# GEODESICS AND ALMOST GEODESIC CYCLES IN RANDOM REGULAR GRAPHS 

ITAI BENJAMINI, CARLOS HOPPEN, ERAN OFEK, PAWEŁ PRA£AT, AND NICK<br>WORMALD


#### Abstract

A geodesic in a graph $G$ is a shortest path between two vertices of $G$. For a specific function $e(n)$ of $n$, we define an almost geodesic cycle $C$ in $G$ to be a cycle in which for every two vertices $u$ and $v$ in $C$, the distance $d_{G}(u, v)$ is at least $d_{C}(u, v)-e(n)$. Let $\omega(n)$ be any function tending to infinity with $n$. We consider a random $d$-regular graph on $n$ vertices. We show that almost all pairs of vertices belong to an almost geodesic cycle $C$ with $e(n)=\log _{d-1} \log _{d-1} n+\omega(n)$ and $|C|=$ $2 \log _{d-1} n+O(\omega(n))$. Along the way, we obtain results on near-geodesic paths. We also give the limiting distribution of the number of geodesics between two random vertices in this random graph.


## 1. Introduction

A geodesic in a graph $G$ is a shortest path between two vertices of $G$. Let $\omega(n)$ be any function tending to infinity with $n$, and put $e(n)=\log _{d-1} \log _{d-1} n+\omega(n)$. We define an almost geodesic cycle $C$ in $G$ to be a cycle in which for every two vertices $u$ and $v$ in $C$, the distance $d_{G}(u, v)$ is at least $d_{C}(u, v)-e(n)$. We investigate the existence of almost geodesic cycles through random pairs of vertices in a random graph, and related questions on geodesics and paths that are nearly geodesic, in a sense to be made precise. Our results refer to the probability space of random $d$-regular graphs with uniform probability distribution. This space is denoted $\mathcal{G}_{n, d}$, and asymptotics (such as "asymptotically almost surely", which we abbreviate to a.a.s.) are for $n \rightarrow \infty$ with $d \geq 3$ fixed, and $n$ even if $d$ is odd.

Some related previous research focussed on finding (edge/internally)-disjoint paths with many sources and targets. Frieze and Zhao [4] showed that for sufficiently large $d$ there exist fixed positive constants $\alpha$ and $\beta$ such that a graph $G$ taken from $\mathcal{G}_{n, d}$ a.a.s. has the following property: for any choice of $k$ pairs $\left\{\left(a_{i}, b_{i}\right) \mid i=1, \ldots, k\right\}$, satisfying
(i) $k \leq\left\lceil\alpha d n / \log _{d} n\right\rceil$, and
(ii) for each vertex $v:\left|i: a_{i}=v\right|+\left|i: b_{i}=v\right| \leq \beta d$,
there exist edge-disjoint paths in $G$ connecting $a_{i}$ to $b_{i}$ for all $i=1,2, \ldots, k$. This result is optimal up to constant factors. The paths returned by their algorithm are of length of at least $10 \log _{d} n$.

Our focus is different as it comes from different motivation: studying almost geodesic cycles in $\mathcal{G}_{n, d}$. Our result on internally disjoint paths refers to one pair of vertices fixed before the graph is chosen. This is a much weaker model than the model of [4], that dealt with $\Theta(n / \log n)$ pairs given by an adversary after the graph is chosen. However, we show the existence of disjoint paths that approximate the optimal path (whose length is a.a.s. in $\left.\left[\log _{d-1} n-\omega(n), \log _{d-1} n+\omega(n)\right]\right)$ by an additive factor of $\log _{d-1} \log _{d-1} n$,

[^0]whereas the result of [4] give at best a constant multiplicative factor. Additionally, our result holds for all $d \geq 3$, and that we find the maximum possible number of internally disjoint paths, $d$, that there can possibly be between two vertices.

Theorem 1.1. Take any integer $d \geq 3$ and any function $\omega(n)$ with $\omega(n) \rightarrow \infty$. Let $G \in \mathcal{G}_{n, d}$ and choose vertices $u$ and $v$ in $V(G)$ independently with uniform probability. Then a.a.s. the following hold:
(i) $\left|d(u, v)-\log _{d-1} n\right|<\omega(n)$,
(ii) there are $d$ paths connecting $u$ and $v$ such that the subgraph induced by each pair of these paths is an almost geodesic cycle.

Note that the $d$ paths in (ii) theorem are internally disjoint because each pair of them induces a cycle.

In a slightly different direction, we also investigate the distribution of the number of geodesics joining two vertices (see Theorem 2.6).

We may obtain the lower bound in part (i) of the theorem from an elementary observation. Note that, given $G \in \mathcal{G}_{n, d}$, the number of vertices at distance at most $i$ from a vertex $u$ is bounded above by

$$
1+d+d(d-1)+\ldots+d(d-1)^{i-1}=O\left((d-1)^{i}\right) .
$$

So, there are $O\left(n(d-1)^{-\omega(n)}\right)$ vertices at distance $i=\log _{d-1} n-\omega(n)$ from any given vertex of $G$, where $\omega(n) \rightarrow \infty$. As a consequence, if two vertices of $G$ are chosen independently with uniform probability, then the probability that the second vertex is at distance at most $i=\log _{d-1} n-\omega(n)$ from the first is at most

$$
\frac{1}{n} O\left(n(d-1)^{-\omega(n)}\right)=O\left((d-1)^{-\omega(n)}\right),
$$

and therefore, a.a.s.

$$
\begin{equation*}
d(u, v) \geq \log _{d-1} n-\omega(n) \tag{1}
\end{equation*}
$$

if $u, v$ are vertices chosen independently with uniform probability in $G \in \mathcal{G}_{n, d}$ and $\omega(n)$ is a function satisfying $\omega(n) \rightarrow \infty$. The fact that a.a.s. $d(u, v) \leq \log _{d-1} n+\omega(n)$ will follow from our study of the distribution of the number of geodesics in $G \in \mathcal{G}_{n, d}$.

The rest of the proof requires more sophisticated arguments. Instead of working directly in the uniform probability space of random regular graphs on $n$ vertices $\mathcal{G}_{n, d}$, we use the pairing model of random regular graphs, first introduced by Bollobás [2], which is described next. Suppose that $d n$ is even, as in the case of random regular graphs, and consider $d n$ points partitioned into $n$ labelled cells $v_{1}, \ldots, v_{n}$ of $d$ points each. A pairing of these points is a perfect matching of them into $d n / 2$ pairs. Given a pairing $P$, we may construct a multigraph $G(P)$, with loops allowed, as follows: the vertices are the cells $v_{1}, \ldots, v_{n}$, and a pair $\{x, y\}$ in $P$ corresponds to an edge $v_{i} v_{j}$ in $G(P)$ if $x$ and $y$ are contained in the cells $v_{i}$ and $v_{j}$, respectively. It is an easy fact that the probability of a random pairing corresponding to a given simple graph $G$ is independent of the graph, hence the restriction of the probability space of random pairings to simple graphs is precisely $\mathcal{G}_{n, d}$. Moreover, it is well known that a random pairing generates a simple graph with probability asymptotic to a constant depending on $d$, so that any event holding a.a.s. over a probability space of random pairings also holds a.a.s. over the corresponding space $\mathcal{G}_{n, d}$. For this reason, asymptotic results over random pairings suffice for our purposes. The advantage of using this model is that the
pairs may be chosen sequentially so that the next pair is chosen uniformly at random over the remaining (unchosen) points. For more information on this model, see [5.

The numbers of geodesics are investigated in Section [2] Theorem [.1] is proved in Section 3) Some final remarks are in Section 4

## 2. Distribution of the number of geodesics

The first portion of our argument is a simplified version of part of the argument of Bollobás and Fernandez de la Vega [3]. We consider the process in which the neighbourhoods of $u$ and $v$ are exposed step by step. First, the neighbours of $u$ and $v$ are revealed, then the vertices at distance two, and so on. This sequential exposure of the random regular graph is analysed using the random pairing model mentioned in the Introduction.

