Hamiltonian Connectedness in 4-Connected Hourglass-free Claw-free Graphs

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Abstract: An hourglass is the only graph with degree sequence 4,2,2,2,2 (i.e. two triangles meeting in exactly one vertex). There are infinitely many claw-free graphs G such that G is not hamiltonian connected while its

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Ryjáček closure $cl(G)$ is hamiltonian connected. This raises such a problem what conditions can guarantee that a claw-free graph G is hamiltonian connected if and only if $cl(G)$ is hamiltonian connected. In this paper, we will do exploration toward the direction, and show that a 3-connected {claw, $(P_6)^2$, hourglass}-free graph G with minimum degree at least 4 is hamiltonian connected if and only if $cl(G)$ is hamiltonian connected, where $(P_6)^2$ is the square of a path P_6 on 6 vertices. Using the result, we prove that every 4-connected $\{claw, (P_6)^2, hourglass\}$-free graph is hamiltonian connected, hereby generalizing the result that every 4-connected hourglass-free line graph is hamiltonian connected by Kriesell [J Combinatorial Theory (B) 82 (2001), 306–315]. J Graph Theory 68: 285–298, 2011

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1. INTRODUCTION

Graphs considered in this paper are simple and finite graphs. We use [3] as a source for undefined terms and notations. An (x_1,x_n)-path is a path $P[x_1,x_n]=x_1x_2\dots x_n$ whose end-vertices are x_1 and x_n. $P[x_i,x_j]$ denotes the sub-path $x_ix_{i+1}\dots x_j$ for $i<j$, and $P^-[x_j,x_i]$ denotes the sub-path $x_jx_{j-1}\dots x_i$ for $i<j$. A path P on n vertices is also denoted by P_n. For graphs G and H, write $G=H$ to mean that the graphs G and H are isomorphic. The line graph of a graph H, denoted by $L(H)$, is a graph whose vertex set $V(L(H))$ is $E(H)$, where two vertices in $L(H)$ are adjacent if and only if the corresponding edges are adjacent in H. Given a set of graphs S, we say that a graph G is S-free if G contains no induced subgraph isomorphic to any graph in the set S. An induced subgraph isomorphic to $K_{1,3}$ is called a claw, and the only vertex of degree three in the claw is called the center of the claw. The classical results on line graphs are surveyed by Hemminger and Beineke [7]. An hourglass is the only graph with degree sequence 4,2,2,2,2 (i.e. two triangles meeting in exactly one vertex) (Fig. 1(A)). The vertex of degree 4 is called the center of the hourglass. G_8 (Fig. 1(B)) is the graph on 6 vertices u_1,u_2,u,v,v_1,v_2 obtained from K_4 by deleting one edge v_1u_2 and adding two paths u_2v_2v and v_1u_1u of length 2, where $V(K_4)=\{u,u_2,v,v_1\}$. Thus G_8 could be easier described as the square of a path P_6 on six vertices, where the square of a graph G is the graph (denoted by G^2) obtained by inserting new edges into G joining all pairs of vertices at distance 2 in G. Hemminger and Beineke [7] defined nine forbidden subgraphs $\{G_1=K_{1,3},G_2=K_5-e,G_3,G_4,G_5,G_6,G_7,G_8=(P_6)^2,G_9\}$ (Fig. 2) to characterize line graphs. One of the major results on line graphs is the following fundamental theorem.

Theorem 1 (Hemminger and Beineke [7]). *A connected graph is a line graph if and only if it is* $\{G_1,G_2,\dots,G_9\}$*-free.*

For hamiltonian connectedness in claw-free graphs, many authors are interested in it, and there exist many results (see [1–11]). Brandt [4] proved the following result.

Theorem 2 (Brandt [4]). *Every 9-connected claw-free graph is hamiltonian connected.*

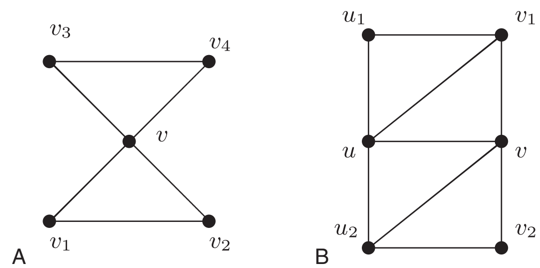

FIGURE 1. Forbidden subgraphs: (A) Hourglass and (B) $G_8=(P_6)^2$.

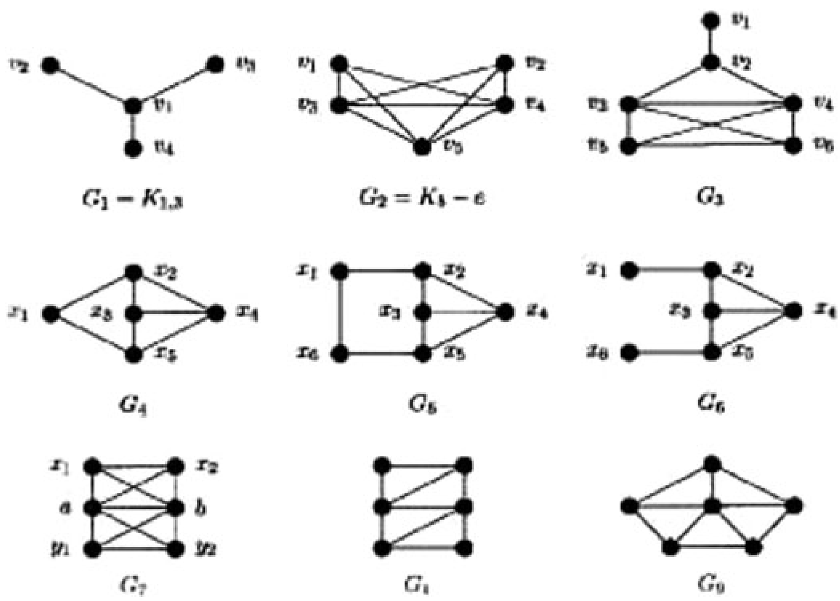

FIGURE 2. Nine forbidden induced subgraphs for line graphs.

Recently, Hu et al. improved Theorem 2 as follows.

Theorem 3 (Hu et al. [8]). *Every* 8*-connected claw-free graph is hamiltonian connected.*

Lai and Soltes [10] proved the following result.

Theorem 4 (Lai and Soltes [10]). *Every* 7*-connected* $\{claw, K_4-e, G_3\}$*-free graph is hamiltonian connected.*

Theorem 5 (Kriesell [9]). *Every* 4*-connected hourglass-free line graph is hamiltonian connected.*

In this paper, one motivation of ours is to strengthen Theorem 5, and improve Theorem 4 by reducing connectivity. We show the following result.

