle incident with v_1. Let b_{13} be the other end vertex of P_{13} (and hence we have $a_{13} = b_{13}$ if the path is just one onerous 4-vertex). Similarly, we can find a separating path P_{14} between f_1 and f_4 with end vertices a_{14}, b_{14}, a separating path P_{23} between f_2 and f_3 with end vertices a_{23}, b_{23}, and a separating path P_{24} between f_2 and f_4 with end vertices a_{24}, b_{24}.

Note that if $a_{13} \neq v_1$, then a_{13} is an onerous vertex, and hence all its cyclic neighbours are in $V_G(f_1) \cup V_G(f_3)$. The same holds for any internal vertex of P_{13}, if such a vertex exists, and for the other paths too.

Put $X = V_G(P_{12})$, $Y_3 = V_G(P_{13}) \setminus (X \cup \{b_{13}\})$, $W_3 = V_G(P_{23}) \setminus (X \cup \{b_{23}\})$, $Y_4 = V_G(P_{14}) \setminus (X \cup \{b_{14}\})$, and $W_4 = V_G(P_{24}) \setminus (X \cup \{b_{24}\})$. From Proposition 3.9 it follows that there is a vertex $x \in X$ which is either separating or onerous. Therefore, the face f_2 is admissible, and Lemma 3.8 shows that $\dim(f_2) \geq 3$. Although X is not empty, any of Y_3, W_3, Y_4, W_4 may be empty. Also, since both f_1 and f_2 have dimension at least three, all these sets are disjoint. Finally, from the previous paragraph we obtain that all vertices in Y_3 have cyclic neighbors in $V_G(f_1) \cup V_G(f_3)$, and similarly for W_3, Y_4, W_4.

Let the neighbors of the vertex x be $u_1, u_2, \ldots, u_d$ in a cyclic order. We form the plane graph H by removing the vertex x and adding edges $u_1u_2, u_2u_3, \ldots, u_{d-1}u_d$, u_du_1. Then H has fewer vertices than G. Also, the new face formed by the edges u_iu_{i+1} has degree at most four and hence has at most four vertices in common with any other face. This means that $\Delta^*_H \leq \max\{\Delta^*_G, 4\}$ and $k^*_H \leq \max\{k^*_G, 4\}$. So H has a cyclic coloring using at most M^*_G colors. This also gives a cyclic coloring of G with at most M^*_G colors where x is not colored yet.

Proposition 3.10. *There exist vertices in Y_3 and in Y_4 whose colors do not appear on vertices of f_2. (In particular, Y_3 and Y_4 are not empty.)*

Proof. Suppose all the colors of vertices in Y_3 also appear at f_2. Then the number of colors appearing on the cyclic neighbors of x is at most

$$|V_G(f_2)| - 1 + |V_G(f_1) \setminus (X \cup Y_3)| \leq \Delta^*_G - 1 + 3\,k^*_G + 1 < M^*_G - 1.$$

Here, we use that $\dim(f_1) = 5$, each irregular path of f_1 has at most one edge, and $X \cup Y_3 = V_G(P_{12}) \cup V_G(P_{13}) \setminus \{b_{13}\}$ contains all but one of the vertices of two separating paths. Thus, x can be colored with a color different from the colors of its cyclic neighbors, a contradiction.

The same argument works for Y_4. ■

Proposition 3.11. *The color of every vertex in $W_3 \cup W_4$ also appears at f_1.*

Proof. Suppose there is a vertex $w_3 \in W_3$ whose color c_w does not appear at f_1. Then after removing the color from w_3, we can color x with c_w. Now we can not find a new color for w_3 only if its cyclic neighbors use all $M_G^* \geq \Delta_G^* + 3\,k_G^* + 2$ colors. Since w_3 has at most $\Delta_G^* - 1$ cyclic neighbors from f_3, there is a set C of at least $3\,k_G^* + 2$ colors that appear on vertices in $V_G(f_2) \setminus \{x, w_3\}$ but not appear at f_3.

By Proposition 3.10 there is a vertex $y_3 \in Y_3$ whose color c_y does not appear at f_2. So after removing the color from y_3, we can color x with c_y. Exactly as in the previous paragraph we conclude that there is the same set C of at least $3\,k_G^* + 2$ colors appearing on vertices in $V_G(f_1) \setminus \{x, y_3\}$. Hence, the number of colors used for the cyclic neighbors of x is at most

$$|V_G(f_2)| - 1 + |V_G(f_1)| - |C| \leq \Delta_G^* - 1 + 5\,k_G^* - (3\,k_G^* + 2) < M_G^* - 1.$$

Thus, x can be colored with a color different from any of its cyclic neighbors, a contradiction.

The same argument works for W_4. ∎

By Proposition 3.11, every color of a vertex in $W_3 \cup W_4$ appears at f_1. Recall that $\dim(f_2) \leq 6$ and f_2 has at most four irregular edges that are not incident with the end vertices of P_{12}. Since the colors of the vertices in $X \cup W_3 \cup W_4$ occur on f_1, and since $X \cup W_3 \cup W_4$ contains all but two of the vertices of three separating paths of f_2, it follows that the maximal number of colors appearing on cyclic neighbors of x is

$$\begin{aligned} &|V_G(f_1)| - 1 + |V_G(f_2) \setminus (X \cup W_3 \cup W_4)| \\ &\quad \leq \Delta_G^* - 1 + 3\,k_G^* + 4 - 4 + 2 \leq M_G^* - 1. \end{aligned}$$

So again we can find a suitable color for x, the final contradiction in the proof of Theorem 1.1. ∎

We do not think that our proof approach can be extended to prove Conjecture 1.2, but we hope that our article opens new perspectives towards proving that $\chi^c \leq \lfloor \frac{3}{2}\,\Delta^* \rfloor$ for plane graphs in general, and $\chi^c \leq \Delta^* + 1$ for 3-connected plane graphs.

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