Let $N_{i}(u)$ denote the set of vertices at distance at most $i$ from $u$. Note that, in the early stages of this process, the graphs grown from $u$ and $v$ tend to be trees, hence the number $n_{i}$ of elements in $N_{i}(u)$ is approximately

$$
n_{i-1}+(d-1)\left(n_{i-1}-n_{i-2}\right) .
$$

Let $f_{i}$ denote the number of vertices in a balanced $d$-regular tree, that is,

$$
f_{i}=1+d \sum_{j=0}^{i-1}(d-1)^{j}=1+\frac{d\left((d-1)^{i}-1\right)}{d-2}
$$

and let

$$
i_{0}=\left\lfloor\frac{1}{2} \log _{d-1} n\right\rfloor .
$$

Lemma 2.1. Let $\omega(n)$ be any function of $n$ such that $\omega(n) \rightarrow \infty$. For $i \leq i_{0}-\omega(n)$ a.a.s. the cardinality $n_{i}$ of $N_{i}(u)$ equals $f_{i}$. Moreover, for $i \leq i_{0}+\omega(n)$ a.a.s.

$$
n_{i}=f_{i}-O\left(\omega(n)(d-1)^{3\left(i-i_{0}\right)+\omega(n)}\right) .
$$

Proof. First note that it is sufficient to consider the case when $\omega(n)=o(\log n)$.
Since $f_{i}$ denotes the number of vertices in a balanced tree where every non-leaf vertex has degree $d$, the first assertion follows if we show that a.a.s. the set of vertices at distance at most $i \leq i_{0}-\omega(n)$ of a vertex $u$ induces a tree. In other words, if we expose, step by step, the vertices at distance $1,2, \ldots, i$ from $u$, we have to avoid, at step $j$, edges that induce cycles. So, we wish not to find edges between any two vertices at distance $j$ from $u$ or edges that join any two vertices at distance $j$ to a same vertex at distance $j+1$ from $u$. We shall refer to edges of this form as 'bad edges'. Note that the expected number of 'bad edges' at step $i+1$ is equal to $O\left(n_{i}^{2} / n\right)=O\left(f_{i}^{2} / n\right)=O\left((d-1)^{2 i} / n\right)$.

Consider $i_{1}=\left\lfloor\frac{1}{2} \log _{d-1} n-\omega(n)\right\rfloor$. The expected number of 'bad edges' found up to step $i_{1}$ is equal to

$$
\sum_{j=0}^{i_{1}-1} O\left((d-1)^{2 j} / n\right)=O\left((d-1)^{2 i_{1}} / n\right)=O\left((d-1)^{-2 \omega(n)}\right)=o(1)
$$

Thus, by Markov's inequality, a.a.s. there are no 'bad edges' until step $i_{1}$, hence a.a.s. $N_{i_{1}}(u)$ is a tree and $n_{i}=f_{i}$ for $i \leq i_{1}$.

In order to prove the second assertion, notice that the expected number of 'bad edges' added between step $i_{1}+1$ and step $i, i \leq\left\lfloor i_{0}+\omega(n)\right\rfloor=\left\lfloor\frac{1}{2} \log _{d-1} n+\omega(n)\right\rfloor$ is equal to

$$
\sum_{j=i_{1}}^{i-1} O\left((d-1)^{2 j} / n\right)=O\left((d-1)^{2 i} / n\right)=O\left((d-1)^{2\left(i-i_{0}\right)}\right) .
$$

Thus, again by Markov's inequality, a.a.s. the total number of 'bad edges' at time $i$ is at most $O\left(\omega(n)(d-1)^{2\left(i-i_{0}\right)}\right)$. Notice that one 'bad edge' added in this time interval can destroy a tree branch of size $O\left((d-1)^{i-i_{0}+\omega(n)}\right)$. This occurs because the 'bad edge' creates a cycle instead of exposing a new vertex $v$. The branch of descendants of $v$, which would appear had $v$ been exposed and had the process continued as a balanced $d$ regular tree, is therefore destroyed and has size at most $1+(d-1)+\cdots+(d-1)^{i-i_{0}+\omega(n)}=$ $O\left((d-1)^{i-i_{0}+\omega(n)}\right)$.

Thus, we have a.a.s.

$$
n_{i}=f_{i}-O\left(\omega(n)(d-1)^{2\left(i-i_{0}\right)}\right) \cdot O\left((d-1)^{i-i_{0}+\omega(n)}\right) .
$$

This completes the proof of the lemma.
Immediately from this lemma, we have
Corollary 2.2. For $i=i_{0}+o(\log n)$ a.a.s.

$$
n_{i}=f_{i}-n^{o(1)}=n^{1 / 2+o(1)} .
$$

In the remainder of this notes, let $u, v$ be vertices chosen independently with uniform probability in a graph $G \in \mathcal{G}_{n, d}$ and consider the process of exposing the neighbourhoods of $u$ and $v$ introduced in Lemma [2.1. We say that $N_{i}(u)$ and $N_{i}(v)$ join at time $i$ if $N_{i-1}(u) \cap N_{i-1}(v)=\emptyset$ and $N_{i}(u) \cap N_{i}(v) \neq \emptyset$.

Also, whenever a result that holds for any $\omega(n)$ satisfying $\lim _{n \rightarrow \infty} \omega(n)=\infty$ is proven, we shall assume without loss of generality that $\omega(n)=o(\log n)$.
Lemma 2.3. Let $k$ be a fixed integer and define $\gamma(n, d)=\frac{1}{2} \log _{d-1} n-i_{0}$, the fractional part of $\frac{1}{2} \log _{d-1} n$. Then

$$
\mathbb{P}\left(N_{i_{0}+k}(u) \cap N_{i_{0}+k}(v)=\emptyset\right) \sim \exp \left(-\frac{d(d-1)^{2 k-2 \gamma(n, d)}}{d-2}\right) .
$$

Proof. Denote $S_{i}$ the event that the neighbourhoods of $u$ and $v$ are separate at time $i$, that is, $N_{j}(u)$ and $N_{j}(v)$ did not join up to time $i$. We claim that

$$
\mathbb{P}\left(S_{i_{0}+k} \mid S_{i_{0}+k-1}\right) \sim \exp \left(-d^{2}(d-1)^{2 k-2 \gamma(n, d)-2}\right)
$$

This implies our result for the following reasons. If $M$ is a positive integer, $-M<k$,

$$
\begin{aligned}
\mathbb{P}\left(S_{i_{0}+k}\right)= & \mathbb{P}\left(S_{i_{0}-M}\right) \\
& \times \prod_{l=-M+1}^{k} \mathbb{P}\left(S_{i_{0}+l} \mid S_{i_{0}+l-1}\right) .
\end{aligned}
$$

Furthermore, equation (11) establishes that a.a.s. $d(u, v)>2 i_{0}-\omega(n)$ for any function $\omega(n)$ with $\lim _{n \rightarrow \infty} \omega(n)=\infty$. In particular, given $\epsilon>0$, we can choose $M=M_{\epsilon}>0$


Figure 1. First case - odd length.


Figure 2. Second case - even length.
sufficiently large so that $\mathbb{P}\left(S_{i_{0}-M}\right)>1-\epsilon$. Given such an $M$, we use the previous equation to derive

$$
\begin{aligned}
\mathbb{P}\left(S_{i_{0}+k}\right) & >(1-\epsilon) \prod_{l=-M+1}^{k} \exp \left(-d^{2}(d-1)^{2 l-2 \gamma(n, d)-2}\right)(1-o(1)) \\
& \sim(1-\epsilon) \exp \left(\sum_{l=-M+1}^{k}-\frac{d^{2}(d-1)^{2 l}}{(d-1)^{2+2 \gamma(n, d)}}\right) \\
& =(1-\epsilon) \exp \left(-\frac{d\left((d-1)^{2 k+2 M}-1\right)}{(d-1)^{2 M+2 \gamma(n, d)}(d-2)}\right) .
\end{aligned}
$$

The same calculations also lead us to

$$
\begin{aligned}
\mathbb{P}\left(S_{i_{0}+k}\right) & <\prod_{l=-M+1}^{k} \exp \left(-\frac{d^{2}(d-1)^{2 l}}{(d-1)^{2+2 \gamma(n, d)}}\right)(1-o(1)) \\
& \sim \exp \left(-\frac{d\left((d-1)^{2 k+2 M}-1\right)}{(d-1)^{2 M+2 \gamma(n, d)}(d-2)}\right) .
\end{aligned}
$$

Putting the last two equations together and letting $\epsilon \rightarrow 0$, during which we may assume $M_{\epsilon} \rightarrow \infty$, we have

$$
\mathbb{P}\left(S_{i_{0}+k}\right) \sim \exp \left(-\frac{d(d-1)^{2 k-2 \gamma(n, d)}}{d-2}\right),
$$

as required.
We now focus on proving the claim. First we would like to find the expected number of joins at time $i=i_{0}+k$ given that $N_{i-1}(u) \cap N_{i-1}(v)=\emptyset$. Let $U_{i-1}=N_{i-1}(u) \backslash N_{i-2}(u)$ and $V_{i-1}=N_{i-1}(v) \backslash N_{i-2}(v)$. These are the sets of vertices at distance $i-1$ from $u$ and $v$, respectively. We have to consider two types of join at time $i$. The first type (see Figure (1) consists of edges that join one vertex in $U_{i-1}$ to a vertex in $V_{i-1}$ and therefore create $u v$-paths of odd length. The second type (see Figure 2) contains joins such that a vertex in $V(G) \backslash\left(N_{i-1}(u) \cup N_{i-1}(v)\right)$ has neighbours in each of $U_{i-1}$ and $V_{i-1}$. This generates a path of even length between $u$ and $v$.