Theorem 6. *Every* 4*-connected* $\{claw, (P_6)^2, hourglass\}$*-free graph is hamiltonian connected.*

Obviously, Theorem 5 is a corollary of Theorem 6 because connected line graphs are $\{claw, (P_6)^2\}$-free by Theorem 1. The condition of "4-connectedness" in Theorems 5 and 6 is necessary. A vertex x is locally connected if its neighborhood $N(x)$ is connected. In [12], Ryjáček defined the *closure* $cl(G)$ of a claw-free graph G to be one obtained by recursively adding edges to join two nonadjacent vertices in the neighborhood of any locally connected vertex of G, as long as this is possible. As we know, there are infinitely many claw-free graphs G such that G is not hamiltonian connected but $cl(G)$ is hamiltonian connected. This raises such a problem what conditions can guarantee that a claw-free graph G is hamiltonian connected if and only if $cl(G)$ is hamiltonian connected. In this paper, the other motivation of ours is to explore this direction. We show the following result.

Theorem 7. *Let G be a* 3*-connected* $\{claw, (P_6)^2, hourglass\}$*-free graph with minimum degree at least* 4*. Then G is hamiltonian connected if and only if $cl(G)$ is hamiltonian connected.*

Now we guess that the condition of $(P_6)^2$-freeness in Theorem 7 may be dropped, and so make the following conjectures.

Conjecture 8. *Let G be a* 3*-connected* $\{claw, hourglass\}$*-free graph with minimum degree at least* 4*. Then G is hamiltonian connected if and only if $cl(G)$ is hamiltonian connected.*

2. PROOFS OF THEOREMS 6 AND 7

In this section, we will provide the proofs of Theorems 6 and 7. If x is a locally connected vertex of G, then the local completion at x is the operation of adding all possible edges between vertices in $N(x)$. The resulting graph, denoted by G'_x, is easily shown to be claw-free again. Iterating local completions, we finally arrive at a graph in which all locally connected vertices have complete neighborhoods. This graph $cl(G)$ does not depend on the order of local completions. Ryjáček [12] proved the following result.

Theorem 9 (Ryjáček [12]). *Let G is a connected claw-free graph. Then the closure $cl(G)$ of G is the line graph of some triangle-free graph.*

The following proposition will be used in the proofs of Proposition 12 and Theorem 7.

Proposition 10. *Let G be a connected* $\{claw, (P_6)^2, hourglass\}$*-free graph and x a locally connected vertex in G. Then G'_x is also* $\{claw, (P_6)^2, hourglass\}$*-free.*

Proof. Obviously, G'_x is claw-free. First we prove that G'_x is hourglass-free. Suppose that G'_x has an hourglass $H = G'_x[v_3, v_1, v_2, v_4, v_5]$, where v_3 is the center of H. Then $v_1v_4, v_1v_5, v_2v_4, v_2v_5 \notin E(G'_x)$ and we have the following claim. ■

Claim 1. *If $v_4v_5 \notin E(G)$, then either $v_3v_5 \notin E(G)$ or $v_3v_4 \notin E(G)$.*

Proof. If $v_4v_5 \notin E(G)$, then $xv_4, xv_5 \in E(G)$, and $v_1x \notin E(G)$ from $G[x, v_4, v_5, v_1] \neq K_{1,3}$. Thus, $v_1v_3 \in E(G)$. If $v_3v_5, v_3v_4 \in E(G)$, then $G[v_3, v_4, v_5, v_1] = K_{1,3}$. This contradiction shows that $v_3v_5 \notin E(G)$ or $v_3v_4 \notin E(G)$. Thus $xv_3 \in E(G)$. If $v_3v_5 \notin E(G)$ and $v_3v_4 \notin E(G)$ then $G[v_3, v_4, v_5, x] = K_{1,3}$, a contradiction. Thus Claim 1 is true. ■

Claim 2. *$v_4v_5 \in E(G)$ and $v_1v_2 \in E(G)$.*

Proof. If $v_4v_5 \notin E(G)$, then, by Claim 1, assume that $v_3v_5 \notin E(G)$ and $v_3v_4 \in E(G)$. Thus $xv_4, xv_5, xv_3 \in E(G)$. Obviously, $xv_1, xv_2 \notin E(G)$. Thus $G[v_3, v_1, v_2, x, v_4]$ is an hourglass in G. This contradiction shows that $v_4v_5 \in E(G)$. Similarly, $v_1v_2 \in E(G)$. ■

Since G is hourglass-free, there is at least one edge (say v_3v_5) in $\{v_1v_3, v_2v_3, v_3v_4, v_3v_5\}$ such that $v_3v_5 \notin E(G)$. Thus, $xv_3, xv_5 \in E(G)$. Note that $v_2x, v_1x \notin E(G)$ since otherwise $v_2v_5 \in E(G'_x)$, a contradiction. Obviously $v_3v_4 \notin E(G)$ since otherwise $xv_4 \in E(G)$ from $G[v_3, v_2, v_4, x] \neq K_{1,3}$, and so $G[v_3, v_1, v_2, x, v_4]$ is an hourglass in G, a contradiction. Thus, $xv_4 \in E(G)$. Since $d_G(x) \geq 4$, there is a vertex v_6 such that $xv_6 \in E(G)$. Since x is a locally connected vertex, there is a vertex $y \in N(x)$ such that $yv_3 \in E(G)$. Without loss of generality assume that $y = v_6$. Then

Claim 3. *Either $v_1v_6 \in E(G)$ or $v_2v_6 \in E(G)$, and either $v_4v_6 \in E(G)$ or $v_5v_6 \in E(G)$.*

Proof. If $v_1v_6, v_2v_6 \notin E(G)$, then $G[v_3, v_1, v_2, x, v_6]$ is an hourglass in G. Thus $v_1v_6 \in E(G)$ or $v_2v_6 \in E(G)$. Similarly, $v_4v_6 \in E(G)$ or $v_5v_6 \in E(G)$. If $v_1v_6 \in E(G)$ and $v_2v_6 \in E(G)$, then $G[v_6, v_1, v_2, v_5, x]$ is an hourglass in G if $v_5v_6 \in E(G)$ and $G[v_6, v_1, v_2, v_4, x]$ is an hourglass in G if $v_4v_6 \in E(G)$. This contradiction shows that $v_1v_6 \notin E(G)$ or $v_2v_6 \notin E(G)$. Without loss of generality assume that $v_1v_6 \in E(G)$ but $v_2v_6 \notin E(G)$. If $v_4v_6 \in E(G)$ and $v_5v_6 \in E(G)$, then $G[v_6, v_4, v_5, v_1, v_3]$ is an hourglass in G. Thus $v_4v_6 \notin E(G)$ or $v_5v_6 \notin E(G)$. Thus Claim 3 is true. ■

Without loss of generality assume that $v_1v_6 \in E(G)$ but $v_2v_6 \notin E(G)$, and $v_4v_6 \in E(G)$ but $v_5v_6 \notin E(G)$. Thus $G[v_1, v_2, v_3, x, v_4, v_6]$ is $(P_6)^2$. This contradiction shows that G'_x is hourglass-free.

Now we prove that G'_x is $(P_6)^2$-free. Suppose that G'_x has a subgraph H isomorphic to $(P_6)^2$ such that $V(H) = \{u, v, u_1, u_2, v_1, v_2\}$ and $E(H) = \{u_1v_1, u_1u, uu_2, u_2v_2, v_2v, vv_1, uv_1, uv, u_2v\}$ (Fig. 1(B)). Obviously, $x \notin \{u_1, v_2\}$, since otherwise, if $x = u_1$, then $u_1v_1, u_1u \in E(G)$, and $uv_1 \in E(G)$ since otherwise $G[v, v_1, v_2, u] = K_{1,3}$. So $E(H)$ is contained in $E(G)$, a contradiction. Thus $x \neq u_1$. Similarly, $x \neq v_2$. We have the following claim.

