# A characterization of b-perfect graphs* 

Chính T. Hoàng ${ }^{\dagger} \quad$ Frédéric Maffray ${ }^{\ddagger}$<br>Meriem Mechebbek ${ }^{\S}$

October 9, 2018


#### Abstract

A b-coloring is a coloring of the vertices of a graph such that each color class contains a vertex that has a neighbor in all other color classes, and the b-chromatic number of a graph $G$ is the largest integer $k$ such that $G$ admits a b-coloring with $k$ colors. A graph is b-perfect if the b-chromatic number is equal to the chromatic number for every induced subgraph of $G$. We prove that a graph is b-perfect if and only if it does not contain as an induced subgraph a member of a certain list of twenty-two graphs. This entails the existence of a polynomial-time recognition algorithm and of a polynomial-time algorithm for coloring exactly the vertices of every b-perfect graph.


Keywords: Coloration, b-coloring, a-chromatic number, b-chromatic number.

## 1 Introduction

A proper coloring of a graph $G$ is a mapping $c$ from the vertex-set $V(G)$ of $G$ to the set $\{1,2, \ldots\}$ of positive integers (colors) such that any two adjacent vertices are mapped to different colors. Each set of vertices colored with one color is a stable set of vertices of $G$, so a coloring is a partition of $V(G)$

[^0]into stable sets. The smallest number $k$ for which $G$ admits a coloring with $k$ colors is the chromatic number $\chi(G)$ of $G$ [1].

Many graph invariants related to colorings have been defined. Most of them try to minimize the number of colors used to color the vertices under some constraints. For some other invariants, it is meaningful to try to maximize this number. The b-chromatic number is such an example. When we try to color the vertices of a graph, we can start from a given coloring and try to decrease the number of colors by eliminating color classes. One possible such procedure consists in trying to reduce the number of colors by transferring every vertex from a fixed color class to a color class in which it has no neighbour, if any such class exists. A b-coloring is a proper coloring in which this is not possible, that is, every color class $i$ contains at least one vertex that has a neighbor in all the other classes. Any such vertex will be called a $b$-vertex of color $i$. The $b$-chromatic number $b(G)$ is the largest integer $k$ such that $G$ admits a b-coloring with $k$ colors. Irving and Manlove [11, 13] proved that deciding whether a graph $G$ admits a b-coloring with a given number of colors is an NP-complete problem, even when it is restricted to the class of bipartite graphs [10]. On the other hand, they gave a polynomial-time algorithm that solves this problem for trees. The NP-completeness results has incited researchers to establish bounds on the b-chromatic number in general or to find its exact values for subclasses of graphs.

Clearly every $\chi(G)$-coloring of a graph $G$ is a b-coloring, and so every graph $G$ satisfies $\chi(G) \leq b(G)$. As usual with such an inequality, it may be interesting to look at the graphs that satisfy it with equality. However, graphs such that $\chi(G)=b(G)$ do not have a specific structure; to see this, we can take any arbitrary graph $G$ and add a component that consists of a clique of size $b(G)$; we obtain a graph $G^{\prime}$ that satisfies $\chi\left(G^{\prime}\right)=b\left(G^{\prime}\right)=b(G)$. This led Hoàng and Kouider [8] to introduce the class of b-perfect graphs: a graph $G$ is called b-perfect if every induced subgraph $H$ of $G$ satisfies $\chi(H)=b(H)$. Hoàng and Kouider [8] proved the b-perfectness of some classes of graphs, and asked for a good characterization of the whole class of b-perfect graphs. Hoàng, Linhares Sales and Maffray [9] proposed the conjecture below. Here we solve the problem by establishing the validity of the conjecture. For a fixed graph $F$, we say that a graph $G$ is $F$-free if it does not have an induced subgraph that is isomporphic to $F$. For a set $\mathcal{F}$ of graphs, we say that a graph $G$ is $\mathcal{F}$-free if it does not have an induced subgraph that is isomporphic to a member of $\mathcal{F}$. Let $\mathcal{F}=\left\{F_{1}, \ldots, F_{22}\right\}$ be the set of graphs depicted in Figure 1 .


Figure 1: Class $\mathcal{F}=\left\{F_{1}, \ldots, F_{22}\right\}$
Conjecture 1 (Hoàng, Linhares Sales, Maffray [9]) A graph is b-perfect if and only if it is $\mathcal{F}$-free.

Our main result is the following.
Theorem 1.1 Conjecture $\mathbb{1}$ is true.
The following theorem was proved before Conjecture 1 was formulated, but it can be seen as evidence for its validity.

Theorem 1.2 (Hoàng and Kouider [8])
A bipartite graph is b-perfect if and only if it contains no $F_{1}, F_{2}$ or $F_{3}$.
A $P_{4}$-free graph is b-perfect if and only if it contains no $F_{3}$ or $F_{6}$.

Moreover, some other partial results were obtained.
Theorem 1.3 (Hoàng, Linhares Sales, Maffray [9]) Conjecture $\mathbb{\square}$ holds for 3-colorable graphs and for diamond-free graphs.

Theorem 1.4 (Maffray, Mechebbek [12]) Conjecture 1 holds for chordal graphs.

In the remainder of this section, we introduce some definitions and notation. For any vertex $v$ of a graph $G$, the neighborhood of $v$ is the set $N(v)=\{u \in V(G) \mid u v \in E\}$ and the degree of $v$ is $\operatorname{deg}(v)=|N(v)|$. If a vertex $x$ is adjacent to a vertex $y$, then we will say that $x$ sees $y$, otherwise we will say $x$ misses $y$. Let us say that a set $A$ of vertices is complete (respectively, anti-complete) to a set $B$ if every vertex of $A$ sees (respectively, misses) every vertex of $B$.

A component of a graph $G$ is a maximal connected subgraph of $G$, and a co-component of $G$ is a component of its complementary graph. Two vertices $x, y$ are twins if $N(x)-\{x, y\}=N(y)-\{x, y\}$; in addition, if $x$ sees $y$ then they are true twins, otherwise they are false twins. A vertex $x$ dominates a vertex $y$ if $N(y) \subseteq N(x) \cup\{x\} ; x$ and $y$ are comparable if $x$ dominates $y$, or vice versa.

For any integer $k \geq 1$, we denote by $P_{k}$ the chordless path with $k$ vertices. For integer $k \geq 3$, we denote by $C_{k}$ the chordless cycle with $k$ vertices. A diamond is a graph with four vertices that consists in a clique minus an edge.

## 2 Some lemmas

We say that a graph $G$ is $b$-imperfect if it is not b-perfect, and minimally $b$-imperfect if $G$ is b-imperfect and every proper induced subgraph of $G$ is bperfect. We say that a graph $G$ is a minimal counterexample to Conjecture 1 if it is a counterexample (i.e., a b-imperfect $\mathcal{F}$-free graph) with the smallest number of vertices and, among all such graphs, with the smallest number of edges. Note that every minimal counterexample is minimally b-imperfect.

Let $\omega(G)$ denote the number of vertices in a largest clique of $G$.

Lemma 2.1 (Hoàng and Kouider [8]) Let $G$ be a minimally b-imperfect graph. Then no component of $G$ is a clique.

Lemma 2.2 ([9]) Let $G$ be graph and $x$ be any simplicial vertex of $G$. Let $c$ be any b-coloring of $G$ with $k$ colors, where $k>\omega(G)$. Then $x$ is not $a$ $b$-vertex for $c$.

Proof. Suppose that $x$ is a b-vertex for $c$. Then all $k$ colors of $c$ appear in the clique formed by $x$ and its neighbours. Thus $\omega(G) \geq k$, a contradiction.

Lemma 2.3 ([9]) Let $G$ be a minimally b-imperfect graph, and let $c$ be any $b$-coloring with $b(G)$ colors. Let $u, v$ be two non-adjacent vertices of $G$ such that $N(u) \subseteq N(v)$. Then $c(u) \neq c(v)$, and $u$ is not a b-vertex. In particular, if $N(u)=N(v)$, then none of $u, v$ is a b-vertex.

Proof. Assume without loss of generality that $c(u)=c(v)=1$. Consider the restriction of $c$ to $G \backslash u$. Every b-vertex $z$ of color $i \geq 2$ in $G$ is still a b-vertex in $G \backslash u$, because it cannot be that $u$ is the only neighbour of $z$ of color 1 . Moreover, it cannot be that $u$ is the only b-vertex of $G$ of color 1, because if it is a b-vertex then $v$ is also a b-vertex. But then $b(G \backslash u) \geq b(G)>\chi(G) \geq \chi(G \backslash u)$, so $G \backslash u$ is b-imperfect, a contradiction. Thus $c(u) \neq c(v)$. This implies that $u$ cannot be a b-vertex, because it has no neighbour of color $c(v)$. In particular, if $N(u)=N(v)$, then the preceding argument works both ways, which leads to the desired conclusion.

Lemma 2.4 Let $G$ be a minimally b-imperfect graph, and let $u, v$ be two true twins of $G$. Then in any b-coloring of $G, u$ is $a b$-vertex if and only if $v$ is.