Consider joins of the first type. Recall that we are considering the process of exposing the neighbourhoods of $u$ and $v$ step by step. After the first $i-1$ steps, we have exposed
the sets $N_{i-1}(u)$ and $N_{i-1}(v)$, which are assumed to be disjoint. Recall that, according to the pairing model (see Introduction), any vertex in $U_{i-1}$ and $V_{i-1}$ can be regarded as a cell of distinct points, where the number of points corresponds to the number of unexposed neighbours of this vertex. The probability that one given point joins another is then asymptotic to $1 /(d n)$, since any pair of unmatched points is equally likely to be paired and the whole process has, by Corollary 2.2, matched at most $n^{1 / 2+o(1)}=o(n)$ pairs of points to this moment. Asymptotically, there are $\left|U_{i-1}\right|\left|V_{i-1}\right|(d-1)^{2}$ pairs of points such that one is associated with a vertex in $U_{i-1}$ and the other with a vertex in $V_{i-1}$. This is because the hypothesis $i=i_{0}+k$ implies, by Lemma 2.1, that the number of vertices in $U_{i-1}$ or $V_{i-1}$ incident with 'bad edges' created at step $i-2$ is a.a.s. at most $O(\omega(n))$ for any $\omega(n) \rightarrow \infty$, and it is clear that each vertex in $U_{i-1}$ or $V_{i-1}$ with degree larger than 1 in $G\left[N_{i-1}(u)\right] \cup G\left[N_{i-1}(v)\right]$ is incident with a 'bad edge'.

Thus, the expected number of edges of the first type joining the neighbourhoods of $u$ and $v$ at time $i-1$, that is, the number of pairs of points consisting of one point associated with a vertex in $U_{i-1}$ and one point associated with a vertex of $V_{i-1}$ exposed at time $i$ is asymptotic to

$$
\frac{(d-1)^{2}}{d n}\left|U_{i-1}\right|\left|V_{i-1}\right|
$$

A similar argument shows that the expected number of edges of the second type is asymptotic to

$$
\frac{d(d-1)(d-1)^{2}}{d^{2} n^{2}} n\left|U_{i-1}\right|\left|V_{i-1}\right|=\frac{(d-1)^{3}}{d n}\left|U_{i-1}\right|\left|V_{i-1}\right|
$$

Summing these,

$$
\begin{aligned}
& \left.\mathbb{E} \text { (number of joins at time } i=i_{0}+k \mid S_{i-1}\right) \\
& \qquad \sim \frac{(d-1)^{2}}{n}\left|U_{i-1}\right|\left|V_{i-1}\right| \sim \frac{d^{2}(d-1)^{2 i-2}}{n}=\frac{d^{2}(d-1)^{2 k}}{(d-1)^{2+2 \gamma(n, d)}} .
\end{aligned}
$$

We wish to apply the method of moments to establish

$$
\mathbb{P}\left(S_{i_{0}+k} \mid S_{i_{0}+k-1}\right) \sim \exp \left(-d^{2}(d-1)^{2 k-2 \gamma(n, d)-2}\right)
$$

so we have to verify that the $j$-th factorial moment of the random variable $Z$ counting the number of joins at time $i=i_{0}+k$ satisfies $\mathbb{E}\left([Z]_{j}\right)=\frac{\mathbb{E}(Z)^{j}}{j!}$, for all $j \geq 2$.

Let $j \geq 2$ and suppose that the subgraphs induced by $N_{i-1}(u)$ and $N_{i-1}(v)$ are disjoint. As before, let $U_{i-1}=N_{i-1}(u) \backslash N_{i-2}(u), V_{i-1}=N_{i-1}(v) \backslash N_{i-2}(v)$, and, given $r \in U_{i-1}, s \in V_{i-1}, t \in V(G) \backslash\left(N_{i-1}(v) \cup N_{i-1}(u)\right)$, introduce indicator random variables $X_{(r, s)}$ for the event that $r s$ is an edge in $G$ and $Y_{(r, s, t)}$ for the event that $r t$ and st are both edges in $G$. So,

$$
Z=\sum_{r \in U_{i-1}, s \in V_{i-1}} X_{(r, s)}+\sum_{\substack{r \in U_{i-1}, s \in V_{i-1} \\ t \in V(G) \backslash\left(N_{i-1}(v) \cup N_{i-1}(u)\right)}} Y_{(r, s, t)}
$$

is the random variable counting the number of joins that appear between the neighbourhoods of $u$ and $v$ at step $i$.

The $j$-th factorial moment of $Z$ is given by

$$
\begin{equation*}
\mathbb{E}\left([Z]_{j}\right)=\sum_{l=0}^{j} \sum_{\star} \mathbb{P}\left(\left(X_{\left(r_{m}, s_{m}\right)}=1,1 \leq m \leq l\right) \wedge\left(Y_{\left(r_{m}, s_{m}, t_{m}\right)}=1, l+1 \leq m \leq j\right)\right), \tag{2}
\end{equation*}
$$

where, for any given $l, \sum_{\star}$ denotes the sum over all distinct ordered pairs $\left(r_{m}, s_{m}\right)$, $1 \leq m \leq l$, and $\left(r_{m}, s_{m}, t_{m}\right), l+1 \leq m \leq j$.

We shall prove later that the relevant terms in this sum are the ones for which all the ordered pairs are disjoint, that is, there is no repetition of vertices among the $j$ events. Assuming this, we obtain

$$
\begin{aligned}
\mathbb{E}\left([Z]_{j}\right)=\sum_{l=0}^{j}\binom{\left|U_{i-1}\right|}{j}\binom{\left|V_{i-1}\right|}{j}\binom{n-o(n)}{j-l} & \binom{j}{l}^{2} l![(j-l)!]^{2} \times \\
& \times\left(\frac{(d-1)^{2}}{d n-o(n)}\right)^{l}\left(\frac{(d-1)^{3}}{d n^{2}-o\left(n^{2}\right)}\right)^{j-l} .
\end{aligned}
$$

This is because there are $\binom{\left|U_{i-1}\right|}{j}\binom{\left|V_{i-1}\right|}{j}\binom{n-o(n)}{j-l}$ ways of choosing $j$ vertices in each of $U_{i}$ and $V_{i}$, and of choosing $j-l$ vertices in $V(G) \backslash\left(N_{i-1}(u) \cup N_{i-1}(v)\right)$. Moreover, pairing $l$ of the chosen vertices in $U_{i}$ with $l$ of the chosen vertices in $V_{i}$ can be done in $\binom{j}{l}^{2} l!$ ways, whereas there are $[(j-l)!]^{2}$ ways of creating triples on the remaining chosen vertices in $U_{i}, V_{i}$ and the vertices chosen in $V(G) \backslash\left(N_{i-1}(u) \cup N_{i-1}(v)\right)$. Now that we fixed distinct ordered pairs $\left(r_{m}, s_{m}\right), 1 \leq m \leq l$, and $\left(r_{m}, s_{m}, t_{m}\right), l+1 \leq m \leq j$, the term $\left(\frac{(d-1)^{2}}{d n-o(n)}\right)^{l}\left(\frac{(d-1)^{3}}{d n^{2}-o\left(n^{2}\right)}\right)^{j-l}$ corresponds to the probability that all the events $X_{\left(r_{m}, s_{m}\right)}=1$ and $Y_{\left(r_{m}, s_{m}, t_{m}\right)}=1$ occur simultaneously, since there is only a finite number of them.