Claim 4. *$u_1v_1, u_2v_2 \in E(G)$.*

Proof. If $u_1v_1 \notin E(G)$, then $xu_1, xv_1 \in E(G)$ by the definition of G'_x. We also have that $u_1u \in E(G)$ or $uv_1 \in E(G)$ since otherwise $xu \in E(G)$ and so $G[x, u_1, v_1, u] = K_{1,3}$, a contradiction.

If $xu \in E(G)$, then $uv_1 \notin E(G)$ since otherwise $G[x, u, u_2, v_2, v, v_1]$ is isomorphic to $(P_6)^2$. Thus $u_1u \in E(G)$. Obviously, $xv, xu_2 \notin E(G)$. Thus, $G[u, u_1, x, v, u_2]$ is an hourglass. This contradiction shows that $xu \notin E(G)$. So $u_1u, uv_1 \in E(G)$. It follows that $G[u, u_1, v_1, u_2]$ is a claw. This contradiction shows that $u_1v_1 \in E(G)$. Similarly, $u_2v_2 \in E(G)$. Thus Claim 4 is true. ■

If $uu_1 \notin E(G)$, then $xu_1, xu \in E(G)$, and so $xv, xv_2, xu_2 \notin E(G)$. If $uv_1 \in E(G)$, then $xv_1 \in E(G)$ since otherwise $G[u,x,v_1,u_2] = K_{1,3}$. Thus $G[x,u,u_2,v_2,v,v_1]$ is isomorphic to $(P_6)^2$. This contradiction shows that $uv_1 \notin E(G)$, and so $xv_1 \in E(G)$ by the definition of G'_x. Note that $xv, xv_2 \notin E(G)$ and so $vv_2, uv \in E(G)$. Thus $G[v,v_1,u,v_2] = K_{1,3}$. This contradiction shows that $uu_1 \in E(G)$. Similarly, $vv_2 \in E(G)$.

If $uv \notin E(G)$, then $xu, xv \in E(G)$. Since $u_1 v \notin E(G'_x)$, $xu_1 \notin E(G)$. Since $v_1 u_2 \notin E(G'_x)$, $xv_1 \notin E(G)$ or $xu_2 \notin E(G)$ (say $xu_2 \notin E(G)$). It follows that $uu_2 \in E(G)$ and so $G[u,u_1,u_2,x] = K_{1,3}$ since $u_1 u_2 \notin E(G)$. Thus $uv \in E(G)$.

If $uu_2 \notin E(G)$, then $xu_2, xu \in E(G)$. Since $u_1 u_2 \notin E(G)$, $u_1 x \notin E(G)$. Since $u_1 v \notin E(G)$ and $G[u,u_1,v,x] \neq K_{1,3}$, $xv \in E(G)$. Since $v_1 u_1 \notin E(G'_x)$, $xv_1 \notin E(G)$ and so $v_1 v \in E(G)$. Thus $G[v,x,v_1,v_2] = K_{1,3}$. This contradiction shows $uu_2 \in E(G)$. Similarly, $vv_1 \in E(G)$. Since $G[u,u_1,v,u_2] \neq K_{1,3}$ and $G[v,u,v_2,v_1] \neq K_{1,3}$, $uv_1, u_2 v \in E(G)$. Thus all edges in H belong to $E(G)$. That is, H is isomorphic to $(P_6)^2$ in G. This contradiction shows that G'_x is $(P_6)^2$-free. Thus, Proposition 10 is true. ■

In order to prove Proposition 12, we need the following lemma.

Lemma 11. *Let x_1 and x_n be two vertices in a connected claw-free graph G of order n such that G has no any hamiltonian path between them, and let x be a locally connected vertex in G such that G'_x has a hamiltonian path between x_1 and x_n. Assume that $P = x_1 x_2 \ldots x_i x_{i+1} \ldots x_{n-1} x_n$ is a hamiltonian path in G'_x connecting x_1 and x_n such that $|E(P) - E(G)|$ is minimal. Let $x_i x_{i+1} \in E(P) - E(G)$, $x_j = x$ and $1 < j < n$. Then*

(1) *$x_i x, x_{i+1} x \in E(G)$ and $x_{j-1} x_{j+1}, x_i x_{i+1} \notin E(G)$.*
(2) *$x_i x_{j-1}, x_{i+1} x_{j+1} \notin E(G)$ and $x_{i+1} x_{j-1}, x_i x_{j+1} \in E(G)$.*
(3) *P contains exactly one edge $x_i x_{i+1} \in E(G'_x) - E(G)$.*

Proof. (1) From the definition of G'_x, we have $x_i x_{i+1} \notin E(G)$ and $x_i x, x_{i+1} x \in E(G)$. If $x_{j-1} x_{j+1} \in E(G)$, and without loss of generality assume that $j < i$, then there is a hamiltonian (x_1, x_n)-path

$$P' = P[x_1, x_{j-1}] P[x_{j+1}, x_i] x_j P[x_{i+1}, x_n]$$

in G'_x containing fewer edges of $E(G'_x) - E(G)$ than P since $x_i x_{i+1} \notin E(P')$, a contradiction. Thus (1) is true.

(2) By symmetry, without loss of generality assume that $i < j$. Then $x_i x_{j-1} \notin E(G)$, since otherwise there exists a hamiltonian (x_1, x_n)-path $P' = P[x_1, x_i] P^-[x_{j-1}, x_{i+1}] P[x_j, x_n]$ in G'_x containing fewer edges of $E(G'_x) - E(G)$ than P since $x_i x_{i+1} \notin E(P')$, a contradiction. Similarly, $x_{i+1} x_{j+1} \notin E(G)$.

From $G[x_j, x_{j-1}, x_i, x_{i+1}] \neq K_{1,3}$, $x_{i+1} x_{j-1} \in E(G)$. Similarly, $x_i x_{j+1} \in E(G)$. Thus (2) is true.

(3) If P contains at least two edges $x_i x_{i+1}$, $x_k x_{k+1}$ in $E(G'_x) - E(G)$ (Fig. 3). If $i < j < k$, then, by (2), $x_{j+1} x_i \in E(G)$. Since $G[x_j, x_k, x_{k+1}, x_{i+1}]$ is not a claw, $x_{i+1} x_k \in E(G)$ or $x_{i+1} x_{k+1} \in E(G)$. If $x_{i+1} x_k \in E(G)$, then there is a hamiltonian path $P' = P[x_1, x_i] P[x_{j+1}, x_k] P[x_{i+1}, x_j] P[x_{k+1}, x_n]$ in G'_x containing fewer edges of $E(G'_x) - E(G)$ than P since $x_i x_{i+1}, x_k x_{k+1} \notin E(P')$. Thus $x_{i+1} x_{k+1} \in E(G)$, and so G'_x has a hamiltonian path $P' = P[x_1, x_i] P[x_{j+1}, x_k] P^-[x_j, x_{i+1}] P[x_{k+1}, x_n]$ containing fewer edges of $E(G'_x) - E(G)$ than P since $x_i x_{i+1}, x_k x_{k+1} \notin E(P')$. This contradiction shows that the case $i < j < k$ cannot occur. Without loss of generality assume that $i < k < j$

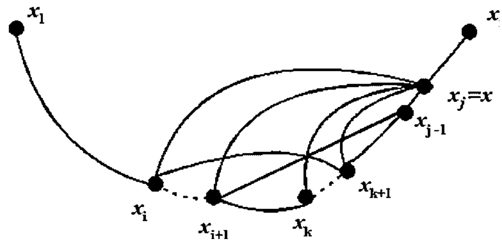

FIGURE 3. A hamiltonian path with $x_ix_{i+1}, x_kx_{k+1} \in E(G_x')$ connecting x_1 and x_n in G_x'.