Proof. Let $c$ be any b-coloring of $G$ with $k$ colors. If $u$ is a b-vertex for $c$, then $k-2$ colors appear in $N(u)-\{v\}$. Since $N(u)-\{u, v\}=N(v)-\{u, v\}$, $k-1$ colors (including the color of $u$ ) appear in $N(v)$. Thus, $v$ is a b-vertex.

Lemma 2.5 ([9]) Let $G$ be a minimally b-imperfect $\mathcal{F}$-free graph. Then $G$ is connected.

Proof. Suppose that $G$ has several components $G_{1}, \ldots, G_{p}, p \geq 2$. By Lemma 2.1, each $G_{i}$ has a subset $S_{i}$ of three vertices that induce a chordless path. Then $G$ is $P_{4}$-free, for otherwise, since a $P_{4}$ is in one component of $G$, $G$ contains an $F_{2}$. But then Theorem 1.2 is contradicted. Thus the lemma holds.

Lemma 2.6 Let $G$ be a minimally b-imperfect graph, and let $c$ be $a b$ coloring of $G$ with $b(G)$ colors. If a vertex $x$ is not a b-vertex, then there is a color $f(x)$ such that $x$ is the only neighbor of color $c(x)$ of every b-vertex of color $f(x)$.

Proof. The definition of $G$ implies $b(G)>\chi(G) \geq \chi(G-x)=b(G-x)$. If the lemma is false, then the coloring $c$ restricted to $G-x$ still has all $b$-vertices of all colors from 1 to $b(G)$, implying $b(G-x)=b(G)$, a contradiction.

Lemma 2.7 Suppose that a graph $G$ has a coloring (not necessarily a $b$ coloring) with $k$ colors where $k>\omega(G)$. Then:

- If there is a b-vertex, then $G$ contains a $P_{3}$.
- If there are two b-vertices of different colors, then $G$ contains as an induced subgraph a $P_{4}, a 2 P_{3}$ or a diamond.

Proof. To prove the first part, suppose on the contrary that $G$ contains no $P_{3}$. Then every component of $G$ is a clique. Let $G_{0}$ be a component that contains a b-vertex. Then $G_{0}$ has vertices of all colors, so $G$ has a clique of size $k$, a contradiction.

To prove the second part, let $x_{i}$ be a b-vertex of color $i$ for each $i=1,2$. Let $Q=N\left(x_{1}\right) \cap N\left(x_{2}\right)$.

First assume that $x_{1}$ sees $x_{2}$. If $Q$ contains non-adjacent vertices $u, v$, then $x_{1}, x_{2}, u, v$ induce a diamond. So, $Q$ is a clique. Then $Q \cup\left\{x_{1}, x_{2}\right\}$ is a clique, and since $k>\omega(G)$, there is a color $j$ that does not appear in $Q \cup\left\{x_{1}, x_{2}\right\}$. Since $x_{1}$ is a b-vertex, it has a neighbor $y$ of color $j$, and similarly $x_{2}$ has a neighbor $z$ of color $j$. By the definition of $j$, vertices $y, z$ are not in $Q$, so $y$ misses $x_{2}, z$ misses $x_{1}$, and $y \neq z$. Then $y, x_{1}, x_{2}, z$ induce $P_{4}$ in $G$.

Now, we know $x_{1}$ misses $x_{2}$. Suppose that $Q$ contains non-adjacent vertices $u, v$. Since $x_{1}$ is a b-vertex, it has a neighbor $y$ of color 2. Thus $y$ misses $x_{2}$ and is not in $Q$. If $y$ misses $u$, then $y, x_{1}, u, x_{2}$ induce a $P_{4}$.

So, $y$ sees $u$, and similarly $v$. Then $y, x_{1}, u, v$ induce a diamond. So, $Q$ is a (possibly empty) clique. By Lemma [2.2, for $i=1,2$, the vertex $x_{i}$ is not simplicial, so it has non-adjacent neighbors $u_{i}, v_{i}$. If $x_{1}, x_{2}$ are not in the same component of $G$, then $x_{1}, u_{1}, v_{1}, x_{2}, u_{2}, v_{2}$ induce a $2 P_{3}$. Thus, $x_{1}, x_{2}$ are in the same component of $G$, and so there is a chordless path between them. If this path has length at least three, then $G$ contains a $P_{4}$. So let this path be $x_{1}-z-x_{2}$. We may assume that $z \neq u_{1}$ (or else, symmetrically, $z \neq v_{1}$ ). If $u_{1}$ misses $z$, then either $u_{1}$ sees $x_{2}$, and then $u_{1}, z$ are non-adjacent members of $Q$, a contradiction, or $u_{1}$ misses $x_{2}$, and then $u_{1}-x_{1}-z-x_{2}$ is a $P_{4}$. Thus we may assume that $u_{1}$ sees $z$. It follows that $z \neq v_{1}$, which restores the symmetry between $u_{1}$ and $v_{1}$, and so $v_{1}$ too sees $z$, and $u_{1}, x_{1}, v_{1}, z$ induce a diamond. Thus the lemma holds.

Lemma 2.8 Let $G$ be a $P_{4}$-free graph with $V(G)=X \cup Y$, where $X=\left\{x_{1}\right.$, $\left.\ldots, x_{j-1}\right\}$ and $Y=\left\{y_{2}, \ldots, y_{j}\right\}$ are cliques of size $j-1 \geq 2$, vertices $x_{1}$ and $y_{j}$ are adjacent, and for each $i=2, \ldots, j-1$, vertices $x_{i}$ and $y_{i}$ are either different and not adjacent or equal. Then $G$ has a clique on $j$ vertices that consists of $x_{1}, y_{j}$ and one of $x_{i}, y_{i}$ for each $i=2, \ldots, j-1$.

Proof. We prove the lemma by induction on the number $n$ of indices $i$ such that $x_{i} \neq y_{i}$. If $n=0$, then $G$ itself is the desired clique. Now let $n>0$, so $j \geq 3$ and, up to symmetry, we may assume that $x_{2} \neq y_{2}$. Let $G^{\prime}$ be the graph obtained by contracting $x_{2}$ and $y_{2}$, that is, replacing them by a vertex $z$ adjacent to all other vertices. It is easy to see that $G^{\prime}$ satisfies the conditions of the lemma; in particular, $G^{\prime}$ is $P_{4}$-free because $G^{\prime} \backslash z$ is equal to $G \backslash\left\{x_{2}, y_{2}\right\}$ and $z$ is adjacent to every vertex of $G^{\prime} \backslash z$. Thus, by the induction hypothesis, $G^{\prime}$ contains a clique $K^{\prime}=\left\{x_{1}, z, z_{3}, \ldots, z_{j-1}, y_{j}\right\}$, where, for $i=3, \ldots, j-1$, vertex $z_{i}$ is either $x_{i}$ or $y_{i}$. Let $K_{x}=\left(K^{\prime} \backslash z\right) \cup\left\{x_{2}\right\}$ and $K_{y}=\left(K^{\prime} \backslash z\right) \cup\left\{y_{2}\right\}$. If none of $K_{x}, K_{y}$ is a clique, then $x_{2}$ misses some vertex $y_{h}$ of $K^{\prime}$ and $y_{2}$ misses some vertex $x_{g}$ of $K^{\prime}$, but then $\left\{x_{2}, x_{g}, y_{h}, y_{2}\right\}$ induces a $P_{4}$ in $G$. Thus one of $K_{x}, K_{y}$ is the desired clique.

A set $H$ of vertices of $G$ is homogeneous if every vertex in $G \backslash H$ either sees all or misses all vertices of $H$. We say that a homogeneous set $H$ of $G$ is proper if $H \neq V(G)$.

Lemma 2.9 Let $G$ be a minimal counterexample to Conjecture 1. If $H$ is a proper homogeneous set in $G$, then $H$ is a clique or a stable set.

Proof. We prove this lemma by induction on the size of $H$. Suppose that $H$ is not a clique or a stable set. Let $T$ be the set of vertices of $G \backslash H$ that
see all of $H$, and $Z$ be the set of vertices of $G \backslash H$ that miss all of $H$. So $H, T, Z$ form a partition of $V(G)$. Note that $T \cup Z \neq \emptyset$ by the definition of a proper homogeneous set. Now, by Lemma [2.5, we have $T \neq \emptyset$.