The previous sum is asymptotic to

$$
\begin{aligned}
& \sum_{l=0}^{j} \frac{\left|U_{i-1}\right|^{j}}{j!} \frac{\left|V_{i-1}\right|^{j}}{j!} \frac{n^{j-l}}{(j-l)!} \frac{j^{2}}{l!}\left(\frac{(d-1)^{2}}{d n}\right)^{l}\left(\frac{(d-1)^{3}}{d n^{2}}\right)^{j-l} \\
& =\frac{\left|U_{i-1}\right|^{j}\left|V_{i-1}\right|^{j}(d-1)^{2 j}}{n^{j} d^{j} j!} \sum_{l=0}^{j} \frac{(d-1)^{j-l} j!}{l!(j-l)!} \\
& =\frac{\left|U_{i-1}\right|^{j}\left|V_{i-1}\right|^{j}(d-1)^{2 j}}{n^{j} d^{j} j!} \sum_{l=0}^{j}\binom{j}{l}(d-1)^{j-l} \\
& =\frac{\left|U_{i-1}\right|^{j}\left|V_{i-1}\right|^{j}(d-1)^{2 j}}{n^{j} j!}=\frac{1}{j!}(\mathbb{E} Z)^{j}=\frac{1}{j!}\left(\frac{d^{2}(d-1)^{2 k}}{(d-1)^{2+2 \gamma(n, d)}}\right)^{j} .
\end{aligned}
$$

It remains to show that indeed the sum over all disjoint ordered pairs $\left(r_{m}, s_{m}\right)$, $1 \leq m \leq l$, and $\left(r_{m}, s_{m}, t_{m}\right), l+1 \leq m \leq j$, is asymptotic to the sum over all distinct ordered pairs. Suppose that there are $j-a$ distinct elements appearing in the first coordinate, $j-b$ in the second and $j-l-c$ in the third, where $a+b+c \geq 1$. The
terms of this form in equation (2) are bounded above by

$$
\begin{aligned}
\sum_{l=0}^{j} \sum_{* *}\binom{j-1}{a}\binom{\left|U_{i-1}\right|}{j-a} & \binom{j-1}{b}\binom{\left|V_{i-1}\right|}{j-b}\binom{j-l-1}{c}\binom{n-o(n)}{j-l-c} \times \\
& \times\binom{ j}{l}^{2} l![(j-l)!]^{2}\left(\frac{(d-1)^{2}}{d n-o(n)}\right)^{l}\left(\frac{(d-1)^{3}}{d n^{2}-o\left(n^{2}\right)}\right)^{j-l}
\end{aligned}
$$

where $\sum_{* *}$ denotes the sum over all triples $(a, b, c) \in\{0, \ldots, j-1\}^{2} \times\{0, \ldots, j-l-1\}$ satisfying $a+b+c \geq 1$. This is because there are $\binom{\left|U_{i-1}\right|}{j-a}$ ways of choosing $j-a$ vertices in $U_{i-1}$ and $\binom{j-1}{a}$ ways of building a multi-set of cardinality $j$ with $j-a$ given elements (and using all of them). The same is true for choosing vertices in $V_{i-1}$ and $V(G) \backslash\left(N_{i-1}(u) \cup N_{i-1}(v)\right)$. Our last expression is smaller or equal to

$$
\begin{aligned}
\sum_{l=0}^{j} \sum_{* *} \frac{(j-1)^{a+b}(j-l-1)^{c}}{a!b!c!} & \frac{\left|U_{i-1}\right|^{j-a}}{(j-a)!} \frac{\left|V_{i-1}\right|^{j-b}}{(j-b)!} \frac{n^{j-l-c}}{(j-l-c)!} \times \\
& \times \frac{j!^{2}}{l!}\left(\frac{(d-1)^{2}}{d n-o(n)}\right)^{l}\left(\frac{(d-1)^{3}}{d n^{2}-o\left(n^{2}\right)}\right)^{j-l}
\end{aligned}
$$

If we divide this by $\frac{\left|U_{i-1}\right|^{j}\left|V_{i-1}\right|^{j}(d-1)^{2 j}}{n^{j} j!}$, this is asymptotic (with respect to $n$ ) to

$$
\sum_{l=0}^{j} \sum_{* *} \frac{\mathcal{K}(a, b, c, j, l, d)}{\left|U_{i-1}\right|^{a}\left|V_{i-1}\right|^{b} n^{c}}
$$

where $\mathcal{K}(a, b, c, j, l, d)$ does not depend on $n$. Since $\left|U_{i-1}\right|^{a}\left|V_{i-1}\right|^{b} n^{c} \rightarrow \infty$ as $n \rightarrow \infty$ for every $a+b+c \geq 1$, we conclude that the above sum tends to zero as $n$ tends to infinity and therefore the terms related to non-disjoint tuples in equation (2) can indeed be ignored to compute $\mathbb{E}\left([Z]_{j}\right)$.

Given this, the method of moments implies that
$\mathbb{P}\left(\right.$ no joins at time $\left.i=i_{0}+k \mid N_{i-1}(v) \cap N_{i-1}(u)=\emptyset\right) \sim \exp \left(-d^{2}(d-1)^{2 k-2 \gamma(n, d)-2}\right)$, which completes the proof of the claim and therefore establishes the lemma.

We are now prepared to prove the result mentioned at the end of the last section.
Corollary 2.4. Let $u, v$ be vertices chosen independently with uniform probability in $G \in \mathcal{G}_{n, d}$. For any function $\omega(n)$ such that $\omega(n) \rightarrow \infty$, the assertion $d(u, v)<\log _{d-1} n+$ $\omega(n)$ holds a.a.s.

Proof. Let $\epsilon>0$. Lemma 2.3 implies that the probability of the event $E_{k}$ that $u$ and $v$ are at distance greater than $\log _{d-1} n+k$ is asymptotic to

$$
\exp \left(-d^{2}(d-1)^{2 k-2 \gamma(n, d)-2}\right)
$$

where $k$ a fixed integer and $\gamma(n, d)$ is the fractional part of $\frac{1}{2} \log _{d-1} n$. So, $\mathbb{P}\left(E_{k}\right)<\epsilon$ for $k$ sufficiently large, and our result follows.

Lemma 2.5. Let $k$ be an integer and let $\gamma(n, d)$ be the fractional part of $\frac{1}{2} \log _{d-1} n$. Define $O_{i}$ to be the random variable counting the number of uv-paths of odd length, that
is, paths of the first case, created at step $i$. Let $E_{i}$ be the equivalent random variable for paths of even length. Then
(i) With $\mu_{k}=d(d-1)^{2 k-2 \gamma(n, d)-2}$,

$$
\mathbb{P}\left(O_{i_{0}+k}=j \mid N_{i_{0}+k-1}(u) \cap N_{i_{0}+k-1}(v)=\emptyset\right) \sim \frac{\mu_{k}^{j}}{j!} \exp \left(-\mu_{k}\right)
$$

(ii) With $\nu_{k}=d(d-1)^{2 k-2 \gamma(n, d)-1}$,

$$
\mathbb{P}\left(E_{i_{0}+k}=j \mid N_{i_{0}+k-1}(u) \cap N_{i_{0}+k-1}(v)=\emptyset \wedge O_{i_{0}+k}=0\right) \sim \frac{\nu_{k}^{j}}{j!} \exp \left(-\nu_{k}\right)
$$

Proof. This can be proven by the method of moments using calculations very similar to the ones in the previous lemma, proceeding separately for joins of the first type and joins of the second type. For the former, we condition on the event that no joins occurred in previous steps of the process, and, for the latter, we further assume that no joins of the first type occurred in the current step. The details are omitted.

We observe that, alternatively, the proofs of the previous lemma and of Lemma 2.3 could be unified by considering joint factorial moments of random variables for joins of the first type and of the second type.

We are now ready to deduce one of the main results.
Theorem 2.6. Fix an integer $l \geq 1$. The probability that two vertices $u, v$ chosen independently with uniform probability in $G \in \mathcal{G}_{n, d}$ are joined by exactly $l$ distinct geodesics is asymptotic to

$$
\begin{aligned}
\sum_{k=-\infty}^{\infty} \frac{\left(d(d-1)^{2 k-2 \gamma(n, d)-2}\right)^{l}}{l!} \exp & \left(-\frac{d(d-1)^{2 k-2 \gamma(n, d)-1}}{d-2}\right) \times \\
& \times\left(1+(d-1)^{l} \exp \left(-d(d-1)^{2 k-2 \gamma(n, d)-1}\right)\right)
\end{aligned}
$$

Proof. Let $Z_{l}$ be the event that $u$ and $v$ are joined by exactly $l$ geodesics, and let $J_{i}$ denote the event that the first join occurs at time $i$. Then, given a positive integer $M$,

$$
\mathbb{P}\left(Z_{l}\right)=\mathbb{P}\left(Z_{l} \wedge \bigcup_{k=1}^{i_{0}-M} J_{k}\right)+\sum_{k=-M+1}^{M-1} \mathbb{P}\left(Z_{l} \wedge J_{i_{0}+k}\right)+\mathbb{P}\left(Z_{l} \wedge \bigcup_{k \geq i_{0}+M} J_{k}\right)
$$