(and the proofs of other cases are similar). By (2), $x_{j-1}x_{i+1} \in E(G)$. If $i+1=k$, then $x_ix_{i+2} \in E(G)$ since $G[x, x_i, x_{i+1}, x_{i+2}] \neq K_{1,3}$. Thus

$$P' = P[x_1, x_i]P[x_{i+2}, x_{j-1}]x_{i+1}P[x_j(=x), x_n]$$

is a hamiltonian path connecting x_1 and x_n, but contains fewer edges of $E(G_x') - E(G)$ than P since $x_ix_{i+1}, x_kx_{k+1} \notin E(P')$. This contradiction shows that $i+1 \neq k$. Since $G[x, x_i, x_k, x_{k+1}] \neq K_{1,3}$, $x_ix_k \in E(G)$ or $x_ix_{k+1} \in E(G)$. If $x_ix_k \in E(G)$, then

$$P' = P[x_1, x_i]P^-[x_k, x_{i+1}]P^-[x_{j-1}, x_{k+1}]P[x_j, x_n]$$

is a hamiltonian path connecting x_1 and x_n in G_x', but contains fewer edges of $E(G_x') - E(G)$ than P since $x_ix_{i+1}, x_kx_{k+1} \notin E(P')$, a contradiction. Thus $x_ix_k \notin E(G)$ and so $x_ix_{k+1} \in E(G)$ (Fig. 3). It follows that

$$P' = P[x_1, x_i]P[x_{k+1}, x_{j-1}]P[x_{i+1}, x_k]P[x_j(=x), x_n]$$

is a hamiltonian path connecting x_1 and x_n in G_x' containing fewer edges of $E(G_x') - E(G)$ than P since $x_ix_{i+1}, x_kx_{k+1} \notin E(P')$, a contradiction. Thus P contains exactly one edge in $E(G_x') - E(G)$. Therefore the proof of Lemma 11 is completed. ∎

Proposition 12. *Let G be a 3-connected $\{claw, (P_6)^2, hourglass\}$-free graph with minimum degree at least 4 and x a locally connected vertex in G. Then G is hamiltonian connected if and only if G_x' is hamiltonian connected.*

Proof. It is easy to see that we only need to prove that if G_x' is hamiltonian connected then G is hamiltonian connected. Assume that G is a non-hamiltonian connected graph of order n which satisfies the conditions of Proposition 12 and G_x' is hamiltonian connected. Then G_x' is also $\{claw, (P_6)^2, hourglass\}$-free by Proposition 10, and there is a pair of vertices (say x_1 and x_n) such that G_x' has a hamiltonian path connecting x_1 and x_n, but there is no a hamiltonian path connecting x_1 and x_n in G. Let $P = x_1x_2 \cdots x_{n-1}x_n$ be a hamiltonian path connecting x_1 and x_n in G_x' such that $|E(P) - E(G)|$ is minimal. By Lemma 11(3), $|E(P) - E(G)| = 1$. Let $x_j = x$ and $E(P) - E(G) = \{x_ix_{i+1}\}$. Then, by Lemma 11(1), $x_ix, x_{i+1}x \in E(G)$, that is, $x_ix_j, x_{i+1}x_j \in E(G)$. Furthermore, we have the following fact.

Claim 1. *$j = 1$ or $j = n$.*

Otherwise, by symmetry, without loss of generality assume that $i+2 \leq j \leq n-1$, then $x_{j+1}x_{j-1} \notin E(G)$ by Lemma 11(1). By Lemma 11(2), $x_i x_{j-1}, x_{i+1}x_{j+1} \notin E(G)$. If $j \neq i+2$, then, from Lemma 11(2), $x_{j-1}x_{i+1} \in E(G)$ and $x_{j+1}x_i \in E(G)$. It follows that $G[x_j, x_{j-1}, x_{j+1}, x_i, x_{i+1}]$ is an hourglass, a contradiction. Thus $j = i+2$.

By Lemma 11(2), $x_i x_{i+3} \in E(G)$ and $x_{i+1}x_{i+3} \notin E(G)$. Since x_{i+2} is locally connected and $x_i x_{i+1} \notin E(G)$, there is a shortest path with at most 4 vertices connecting x_i and x_{i+1} in $N(x_{i+2})$, which implies that there is a vertex x_r ($\neq x_i$, x_{i+3}) on P such that $x_{i+1}x_r, x_{i+2}x_r \in E(G)$.

If $i+3 < r < n$, then $x_{i+1}x_{r-1}, x_{i+1}x_{r+1} \notin E(G)$ since otherwise G has a hamiltonian (x_1, x_n)-path, and so $x_{r+1}x_{r-1} \in E(G)$ since $G[x_r, x_{r+1}.x_{r-1}, x_{i+1}] \neq K_{1,3}$. We have $x_{i+2}x_{r-1} \notin E(G)$ since otherwise G has a hamiltonian (x_1, x_n)-path $P[x_1, x_i]$ $P[x_{i+3}, x_{r-1}]x_{i+2}x_{i+1}P[x_r, x_n]$, a contradiction. Similarly, $x_{i+2}x_{r+1} \notin E(G)$. Thus $G[x_r, x_{r+1}, x_{r-1}, x_{i+1}, x_{i+2}]$ is an hourglass, a contradiction.

If $1 < r < i$, then $x_{r+1}x_{i+1}, x_{r-1}x_{i+1} \notin E(G)$ since otherwise, say $x_{r-1}x_{i+1} \in E(G)$, G has a hamiltonian path $P[x_1, x_{r-1}]x_{i+1}P[x_r, x_i]P[x_{i+2}, x_n]$, a contradiction. Thus $x_{r+1}x_{r-1} \in E(G)$ from $G[x_r, x_{i+1}, x_{r-1}, x_{r+1}] \neq K_{1,3}$. Recall that $x_i x_{i+3} \in E(G)$. Note that $x_{i+2}x_{r+1}, x_{i+2}x_{r-1} \notin E(G)$ since otherwise, say $x_{i+2}x_{r+1} \in E(G)$, G has a hamiltonian (x_1, x_n)-path

$$P[x_1, x_r]x_{i+1}x_{i+2}P[x_{r+1}, xi]P[x_{i+3}, x_n].$$

Thus $G[x_r, x_{r-1}, x_{r+1}, x_{i+1}, x_{i+2}]$ is an hourglass. This contradiction shows that $r = n$ or $r = 1$.