Let $c$ be any b-coloring of $G$ with $k>\chi(G)$ colors. We may assume that the colors that appear in $H$ are $1, \ldots, h$ and that those that have a b-vertex in $H$ are colors $1, \ldots, h_{b}$. Clearly we have $h_{b} \leq h$ and $\chi(H) \leq h$. We claim that

$$
\begin{equation*}
h_{b} \leq \chi(H) . \tag{1}
\end{equation*}
$$

For suppose that $\chi(H)<h_{b}$. Let $H^{\prime}$ be the subgraph of $G$ induced by the vertices of $H$ that have colors $1, \ldots, h_{b}$, and let $c^{\prime}$ be the restriction of $c$ to $H^{\prime}$. Then the graph $H^{\prime}$ has strictly fewer vertices than $G$ (because $T \cup Z \neq \emptyset$ ), it is $\mathcal{F}$-free, and $c^{\prime}$ is a b-coloring of $H^{\prime}$ with $h_{b}$ colors, where $h_{b}>\chi(H) \geq \chi\left(H^{\prime}\right)$, so $H^{\prime}$ contradicts the minimality of $G$. Thus (11) holds.

$$
\begin{equation*}
h_{b}>0 . \tag{2}
\end{equation*}
$$

For suppose that $h_{b}=0$. Let $G^{\prime}$ be the graph obtained from $G$ by replacing $H$ with a stable set $S$ of size $h$ (so that all vertices of $S$ see all of $T$ and none of $Z$ in $G^{\prime}$ ). We establish four properties (i)-(iv) of $G^{\prime}$.
(i) $G^{\prime}$ is $\mathcal{F}$-free. For suppose that $G^{\prime}$ contains a member $F$ of $\mathcal{F}$. If $F$ has three or more vertices of $S$, then these vertices are pairwise twins in $F$; but no member of $F$ has three pairwise twins. So $F$ has at most two vertices of $S$. Then, since $H$ is not a clique, these vertices can be replaced by the same number of non-adjacent vertices of $H$ so that we obtain a copy of $F$ that is an induced subgraph of $G$, a contradiction.
(ii) Consider the coloring $c^{\prime}$ of $G^{\prime}$ that is obtained by setting $c^{\prime}(x)=c(x)$ for every $x \in T \cup Z$ and by giving colors $1, \ldots, h$ to the vertices of $S$. Then $c^{\prime}$ is a b-coloring of $G^{\prime}$ with $k$ colors, because every b-vertex in $G$ is still a b-vertex in $G^{\prime}$.
(iii) $\chi\left(G^{\prime}\right) \leq \chi(G)$, because every coloring of $G$ with $\chi(G)$ colors can be transformed into a $\chi(G)$ coloring of $G^{\prime}$ by maintaining the color of the vertices in $V \backslash H$ and giving to all vertices of $S$ the color of a fixed vertex of $H$.
(iv) $\left|V\left(G^{\prime}\right)\right| \leq|V(G)|$; and if $\left|V\left(G^{\prime}\right)\right|=|V(G)|$ then, since $H$ is not a stable set, we have $\left|E\left(G^{\prime}\right)\right|<|E(G)|$.
It follows from Properties (i)-(iv) that $G^{\prime}$ contradicts the minimality of $G$. Thus (2) holds.

Note that, since $T \neq \emptyset$, we have $k \geq h+1$. By (2), all colors $h+1, \ldots, k$ appear in $T$. We claim that

$$
\begin{equation*}
h>\chi(H) . \tag{3}
\end{equation*}
$$

For suppose on the contrary that $h=\chi(H)$. Let $G^{\prime}$ be the graph obtained from $G$ by replacing $H$ with a clique $K$ of size $h$ (so that all vertices of $K$ see all of $T$ and none of $Z$ in $G^{\prime}$ ). We establish four properties (i)-(iv) of $G^{\prime}$.
(i) $G^{\prime}$ is $\mathcal{F}$-free. For suppose that $G^{\prime}$ contains a member $F$ of $\mathcal{F}$. If $F$ has three or more vertices of $K$, then these vertices are pairwise adjacent twins in $F$; but no member of $F$ has three pairwise adjacent twins. So $F$ has at most two vertices of $K$. Then these vertices can be replaced by the same number of adjacent vertices of $H$, so that we obtain a copy of $F$ that is an induced subgraph of $G$, a contradiction.
(ii) Consider the coloring $c^{\prime}$ of $G^{\prime}$ obtained by setting $c^{\prime}(x)=c(x)$ for every $x \in T \cup Z$ and by giving colors $1, \ldots, h$ to the vertices of $K$. Then $c^{\prime}$ is a b-coloring of $G^{\prime}$ with $k$ colors, because every b-vertex of color $>h$ in $G$ is still a b-vertex in $G^{\prime}$, and every vertex of $K$ is a b-vertex in $G^{\prime}$ since all colors $h+1, \ldots, k$ appear in $T$.
(iii) $\chi\left(G^{\prime}\right) \leq \chi(G)$, because every coloring of $G$ with $\chi(G)$ colors must use at least $\chi(H)=h$ colors on $H$ and can be turned into a $\chi(G)$ coloring of $G^{\prime}$ by giving colors $1, \ldots, h$ to the vertices of $K$.
(iv) Since $H$ is not a clique, we have $|H|>\chi(H)$, so $G^{\prime}$ has strictly fewer vertices than $G$.
It follows from Properties (i)-(iv) that $G^{\prime}$ contradicts the minimality of $G$. Thus (3) holds.

Now, we can apply the first part of Lemma 2.7 to $H$ and to the restriction of $c$ to $H$, which implies that $H$ contains a $P_{3}$. Note that (1) and (3) imply $h_{b}<h$.

Now we claim that:

$$
\begin{equation*}
H \text { contains no } P_{4} \text { and no } 2 P_{3} \text {. } \tag{4}
\end{equation*}
$$

For suppose that $H$ contains a $P_{4}$ or a $2 P_{3}$, with vertex-set $X$. We distinguish between two cases.
Case 1: $h_{b}=1$. Let $x$ be a b-vertex of $H$ with $c(x)=1$. Let $G^{\prime}$ be the graph obtained from $G$ by removing every edge whose two endvertices are in $H \backslash\{x\}$. We establish three properties (i)-(iii) of $G^{\prime}$.
(i) $G^{\prime}$ is $\mathcal{F}$-free. For suppose that $G^{\prime}$ contains a member $F$ of $\mathcal{F}$. Then $F \cap H$
is a homogeneous set of $F$; and since $H$ (in $G^{\prime}$ ) contains no $P_{4}$, no $2 P_{3}$ and no diamond, this is possible only if $F \cap H$ either is a $P_{3}$ or has at most two vertices; and in either case it is possible to replace $F \cap H$ by a subgraph of $H$ in $G$ that is isomorphic to $F \cap H$, so that we obtain a copy of $F$ that is an induced subgraph of $G$, a contradiction.
(ii) $c$ is a b-coloring of $G^{\prime}$ with $k$ colors (because every b-vertex in $G$ is still a b-vertex in $G^{\prime}$ ).
(iii) $\chi\left(G^{\prime}\right) \leq \chi(G)$, clearly.

It follows from Properties (i)-(iii) that if $G^{\prime}$ has strictly fewer edges than $G$, then $G^{\prime}$ contradicts the minimality of $G$. So it must be that $G^{\prime}=G$. This means that every edge in $H$ is adjacent to $x$. Thus $H$ contains no $P_{4}$ and no $2 P_{3}$ as desired.
Case 2: $h_{b} \geq 2$. So $h \geq 3$. Let $u$ be any b-vertex of color $h$. Since $h_{b}<h$, vertex $u$ is not in $H$, and since color $h$ appears in $H$ it is not in $T$, so we have $u \in Z$. Note that $Z$ contains no $P_{3}$, for otherwise, if $X^{\prime}$ is the vertex-set of a $P_{3}$ in $Z$, then $X \cup X^{\prime}$ induces an $F_{2}$ or $F_{3}$. Therefore every component of $Z$ is a clique. Let $Y$ be the component of $Z$ that contains $u$. Since $u$ is a b-vertex, all colors $1, \ldots, h$ must appear in $Y$. So $|Y| \geq h \geq 3$. Suppose that $Y$ is not homogeneous. Then there are vertices $y_{1}, y_{2} \in Y$ and a vertex $t$ that sees $y_{1}$ and misses $y_{2}$. Clearly $t \in T$. Let $y_{3}$ be a vertex of $Y \backslash\left\{y_{1}, y_{2}\right\}$. If $t$ misses $y_{3}$, then $X \cup\left\{t, y_{1}, y_{2}, y_{3}\right\}$ induces an $F_{8}$ or $F_{9}$. If $t$ sees $y_{3}$ then, letting $X^{\prime \prime}$ be the vertex-set of a $P_{3}$ in $H$, we obtain that $X^{\prime \prime} \cup\left\{t, y_{1}, y_{2}, y_{3}\right\}$ induces an $F_{5}$. Therefore $Y$ is a homogeneous set. It follows that all vertices of $Y$ are b-vertices, and so $Y$ contains b-vertices of colors $1, \ldots, h$. Let $G^{\prime}$ be the graph obtained from $G$ by removing every edge whose two endvertices are in $H$. We establish four properties (i)-(iv) of $G^{\prime}$.
(i) $G^{\prime}$ is $\mathcal{F}$-free. For suppose that $G^{\prime}$ contains a member $F$ of $\mathcal{F}$. If $F$ has three or more vertices of $H$, then these vertices are pairwise non-adjacent twins in $F$; but no member of $F$ has three pairwise non-adjacent twins. So $F$ has at most two vertices of $H$. Then these vertices can be replaced by the same number of non-adjacent vertices of $H$ in $G$, so that we obtain a copy of $F$ that is an induced subgraph of $G$, a contradiction.
(ii) $c$ is a b-coloring of $G^{\prime}$ with $k$ colors (because every b-vertex of color $>h$ in $G$ is still a b-vertex in $G^{\prime}$, and $Y$ contains b-vertices of colors $\left.1, \ldots, h\right)$. (iii) $\chi\left(G^{\prime}\right) \leq \chi(G)$, obvious.
(iv) Since $H$ is not a stable set, $G^{\prime}$ has strictly fewer edges than $G$.