The first and last element in the right-hand side can be made less than $\epsilon$, for any given $\epsilon>0$, by choosing $M=M_{\epsilon}$ sufficiently large, as ensured by Corollary 2.4 and by the fact that equation (11) holds a.a.s. Also, each of the terms $\mathbb{P}\left(Z_{l} \wedge J_{i_{0}+k}\right)$, for $-M+1 \leq k \leq M-1$, is equal to

$$
\begin{aligned}
& \mathbb{P}\left(N_{i_{0}+k-1}(u) \cap N_{i_{0}+k-1}(v)=\emptyset\right)\left[\mathbb{P}\left(O_{i_{0}+k}=l \mid N_{i_{0}+k-1}(u) \cap N_{i_{0}+k-1}(v)=\emptyset\right)+\right. \\
& \quad+\mathbb{P}\left(O_{i_{0}+k}=0 \mid N_{i_{0}+k-1}(u) \cap N_{i_{0}+k-1}(v)=\emptyset\right) \times \\
& \left.\quad \times \mathbb{P}\left(E_{i_{0}+k}=l \mid O_{i_{0}+k}=0 \wedge N_{i_{0}+k-1}(u) \cap N_{i_{0}+k-1}(v)=\emptyset\right)\right] .
\end{aligned}
$$

By our previous lemmas, this is asymptotic to

$$
\begin{aligned}
& \exp \left(-\frac{d(d-1)^{2 k-2 \gamma(n, d)-2}}{d-2}\right)\left(\frac{\left(d(d-1)^{2 k-2 \gamma(n, d)-2}\right)^{l}}{l!} \exp \left(-d(d-1)^{2 k-2 \gamma(n, d)-2}\right)+\right. \\
& \left.\quad+\frac{\left(d(d-1)^{2 k-2 \gamma(n, d)-1}\right)^{l}}{l!} \exp \left(-d(d-1)^{2 k-2 \gamma(n, d)-2}-d(d-1)^{2 k-2 \gamma(n, d)-1}\right)\right)
\end{aligned}
$$

Hence, if we let $\epsilon$ tend to zero,

$$
\begin{aligned}
\mathbb{P}\left(Z_{l}\right) \sim & \sum_{k=-\infty}^{\infty} \frac{\left(d(d-1)^{2 k-2 \gamma(n, d)-2}\right)^{l}}{l!} \exp \left(-\frac{d(d-1)^{2 k-2 \gamma(n, d)-1}}{d-2}\right) \times \\
& \times\left(1+(d-1)^{l} \exp \left(-d(d-1)^{2 k-2 \gamma(n, d)-1}\right)\right)
\end{aligned}
$$

as required.
An interesting special case is when $l=1$, since this theorem provides the probability of $u$ and $v$ being joined by a unique geodesic. This probability is given by

$$
\begin{aligned}
\sum_{k=-\infty}^{\infty} d(d-1)^{2 k-2 \gamma(n, d)-2} \exp & \left(-\frac{d(d-1)^{2 k-2 \gamma(n, d)-1}}{d-2}\right) \times \\
& \times\left(1+(d-1) \exp \left(-d(d-1)^{2 k-2 \gamma(n, d)-1}\right)\right)
\end{aligned}
$$

The probability here is a function of $\gamma(n, d)$ and oscillates as $\gamma(n, d)$ varies from 0 to 1 .
We include some numerical results in the table below for some values of $d$, where prob is the probability of a unique geodesic as $\gamma(n, d)=0$ and osc is the maximum variation with respect to $\gamma=0$ as $\gamma$ varies from 0 to 1 .

| $d$ | prob | osc |
| :---: | :---: | :---: |
| 3 | 0.7213 | $8.6 \times 10^{-6}$ |
| 4 | 0.6073 | $1.4 \times 10^{-3}$ |
| 5 | 0.5444 | $7.9 \times 10^{-3}$ |
| 10 | 0.4411 | $7.6 \times 10^{-2}$ |
| 100 | 0.3743 | 0.3 |

The magnitude of the oscillations depends on $d$. We justify why it is small when $d$ is small. Note that the probability of a unique geodesic is equal to

$$
\begin{align*}
& \frac{d-2}{d-1} S_{(d-1)^{2}}\left(-\gamma(n, d)+\log _{(d-1)^{2}} \frac{d}{(d-1)(d-2)}\right)+ \\
& +\frac{d-2}{d-1} S_{(d-1)^{2}}\left(-\gamma(n, d)+\log _{(d-1)^{2}} \frac{d}{d-2}\right) \tag{3}
\end{align*}
$$

where $S_{c}(x)=\sum_{m=-\infty}^{\infty} c^{m+x} \exp \left(-c^{m+x}\right)$, a function with period 1. The classical Poisson summation formula gives us that

$$
S_{c}(x)=\sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} c^{t+x} \exp \left(-c^{t+x}\right) \exp (2 \pi i m t) d t
$$

Setting $z=c^{t+x}$ gives

$$
\begin{equation*}
S_{c}(x)=\frac{1}{\log c} \sum_{m=-\infty}^{\infty} \exp (-2 \pi i m x) \int_{0}^{\infty} \exp (-z+2 \pi i m \log z / \log c) d z \tag{4}
\end{equation*}
$$

and the integral is just $\Gamma(2 \pi i m / \log c+1)$.
By properties of the gamma function (see for instance [1]), we have

$$
|\Gamma(1+y i)|=|i y \Gamma(y i)|=|y| \sqrt{\frac{\pi}{y \sinh (\pi y)}},
$$

so given $m$ in the previous summation,

$$
\begin{align*}
|\Gamma(2 \pi m i / \log c+1) \exp (-2 \pi i m x)| & =|\Gamma(2 \pi i m / \log c+1)| \\
& =\left(\frac{2 \pi^{2}|m|}{\log c\left|\sinh \left(2 \pi^{2} m / \log c\right)\right|}\right)^{1 / 2} \tag{5}
\end{align*}
$$

The term for $m=0$ in the sum (4) is independent of $x$, hence it yields terms independent of $n$ in equation (3). In the special case $d=3$, equation (5) leads to the following bounds on the other terms of the sum (4). For $|m|=1$, the bound is approximately $4.32 \times 10^{-3}$, for $|m|=2$, it is approximately $4.94 \times 10^{-6}$, and for larger $|m|$ the bounds are even smaller. Similar observations explain the small oscillations when $d$ is small.

## 3. Almost geodesic cycles

In the proof of Theorem 1.1 we shall use the following auxiliary result.
Lemma 3.1. Let $G \in \mathcal{G}_{n, d}$ and let $u, v$ be vertices chosen independently at random in $G$. Consider functions $\alpha(n), \beta(n)$ such that $\alpha(n), \beta(n) \rightarrow \infty, \alpha(n)=o\left(\log _{d-1} n\right)$ and $\beta(n)=o(\alpha(n))$. Then a.a.s. every vertex at distance $\lfloor\alpha(n)\rfloor$ from $u$ or $v$ lies on at most one uv-path with length less than or equal to $\log _{d-1} n+\beta(n)$.

Proof. We prove this result for vertices at distance $\lfloor\alpha(n)\rfloor$ from $u$, and by a similar argument the same result holds for vertices at distance $\lfloor\alpha(n)\rfloor$ from $v$. As in Section [2, we consider the process of exposing the neighbourhoods of $u$ and $v$ based on the pairing model. Here, $N(u)$, the neighbourhood of $u$, is exposed for $\lfloor\alpha(n)\rfloor$ steps while $N(v)$, the neighbourhood of $v$, is exposed for $\left\lfloor\frac{1}{2} \log _{d-1} n-\beta(n)\right\rfloor$ steps. By Lemma [2.1] a.a.s. $N(u)$ and $N(v)$ are both trees. Moreover, Lemma 2.3 ensures that $N(u) \cap N(v)=\emptyset$ holds a.a.s.