If $r = n$, then $x_i x_{n-1} \notin E(G)$ since otherwise G has a hamiltonian (x_1, x_n)-path $P[x_1, x_i]$ $P^{-}[x_{n-1}, x_{i+1}]x_n$. Obviously, $x_{i+1}x_{n-1} \notin E(G)$ since otherwise G has a hamiltonian (x_1, x_n)-path

$$P[x_1, x_i]P[x_{i+2}, x_{n-1}]x_{i+1}x_n,$$

a contradiction. We have $x_i x_n \notin E(G)$ since otherwise $G[x_n, x_i, x_{i+1}, x_{n-1}] = K_{1,3}$, a contradiction. Since $x_i x_{i+3}, x_i x_{i+2}, x_n x_{i+1}, x_n x_{i+2} \in E(G)$ and $G[x_{i+2}, x_i, x_{i+3}, x_{i+1}, x_n]$ is not an hourglass, $x_{i+3}x_n \in E(G)$. It follows that $x_{i+3}x_{n-1} \in E(G)$ since $G[x_n, x_{i+1}, x_{i+3}, x_{n-1}] \neq K_{1,3}$.

If $i+4 = n-1$, then x_{n-1} and x_{i+1} have exactly two neighbors in $P[x_{i+1}, x_n]$, respectively, and so there are at least two vertices x_p and x_q in $P[x_1, x_i]$ such that $x_{n-1}x_p, x_{n-1}x_q \in E(G)$ since $\delta(G) \geq 4$. Let $p < q$. Then $q \neq i$ since otherwise we easily construct a hamiltonian (x_1, x_n)-path in G. Obviously $x_{q+1}, x_{q-1} \notin N(x_{n-1})$ since otherwise G has a hamiltonian (x_1, x_n)-path by inserting x_{n-1} into $P[x_1, x_i]$. Thus $p+1 \neq q$ and $x_{q+1}x_{q-1} \in E(G)$ from $G[x_q, x_{q+1}, x_{q-1}, x_{n-1}] \neq K_{1,3}$. Similarly, $x_{p+1}, x_{p-1} \notin N(x_{n-1})$ and $x_{p+1}x_{p-1} \in E(G)$ if $x_1 \neq x_p$.

Assume that $x_1 \neq x_p$. Then we similarly prove that $p+2 \neq q-1$. We have $x_p x_{q-1} \notin E(G)$ since otherwise G has a hamiltonian path

$$P[x_1, x_{p-1}]P[x_{p+1}, x_{q-1}]x_p x_{n-1}P[x_q, x_i]x_{i+3}x_{i+2}x_{i+1}x_n.$$

Similarly, $x_p x_{q+1} \notin E(G)$. Thus $x_p x_q \notin E(G)$ since otherwise $G[x_q, x_{q-1}, x_{q+1}, x_p, x_{n-1}]$ is an hourglass, a contradiction. Since $G[x_{n-1}, x_{i+3}, x_p, x_q] \neq K_{1,3}$, $x_p x_{i+3} \in E(G)$ or $x_q x_{i+3} \in E(G)$. It is easy to prove that $x_{p+1}, x_{p-1}, x_{q+1}, x_{q-1} \notin N(x_{i+3})$. If $x_p x_{+3} \in E(G)$, then $G[x_p, x_{p-1}, x_{p+1}, x_{n-1}, x_{i+3}]$ is an hourglass. If $x_q x_{i+3} \in E(G)$, then

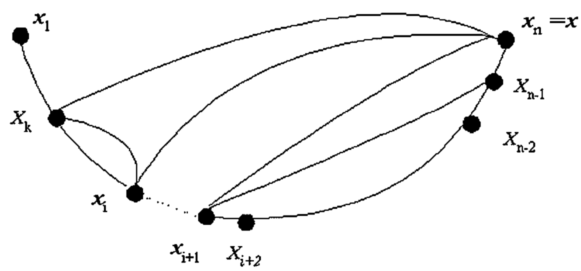

FIGURE 4. A hamiltonian path with $x_j = x_n = x$ connecting x_1 and x_n in G'_x.

$G[x_q, x_{q-1}, x_{q+1}, x_{n-1}, x_{i+3}]$ is an hourglass. This contradiction shows that $x_1 = x_p$. That is, $x_1x_{n-1} \in E(G)$.

It is easy to see that x_{i+1} has at least two neighbors in $P[x_1, x_i]$. By a similar argument to the above, we can prove that $x_1x_{i+1} \in E(G)$. Note that $x_{n-1}x_{i+1} \notin E(G)$ and $x_2x_{i+1}, x_2x_{n-1} \notin E(G)$ since otherwise, say $x_2x_{i+1} \in E(G)$, G has a hamiltonian path $x_1x_{i+1}P[x_2, x_i]P[x_{i+2}, x_n]$, a contradiction. Thus $G[x_1, x_2, x_{i+1}, x_{n-1}] = K_{1,3}$. This contradiction shows that $i+4 \neq n-1$.

Note that $x_{i+4}x_i \notin E(G)$ since otherwise G has a hamiltonian (x_1, x_n)-path

$$P[x_1, x_i]P[x_{i+4}, x_{n-1}]x_{i+3}x_{i+2}x_{i+1}x_n.$$

Similarly, $x_{i+2}x_{i+4} \notin E(G)$. From $G[x_{i+3}, x_{i+4}, x_{n-1}, x_i] \neq K_{1,3}$, $x_{i+4}x_{n-1} \in E(G)$. Obviously, $x_{i+2}x_{n-1} \notin E(G)$. Thus $G[x_{i+3}, x_{i+4}, x_{n-1}, x_{i+2}, x_i]$ is an hourglass. This contradiction shows that $r \neq n$, and so $r = 1$.

By Lemma 11(2), $x_{i+1}x_{i+3} \notin E(G)$ and $x_ix_{i+3} \in E(G)$. Note that $x_{i+m} \notin N(x_2)$ for $m = 1, 2, 3, 4$ since otherwise, say $x_2x_{i+3} \in E(G)$, G has a hamiltonian path $x_1x_{i+1}x_{i+2}$ $P^-[x_2, x_i]P[x_{i+3}, x_n]$. Thus $x_1x_{i+3} \notin E(G)$ since otherwise $G[x_1, x_2, x_{i+3}, x_{i+1}] = K_{1,3}$. Since $G[x_{i+2}, x_{i+3}, x_i, x_1, x_{i+1}]$ is not an hourglass, $x_1x_i \in E(G)$, which implies that $x_2x_i \in E(G)$ from $G[x_1, x_2, x_i, x_{i+1}] \neq K_{1,3}$. Note that $x_{i-1}x_{i+2} \notin E(G)$ since otherwise G has a hamiltonian path $x_1x_{i+1}x_{i+2}P^-[x_{i-1}, x_2]x_iP[x_{i+3}, x_n]$. It follows that $x_2x_{i-1} \in E(G)$ from $G[x_i, x_2, x_{i-1}, x_{i+2}] \neq K_{1,3}$. Also, $x_{i-1}x_{i+3} \notin E(G)$ since otherwise G has a hamiltonian path $x_1x_{i+1}x_{i+2}x_iP[x_2, x_{i-1}]P[x_{i+3}, x_n]$. Thus $G[x_i, x_{i+2}, x_{i+3}, x_2, x_{i-1}]$ is an hourglass. This contradiction shows that $j = 1$ or $j = n$. Thus Claim 1 is true. ■

By symmetry, without loss of generality assume that $j = n$ (Fig. 4). Then we have the following fact.

