# A NOTE ON COMPLETE SUBDIVISIONS IN DIGRAPHS OF LARGE OUTDEGREE 

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#### Abstract

Mader conjectured that for all $\ell$ there is an integer $\delta^{+}(\ell)$ such that every digraph of minimum outdegree at least $\delta^{+}(\ell)$ contains a subdivision of a transitive tournament of order $\ell$. In this note we observe that if the minimum outdegree of a digraph is sufficiently large compared to its order then one can even guarantee a subdivision of a large complete digraph. More precisely, let $\vec{G}$ be a digraph of order $n$ whose minimum outdegree is at least $d$. Then $\vec{G}$ contains a subdivision of a complete digraph of order $\left\lfloor d^{2} /\left(8 n^{3 / 2}\right)\right\rfloor$.


## 1. Introduction

A fundamental result of Mader [4] states that for every integer $\ell$ there is a smallest $d=d(\ell)$ so that every graph of average degree at least $d$ contains a subdivision of a complete graph on $\ell$ vertices. Bollobás and Thomason [1] as well as Komlós and Szemerédi (3) showed that $d(\ell)$ is quadratic in $\ell$. In [6], Mader made the following conjecture, which would provide a digraph analogue of these results (a transitive tournament is a complete graph whose edges are oriented transitively).

Conjecture 1 (Mader [6]). For every integer $\ell>0$ there is a smallest integer $\delta^{+}(\ell)$ such that every digraph $\vec{G}$ with minimum outdegree at least $\delta^{+}(\ell)$ contains a subdivision of the transitive tournament on $\ell$ vertices.

It is easy to see that $\delta^{+}(\ell)=\ell-1$ for $\ell \leq 3$. Mader [7] showed that $\delta^{+}(4)=3$. Even the existence of $\delta^{+}(5)$ is not known. One might be tempted to conjecture that large minimum outdegree would even force the existence of a subdivision of a large complete digraph (a complete digraph has a directed edge from $v$ to $w$ for any ordered pair $v, w$ of vertices). However, for all $n$ Thomassen [9] constructed a digraph on $n$ vertices whose minimum outdegree is at least $\frac{1}{2} \log _{2} n$ but which does not contain an even directed cycle (and thus no complete digraph on 3 vertices). The additional assumption of large minimum indegree in Conjecture 1 does not help either. Mader [6 modified the construction in 9] to obtain digraphs having arbitrarily large minimum indegree and outdegree without a subdivision of a complete digraph on 3 vertices.

The fact that one certainly cannot replace the minimum outdegree in Conjecture $\square$ by the average degree is easy to see: consider the complete bipartite graph with equal size vertex classes and orient all edges from the first to the second class. The resulting digraph $\vec{B}$ has average degree $|\vec{B}| / 2$ but not even a directed cycle or a transitive tournament on 3 vertices. (On the other hand, Jagger [2] showed that if the average degree of a digraph $\vec{G}$ is a little larger than $|\vec{G}| / 2$, then $\vec{G}$ does contain a subdivision of a large complete digraph.)

So in some sense, the above examples and constructions show that Conjecture 1 is the only possible analogue of the result in 4 mentioned above. Our main result is that if the minimum outdegree of a digraph is sufficiently large compared to its order, then Conjecture $\square$ is true. In fact, we show that in this case, one can even guarantee a subdivision of a complete digraph.
Theorem 2. Let $\vec{G}$ be a digraph of order $n$ whose minimum outdegree is at least d. Then $\vec{G}$ contains a subdivision of the complete digraph of order $\left\lfloor d^{2} /\left(8 n^{3 / 2}\right)\right\rfloor$.

Note that the bound is nontrivial as soon as $d$ is a little larger than $n^{3 / 4}$. Also, recall that the result of Thomassen (9) mentioned above implies that we cannot have a subdivision of a complete digraph of order at least 3 if $d \leq \frac{1}{2} \log _{2} n$. Furthermore, note that if $d=c n$, then Theorem 2 guarantees a subdivision of a complete digraph of order $\left\lfloor c^{\prime} \sqrt{n}\right\rfloor$, where $c^{\prime}=c^{2} / 8$. It is easy to see that this is best possible up to the value of $c^{\prime}$ (consider the complete bipartite digraph with vertex classes of equal size).

The main ingredient in the proof of Theorem 2 is Lemma 4 It states that if $\vec{G}$ has $n$ vertices and its minimum outdegree is $\gg \sqrt{n}$, then $\vec{G}$ has a subdigraph $\vec{H}$ which is highly connected in the following sense: if $x$ is any vertex of $\vec{H}$ and $y$ is a vertex of large indegree, then there are many internally disjoint dipaths from $x$ to $y$ in $\vec{H}$. Lemma 4 also guarantees the existence of many such vertices $y$. For undirected graphs, there is a much stronger result of Mader [5] which implies that every graph of minimum degree at least $4 k$ has a $k$-connected subgraph. Since a digraph version of this result is not known, Lemma 4 may be of independent interest. There are also several related results of Mader [6, 8] which investigate the existence of pairs of vertices with large local connectivity in digraphs of large minimum outdegree. The proof of Lemma 4 is quite elementary: if the current subdigraph $\vec{H}$ does not satisfy the requirements, then we can use Menger's theorem to find a significantly smaller subdigraph whose minimum outdegree is almost as large as that of $\vec{H}$. Since this means that the density of the successive subdigraphs increases, this process must eventually terminate.