Let $U_{\alpha}$ be the set of vertices at distance $\lfloor\alpha(n)\rfloor$ from $u$. Given a vertex $w \in U_{\alpha}$, let $Y_{w}$ be the indicator random variable for the event that $w$ is connected to $N(v)$ by at least two distinct paths of length less than or equal to $\frac{1}{2} \log _{d-1} n-\alpha(n)+2 \beta(n)+2$. Define $Y=\sum_{w \in U_{\alpha}} Y_{i}$. It is clear that this lemma follows if we prove that a.a.s. $Y=0$. We shall do this by using

$$
\mathbb{P}(Y \geq 1) \leq \sum_{w \in U_{\alpha}} \mathbb{P}\left(Y_{w}=1\right)
$$

and by showing that the right-hand side goes to zero as $n$ tends to infinity.
For a fixed $w$, define the set $N_{w}^{\prime}$ obtained by the exposure of the neighbourhood of $w$ for $\frac{1}{2} \log _{d-1} n-\alpha(n)+2 \beta(n)+2$ steps. This time, however, the neighbour of $w$ in $N(u)$
is not added to $N_{w}^{\prime}$ at the first step of the process, that is, only the "new" neighbours of $w$ are exposed. As in Lemma [2.1] we use the term "bad edges" for edges that yield cycles in $N_{w}^{\prime}$. Consider the random variable $X_{w}$ counting the number of "bad edges" in $N_{w}^{\prime}$. Then, calculations analogous to the ones in Lemma 2.1 establish that

$$
\begin{aligned}
\mathbb{E}\left(X_{w}\right) & =\sum_{s=0}^{\left\lfloor\frac{1}{2} \log _{d-1} n-\alpha(n)+2 \beta(n)+2\right\rfloor} O\left(\frac{(d-1)^{2 s}}{n}\right) \\
& =O\left(\frac{(d-1)^{\log _{d-1} n-2 \alpha(n)+4 \beta(n)}}{n}\right)=O\left((d-1)^{4 \beta(n)-2 \alpha(n)}\right) .
\end{aligned}
$$

Thus Markov's inequality implies

$$
\mathbb{P}\left(X_{w} \geq 1\right)=O\left((d-1)^{4 \beta(n)-2 \alpha(n)}\right) .
$$

Now, note that

$$
\mathbb{P}\left(Y_{w}=1\right)=\mathbb{P}\left(X_{w} \geq 1\right) \mathbb{P}\left(Y_{w}=1 \mid X_{w} \geq 1\right)+\mathbb{P}\left(X_{w}=0\right) \mathbb{P}\left(Y_{w}=1 \mid X_{w}=0\right)
$$

We have a bound for the first term in this sum. For the second term, we use the definition of conditional probability and observe that the event $\left(Y_{w}=1\right) \wedge\left(X_{w}=0\right)$ occurs only if there is a pair of distinct paths joining $w$ to $N(v)$ with length at most $\frac{1}{2} \log _{d-1} n-\alpha(n)+2 \beta(n)+2$ and with the property that, after they first split, they do not join again.

So, a bound on $\mathbb{P}\left(Y_{w}=1 \wedge X_{w}=0\right)$ may be obtained by counting the number of possible pairs of distinct paths $P$ and $Q$ joining $u_{i}$ to $N(v)$ with lengths $r$ and $s$, where $r \leq s \leq \frac{1}{2} \log _{d-1} n-\alpha(n)+2 \beta(n)+2$, and the first $j$ vertices are shared by both paths, while the remainder of the paths are internally disjoint. So, if $i_{0}=\left\lfloor\frac{1}{2} \log _{d-1} n\right\rfloor$,

$$
\begin{aligned}
\mathbb{P}\left(Y_{w}=1 \wedge X_{w}=0\right)= & \sum_{s=1}^{\left\lfloor i_{0}-\alpha(n)+2 \beta(n)+2\right\rfloor} \sum_{r=1}^{\left\lfloor i_{0}-s\right\rfloor} \sum_{j=0}^{r-1} \\
& \binom{(d-1)^{\left\lfloor i_{0}-\beta(n)\right\rfloor}}{2}\binom{n-o(n)}{r+s-j-2}\binom{r+s-j-2}{j}\binom{r+s-2}{r-1} \\
& \times j!(r-1)!(s-1)!O\left(\left(\frac{(d-1)}{n-o(n)}\right)^{r+s-j}\right) \\
= & \sum_{s=1}^{\left\lfloor i_{0}-\alpha(n)+2 \beta(n)+2\right\rfloor} \sum_{s=1}^{\left\lfloor i_{0}-s\right\rfloor} \sum_{j=0}^{r-1} O\left(\frac{(d-1)^{2 i_{0}-2 \beta(n)}(d-1)^{r+s-j}}{n^{2}}\right) \\
& =O\left((d-1)^{2 \beta(n)-2 \alpha(n)}\right) .
\end{aligned}
$$

Note that the formula holds because there are at most $\binom{(d-1)^{\left\lfloor i_{0}-\beta(n)\right\rfloor}}{2}$ ways of choosing two vertices in $N(v)$ and there are $\binom{n-o(n)}{r+s-j-2}$ ways of choosing vertices in the graph to include in the two paths. Moreover, these vertices can be divided into vertices of $P \cap Q, P-Q$ and vertices of $Q-P$ in $\binom{r+s-j-2}{j}\binom{r+s-2}{r-1}$ ways and can then be ordered to form the paths in $j!(r-1)!(s-1)$ ! ways. Finally, each edge on the path appears with probability at most $\frac{(d-1)}{n-o(n)}$ conditional on the fact that all previous edges on the path have appeared.

We conclude that

$$
\begin{aligned}
\mathbb{P}\left(Y_{w}=1\right) & =\mathbb{P}\left(X_{w} \geq 1\right) \mathbb{P}\left(Y_{w}=1 \mid X_{w} \geq 1\right)+\mathbb{P}\left(X_{w}=0\right) \mathbb{P}\left(Y_{w}=1 \mid X_{w}=0\right) \\
& \leq \mathbb{P}\left(X_{w} \geq 1\right)+\mathbb{P}\left(Y_{w}=1 \mid X_{w}=0\right) \\
& =O\left((d-1)^{4 \beta(n)-2 \alpha(n)}\right)+\frac{O\left((d-1)^{2 \beta(n)-2 \alpha(n)}\right)}{1-O\left((d-1)^{4 \beta(n)-2 \alpha(n)}\right)} \\
& =O\left((d-1)^{4 \beta(n)-2 \alpha(n)}\right) .
\end{aligned}
$$

Now, because there are $O\left((d-1)^{\alpha(n)}\right)$ vertices at distance $\lfloor\alpha(n)\rfloor$ of $u$, we have

$$
\begin{aligned}
\mathbb{P}(Y \geq 1) & \leq \sum_{w \in U_{\alpha}} \mathbb{P}\left(Y_{w}=1\right) \\
& =O\left((d-1)^{\alpha(n)}\right) O\left((d-1)^{4 \beta(n)-2 \alpha(n)}\right)=O\left((d-1)^{4 \beta(n)-\alpha(n)}\right) .
\end{aligned}
$$

Because $\beta(n)=o(\alpha(n))$, this term goes to zero as $n$ tends to infinity and indeed $\mathbb{P}(Y \geq 1) \rightarrow 0$. The lemma follows.

We are now ready to prove the main theorem.
Proof of Theorem 1.1, Part (i) of the theorem follows from Corollary 2.4 and equation (II).

Before proving part (ii), it is convenient to deal first with the following simpler goal. To state this we need two definitions. A $k$-near-geodesic is a path that is a geodesic between the two vertices at distance $k$ from its ends. A vertex $p$ on a path $P$ between vertices $u$ and $v$ is said to be a midpoint of $P$ if $\left|d_{P}(u, p)-d_{P}(v, p)\right| \leq 1$, where $d_{P}$ denotes the distance along path $P$.

Lemma 3.2. Asymptotically almost surely, for every two distinct $k$-near-geodesics, $P$ and $Q$, between $u$ and $v$ with midpoints $p$ and $q$, respectively,

$$
\log _{d-1}(n)-\omega(n)<d(p, q)<\log _{d-1}(n)+\omega(n)
$$

Proof. For the upper bound, we know as in part (i) of the theorem that a.a.s. $d(u, v)<$ $\log _{d-1} n+\omega(n)$, hence there is a sufficiently short path connecting $p$ to $q$ through $u$ or $v$.

We turn to the lower bound. Given a function $\omega(n)$ satisfying $\omega(n) \rightarrow \infty$, we know that a.a.s. $d(u, v) \geq \log _{d-1} n-\omega(n)$ (see (11). Consider distinct $k$-near geodesics $P$ and $Q$.
Claim 1: $P$ and $Q$ a.a.s. do not have a vertex in common at distance at least $\frac{\omega(n)}{3}$ from their endpoints.

We prove the claim by contradiction. Suppose without loss of generality that such a vertex is closer to $u$ on $P$ and let $w$ be the vertex on $P$ at distance $\left\lfloor\frac{\omega(n)}{3}\right\rfloor$ from $u$. Note that $P$ and $Q$ a.a.s. differ at some vertex or edge after $w$, since the set of vertices at distance at most $\frac{\omega(n)}{3}$ a.a.s. induces a tree. But then, $w$ lies on at least two distinct $u, v$ paths with length less than or equal to $\log _{d-1} n+\log \omega(n)$, which a.a.s. does not occur by Lemma 3.1 with $\alpha(n)=\frac{\omega(n)}{3}$ and $\beta(n)=\log \omega(n)$. (Note that the lengths of both $P$ and $Q$ are a.a.s. bounded by $\log _{d-1} n+\log \omega(n)$ because any $k$-near geodesic between $u$ and $v$ has length at most $d(u, v)+4 k$.) This proves the claim.