Claim 2. $1 < i < n-1$.

Proof. It is easy to see that $i \neq n-1$ since $j = n$. If $i = 1$, then, by Lemma 11(2), $x_1x_{n-1} \notin E(G)$ and $x_2x_{n-1} \in E(G)$. Since x_n is locally connected, there is a vertex x_k in P such that $x_kx_1, x_kx_n \in E(G)$.

Note that $x_1x_{k-1} \notin E(G)$ since otherwise $x_1P^-[x_{k-1}, x_2]P^-[x_{n-1}, x_k]x_n$ is a hamiltonian (x_1, x_n)-path in G, a contradiction. Again, $x_1x_{k+1} \notin E(G)$ since otherwise G has a hamiltonian (x_1, x_n)-path

$$x_1P[x_{k+1}, x_{n-1}]P[x_2, x_k]x_n,$$

a contradiction. From $G[x_k,x_{k-1},x_{k+1},x_1]\neq K_{1,3}$, we have that $x_{k+1}x_{k-1}\in E(G)$. It is easy to check that $x_{k+1}x_n\notin E(G)$ since otherwise

$$x_1P^-[x_k,x_2]P^-[x_{n-1},x_{k+1}]x_n$$

is a hamiltonian (x_1,x_n)-path in G. Note that $x_{k-1}x_n\notin E(G)$ since otherwise G has a hamiltonian (x_1,x_n)-path

$$x_1P[x_k,x_{n-1}]P[x_2,x_{k-1}]x_n.$$

Thus $G[x_k,x_{k-1},x_{k+1},x_1,x_n]$ is an hourglass. This contradiction shows that $i>1$. Thus Claim 2 is true. ∎

Since x_n is locally connected, there is a vertex x_k such that $x_kx_i,x_kx_n\in E(G)$ (Fig. 4). We have the following fact.

Claim 3. $1\leq k\leq i-1$.

Proof. Otherwise, $x_k\in V(P[x_{i+2},x_{n-1}])$. By Lemma 11(2), $x_{i+1}x_{n-1}\in E(G)$. It follows that $x_{k-1}x_i,x_{k+1}x_i\notin E(G)$ since otherwise, for example, $x_{k-1}x_i\in E(G)$, G has a hamiltonian (x_1,x_n)-path

$$P[x_1,x_i]P^-[x_{k-1},x_{i+1}]P^-[x_{n-1},x_k]x_n,$$

a contradiction. Similarly, $x_{k-1}x_n,x_{k+1}x_n\notin E(G)$. Thus we have from $G[x_k,x_{k-1},x_{k+1},x_i]\neq K_{1,3}$ that $x_{k+1}x_{k-1}\in E(G)$. It follows that $G[x_k,x_{k-1},x_{k+1},x_i,x_n]$ is an hourglass. This contradiction shows that $1\leq k\leq i-1$. Thus Claim 3 is true. ∎

Furthermore, we have the following fact.

Claim 4. $x_{i+1}x_{k+1},x_{k+1}x_{n-1},x_ix_{n-1}\notin E(G)$; *and* $x_{n-2}x_{k+1},x_{i+2}x_{k+1},x_ix_{i+2}\notin E(G)$ *if* $x_{i+2}\neq x_n$.

Proof. Assume that $x_{k+1}x_{i+1}\in E(G)$. Then G has a hamiltonian (x_1,x_n)-path

$$P[x_1,x_k]P^-[x_i,x_{k+1}]P[x_{i+1},x_n].$$

This contradiction shows that $x_{k+1}x_{i+1}\notin E(G)$. Similarly, $x_{k+1}x_{n-1},x_ix_{n-1}\notin E(G)$. By Lemma 11(2), $x_{i+1}x_{n-1}\in E(G)$.

Assume that $x_{i+2}\neq x_n$. If $x_{k+1}x_{i+2}\in E(G)$, then G has a hamiltonian (x_1,x_n)-path

$$P[x_1,x_k]P^-[x_i,x_{k+1}]P[x_{i+2},x_{n-1}]x_{i+1}x_n.$$

Similarly, $x_ix_{i+2}\notin E(G)$. If $x_{n-2}x_{k+1}\in E(G)$, then G has a hamiltonian (x_1,x_n)-path

$$P[x_1,x_k]P^-[x_i,x_{k+1}]P^-[x_{n-2},x_{i+1}]x_{n-1}x_n.$$

This contradiction shows that Claim 4 is true. ∎

Claim 5. $i+2\neq n$.

Proof. Assume that $i+2=n$, then we have the following facts:

(1) If $x_tx_{i+1}\in E(G)$ for $1<t<i$, then $x_{t-1}x_{i+1}, x_{t+1}x_{i+1}\notin E(G)$ and $x_{t-1}x_{t+1}\in E(G)$. For example, $x_{t-1}x_{i+1}\in E(G)$, we easily construct a hamiltonian (x_1,x_n)-path in G by replacing the edge $x_{t-1}x_t$ by $x_{t-1}x_{i+1}x_t$. From $G[x_t,x_{t-1},x_{t+1},x_{i+1}]\neq K_{1,3}$, we have $x_{t-1}x_{t+1}\in E(G)$.

(2) There does not exist such a vertex x_t $(1<t<i)$ that $x_tx_{i+1}, x_tx_i\in E(G)$ since otherwise $x_{t-1}x_i\in E(G)$ from $G[x_t,x_{t-1},x_i,x_{i+1}]\neq K_{1,3}$, and so G has a hamiltonian path $P[x_1,x_{t-1}]P^-[x_i,x_t]x_{i+1}x_n$.

Since $\delta(G)\geq 4$, there are at least two vertices x_p and x_q such that $x_px_{i+1}, x_qx_{i+1}\in E(G)$ and $1<p<q<i$. By (1), $q\neq p+1$, and $x_{p+1}x_{i+1}, x_{p-1}x_{i+1}, x_{q+1}x_{i+1}, x_{q-1}x_{i+1}\notin E(G)$ and $x_{p+1}x_{p-1}, x_{q+1}x_{q-1}\in E(G)$.

We have $x_{p+1}x_q\notin E(G)$ since otherwise G has a hamiltonian (x_1,x_n)-path

$$P[x_1,x_p]x_{i+1}x_qP[x_{p+1},x_{q-1}]P[x_{q+1},x_i]x_n,$$

a contradiction. Similarly, $x_{p-1}x_q\notin E(G)$. Since $G[x_p,x_{p+1},x_{p-1},x_q,x_{i+1}]$ is not an hourglass, $x_px_q\notin E(G)$. From $G[x_{i+1},x_p,x_q,x_n]\neq K_{1,3}$, $x_nx_p\in E(G)$ or $x_nx_q\in E(G)$.