It follows from Properties (i)-(iv) that $G^{\prime}$ contradicts the minimality of $G$. Thus (4) holds.

Suppose that $H$ has at least two components. Since $H$ contains no $2 P_{3}$,
one of these components $K$ is a clique. Let $x$ be a b-vertex in $H$. If $x \in K$, then all colors $1, \ldots, h$ appear in $K$ and all vertices of $K$ are b-vertices; thus $h_{b}=h$, a contradiction. So $K$ contains no b-vertex, and $x$ is in another component of $H$. It follows that all colors that appear in $K$ also appear in $H \backslash K$. Consider the graph $G \backslash K$ and the restriction $c^{\prime}$ of $c$ to that graph. Then $c^{\prime}$ is a b-coloring of $G \backslash K$ with $k$ colors, and $k>\chi(G) \geq \chi(G \backslash K)$, so $G \backslash K$ contradicts the minimality of $G$. So $H$ is connected.

Since $H$ is $P_{4}$-free, connected, and has more than one vertex, a classical theorem of Seinsche [16] states that $H$ can be partitioned into two nonempty sets $Q, S$ such that every vertex of $Q$ is adjacent to every vertex of $S$. Now, each of $Q, S$ is a homogeneous set that is strictly smaller than $H$. By the induction hypothesis, each of $Q, S$ is a clique or a stable set. If $Q$ and $S$ are two cliques, then $H$ is a clique as desired. If $Q$ and $S$ are two stable sets (of size at least two), then Lemma 2.3 implies that no vertex of $H$ is a b-vertex, which contradicts $h_{b}>0$. Therefore we may assume up to symmetry that $Q$ is a clique and $S$ is a stable set of size at least two. Let $\ell=|S|$ and $S=\left\{s_{1}, \ldots, s_{\ell}\right\}$. By Lemma [2.3, all vertices of $S$ have different colors and are not b-vertices. Up to renaming colors, we will assume that $s_{i}$ has color $i$ for each $i=1, \ldots, \ell$. Since $H$ is complete to $T$, the set $T$ contains no vertex of color $1, \ldots, \ell$; thus we know that:

$$
\begin{equation*}
Z \text { contains b-vertices of all colors } 1, \ldots, \ell . \tag{5}
\end{equation*}
$$

We claim that:
Each vertex of $Q$ is the only b-vertex of its color.
First note that, since $h_{b}>0$ and the vertices of $S$ are not b-vertices, some vertex of $Q$ is a b-vertex; and since the vertices of $Q$ are pairwise adjacent twins, by Lemma 2.4 they are all b-vertices. Now suppose that some vertex $q$ of $Q$ is not the only b-vertex of its color, say color $\ell+1$. Let $G^{\prime}$ be the graph obtained from $G$ by removing every edge between $q$ and $S$. Note that the subgraph $G^{\prime}[H]$ contains no $P_{4}$ or $2 P_{3}$. We establish four properties (i)-(iv) of $G^{\prime}$.
(i) $G^{\prime}$ is $\mathcal{F}$-free. For suppose that $G^{\prime}$ contains a member $F$ of $\mathcal{F}$. If $F$ has at most two vertices of $H$, then these vertices can be replaced by the same number of vertices of $H$ in $G$, so that we obtain a copy of $F$ that is an induced subgraph of $G$, a contradiction. If $F$ has three or more vertices of $H$, then, since $G^{\prime}[H]$ contains no $P_{4}$ or $2 P_{3}$, it must be (by examination of the list $\mathcal{F}$ ) that $F \cap H$ is either a $P_{3}$ or diamond; but this implies that there
is a $P_{3}$ or diamond in $H$, and so, $G$ contains a copy of $F$.
(ii) $c$ is a b-coloring of $G^{\prime}$ with $k$ colors, because every b-vertex of color $\neq \ell+1$ in $G$ is still a b-vertex in $G^{\prime}$ and there is a b-vertex of color $\ell+1$ different from $q$.
(iii) $\chi\left(G^{\prime}\right) \leq \chi(G)$, clearly.
(iv) $G^{\prime}$ has strictly fewer edges than $G$, clearly.

It follows from Properties (i)-(iv) that $G^{\prime}$ contradicts the minimality of $G$. Thus (6) holds.

Pick a vertex $q \in Q$, and assume that its color is $\ell+1$. We note that:

$$
\begin{equation*}
Z \text { contains no } P_{4} \text { and no } 2 P_{3} . \tag{7}
\end{equation*}
$$

For otherwise the union of the vertices of such a subgraph of $Z$ with vertices $q, s_{1}, s_{2}$ induces an $F_{2}$ or $F_{3}$. Thus (7) holds.

Let $C$ be a clique in $Z$ and let $i$ be a color not in $C$ such that every vertex of $C$ has a neighbor of color $i$. Then there is a vertex of color $i$ that is adjacent to all of $C$.

Pick a vertex $u$ of color $i$ that has the most neighbors in $C$. Suppose that $u$ has a non-neighbor $y$ in $C$. By the hypothesis we know that $y$ has a neighbor $v$ of color $i$. By the choice of $u$, there exists a vertex $x$ of $C$ that sees $u$ and misses $v$. So $\{u, v, x, y\}$ induce a $P_{4}$. By (7), one of $u, v$ is not in $Z$ and thus is in $T$. If exactly one of $u, v$ is in $T$, then $\{u, v, x, y, q\}$ induces an $F_{1}$; if both $u, v$ are in $T$, then $\left\{u, v, x, y, q, s_{1}\right\}$ induces an $F_{16}$, a contradiction. So $u$ is adjacent to all of $C$. Thus (8) holds.

For every set $J \subseteq\{1, \ldots, \ell\}, Z$ contains a clique of b-vertices of all colors from $J$.

