d-Regular Graphs of Acyclic Chromatic Index at least d+2

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Abstract

An $acyclic$ edge coloring of a graph is a proper edge coloring such that there are no bichromatic cycles. The *acyclic chromatic index* of a graph is the minimum number k such that there is an acyclic edge coloring using k colors and is denoted by $a'(G)$. It was conjectured by Alon, Sudakov and Zaks (and earlier by Fiamcik) that $a'(G) \leq \Delta + 2$, where $\Delta = \Delta(G)$ denotes the maximum degree of the graph. Alon et.al also raised the question whether the complete graphs of even order are the only regular graphs which require $\Delta + 2$ colors to be acyclically edge colored. In this paper, using a simple counting argument we observe not only that this is not true, but infact all d-regular graphs with $2n$ vertices and $d > n$, requires at least $d + 2$ colors. We also show that $a'(K_{n,n}) \geq n + 2$, when n is odd using a more non-trivial argument(Here $K_{n,n}$ denotes the complete bipartite graph with n vertices on each side). This lower bound for $K_{n,n}$ can be shown to be tight for some families of complete bipartite graphs and for small values of n. We also infer that for every d, n such that $d \geq 5$, $n \geq 2d + 3$ and dn even, there exist d-regular graphs which require at least $d + 2$-colors to be acyclically edge colored.

Keywords: Acyclic edge coloring, acyclic edge chromatic index, matching, perfect 1-factorization, complete bipartite graphs.

All graphs considered in this paper are finite and simple. A proper *edge coloring* of $G = (V, E)$ is a map $c : E \rightarrow C$ (where C is the set of available $colors$) with $c(e) \neq c(f)$ for any adjacent edges e,f. The minimum number of colors needed to properly color the edges of G, is called the chromatic index of G and is denoted by $\chi'(G)$. A proper edge coloring c is called acyclic if there are no bichromatic cycles in the graph. In other words an edge coloring is acyclic if the union of any two color classes induces a set of paths (i.e., linear forest) in G. The *acyclic edge chromatic number* (also called *acyclic chromatic index*), denoted by $a'(G)$, is the minimum number of colors required to acyclically edge color G. The concept of *acyclic coloring* of a graph was introduced by Grünbaum [6]. Let $\Delta = \Delta(G)$ denote the maximum degree of a vertex in graph G. By Vizing's theorem, we have $\Delta \leq \chi'(G) \leq \Delta + 1$(see [4] for proof). Since any acyclic edge coloring is also proper, we have $a'(G) \geq \chi'(G) \geq \Delta$.

It has been conjectured by Alon, Sudakov and Zaks [2] that $a'(G) \leq \Delta + 2$ for any G. We were informed by Alon that the same conjecture was raised earlier by Fiamcik [5]. Using probabilistic arguments Alon, McDiarmid and Reed [1] proved that $a'(G) \leq 60\Delta$. The best known result up to now for arbitrary graph, is by Molloy and Reed [7] who showed that $a'(G) \leq 16\Delta$.

The complete graph on n vertices is denoted by K_n and the complete bipartite graph with n vertices on each side is denoted by $K_{n,n}$. We denote the sides of the bi-partition by A and B. Thus $V(K_{n,n}) = A \cup B$.

Our Result: Alon, Sudakov and Zaks [2] suggested a possibility that complete graphs of even order are the only regular graphs which require $\Delta + 2$ colors to be acyclically edge colored. Nešetřil and Wormald [8] supported the statement by showing that the acyclic edge chromatic number of a random d-regular graph is asymptotically almost surely equal to $d + 1$ (when $d \geq 2$). In this paper, we show that this is not true in general. More specifically we prove the following Theorems :

Theorem 1. *Let G be a d-regular graph with $2n$ vertices and $d > n$, then $a'(G) \geq d + 2 = \Delta(G) + 2$.*

Theorem 2. *For any d and n such that dn is even and $d \geq 5, n \geq 2d + 3$, there exists a connected d-regular graphs that require $d + 2$ colors to be acyclically edge colored.*

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Theorem 3. *$a'(K_{n,n}) \geq n + 2 = \Delta + 2$, when n is odd.*

Remarks:

1. It is interesting to compare the statement of Theorem 1 to the result of [8], namely that almost all d-regular graphs for a fixed d, require only $d + 1$ colors to be acyclically edge colored. From the introduction of [8], it appears that the authors expect their result for random d-regular graphs would extend to all d-regular graphs except for K_n, n even. From Theorem 1 and Theorem 2 it is clear that this is not true: There exists a large number of d-regular graphs which require $d + 2$ colors to be acyclically adge colored, even d is fixed.