Assume that $x_nx_p\in E(G)$. We have that $x_{p-1}x_i, x_{q-1}x_i, x_{p-1}x_{i+1}\notin E(G)$ since otherwise, say $x_{p-1}x_i\in E(G)$, G has a hamiltonian (x_1,x_n)-path $P[x_1,x_{p-1}]P^-[x_i,x_p]x_{i+1}x_n$, a contradiction. Thus, $x_{p-1}x_n, x_{q-1}x_n\notin E(G)$ since otherwise, say $x_{p-1}x_n\in E(G)$, $G[x_n,x_{p-1},x_i,x_{i+1}]=K_{1,3}$, a contradiction. We have that $x_nx_{p+1}\in E(G)$ since $G[x_p,x_{p-1},x_{p+1},x_{i+1},x_n]$ is not an hourglass and $x_{p+1}x_{i+1}, x_{p-1}x_{i+1}\notin E(G)$. Since $x_{i+1}x_{p+1}, x_ix_{i+1}\notin E(G)$ and $G[x_n,x_{p+1},x_i,x_{i+1}]\neq K_{1,3}$, $x_{p+1}x_i\in E(G)$. Note that $x_px_{p+2}\notin E(G)$ since otherwise G has a hamiltonian (x_1,x_n)-path

$$P[x_1,x_{p-1}]x_{p+1}P^-[x_i,x_{p+2}]x_px_{i+1}x_n.$$

Similarly, $x_{p-1}x_{p+2}, x_px_i, x_{p-1}x_i\notin E(G)$. Since $G[x_{p+1},x_p,x_{p-1},x_i,x_{p+2}]$ is not an hourglass, $x_ix_{p+2}\notin E(G)$. Thus $G[x_{p+1},x_{p+2},x_{p-1},x_i]=K_{1,3}$. This contradiction shows that $x_nx_p\notin E(G)$. It follows that $x_nx_q\in E(G)$. By a similar argument to the above, we can get a contradiction. So Claim 5 is true. ■

Claim 6. $i\geq 3$.

Proof. If $i=2$, then we have $k=1$ by Claim 3. By Claim 4, $x_2x_{n-1}\notin E(G)$. Then $x_3x_{n-1}\in E(G)$ since $G[x_n,x_{n-1},x_2,x_3]\neq K_{1,3}$. Since $\delta(G)\geq 4$, there is at least two vertices x_p,x_q on $P[x_3,x_{n-1}]$ such that $x_2x_p, x_2x_q\in E(G)$. Since $x_2x_3, x_2x_{n-1}\notin E(G)$, $3<p$, $q<n-1$.

Note that if $x_2x_t\in E(G)$ for some $3<t<n-1$, then $(N(x_1)\cup N(x_n))\cap\{x_{t-1},x_{t+1}\}=\emptyset$, otherwise, for example, $x_{t-1}x_n\in E(G)$, G has a hamiltonian (x_1,x_n)-path $x_1x_2P[x_t,x_{n-1}]$ $P[x_3,x_{t-1}]x_n$. If $x_2x_{t-1}\in E(G)$ or $x_2x_{t+1}\in E(G)$, then $x_1x_{n-1}\notin E(G)$ since otherwise G has a hamiltonian path by inserting x_2 into $P[x_3,x_n]$ and using these edges $\{x_1x_{n-1}, x_3x_n, x_2x_t\}$. Thus, we have that $x_1x_{p-1}, x_1x_{p+1}, x_nx_{p+1}, x_nx_{p-1}, x_1x_{q-1}, x_1x_{q+1}$, $x_nx_{q+1}, x_nx_{q-1}\notin E(G)$.

Assume that $p<q$. Then $p+1\neq q$, otherwise we obtain, from the above, that $x_px_1, x_px_n, x_qx_1, x_qx_n\notin E(G)$. Thus $x_1x_n\in E(G)$ from $G[x_2,x_1,x_n,x_p]\neq K_{1,3}$, and so $G[x_1,x_n,x_2,x_p,x_q]$ is an hourglass, a contradiction.

We further have that $x_{p-1}x_{p+1}, x_{q-1}x_{q+1}\in E(G)$, since otherwise, for example, $x_{p-1}x_{p+1}\notin E(G)$, we have $x_2x_{p-1}\in E(G)$ or $x_2x_{p+1}\in E(G)$ (say $x_2x_{p-1}\in E(G)$) since

$G[x_p,x_{p-1},x_{p+1},x_2] \neq K_{1,3}$. From $G[x_2,x_1,x_n,x_{p-1}] \neq K_{1,3}$, $x_1x_n \in E(G)$. Note that $x_1x_3 \notin E(G)$ since otherwise G has a hamiltonian path $x_1P[x_3,x_{p-1}]x_2P[x_p,x_n]$. Since $G[x_n,x_3,x_{n-1},x_1,x_2]$ is not an hourglass, $x_1x_{n-1} \in E(G)$. Thus G has a hamiltonian path $x_1P^-[x_{n-1},x_p]x_2P^-[x_{p-1},x_3]x_n$, a contradiction.

Note that $x_1x_p, x_1x_q \notin E(G)$, since otherwise, for example, $x_1x_p \in E(G)$, we have $x_2x_{p-1}, x_2x_{p+1} \notin E(G)$ since otherwise, say $x_2x_{p-1} \in E(G)$, G has a hamiltonian (x_1,x_n)-path $x_1P[x_p,x_{n-1}]P[x_3,x_{p-1}]x_2x_n$. Recall that $x_{p+1}x_1, x_{p-1}x_1 \notin E(G)$. Thus $G[x_p,x_{p-1},x_{p+1},x_2,x_1]$ is an hourglass. From $G[x_2,x_1,x_p,x_q] \neq K_{1,3}$, $x_px_q \in E(G)$.

Note that $x_2x_{p-1} \notin E(G)$ since otherwise we have $x_1x_n \in E(G)$ from $G[x_2,x_1,x_{p-1},x_n] \neq K_{1,3}$, and so $G[x_n,x_1,x_2,x_3,x_{n-1}]$ is an hourglass (note that $x_1x_{n-1} \notin E(G)$), a contradiction. Similarly, $x_2x_{p+1}, x_2x_{q-1}, x_2x_{q+1} \notin E(G)$. It follows that $x_nx_p \notin E(G)$ since otherwise $G[x_p,x_{p-1},x_{p+1},x_2,x_n]$ is an hourglass. Similarly, $x_nx_q \notin E(G)$. From $G[x_2,x_1,x_n,x_p] \neq K_{1,3}$, $x_1x_n \in E(G)$. Thus $G[x_2,x_p,x_q,x_1,x_n]$ is an hourglass. This contradiction shows that $i \geq 3$. ∎

Claim 7. $k < i-1$.