We prove (9) by induction on $|J|$. The assertion holds when $|J|=1$ by (5). Let us now assume that $|J| \geq 2$. To simplify notation put $J=\{1,2, \ldots, j\}$. By the induction hypothesis, $Z$ contains a clique $X=\left\{x_{1}, \ldots, x_{j-1}\right\}$ where each $x_{i} \in X$ is a b-vertex of color $i$. For each $i=1, \ldots, j-1$, vertex $x_{i}$ has a neighbor of color $j$, so, by (8), there exists a vertex $u_{j}$ of color $j$ that is adjacent to all of $X$; moreover $u_{j}$ is in $Z$ since color $j$ does not appear in $T$. If $u_{j}$ is a b-vertex, then the desired conclusion holds with clique $X \cup\left\{u_{j}\right\}$. So let us assume that there is a color $g \neq j$ such that $u_{j}$ has no neighbor of color $g$. Clearly $g \notin\{1, \ldots, j-1\}$. Every member of $X$ has a neighbor of
color $g$, so, by (8), there is a vertex $v_{g}$ of color $g$ that is adjacent to all of $X$. Note that $v_{g}$ is in $Z \cup T$. Similarly, by the induction hypothesis, $Z$ contains a clique $Y=\left\{y_{2}, \ldots, y_{j}\right\}$ where each $y_{i} \in Y$ is a b-vertex of color $i$. By the same argument as for $X$, there exists in $Z$ a vertex $u_{1}$ that is adjacent to all of $Y$, vertex $u_{1}$ is not a b-vertex, so there is a color $h \notin\{1, \ldots, j\}$ such that $u_{1}$ has no neighbor of color $h$, and there is a vertex $v_{h}$ of color $h$ in $Z \cup T$ that is adjacent to all of $Y$.
If $x_{1}$ sees $y_{j}$, then $X$ and $Y$ satisfy the hypothesis of Lemma [2.8, so there exists a clique that contains $x_{1}, y_{j}$ and one of $x_{i}, y_{i}$ for each $i=2, \ldots, j-1$, which is the desired clique for (9). Let us now assume that $x_{1}$ misses $y_{j}$. Then $u_{1}$ misses $u_{j}$, for otherwise $\left\{u_{1}, x_{1}, u_{j}, y_{j}\right\}$ induces a $P_{4}$ in $Z$, which contradicts (77). We have $v_{g} \neq v_{h}$, for otherwise $\left\{u_{j}, x_{1}, u_{1}, y_{j}, v_{g}\right\}$ induces an $F_{1}$. Suppose that $v_{g}, v_{h}$ are both in $Z$. Then $v_{g}$ misses $u_{1}$, for otherwise $\left\{u_{j}, x_{1}, v_{g}, u_{1}\right\}$ induces a $P_{4}$ in $Z$. Similarly, $v_{g}$ misses $y_{j}$, and $v_{h}$ misses both $u_{j}, x_{1}$. Then $v_{g}$ misses $v_{h}$, for otherwise $\left\{x_{1}, v_{g}, y_{j}, v_{h}\right\}$ induces a $P_{4}$ in $Z$. But then $\left\{u_{j}, x_{1}, v_{g}, u_{1}, y_{j}, v_{h}\right\}$ induces a $2 P_{3}$ in $Z$, which contradicts (7). Therefore we may assume up to symmetry that $v_{g}$ is in $T$. If $j \geq 3$, then $\left\{s_{1}, s_{2}, q, v_{g}, x_{1}, x_{2}, u_{j}\right\}$ induces an $F_{5}$. So $j=2$.
Suppose that $u_{1}, u_{2}$ are in the same component of $Z$. So there is a chordless path in $Z$ between them, and since $Z$ contains no $P_{4}$, there is a vertex $z \in Z$ that sees both $u_{1}, u_{2}$. Then $z$ sees $x_{1}$, for otherwise $\left\{x_{1}, u_{2}\right.$, $\left.z, u_{1}\right\}$ induces a $P_{4}$, and similarly $z$ sees $y_{2}$. Then $v_{g}$ misses $z$, for otherwise $\left\{q, s_{1}, s_{2}, v_{g}, z, x_{1}, u_{2}\right\}$ induces an $F_{5}$. Then $v_{g}$ sees $u_{1}$, for otherwise $\left\{q, v_{g}, x_{1}, z, u_{1}\right\}$ induces an $F_{1}$, and similarly $v_{g}$ sees $y_{2}$. But then $\left\{q, s_{1}, s_{2}, v_{g}, u_{1}, y_{2}, z\right\}$ induces an $F_{5}$. Therefore $u_{1}$ and $u_{2}$ are in different components of $Z$.
Let $X^{\prime}$ be the component of $Z$ that contains $x_{1}$ and $u_{2}$, and let $Y^{\prime}$ be the component that contains $y_{2}$ and $u_{1}$. If both $X^{\prime}$ and $Y^{\prime}$ contain a $P_{3}$, then each of $H, X^{\prime}, Y^{\prime}$ contains a $P_{3}$, so $G$ contains an $F_{3}$, a contradiction. Thus one of $X^{\prime}, Y^{\prime}$ is a clique.
Since $x_{1}$ is a b-vertex, it has a neighbor $r$ of color $\ell+1$ (the color of $q$ ), and this neighbor is in $X^{\prime}$ because color $\ell+1$ does not appear in $T$. Note that if $v_{g}$ has a neighbor in $Y^{\prime}$, then it is $Y^{\prime}$-complete, for otherwise $\left\{u_{2}, x_{1}, v_{g}\right.$, $\left.z, z^{\prime}\right\}$ induces an $F_{1}$ for some adjacent vertices $z, z^{\prime} \in Y^{\prime}$.
Suppose that $Y^{\prime}$ is not a clique (and thus $X^{\prime}$ is a clique); so it contains a $P_{3}$ with vertices $y, y^{\prime}, y^{\prime \prime}$. Then $v_{g}$ sees at least one of $y, y^{\prime}, y^{\prime \prime}$, for otherwise $\left\{q, v_{g}, x_{1}, u_{2}, y, y^{\prime}, y^{\prime \prime}\right\}$ induces an $F_{2}$; and so $v_{g}$ is $Y^{\prime}$-complete. Now, $v_{g}$ misses $r$, for otherwise $\left\{s_{1}, q, s_{2}, v_{g}, x_{1}, r, u_{2}\right\}$ induces an $F_{5}$. But then $\left\{s_{1}, q, s_{2}, v_{g}, y, y^{\prime}, y^{\prime \prime}, x_{1}, u_{2}, r\right\}$ induces an $F_{9}$. Therefore $Y^{\prime}$ is a clique, and thus $v_{h} \in T$. It follows by symmetry that $X^{\prime}$ is also a clique, and if $v_{h}$
has a neighbor in $X^{\prime}$ then it is $X^{\prime}$-complete. Suppose that $v_{g}$ and $v_{h}$ are not adjacent. If $v_{h}$ has no neighbor in $X^{\prime}$, then $\left\{u_{2}, x_{1}, v_{g}, q, v_{h}\right\}$ induces an $F_{1}$. So $v_{h}$ is $X^{\prime}$-complete, and similarly $v_{g}$ is $Y^{\prime}$-complete. But then $\left\{u_{1}, u_{2}, v_{g}, v_{h}, q\right\}$ induces an $F_{1}$. Therefore $v_{g}$ and $v_{h}$ are adjacent. Suppose $v_{g}$ is $Y^{\prime}$-complete. If $v_{h}$ is $X^{\prime}$-anti-complete, then $\left\{q, v_{h}, y_{2}, u_{1}, v_{g}, x_{1}, u_{2}, r\right\}$ induces an $F_{8}$ (note that $v_{g}$ misses $r$, for otherwise $\left\{s_{1}, q, s_{2}, v_{g}, x_{1}, r, u_{2}\right\}$ induces an $F_{5}$.). If $v_{h}$ is $X^{\prime}$-complete, then $\left\{u_{2}, x_{1}, v_{g}, v_{h}, y_{2}, u_{1}\right\}$ induces a $F_{4}$. So $v_{g}$ has no neighbor in $X^{\prime}$, and similarly $v_{h}$ has no neighbor in $Y^{\prime}$. But then $\left\{u_{2}, x_{1}, v_{g}, v_{h}, y_{2}\right\}$ induces an $F_{1}$, a contradiction. Thus (9) holds.

By (91), we know that $Z$ contains a clique of b-vertices $b_{1}, \ldots, b_{\ell}$, which we call $D_{\ell}$. By (8) there is a vertex $u_{\ell+1}$ of color $\ell+1$ (the color of $q$ ) that is adjacent to all of $D_{\ell}$, and $u_{\ell+1}$ is in $Z$ because color $\ell+1$ does not appear in $T$. By (6), $q$ is the only b-vertex of color $\ell+1$, so there exists a color $m$ such that $u_{\ell+1}$ has no neighbor of color $m$. Clearly, $m>\ell$. By (8) there is a vertex $u_{m}$ of color $m$ that is adjacent to all of $D_{\ell}$. If $u_{m}$ is in $T$, then $\left\{q, s_{1}, s_{2}, u_{m}, b_{1}, b_{2}, u_{\ell+1}\right\}$ induces an $F_{5}$. So $u_{m}$ is in $Z$. If $|Q| \geq 2$, pick any vertex $q^{\prime} \in Q$ with $q^{\prime} \neq q$; then $\left\{q, q^{\prime}, s_{1}, s_{2}, b_{1}, b_{2}, u_{\ell+1}, u_{m}\right\}$ induces an $F_{6}$. So $Q=\{q\}$. Since $q$ has a neighbor of color $m$, and $m>\ell$, there is a vertex $t_{m}$ of color $m$ in $T$. Then $t_{m}$ misses one of $b_{1}, b_{2}$, for otherwise $\left\{q, s_{1}, s_{2}, t_{m}, b_{1}, b_{2}, u_{m}\right\}$ induces an $F_{5}$. Say $t_{m}$ misses $b_{1}$. Recall that $u_{\ell+1}$ has no neighbor of color $m$. Then $\left\{q, s_{1}, s_{2}, t_{m}, b_{1}, b_{2}, u_{\ell+1}, u_{m}\right\}$ induces an $F_{6}$ (if $t_{m}$ misses $b_{2}$ ) or $F_{7}$ (if $t_{m}$ sees $b_{2}$ ), a contradiction. This concludes the proof of Lemma 2.9.

## 3 Graphs that contain a $C_{5}$

The following lemma was proved in [9].
Lemma 3.1 ([9]) Let $G$ be an $\mathcal{F}$-free graph that contains a $C_{5}$. Then $V(G)$ can be partitioned into sets $X_{1}, \ldots, X_{6}, T, Z$ such that:

1. Each of $X_{1}, \ldots, X_{5}$ is not empty.
2. For every $j$ modulo $5, X_{j}$ is complete to $X_{j+1}$.
3. For every $j$ modulo 5 and $j \neq 4, X_{j}$ is anti-complete to $X_{j+2}$, and some vertex of $X_{1}$ misses a vertex of $X_{4}$.
4. $X_{6}$ is complete to $X_{2} \cup X_{3} \cup X_{5}$ and anti-complete to $X_{1} \cup X_{4}$.
5. $X_{2}, X_{3}, X_{5}$ are stable sets.
6. The sets $X_{1}^{\prime}=\left\{x \in X_{1} \mid x\right.$ has a non-neighbour in $\left.X_{4}\right\}$ and $X_{4}^{\prime}=$ $\left\{x \in X_{4} \mid x\right.$ has a non-neighbour in $\left.X_{1}\right\}$ are stable sets, and there is no edge between $X_{1}^{\prime}$ and $X_{1} \backslash X_{1}^{\prime}$ and no edge between $X_{4}^{\prime}$ and $X_{4} \backslash X_{4}^{\prime}$.
7. At least one of $X_{1} \backslash X_{1}^{\prime}, X_{4} \backslash X_{4}^{\prime}, X_{6}$ is empty.
8. Any two non-adjacent vertices of $X_{1}$ have inclusionwise comparable neighbourhoods in $V(G) \backslash X_{1}$, and the same holds for $X_{4}$ and $X_{6}$.
9. $T$ is complete to $X_{1} \cup \cdots \cup X_{6}$.
10. $Z$ is anti-complete to $X_{1}^{\prime} \cup X_{2} \cup X_{3} \cup X_{4}^{\prime} \cup X_{5}$; and if $X_{6} \neq \emptyset$, then $Z$ is anti-complete to $X_{1} \cup X_{2} \cup X_{3} \cup X_{4} \cup X_{5}$.
11. Every component of $Z$ is a clique and is a homogeneous set in $G \backslash T$.