Now consider the event that the midpoints $p$ and $q$ of $P$ and $Q$ are at distance at most $\log _{d-1} n-\omega(n)$. One way for this to occur is by the existence of a $p q$-path

14 ITAI BENJAMINI, CARLOS HOPPEN, ERAN OFEK, PAWEŁ PRAŁAT, AND NICK WORMALD


Figure 3. Path $R$
$R$ of length at most $\log _{d-1} n-\omega(n)$ using vertices and edges on $P \cup Q$ only. But $d(u, v) \geq \log _{d-1} n-\omega(n)$ implies that $R$ does not contain vertices at distance less than or equal to $\frac{\omega(n)}{3}$ from $u$ or $v$. Claim 1 shows that no other vertex can be in common. Thus, a.a.s. there is no short path from $p$ to $q$ using edges on $P$ and $Q$ only.

So consider a geodesic $A$ between $p$ and $q$ containing at least one edge outside $P \cup Q$. Using $A$ oriented from $p$ to $q$ as a reference, let $v_{P}$ denote the last vertex on $A \cap P$ and let $v_{Q}$ be the first vertex on $A \cap Q$ after $v_{P}$. The vertices $v_{P}$ and $v_{Q}$ divide the geodesic into three parts, namely from $p$ to $v_{P}$, from $v_{P}$ to $v_{Q}$ and from $v_{Q}$ to $q$. Because $P, Q$ are $k$-near-geodesics between $u$ and $v$ for a fixed $k$ and $A$ is a geodesic between $p$ and $q$, we must have $d_{A}\left(p, v_{P}\right)=d_{P}\left(p, v_{P}\right)$ and $d_{A}\left(v_{Q}, q\right)=d_{Q}\left(v_{Q}, q\right)$. So, for $p$ and $q$ to be at distance at most $\log _{d-1} n-\omega(n)$ for some $\omega(n) \rightarrow \infty$, it must be that

$$
\begin{equation*}
d_{A}\left(v_{P}, v_{Q}\right)<\log _{d-1} n-d_{P}\left(p, v_{P}\right)-d_{Q}\left(v_{Q}, q\right)-\omega(n) . \tag{6}
\end{equation*}
$$

So, a short path between $p$ and $q$ has to be caused by a short path connecting a vertex in $P$ to a vertex in $Q$ which is internally disjoint from $P \cup Q$. More precisely, there must exist vertices $v_{P}, v_{Q}$ on $P$ and $Q$, respectively, and an $v_{P} v_{Q}$-path $R$ satisfying:

$$
\begin{array}{r}
V(R) \cap V(P \cup Q)=\left\{v_{P}, v_{Q}\right\}, \\
|R| \leq \log _{d-1} n-d\left(p, v_{P}\right)-d\left(v_{Q}, q\right)-\omega(n) . \tag{8}
\end{array}
$$

Such a configuration is illustrated by Figure 3.
We prove that a.a.s. $G$ does not contain a path $R$ satisfying (77) and (8). We do this by exposing the neighbourhoods of vertices along $P$ and $Q$ conditional on the particular paths $P$ and $Q$ being in the graph. By relaxing the condition that $P$ and $Q$ are $k$-near geodesics (but retaining the condition that their length is at most $d(u, v)+4 k+\omega(n))$, we may take the rest of the pairing to be random. We will later argue that the number of pairs of such paths $P$ and $Q$ is small enough for our argument to work.

Given a vertex $p_{r}$ at distance $r$ from $p$ along $P$ and a vertex $q_{s}$ at distance $s$ from $q$ along $Q$, let $X_{p_{r}, q_{s}}$ be the event that $p_{r}$ and $q_{s}$ are connected by a path of length at most $\log _{d-1} n-r-s-\omega(n)$ which is internally disjoint from $P$ and $Q$. Define the random variable $Y_{P, Q}=\sum_{r=0}^{\left\lceil\log _{d-1} n-\omega\right\rceil} \sum_{s=0}^{\left\lceil\log _{d-1} n-\omega\right\rceil-r} X_{p_{r}, q_{s}}$, so that $Y_{P, Q}=0$ only if $G$ does not contain a path $R$ satisfying (7) and (8) with respect to $P$ and $Q$.

Once again, we look at the process in which the neighbours of $p_{r}$ and $q_{s}$ are exposed, then their neighbours are exposed, and so on, but we do not consider the neighbours of $p_{r}$ and $q_{s}$ on $P$ or $Q$, so as to expose the sets $N_{i}\left(p_{r}\right)$ and $N_{i}\left(q_{s}\right)$ containing only the vertices at distance at most $i$ from $p_{r}$ and $q_{s}$ that can be reached by paths internally disjoint from $P$ and $Q$. Clearly, $p_{r}$ and $q_{s}$ are joined by a path as in (77) and (8) only if $N_{i}\left(p_{r}\right)$ and $N_{i}\left(q_{s}\right)$ join in at most $\frac{1}{2}\left(\log _{d-1} n-r-s-\omega(n)\right)$ steps. The probability of
this can be calculated as in the earlier sections, and we conclude that

$$
\begin{aligned}
\mathbb{P}\left(Y_{P, Q} \geq 1\right) & \leq \mathbb{E}(Y) \\
& \leq \sum_{r=0}^{\left\lceil\frac{1}{2}\left(\log _{d-1} n-\omega\right)\right\rceil\left\lceil\frac{1}{2}\left(\log _{d-1} n-\omega\right\rceil\right)-r} \sum_{s=0}^{\left\lceil\frac{1}{2}\left(\log _{d-1} n-\omega\right)\right\rceil\left\lceil\frac{1}{2}\left(\log _{d-1} n-\omega\right)\right\rceil-r} \mathbb{P}\left(X_{p_{r}, q_{s}}\right) \\
& =4 \sum_{r=0} O\left((d-1)^{2\left(\frac{1}{2}\left(\log _{d-1} n-\omega\right)-r-s\right)} / n\right) \\
& =O\left((d-1)^{-\omega(n)}\right) .
\end{aligned}
$$

By Lemma 3.1] with $\alpha(n)$ any function tending to infinity sufficiently slowly, and $\beta(n)=$ $o(\alpha(n))$, the number paths of length at most $\log _{d-1} n+\beta(n)$ between $u$ and $v$ is a.a.s. at most $2(d-1)^{\alpha(n)}$, since this is a bound on the number of vertices at distance $\lfloor\alpha(n)\rfloor$ from $u$. Thus, a.a.s. there are at most $\gamma(n)$ pairs of such paths between $u$ and $v$, for any $\gamma(n) \rightarrow \infty$. Let $Z$ denote this asymptotically almost sure event (for some $\alpha$ to be restricted shortly), and let $\bar{Z}$ be its complement.

Let $Y=\sum_{P, Q} Y_{P, Q}$, where the sum is over all pairs of distinct paths between $u$ and $v$ whose length is at most $d(u, v)+4 k+\omega(n)$. Then,

$$
\begin{aligned}
\mathbb{P}(Y \geq 1) & =\mathbb{P}((Y \geq 1) \cap \bar{Z})+\mathbb{P}((Y \geq 1) \cap Z) \\
& \leq \mathbb{P}(\bar{Z})+\sum_{P, Q} \mathbb{P}\left(\left(Y_{P, Q} \geq 1\right) \cap Z\right) \\
& \leq \mathbb{P}(\bar{Z})+O\left(\gamma(n)(d-1)^{-\omega(n)}\right),
\end{aligned}
$$

which tends to 0 provided $\gamma(n)=o\left((d-1)^{\omega(n)}\right)$. As a consequence, a.a.s. there are no configurations satisfying the conditions in (77) and (8). Hence, a.a.s. the inequality $d(p, q)>\log _{d-1} n-\omega(n)$ holds, and we the lemma.

We return to the proof of the theorem. We say that two points on a cycle are diametrically opposite if the distance between them around the cycle is $\lfloor\ell / 2\rfloor$, where the cycle has length $\ell$. Note that part (ii) of the theorem follows immediately from the following, since if there is a "short-cut" for any two vertices on a cycle, there is a short-cut for a pair of diametrically opposite ones.
Claim 2: Asymptotically almost surely, there is a cycle $C$ passing through $u$ and $v$, of length $\ell$ satisfying $\left|2 \log _{d-1} n-\ell\right|<\omega(n)$, such that for every pair of points $p$ and $q$ that are diametrically opposite on $C, d(p, q) \geq \log _{d-1} n-\log _{d-1} \log _{d-1} n-\omega(n)$.