## 2. Proof of Theorem 2

Before we start with the proof of Theorem 2 let us introduce some notation. The digraphs $\vec{G}$ considered in this note do not contain loops and between any ordered vertex pair $x, y \in \vec{G}$ there is at most one edge from $x$ to $y$. (There might also be another edge from $y$ to $x$.) We denote by $\delta^{+}(\vec{G})$ the minimum outdegree of a digraph $\vec{G}$ and by $|\vec{G}|$ its order. We write $d_{\vec{G}}^{+}(x)$ for the outdegree of a vertex $x \in \vec{G}$ and $d_{\vec{G}}^{-}(x)$ for its indegree. A digraph $\vec{H}$ is a subdivision of $\vec{G}$ if $\vec{H}$ can be obtained from $\vec{G}$ by replacing each edge $\overrightarrow{x y} \in \vec{G}$ with a dipath from $x$ to $y$ such that all these dipaths are internally disjoint for distinct edges. The vertices of $\vec{H}$ corresponding to the vertices of $\vec{G}$ are called branch vertices.

Given two vertices $x$ and $y$ of a digraph $\vec{G}$, we define $\kappa_{\vec{G}}(x, y)$ to be the largest integer $1 \leq k \leq|\vec{G}|-2$ such that $\vec{G}-S$ contains a dipath from $x$ to $y$
for every vertex set $S \subseteq V(\vec{G}) \backslash\{x, y\}$ of size $<k$. We define $\kappa_{\vec{G}}(x, y):=0$ if $\vec{G}$ does not contain a dipath from $x$ to $y$. We will use the following version of Menger's theorem for digraphs.

Theorem 3 (Menger's theorem for digraphs). Let $x$ and $y$ be vertices of a digraph $\vec{G}$ such that $\kappa_{\vec{G}}(x, y) \geq k$. Then $\vec{G}$ contains $k$ internally disjoint dipaths from $x$ to $y$.

As mentioned above, the main step in the proof of Theorem 2 is to find a subdigraph $\vec{H}$ of $\vec{G}$ such that the minimum outdegree of $\vec{H}$ is still large and such that every vertex of $\vec{H}$ sends many internally disjoint dipaths to each vertex of $\vec{H}$ which has large indegree.

Lemma 4. Every digraph $\vec{G}$ of order $n$ with $\delta^{+}(\vec{G}) \geq d$ contains a subdigraph $\vec{H}$ such that
(i) $\delta^{+}(\vec{H})>d / 2$,
(ii) $\kappa_{\vec{H}}(x, y) \geq d^{2} /(4 n)$ for all pairs $x, y \in V(\vec{H})$ with $d_{\vec{H}}^{-}(y) \geq d / 2$,
(iii) at least $d^{2} /(4 n)$ vertices of $\vec{H}$ have indegree at least $d / 2$ in $\vec{H}$.

Proof. Put

$$
\alpha:=\frac{d}{n} \quad \text { and } \quad \alpha^{\prime}:=\frac{d^{2}}{4 n^{2}}=\frac{\alpha^{2}}{4}
$$

By Theorem 3 we may assume that $\kappa_{\vec{G}}(x, y)<\alpha^{\prime} n$ for some vertices $x, y$ of $\vec{G}$ with $d_{\vec{G}}^{-}(y) \geq d / 2$. Otherwise we could take $\vec{H}:=\vec{G}$. (It is easy to check that $\vec{H}$ then also satisfies condition (iii) of the lemma.) Let $S \subseteq V(\vec{G}) \backslash\{x, y\}$ be a set of size $<\alpha^{\prime} n$ such that $\vec{G}-S$ does not contain a dipath from $x$ to $y$. Let $Y$ be the set of all those vertices $z$ for which $\vec{G}-S$ contains a dipath from $z$ to $y$. Then $Y \cup S$ contains $y$ as well as all the at least $d / 2=\alpha n / 2$ inneighbours of $y$. Let $C$ denote the component of the undirected graph corresponding to $\vec{G}-(Y \cup S)$ which contains $x$. Let $\vec{G}_{1}$ be the subdigraph of $\vec{G}$ induced by all vertices in $C$. Then $\left|\vec{G}_{1}\right| \leq n-|Y \cup S|<(1-\alpha / 2) n$. Moreover, note that there exists no edge directed from a vertex of $\vec{G}_{1}$ to a vertex outside $V\left(\vec{G}_{1}\right) \cup S$. Thus

$$
\begin{equation*}
\delta^{+}\left(\vec{G}_{1}\right) \geq \delta^{+}(\vec{G})-|S|>\left(\alpha-\alpha^{\prime}\right) n \tag{1}
\end{equation*}
$$