Theorem 3.2 Let $G$ be a minimal counterexample to Conjecture 1. Then $G$ contains no $C_{5}$.

Proof. Suppose on the contrary that $G$ contains a $C_{5}$. So $\chi(G) \geq 3$ and consequently $b(G) \geq 4$. Let $c$ be a b-coloring of $G$ with $b(G)$ colors. For each $i \in\{1, \ldots, b(G)\}$, let $b_{i}$ be a b-vertex of color $i$. Since $G$ contains a $C_{5}$, it has a partition into sets $X_{1}, \ldots, X_{6}, T, Z$ with the notation and properties given in Lemma 3.1. For each $j \in\{2,3,5\}$ let $a_{j}$ be an arbitrary vertex in $X_{j}$, and let $a_{1}, a_{4}$ be non-adjacent vertices of $X_{1}$ and $X_{4}$ respectively. Such vertices exist by item 3 of Lemma 3.1. For each $j \in\{1,4\}$, let $X_{j}^{\prime \prime}=X_{j} \backslash X_{j}^{\prime}$. So every vertex of $X_{j}^{\prime \prime}$ is $X_{5-j}$-complete. Lemmas 3.1 and 2.3 imply easily the following two facts:

For each $j \in\{2,3,5\}$, any two vertices in $X_{j}$ are twins and have
different colors. If $X_{j}$ contains a b-vertex, then $\left|X_{j}\right|=1$.
For each $j \in\{1,4\}$, if $u, v \in X_{j}^{\prime}$, then either $N(u) \subseteq N(v)$ or $N(v) \subseteq N(u)$. If $u \in X_{j}^{\prime}$ and $v \in X_{j}^{\prime \prime}$, then $N(u) \subset N(v)$. If $X_{j}^{\prime}$ contains a b-vertex $u$, then $X_{j}^{\prime \prime}=\emptyset$ and any vertex $v$ of $X_{j}^{\prime} \backslash\{u\}$ satisfies $N(v) \subset N(u)$ and is not a b-vertex.
Let $X^{\prime}=X_{1}^{\prime} \cup X_{2} \cup X_{3} \cup X_{4}^{\prime} \cup X_{5}$. Note that $X^{\prime}$ contains $a_{1}, \ldots, a_{5}$, which induce a $C_{5}$; and that every vertex of $Z$ misses every vertex of $X^{\prime}$, by Lemma 3.1 .

If a component $Y$ of $Z$ has at least three vertices, then $Y$ is a homogeneous set and a clique.

Recall that $Y$ is a clique by item 11 of Lemma 3.1. Suppose that $Y$ is not homogeneous. Then, by item 11 again and by the definition of $Z$, there is a vertex $t \in T$ and vertices $y_{1}, y_{2} \in Y$ such that $t$ sees $y_{1}$ and misses $y_{2}$. Consider a vertex $y_{3} \in Y \backslash\left\{y_{1}, y_{2}\right\}$. If $t$ sees $y_{3}$, then $\left\{t, y_{1}, y_{2}, y_{3}, a_{1}, a_{2}, a_{3}\right\}$ induces an $F_{5}$; if $t$ misses $y_{3}$, then $\left\{t, y_{1}, y_{2}, y_{3}, a_{1}, a_{2}, a_{3}, a_{4}\right\}$ induces an $F_{8}$, a contradiction, so (12) holds.

If a component $Y$ of $Z$ contains a b-vertex, then $Y$ is a homogeneous set and a clique, $|Y| \geq 3$, every vertex of $Y$ is a b-vertex, and every color that appears in $X^{\prime}$ appears in $Y$.

Let $M=N(Y)$. By Lemma 3.1 we have $M \subseteq X_{1}^{\prime \prime} \cup X_{4}^{\prime \prime} \cup X_{6} \cup T$. Since $Y$ contains a b-vertex, all colors appear in $Y \cup M$.
Suppose that $M \cap X_{6} \neq \emptyset$. So there is an edge $y u_{6}$ with $y \in Y$ and $u_{6} \in X_{6}$. Then there is no edge $x_{1} x_{4}$ with $x_{1} \in X_{1}$ and $x_{4} \in X_{4}$, for otherwise, by item 10 of Lemma 3.1. $\left\{y, u_{6}, a_{2}, x_{1}, x_{4}\right\}$ induces an $F_{1}$. It follows that $X_{1}=X_{1}^{\prime}$ and $X_{4}=X_{4}^{\prime}$, and so these are stable sets. Now, by items 4 and 10 of Lemma [3.1, every vertex $x \in X_{1} \cup X_{4}$ satisfies $N(x) \subseteq N\left(x_{6}\right)$ for every $x_{6} \in X_{6}$, and so, by Lemma [2.3, the color of $x$ is different from all colors in $X_{6}$. Clearly the color of every vertex in $X_{2} \cup X_{3} \cup X_{5}$ is different from all colors in $X_{6}$; and the color of every vertex in $X^{\prime}$ is also different from all colors in $T$. Thus every color that appears in $X^{\prime}$ does not appear in $M$, and consequently it appears in $Y$. So $|Y| \geq 3$, since at least three colors appear in $X^{\prime}$ because $X^{\prime}$ contains a $C_{5}$. By (12), $Y$ is a homogeneous set and a clique, and so every vertex of $Y$ is a b-vertex. Thus in this case (13) holds. Now suppose that $M \cap\left(X_{1} \cup X_{4}\right) \neq \emptyset$. So, up to symmetry, there exist adjacent vertices $y \in Y$ and $x_{1} \in X_{1}$. By item 10 of Lemma 3.1, $x_{1}$ sees all of $X_{4}$ (that is, $x_{1} \in X_{1}^{\prime \prime}$ ), and $X_{6}$ is empty. Now every vertex $x$ in $X_{1}^{\prime} \cup X_{3}$ satisfies $N(x) \subseteq N\left(v_{1}\right)$ for every $v_{1} \in X_{1}^{\prime \prime}$, and so, by Lemma [2.3] the color of $x$ is different from all colors in $X_{1}^{\prime \prime}$. Clearly the color of every vertex in $X_{2} \cup X_{4} \cup X_{5}$ is different from all colors in $X_{1}^{\prime \prime}$; and the color of every vertex in $X^{\prime}$ is also different from all colors in $T$. Thus every color that appears in $X^{\prime}$ does not appear in $M$, and consequently it appears in $Y$. As above, this implies that $|Y| \geq 3, Y$ is a homogeneous set and a clique, and every vertex of $Y$ is a b-vertex. Thus in this case too (13) holds.

Finally suppose that $M \subset T$. Then every vertex of $M$ is adjacent to every vertex of $X^{\prime}$, so we conclude immediately as above. Thus (13) holds.

If some component $Y$ of $Z$ contains a b-vertex, then there does not exist vertices $a \in X_{3} \cup X_{4}, b_{h} \in T \cup X_{4}^{\prime \prime}$ such that $c\left(b_{h}\right)=h, a$ is not a b-vertex but $b_{h}$ is, and $f(a)=h$, that is, every b-vertex of color $h$ has $a$ as its unique neighbor of color $c(a)$.