We now prove Claim 2. Note that, by (i), for all $\varepsilon>0$, there is $K$ sufficiently large that $\mathbb{P}\left(d(u, v)>\log _{d-1} n-K\right)>1-\varepsilon$ for all $n$ sufficiently large. From the proof of Lemma [2.3 it is evident that for each neighbour $u^{\prime}$ of $u$ and neighbour $v^{\prime}$ of $v$, there is a.a.s. a path from $u$ to $v$ in which the second and penultimate vertices are $u^{\prime}$ and $v^{\prime}$, and of length at most $\log _{d-1} n+\omega(n)$. Hence we may also assume that $K$ is sufficiently large that the probability that there is such a path for two given neighbours $u^{\prime}$ and $v^{\prime}$ and of length at most $\log _{d-1} n+K$ is at least $1-\varepsilon$. So with probability at least $1-2 \varepsilon$, we can choose two such $u v$-paths $P$ and $Q$, where the neighbours of $u$ and $v$ on $P$ are both different from those on $Q$. In each case we may select a shortest path with these specifications. Then $P$ and $Q$ must be $(K+1)$-near geodesics.

By Claim 1, we may assume there is no vertex in common between $P$ and $Q$ that is more than $\omega(n) / 3$ from $u$ and $v$. For $\omega$ growing slowly enough, there is a.a.s. no point in common that is at most $\omega(n) / 3$ from $u$ and $v$ either, since Lemma 2.1 implies that a.a.s. neither $u$ nor $v$ is in a short cycle. Let $C$ be the union of the paths $P$ and $Q$. From the bounds on $d(u, v), C$ has length at least $2 \log _{d-1} n-2 K$.

To prove the statement about all diametrically opposite points $p$ and $q$ on $C$, we may rework the argument in Lemma [3.2. The Claim proved above shows that every short path of the type we are interested in must use some edge not on $P$ or $Q$. Arguing as before, we only need to eliminate the existence of $A$ such that (6) holds. The same argument as before shows that for any fixed such $p$ and $q$, with $Y_{P, Q}$ defined as before, we again have $\mathbb{P}\left(Y_{P, Q} \geq 1\right)=O\left((d-1)^{-\omega(n)}\right)$.

Now apply this inequality to the $O\left(\log _{d-1} n\right)$ pairs of vertices $p$ and $q$ diametrically opposite on $C$. Also, put

$$
\omega(n)=\log _{d-1} \log _{d-1} n+\gamma(n) .
$$

Then the probability that $Y_{P, Q} \geq 1$ for at least one of these choices of $p$ and $q$ is $O\left((d-1)^{-\gamma(n)}\right)$. Hence, if $\gamma(n) \rightarrow \infty$, we have a.a.s. for all such $p$ and $q, d(p, q) \geq$ $f(n)-\log _{d-1} \log _{d-1} n-\gamma(n)$. Replacing $\gamma$ by $\omega$ gives the final statement in Claim 2, with probability at least $1-2 \varepsilon+o(1)$. This statement is true for all $\varepsilon>0$. That fact implies that the final statement in Claim 2 holds a.a.s. (This can be regarded as "letting $\varepsilon \rightarrow 0$ sufficiently slowly".) Combining this with part (i) proves Claim 2, since, although there may be different functions at the different occurrences of $\omega$, they can be made the same. This completes the proof of Claim 2. To obtain the theorem, we note that the proof of Claim 2 does more: it shows that the two paths can be chosen to use distinct neighbours of $u$ in their initial step, and distinct neighbours of $v$ in their final step.

## 4. Final Remarks

In this article we have examined the "shape" of random regular graphs. This brings up related questions.

Our proof of the main theorem can be seen to give more: a.a.s. for every pair of short (i.e. bounded length) paths, one containing $u$ and one containing $v$, there is an almost geodesic cycle containing both of these paths. We also show that the paths referred to in the theorem each contain a geodesic between the two vertices at distance $k$ from its ends, for any $k$ tending to infinity with $n$.

A geodesic cycle $C$ in $G$ is a cycle in which for every two vertices $u$ and $v$ in $C$, the distance $d_{G}(u, v)$ is equal to $d_{C}(u, v)$. A significant open problem is to determine whether in a random $d$-regular graph, a.a.s. almost all pairs of vertices lie in a geodesic cycle. It is not even known if at least one geodesic cycle of length asymptotic to $\log _{d-1} n$ exists a.a.s.

We may also draw conclusions on how "thin" the topological triangles are in random regular graphs. Consider the proof of Lemma [2.3, which analyses the time at which two simultaneous breadth-first reaches from $u$ and from $v$ join each other. The proof is concerned with an accurate estimate of the probability that there are no joins by a time near $i_{0}$. It is easy to see from the ideas in the proof that for large $K$, the second join is quite likely to occur by time $i_{0}+K$, and furthermore that the first two joins
are quite likely to be in branches that diverged, in the breadth first search from $u$, at time less than $K$, and similarly from $v$. Let $u^{\prime}$ and $v^{\prime}$ be the points of divergence near $u$ and $v$. Then the joins give two paths $P$ and $Q$ from $u^{\prime}$ to $v^{\prime}$, the shorter of which, say $P$, is geodesic, and we can choose another vertex, $w$, on $Q$, of distance $K$ from $u^{\prime}$, such that the resulting two subpaths of $Q$ to $u^{\prime}$ and $v^{\prime}$ from $w$ are both geodesic. Thus $u^{\prime}, v^{\prime}, w$ form a geodesic triangle. By Lemma 3.2 (noting $P$ and $Q$ are $2 K$-near geodesics from $u$ to $v$ ), the distance between the midpoints of $P$ and $Q$ is a.a.s. at least $\log _{d-1}(n)-\omega(n)$, where $\omega(n)$ is any function tending to $\infty$. Hence the midpoint $p$ of $P$ a.a.s. has distance at least $\frac{1}{2} \log _{d-1}(n)-\omega(n)$ from the union of the other two sides of the geodesic triangle $u^{\prime}, v^{\prime}, w$. The probability in the above statements tends to 1 if we let $K \rightarrow \infty$ sufficiently slowly. It follows that a random $d$-regular graph, for $d \geq 3$, is a.a.s. $\delta$-hyperbolic for $\delta=\left(\log _{d-1} n\right) / 2-\omega(n)$. An upper bound on such $\delta$ is obviously half of the diameter of the graph, which is $\left(\log _{d-1} n\right) / 2+O(\log \log n)$ by the main result of (3).

Finally, it would be interesting to see to what extent the geometric properties we have addressed in this paper are preserved if the model of regular graphs changes. Particular models of interest might be random Cayley graphs, random lifts of regular graphs, and one can consider also some deterministic models of expanders.

## References

1. M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, with Formulas, Graphs and Mathematical Tables, National Bureau of Standards, Applied Mathematics Series, 1964.
2. B. Bollobás, A probabilistic proof of an asymptotic formula for the number of labelled regular graphs, European Journal of Combinatorics 1 (1980), 311-316.
3. B. Bollobás and W. Fernandez de la Vega, The diameter of random regular graphs, Combinatorica 2 (1982), 125-134.
4. A. M. Frieze and L. Zhao, Optimal construction of edge-disjoint paths in random regular graphs, Proceedings of SODA (1999), 346-355.
5. N.C. Wormald, Models of random regular graphs, Surveys in Combinatorics, 1999, London Mathematical Society Lecture NoteSeries 267 (J.D. Lamb and D.A. Preece, eds) Cambridge UniversityPress, Cambridge,pp. 239-298, 1999.

Department of Mathematics, The Weizmann Institute of Science, Rehovot 76100, IsRAEL

E-mail address: itai.benjamini@weizmann.ac.il
Department of Combinatorics and Optimization, University of Waterloo, WaterlooON, Canada N2L 3G1

E-mail address: choppen@math.uwaterloo.ca
Department of Computer Science and Applied Mathematics, The Weizmann Institute of Science, Rehovot 76100, Israel

E-mail address: eran.ofek@weizmann.ac.il
Department of Mathematics and Statistics, Dalhousie University, Halifax-NS, Canada B3H 3J5

E-mail address: pralat@mathstat.dal.ca
Department of Combinatorics and Optimization, University of Waterloo, WaterlooON, Canada N2L 3G1

E-mail address: nwormald@math.uwaterloo.ca


[^0]:    The fifth author acknowledges the support of the Canadian Research Chairs Program and NSERC.