If $\vec{G}_{1}$ does not satisfy condition (ii) of the lemma we again apply Theorem 3 to obtain a subdigraph $\vec{G}_{2} \subseteq \vec{G}_{1}$. We continue in this fashion until we obtain a subdigraph $\vec{G}_{r}$ which satisfies condition (ii). We will show that $\vec{G}_{r}$ also satisfies (i) and (iii). Put $\vec{G}_{0}:=\vec{G}$,

$$
\delta_{i}:=\frac{\delta^{+}\left(\vec{G}_{i}\right)}{\left|\vec{G}_{i}\right|} \quad \text { and } \quad \gamma_{i-1}:=\frac{\left|\vec{G}_{i-1}\right|}{\left|\vec{G}_{i}\right|}
$$

for all $i \leq r$. Similarly as in (1) it follows that

$$
\begin{equation*}
\delta^{+}\left(\vec{G}_{i}\right)=\delta_{i}\left|\vec{G}_{i}\right| \geq \delta_{i-1}\left|\vec{G}_{i-1}\right|-\alpha^{\prime} n \geq\left(\alpha-i \alpha^{\prime}\right) n \tag{2}
\end{equation*}
$$

Thus $\delta_{i} \geq \delta_{i-1} \gamma_{i-1}-\alpha^{\prime} n /\left|\vec{G}_{i}\right|=\delta_{i-1} \gamma_{i-1}-\alpha^{\prime} \prod_{j=0}^{i-1} \gamma_{j}$. Using this inequality and induction on $i$ one can show that

$$
\begin{equation*}
\delta_{i} \geq\left(\alpha-i \alpha^{\prime}\right) \prod_{j=0}^{i-1} \gamma_{j}=\left(\alpha-i \alpha^{\prime}\right) \frac{n}{\left|\vec{G}_{i}\right|} \tag{3}
\end{equation*}
$$

Since we delete at least $d / 2=\alpha n / 2$ vertices when going from $\vec{G}_{i-1}$ to $\vec{G}_{i}$ (namely the inneighbours of the vertex playing the role of $y$ ), we have that $\left|\vec{G}_{r}\right| \leq n-r \alpha n / 2$. In particular this shows that $r<2 / \alpha$. However, since (3) implies that $1>\delta_{r} \geq\left(\alpha-r \alpha^{\prime}\right) /(1-r \alpha / 2)$ we even have $r<(1-\alpha) /\left(\alpha / 2-\alpha^{\prime}\right)$. Thus

$$
\begin{equation*}
\delta^{+}\left(\vec{G}_{i}\right) \stackrel{\sqrt[2]{2}}{\geq}\left(\alpha-r \alpha^{\prime}\right) n \geq\left(\alpha-\frac{1-\alpha}{2 / \alpha-1}\right) n=\frac{\alpha n}{2-\alpha}>\frac{d}{2} . \tag{4}
\end{equation*}
$$

Altogether this shows that $\vec{G}_{r}=: \vec{H}$ satisfies conditions (i) and (ii) of the lemma. To check that $\vec{H}$ also satisfies condition (iii) let $\ell$ denote the number of vertices of indegree $\geq d / 2$ in $\vec{H}$. Then

$$
\frac{\alpha n|\vec{H}|}{2-\alpha} \stackrel{4 \pi}{\leq} \delta^{+}(\vec{H})|\vec{H}| \leq|\vec{H}| \frac{d}{2}+\ell|\vec{H}|,
$$

which implies that $\ell \geq \alpha d /(4-2 \alpha) \geq d^{2} /(4 n)$, as required.
Proof of Theorem 2, Let $\ell:=\left\lfloor d^{2} /\left(8 n^{3 / 2}\right)\right\rfloor$. We first apply Lemma 4 to obtain a subdigraph $\vec{H} \subseteq \vec{G}$ as described there. We pick a set $X \subseteq V(\vec{H})$ of $\ell$ vertices having indegree $\geq d / 2$ in $\vec{H}$. (Such a set $X$ exists by condition (iii) of Lemma (4) $X$ will be the set of our branch vertices. For every pair $x, y \in X$ there exist at least $d^{2} /(4 n)$ internally disjoint dipaths from $x$ to $y$. Thus the average number of inner vertices on such a path is at most $4 n^{2} / d^{2}$. Hence $\vec{H}$ contains at least $d^{2} /(8 n)$ internally disjoint dipaths from $x$ to $y$ such that each of these has at most $8 n^{2} / d^{2}$ inner vertices. Let us call such a dipath short. This shows that we can connect all pairs of branch vertices greedily (in both directions) by choosing each time a short dipath which is internally disjoint from all the short dipaths chosen before. In each step we destroy at most $8 n^{2} / d^{2}$ further dipaths. But $\left(|X|^{2}-1\right) 8 n^{2} / d^{2}<8 \ell^{2} n^{2} / d^{2} \leq d^{2} /(8 n)$, so we can connect all pairs of branch vertices by short dipaths.

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