Suppose $Y, a, b_{h}$ exist. Let $y, y^{\prime}$ be two vertices of $Y$. For simplicity, assume $c(a)=1$. So $Y$ satisfies the properties given in (13). By (13), we may assume that $b_{1} \in Y$. Let $M=N(Y)$. By Lemma 3.1 we have $M \subseteq$ $X_{1}^{\prime \prime} \cup X_{4}^{\prime \prime} \cup X_{6} \cup T$. By (13), the vertices of $Y$ must have a neighbor $u_{h}$ of color $h$, so $u_{h} \in M$. If $u_{h} \in X_{6}$, then $b_{h} \in X_{4}^{\prime \prime}$, and so $b_{1}, u_{h}, a_{3}, b_{h}, a_{1}$ induces an $F_{1}$, a contradiction. If $u_{h} \in X_{1}$, then it sees $b_{h}$, a contradiction. So $u_{h}$ is in $T \cup X_{4}^{\prime \prime}$. In other words, $b_{h}$ and $u_{h}$ are in the same co-component $K$ of $T \cup X_{4}^{\prime \prime}$ (with either $K \subset T$ or $K \subseteq X_{4}^{\prime \prime}$ ). Since $u_{h}$ is adjacent to $b_{1}$, it is not a b-vertex, and so it has no neighbor of color $i$ for some $i \neq h$. Since $u_{h}$ is $Y$-complete, color $i$ does not appear in $Y$. The vertices of $Y$ must have a neighbor $u_{i}$ of color $i$, so $u_{i} \in M$. If $u_{i} \in X_{6}$, then $u_{h}, b_{h} \in X_{4}^{\prime \prime}$, and so $b_{1}, u_{i}, a_{3}, b_{h}, a_{1}$ induce an $F_{1}$. It follows that $u_{i} \in K$. Then $b_{h}$ is not adjacent to $u_{i}$, for otherwise $\left\{a_{3}, a_{5}, b_{h}, u_{i}, u_{h}, y, y^{\prime}\right\}$ induces an $F_{11}$. Vertex $b_{h}$ must have a neighbor $v_{i}$ of color $i$. If $v_{i}$ is in $M$, then it can play the role of $u_{i}$ (in particular, $v_{i} \notin X_{1}$ and $v_{i}$ sees $x_{3}, x_{5}$ ) and we find an $F_{11}$ again. So $v_{i} \notin M$. If $v_{i}$ is in $Z$, then $\left\{v_{i}, b_{h}, a_{3}, u_{i}, y\right\}$ induces an $F_{1}$. So $v_{i} \notin Z$. If $v_{i}$ is in $X_{1} \cup X_{2} \cup X_{3} \cup X_{5}$, then it is either $K$-complete or $K$-anticomplete, which is impossible (as $v_{i}$ sees $b_{h}$ and misses $u_{i}$ ). So $v_{i}$ is in $T \cup X_{4}^{\prime \prime}$ (that is, in $K$ ). But then $\left\{a_{3}, a_{5}, b_{h}, u_{i}, u_{h}, v_{i}, y, y^{\prime}\right\}$ induces an $F_{13}$. We have established (14).

$$
\begin{equation*}
Z \text { contains no b-vertex. } \tag{15}
\end{equation*}
$$

For suppose that $Y$ is a component of $Z$ that contains a b-vertex. So $Y$ satisfies the properties given in (13). Let $M=N(Y)$. By Lemma 3.1 we have $M \subseteq X_{1}^{\prime \prime} \cup X_{4}^{\prime \prime} \cup X_{6} \cup T$. There must be an edge $x z$ with $x \in X_{1} \cup \cdots \cup X_{6}$ and $z \in Z$, for otherwise $X_{1} \cup \cdots \cup X_{6}$ is a homogeneous set, which contradicts Lemma 2.9. By item 10 of Lemma 3.1, $x$ is in $X_{1}^{\prime \prime} \cup X_{4}^{\prime \prime} \cup X_{6}$. Up to symmetry we distinguish two cases.
Case 1: $x \in X_{1}^{\prime \prime}$. Then item 10 of Lemma 3.1 implies $X_{6}=\emptyset$. So $M \subseteq$ $X_{1}^{\prime \prime} \cup X_{4}^{\prime \prime} \cup T$. Recall from item 9 of Lemma 3.1 and the definition of $X_{1}^{\prime \prime}, X_{4}^{\prime \prime}$
that the three sets $X_{1}^{\prime \prime}, X_{4}^{\prime \prime}$ and $T$ are complete to each other. Since $X_{6}=\emptyset$, we have $N\left(a_{3}\right) \subset N(x)$, so by Lemma [2.3, $a_{3}$ is not a b-vertex and we can apply Lemma 2.6. Let $c\left(a_{3}\right)=1$ and $f\left(a_{3}\right)=h>1$. By (13), we may assume that $b_{1} \in Y$. The definition of $f\left(a_{3}\right)$ means that every b-vertex of color $h$ is in $N\left(a_{3}\right)$ and not in $N\left(b_{1}\right)$. Recall that $N\left(a_{3}\right)=X_{2} \cup X_{4} \cup T$. Moreover $b_{h}$ is not in $X_{2} \cup X_{4}^{\prime}$, for otherwise, by (13) there would be a bvertex of color $h$ in $Y$, a contradiction. Thus $b_{h}$ is in $T \cup X_{4}^{\prime \prime}$. But, the existence of $a_{3}$ and $b_{h}$ is contradicted by (14).

Case 2: $x \in X_{6}$. Then item 10 of Lemma 3.1implies that there is no edge between $Z$ and $X_{1} \cup X_{4}$. So $M \subseteq X_{6} \cup T$. By item 9 of Lemma 3.1, $X_{6}$ and $T$ are complete to each other. If there is an edge between two vertices $x_{1} \in X_{1}$ and $x_{4} \in X_{4}$, then the five vertices $z, x, a_{3}, x_{4}, x_{1}$ induce an $F_{1}$. Item 6 of Lemma 3.1 implies $X_{4}$ is a stable set. Pick any $a \in X_{4}$; so $a$ has no neighbor in $X_{1}$. We have $N(a) \subset N(x)$, so by Lemma 2.3, $a$ is not a b-vertex and we can apply Lemma 2.6. Let $c(a)=1$ and $f(a)=h>1$. By (13), we may assume that $b_{1} \in Y$. The definition of $f(a)$ means that every b-vertex of color $h$ is in $N(a)$ and not in $N\left(b_{1}\right)$. Recall that $N(a)=X_{3} \cup X_{5} \cup T$. Moreover, $b_{h}$ is not in $X_{3} \cup X_{5}$, for otherwise, by (13) there would be a bvertex of color $h$ in $Y$, a contradiction. Thus, $b_{h}$ is in $T$. But, the existence of $a$ and $b_{h}$ is contradicted by (14).

$$
\begin{equation*}
X_{6}=\emptyset . \tag{16}
\end{equation*}
$$

For suppose the contrary. Pick any $x_{6} \in X_{6}$. By item 10 of Lemma 3.1, there is no edge between $Z$ and $X_{1} \cup \cdots \cup X_{5}$. By item 7, we may assume that $X_{1}^{\prime \prime}=\emptyset$; so $X_{1}$ is a stable set. Suppose that every vertex of $X_{4}^{\prime}$ has a neighbor in $X_{1}$. Consider a vertex $u$ of $X_{1}$ for which $N(u) \cap X_{4}^{\prime}$ is maximal; then, the first line of (11) implies $X_{4}^{\prime} \subset N(u)$, and so $u \in X_{1}^{\prime \prime}$, a contradiction. Therefore some vertex $a \in X_{4}$ has no neighbor in $X_{1}$. Let $c(a)=1$. We have $N(a) \subset N(x)$ for every $x \in X_{6} \cup X_{4} \backslash\{a\}$, so Lemma 2.3 implies that color 1 does not appear in $X_{6} \cup X_{4} \backslash\{a\}$ and that $a$ is not a b-vertex. By (15), we have $b_{1} \in X_{1} \cup X_{2}$. Let $f(a)=3$. So $b_{3}$ is in $N(a)$ and not in $N\left(b_{1}\right) \cup Z$. Note that $N(a)=X_{3} \cup X_{5} \cup T$.
Suppose that $b_{1} \in X_{1}$. So $b_{3} \in X_{3}$, and $X_{3}=\left\{b_{3}\right\}$. We may assume that $c\left(x_{6}\right)=2$. Vertex $b_{1}$ must have a neighbor $u_{2}$ of color 2 , and necessarily $u_{2} \in X_{4}$. Then there is no edge $x z$ with $x \in X_{6}$ and $z \in Z$, for otherwise $\left\{z, x, a_{2}, b_{1}, u_{2}\right\}$ induces an $F_{1}$. Thus, vertices of $X_{6}$ are not b-vertices as they cannot have any neighbor of color 1 . It follows that $b_{2}$ is in $X_{1} \cup X_{4}$; in fact it cannot be in $X_{1}$ by (11) ( $X_{1}^{\prime}$ cannot contain two b-vertices); so $b_{2}$ is in
$X_{4}$. Vertex $b_{1}$ must have a neighbor $u_{3}$ of color 3 , and necessarily $u_{3} \in X_{5}$. The definition of $f(a)$ implies that $u_{3}$ is not a b-vertex, so it has no neighbor of some color $h \neq 3$. So color $h$ does not appear in $X_{1} \cup X_{4} \cup X_{6} \cup T$, and $h \geq 4$. Vertex $b_{2}$ must have a neighbor $u_{h}$ of color $h$, and (since $\left|X_{3}\right|=1$ ) we have $u_{h} \in X_{5}$. But then $\left\{b_{1}, a_{2}, b_{3}, u_{2}, a, x_{6}, u_{3}, u_{h}\right\}$ induces an $F_{21}$, a contradiction.
Therefore $b_{1} \in X_{2}$, so $X_{2}=\left\{b_{1}\right\}$. Then (because $b_{3}$ sees $a$ and not $b_{1}$ ) $b_{3} \in X_{5}$, so $X_{5}=\left\{b_{3}\right\}$. Vertex $b_{1}$ must have a neighbor of color 3 , so we may assume that $c\left(a_{3}\right)=3$. The definition of $f(a)$ implies that $a_{3}$ is not a b-vertex, so we may assume that it misses color 4 . So color 4 does not appear in $X_{6} \cup T$, and we may assume $c\left(x_{6}\right)=2$. Vertex $b_{1}$ must have a neighbor $u_{4}$ of color 4 , and necessarily $u_{4} \in X_{1} \cup X_{3}$. If $u_{4} \in X_{3}$, then $\left|X_{3}\right| \geq 2$, so $u_{4}$ is not a b-vertex by (10), and $b_{4}$ can only be in $X_{1}$. Then $b_{4}$ must have a neighbor $u_{2}$ of color 2 , and $u_{2}$ can only be in $X_{4}$. But then $\left\{b_{1}, b_{3}, b_{4}, a, u_{2}, a_{3}, u_{4}, x_{6}\right\}$ induces an $F_{21}$, a contradiction. So color 4 does not appear in $X_{3}$, and consequently $u_{4} \in X_{1}$ and $b_{4} \in X_{1} \cup X_{4}$.
If $b_{4}$ is in $X_{4}$, then it has no neighbor of color 1 , a contradiction. So, $b_{4}$ is in $X_{1}$. Vertex $b_{4}$ must have a neighbor $u_{2}$ of color 2, and (because of $x_{6}$ ) $u_{2}$ can only be in $X_{4}$. Then there is no edge $x z$ with $x \in X_{6}$ and $z \in Z$, for otherwise $\left\{z, x, b_{1}, b_{4}, u_{2}\right\}$ induces an $F_{1}$. Thus, vertices of $X_{6}$ are not b-vertices as they have neither color 4 nor a neighbor of color 4 . It follows that $b_{2}$ is in $X_{1} \cup X_{4}$; in fact it cannot be in $X_{1}$ by (11) ( $X_{1}^{\prime}$ cannot contain two b-vertices); so $b_{2}$ is in $X_{4}$. Recall that the definition of $f(a)$ implies that $a$ is the only neighbor of color 1 of $a_{5}=b_{3}$. But then $b_{2}$ cannot have any neighbor of color 1 . Thus (16) holds.