2. The complete bipartite graph, $K_{n+2,n+2}$ is said to have a perfect 1-factorization if the edges of $K_{n+2,n+2}$ can be decomposed into $n+2$ disjoint perfect matchings such that the union of any two perfect matchings forms a hamiltonian cycle. It is obvious from Lemma 1 that $K_{n+2,n+2}$ does not have perfect 1-factorization when n is even. When n is odd, some families have been proved to have perfect 1-factorization (see [3] for further details). It is easy to see that if $K_{n+2,n+2}$ has a perfect 1-factorization then $K_{n+2,n+1}$ and therefore $K_{n+1,n+1}$ has a acyclic edge coloring using $n + 2$ colors. Therefore the statement of Theorem 3 cannot be extended to the case when n is even in general.

3. Clearly if $K_{n+2,n+2}$ has a perfect 1-factorization, then $a'(K_{n,n}) = n + 2$. It is known that (see [3]), if $n + 2 \in \{p, 2p - 1, p^2\}$, where p is an odd prime or when $n + 2 < 50$ and odd, then $K_{n+2,n+2}$ has a perfect 1-factorization. Thus the lower bound in Theorem 3 is tight for the above mentioned values of $n + 2$.

Proof of Theorem 1:

Proof. Observe that two different color classes cannot have n edges each, since that will lead to a bichromatic cycle. Therefore at most one color class can have n edges while all other color classes can have at most $n - 1$ edges. Thus the number of edges in the union of $\Delta(G) + 1 = d + 1$ color classes is at most $n + d(n - 1) < dn$, when $d > n$ (Note that dn is the total number of edges in G). Thus G needs at least one more color. Thus $a'(G) \geq d + 2 = \Delta(G) + 2$. □

Remark: It is clear from the proof that if $n + d(n - 1) + x < dn$ then even after removing x edges from the given graph, the resulting graph still would require $d + 2$ colors to be acyclically edge colored.

Proof of Theorem 2:

Proof. If d is odd, let $G' = K_{d+1}$. Else if d is even let G' be the complement of a perfect matching on $d + 2$ vertices. Let H be any d-regular graph on $N = n - n'$ vertices. Now remove an edge (a, a') from G' and an edge (b, b') from H. Now connect a to b and a' to b' to create a d-regular graph G. Clearly G requires $d+2$ colors to be acyclically edge colored since otherwise it would mean that $G' - \{(a, a')\}$ is $d + 1$ colorable, a contradiction in view of the Remark following Theorem 1, for $d \geq 5$. □

Complete bipartite graphs offer a interesting case since they have $d = n$. Observe that the above counting argument fails. We deal with this case in the next section.

Complete Bipartite Graphs

Lemma 1. *If n is even, then $K_{n,n}$ does not contain three disjoint perfect matchings M_1, M_2, M_3 such that $M_i \cup M_j$ forms a hamiltonian cycle for $i, j \in \{1, 2, 3\}$ and $i \neq j$.*

Proof. Observe that a perfect matching of $K_{n,n}$ corresponds to a permutation of $\{1, 2, \ldots, n\}$. Let perfect matching M_i corresponds to permutation π_i. Without loss of generality, we can assume that π_1 is the identity permutation by renumbering the vertices of one side of $K_{n,n}$.

Suppose $K_{n,n}$ contains three perfect matchings M_1, M_2, M_3 such that $M_i \cup M_j$ forms a hamiltonian cycle for $i, j \in \{1, 2, 3\}$ and $i \neq j$.

Now we study the permutation $\pi_i^{-1}\pi_j$. Since $M_i \cup M_j$ induces a hamiltonian cycle in $K_{n,n}$, it is easy to see that the smallest $t \geq 1$ such that $(\pi_i^{-1}\pi_j)^t(1) = 1$ equals n. It follows that, in the cycle structure of $\pi_i^{-1}\pi_j$, there exists exactly one cycle and this cycle is of length n. The sign of a permutation is defined as: $sign(\pi) = (-1)^k$, where k is the number of even cycles in the cycle structure of the permutation π. Recalling that n is even, we have the following claim:

Claim 1. *$sign(\pi_i^{-1}\pi_j) = -1$ for $i, j \in \{1, 2, 3\}$ and $i \neq j$.*

Now with respect to $\pi_i^{-1}\pi_j$, taking $\pi_i = \pi_1$ (the identity permutation) and $\pi_j = \pi_2$ (or π_3), we infer that $sign(\pi_2) = -1$ and $sign(\pi_3) = -1$. Now $sign(\pi_2^{-1}\pi_3) = sign(\pi_2^{-1})sign(\pi_3)$ = (-1)(-1) = 1, a contradiction in view of $Claim$ 1. $\square$

Proof of Theorem 3:

Proof. Since $K_{n,n}$ is a regular graph, $a'(K_{n,n}) \geq \Delta + 1 = n + 1$. Suppose $n+1$ colors are sufficient. This can be achieved only in the following way: One color class contains n edges and the remaining color classes contain $n-1$ edges each. Let α be the color class that has n edges. Thus color α is present at every vertex on each side A and B. Any other color is missing at exactly one vertex on each side.

Observation 1. *Let $\theta \neq \alpha$ be a color class. The subgraph induced by color classes θ and α contains $2n-1$ edges and since there are no bichromatic cycles, the subgraph induced is a hamiltonian path. We call this an (α, θ) hamiltonian path.*

Observation 2. *Let θ_1 and θ_2 be color classes with $n-1$ edges each. The subgraph induced by color classes θ_1 and θ_2 contains $2n-2$ edges. Since there are no bichromatic cycles, the subgraph induced consists of exactly two paths.*

Note that there is a unique color missing at each vertex on each side of $K_{n,n}$. Let $m(u)$ be the color missing at vertex u. For $a_1 \in A$ and $b_1 \in B$, let $m(a_1) = m(b_1) = \beta$. Let the color of the edge $(a_1, b_1) = \gamma$. Clearly $\gamma \neq \alpha$ since otherwise there cannot be a (α, β) hamiltonian path, a contradiction to $Observation$ 1. For $a_2 \in A$ and $b_2 \in B$, let $m(a_2) = m(b_2) = \gamma$. Its clear that $a_1 \neq a_2$ and $b_1 \neq b_2$. Consider the subgraph induced by the colors β and γ. In view of $Observation$ 2 it consists of exactly two paths. One of them is the single edge (a_1, b_1). The other path has length $2n-3$ and has a_2 and b_2 as end points.

Now we construct a $K_{n+1,n+1}$ from the above $K_{n,n}$ by adding a new vertex, a_{n+1} to side A and a new vertex, b_{n+1} to side B. Now for $u \in B$ color each edge (a_{n+1}, u) by the color $m(u)$ and for $v \in A$ color each edge (b_{n+1}, v) by the color $m(v)$. Assign the color α to the edge (a_{n+1}, b_{n+1}). Clearly the coloring thus obtained is a proper coloring.

Now we know that there existed a (α, β) hamiltonian path in $K_{n,n}$ with a_1 and b_1 as end points. Recalling that $m(a_1) = m(b_1) = \beta$, we have $color(a_{n+1}, b_1) = color(b_{n+1}, a_1) = \beta$. It is easy to see that in $K_{n+1,n+1}$ this path along with the edges (a_1, b_{n+1}), (b_{n+1}, a_{n+1}) and (a_{n+1}, b_1) forms a (α, β) hamiltonian cycle. In a similar way, for (α, γ) hamiltonian path that existed in $K_{n,n}$, we can see that in $K_{n+1,n+1}$, we have a corresponding (α, γ) hamiltonian cycle.

Recall that there was a (β, γ) bichromatic path starting from a_2 and ending at b_2 in $K_{n,n}$. In the $K_{n+1,n+1}$ we created, we have $c(a_2, a_{n+1}) = \gamma$, $c(a_1, b_{n+1}) = \beta$, $c(a_{n+1}, b_1) = \beta$ and $c(a_{n+1}, b_2) = \gamma$. Thus the above (β, γ) bichromatic path in $K_{n,n}$ along with the edges (a_2, b_{n+1}), (b_{n+1}, a_1), (a_1, b_1), (b_1, a_{n+1}), (a_{n+1}, b_2) in that order. Thus we have 3 perfect matchings induced by the color classes α, β and γ whose pairwise union gives rise to hamiltonian cycles in $K_{n+1,n+1}$, a contradiction to $Lemma$ 1 since $n+1$ is even. $\square$

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