Proof. If $k = i-1$, by Claim 6, $k \geq 2$. By Lemma 12(2), $x_ix_{n-1} \notin E(G)$, which implies $x_{i+1}x_{n-1} \in E(G)$. Thus $x_{i-1}x_{i+1} \in E(G)$ or $x_ix_{n-1} \in E(G)$, since otherwise, $G[x_n,x_i,x_{i-1},x_{i+1},x_{n-1}]$ is an hourglass. Assume that $x_{i-1}x_{i+1} \in E(G)$. We have that $x_ix_{i-2} \notin E(G)$ since otherwise G has a hamiltonian (x_1,x_n)-path

$$P[x_1,x_{i-2}]x_ix_{i-1}P[x_{i+1},x_n].$$

From $G[x_{i-1},x_i,x_{i-2},x_{i+1}] \neq K_{1,3}$, $x_{i-2}x_{i+1} \in E(G)$. Note that $x_{i-2}x_{n-1}, x_{i+2}x_{i-2} \notin E(G)$. From $G[x_{i+1},x_{i+2},x_{i-2},x_{n-1}] \neq K_{1,3}$, $x_{i+2}x_{n-1} \in E(G)$. Note that $x_{i-1}x_{n-1} \notin E(G)$ since otherwise G has a hamiltonian (x_1,x_n)-path

$$P[x_1,x_{i-2}]P[x_{i+1},x_{n-1}]x_{i-1}x_ix_n.$$

Thus $G[x_{i+1},x_{i+2},x_{n-1},x_{i-1},x_{i-2}]$ is an hourglass, a contradiction. It follows that $x_{i-1}x_{n-1} \in E(G)$. Note that $x_ix_{i-2} \notin E(G)$ since otherwise G has a hamiltonian path $P[x_1,x_i]P[x_{i+2},x_{n-1}]x_{i+1}x_n$. It follows that $x_{i-2}x_{n-1} \in E(G)$ since otherwise $G[x_{i-1},x_i,x_{i-2},x_{n-1}] = K_{1,3}$. Since $x_{i-1}x_{n-2} \notin E(G)$, we have that $x_{i+1}x_{n-2} \in E(G)$, otherwise, $G[x_{n-1},x_{n-2},x_{i+1},x_{i-1}] = K_{1,3}$. $x_{i-2}x_{n-2} \notin E(G)$, otherwise G has a hamiltonian path $P[x_1,x_{i-2}]P^-[x_{n-2},x_{i+1}]x_{n-1}x_{i-1}x_ix_n$, a contradiction. $x_{i-2}x_{i+1} \notin E(G)$, otherwise, G has a hamiltonian path $P[x_1,x_{i-2}]P[x_{i+1},x_{n-1}]x_{i-1}x_ix_n$ a contradiction. It follows that $G[x_{n-1},x_{i-2},x_{i-1},x_{n-2},x_{i+1}]$ is an hourglass, a contradiction. So $x_{i-1}x_{n-1} \notin E(G)$. It follows that $k < i-1$. Thus Claim 7 is true. ∎

Claim 8. *$x_kx_{i+1}, x_{k+1}x_i \in E(G)$ and $x_kx_{n-1}, x_{k+1}x_{n-1}, x_{k-1}x_{n-1} \notin E(G)$ (if $k > 1$).*

Proof. By Claim 4, $x_ix_{n-1} \notin E(G)$. By Lemma 12(2), $x_{i+1}x_{n-1} \in E(G)$. If $x_kx_{i+1} \notin E(G)$, then, since $G[x_n,x_{i+1},x_{n-1},x_i,x_k]$ is not an hourglass, $x_kx_{n-1} \in E(G)$. Note that $x_{k+1}x_{n-1} \notin E(G)$ since otherwise G has a hamiltonian (x_1,x_n)-path

$$P[x_1,x_k]P^-[x_i,x_{k+1}]P^-[x_{n-1},x_{i+1}]x_n,$$

a contradiction. Similarly, if $k > 1$, then $x_ix_{k-1}, x_{k-1}x_{k+1} \notin E(G)$. From $G[x_k,x_{k+1},x_i,x_{n-1}] \neq K_{1,3}$, we have that $x_{k+1}x_i \in E(G)$.

If $k=1$, then it is easy to check that $H=G[x_n,x_1,x_i,x_{n-1},x_{i+1},x_2]$ is isomorphic to $(P_6)^2$ (note that $d_H(x_n)=d_H(x_1)=4$, $d_H(x_{n-1})=d_H(x_i)=3$ and $d_H(x_{i+1})=d_H(x_2)=2$). This contradiction shows $x_kx_{i+1}\in E(G)$. If $k\geq 2$, then, from $G[x_k,x_{k-1},x_{k+1},x_{n-1}]\neq K_{1,3}$, $x_{k-1}x_{n-1}\in E(G)$. Thus $G[x_k,x_{k-1},x_{k+1},x_i,x_{n-1}]$ is an hourglass. This contradiction shows that $x_kx_{i+1}\in E(G)$.

By Claim 4, $x_{k+1}x_{i+1}\notin E(G)$. From $G[x_k,x_{k+1},x_i,x_{i+1}]\neq K_{1,3}$, $x_{k+1}x_i\in E(G)$.

If $x_kx_{n-1}\in E(G)$, then $G[x_k,x_{i+1},x_{n-1},x_i,x_{k+1}]$ is an hourglass, a contradiction. Thus $x_kx_{n-1}\notin E(G)$. If $x_{k+1}x_{n-1}\in E(G)$ then G has a hamiltonian (x_1,x_n)-path

$$P[x_1,x_k]P^-[x_i,x_{k+1}]P^-[x_{n-1},x_{i+1}]x_n.$$

If $k>1$, then $x_{k-1}x_{n-1}\notin E(G)$ since otherwise G has a hamiltonian (x_1,x_n)-path

$$P[x_1,x_{k-1}]P^-[x_{n-1},x_{i+1}]P[x_k,x_i]x_n.$$

This contradiction shows that Claim 8 is true. ■

Now we complete the proof of Proposition 12. By Lemma 12(1), $x_ix_n,x_{i+1}x_n\in E(G)$ and $x_ix_{i+1}\notin E(G)$. By Claims 5 and 6, we have $i+2\neq n$ and $i\geq 3$. Let x_k be chosen as before Claim 3 with $x_kx_i\in E(G)$ and $x_kx_n\in E(G)$. Then, by Claim 7, $k\leq i-2$. By Lemma 12(2), $x_ix_{n-1}\notin E(G)$ and $x_{i+1}x_{n-1}\in E(G)$. By Claim 8, x_ix_{k+1}, $x_kx_{i+1}\in E(G)$ and $x_kx_{n-1}\notin E(G)$. Note that $x_{k+1}x_n\notin E(G)$ since otherwise $G[x_n,x_{k+1},x_i,x_{i+1},x_{n-1}]$ is an hourglass since $x_{i+1}x_i,x_{i+1}x_{k+1},x_{n-1}x_i,x_{n-1}x_{k+1}\notin E(G)$. Thus we can derive that $G[x_k,x_{k+1},x_i,x_{i+1},x_{n-1},x_n]$ is a $(P_6)^2$, a contradiction. Therefore, we complete the proof of Proposition 12.

By Propositions 10 and 12, we immediately deduce Theorem 7.

Proof of Theorem 6. Let G be a 3-connected $\{claw,(P_6)^2,hourglass\}$-free graph. By Theorem 7, we only consider the hamiltonian connectedness in the closure $cl(G)$ of G. By Theorem 9, we know that $cl(G)$ is the line graph of some triangle-free graph. By Proposition 10, we easily obtain that $cl(G)$ is hourglass-free. By Theorem 5, $cl(G)$ is hamiltonian connected. Again from Theorem 7, G is hamiltonian connected. Thus, we complete the proof of Theorem 6. ■

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