$$
\begin{equation*}
X_{1}^{\prime \prime}=\emptyset \text { and } X_{4}^{\prime \prime}=\emptyset . \tag{17}
\end{equation*}
$$

For suppose on the contrary and up to symmetry that there is a vertex $x$ in $X_{1}^{\prime \prime}$. Let $c\left(a_{1}\right)=1$. Every vertex $u \in X_{1}^{\prime} \cup X_{3}$ satisfies $N(u) \subset N(x)$, so by Lemma 2.3, $X_{1}^{\prime} \cup X_{3}$ contains no b-vertex. In particuler $a_{1}$ is not a b-vertex. By (11) and Lemma 2.3, no vertex of $X_{1} \backslash\left\{a_{1}\right\}$ has color 1 . Thus, and by (15), $b_{1}$ is in $X_{4}$, and since it misses $a_{1}$ it is in $X_{4}^{\prime}$ and has no neighbor in $Z$. Since $a_{1}$ is not a b-vertex, color $f\left(a_{1}\right)$ exists, say $f\left(a_{1}\right)=2$. By (15) and the definition of $f\left(a_{1}\right)$, all b-vertices of color 2 must be in $X_{2} \cup X_{4}$. In fact if $b_{2}$ is in $X_{4}$, then by (11) we have $N\left(b_{1}\right) \subset N\left(b_{2}\right)$, which contradicts Lemma [2.3. So, $X_{4}$ contains no b-vertex of color 2, and, by (10), we have $X_{2}=\left\{b_{2}\right\}$. Clearly color 2 does not appear in $X_{3}$, and by the definition of $f\left(a_{1}\right)$, color 1 does not appear in $X_{3}$. Let $c\left(a_{5}\right)=h$. Then $h$ must appear in $X_{3}$, for otherwise $b_{2}$ cannot have any neighbor of color $h$. So we may
assume that $c\left(a_{5}\right)=c\left(a_{3}\right)=3$. Therefore $a_{3}$ and $a_{5}$ are the only vertices of color 3 in $V \backslash Z$, and since $X_{3}$ contains no b-vertices and by (15), we have $b_{3}=a_{5}$, and, by (10), $\left|X_{5}\right|=1$. But now $b_{1}$ cannot have any neighbor of color 2. Thus (17) holds.

$$
\begin{equation*}
Z=\emptyset \text { and } T=\emptyset . \tag{18}
\end{equation*}
$$

By item 10 of Lemma 3.1 and (17) there is no edge between $X_{1} \cup \cdots \cup X_{5}$ and $Z$. So if $Z$ or $T$ is not empty, then $X_{1} \cup \cdots \cup X_{5}$ is a homogeneous set that contradicts Lemma (2.9, Thus (18) holds.

By the preceding points, each $X_{i}(i=1, \ldots, 5)$ is a stable set and contains at most one b-vertex, and $T \cup Z \cup X_{6}=\emptyset$. So $b(G) \leq 5$. Moreover, if $a_{2}, a_{3}, a_{5}$ are b-vertices of three different colors, then, by (10), $a_{2}$ and $a_{3}$ cannot have a neighbor of color $c\left(a_{5}\right)$, a contradiction. It follows that $b(G)=4$ and that we may assume up to symmetry that $b_{1} \in X_{1}, b_{4} \in X_{4}$, and $X_{2}=\left\{b_{2}\right\}$. Vertex $b_{4}$ must have a neighbor $u_{2}$ of color 2 , and $u_{2}$ can only be in $X_{5}$. Then, by (10), $b_{3}$ must be in $X_{3}$, so $X_{3}=\left\{b_{3}\right\}$. Vertex $b_{1}$ must have a neighbor $u_{3}$ of color 3 , and necessarily, $u_{3} \in X_{5}$. Vertex $b_{2}$ must have a neighbor $u_{4}$ of color 4 , and $u_{4}$ can only be in $X_{1}$. Vertex $b_{3}$ must have a neighbor $u_{1}$ of color 1 , and $u_{1}$ can only be in $X_{4}$. If both $b_{1} b_{4}$ and $u_{1} u_{4}$ are edges, then $\left\{b_{1}, b_{2}, b_{4}, u_{1}, u_{4}\right\}$ induces an $F_{1}$. If exactly one is an edge, then $\left\{b_{1}, b_{2}, b_{3}, b_{4}, u_{1}, u_{2}, u_{3}, u_{4}\right\}$ induces an $F_{19}$. So both are non-edges. Then $b_{1}$ must have a neighbor $v_{4}$ of color 4 , which can only be in $X_{4}$; and $\left\{b_{1}, b_{2}, b_{3}, b_{4}, v_{4}, u_{2}, u_{3}, u_{4}\right\}$ induces an $F_{19}$ again. This completes the proof of Theorem 3.2,

## 4 Graphs that contain a boat

Let us call boat any graph whose vertex-set can be partitioned into sets $A_{0}, A_{1}, \ldots, A_{q}, B_{0}, B_{1}, \ldots, B_{q}$ that satisfy the following properties, where $A=A_{0} \cup A_{1} \cup \cdots \cup A_{q}$ and $B=B_{0} \cup B_{1} \cup \cdots \cup B_{q}$ :

- $q \geq 2$ and each of $A_{1}, \ldots, A_{q}, B_{1}, \ldots, B_{q}$ is not empty;
- If $q=2$ then also $A_{0}$ and $B_{0}$ are not empty;
- $A_{0}, A_{1}, \ldots, A_{q}$ are pairwise complete to each other, and $B_{0}, B_{1}, \ldots, B_{q}$ are pairwise complete to each other;
- For $j=1, \ldots, q, A_{j}$ is complete to $B_{j}$ and anticomplete to $B \backslash B_{j}$;
- $A_{0}$ is anticomplete to $B_{0}$.

Note that the smallest boats have six vertices: these are the boat with $q=2$ where each of $A_{0}, A_{1}, A_{2}, B_{0}, B_{1}, B_{2}$ has size one; and the boat with $q=3$ where each of $A_{1}, A_{2}, A_{3}, B_{1}, B_{2}, B_{3}$ has size one and $A_{0}=B_{0}=\emptyset$. We call these two graphs the small boats.

Lemma 4.1 Let $G$ be a graph that contains no $F_{1}, F_{4}, F_{10}$ or $C_{5}$. If $G$ contains a boat, then $V(G)$ can be partitioned into sets $M, T, Z$ such that the subgraph induced by $M$ is a boat and every vertex of $M$ is complete to $T$ and anticomplete to $Z$.

Proof. Since $G$ contains a boat, there is a set $M$ of vertices of $G$ that induces a boat and is maximal with this property. We use the same notation $A_{0}, A_{1}, \ldots, A_{q}, B_{0}, B_{1}, \ldots, B_{q}$ and properties as in the definition of a boat. Let then $T$ be the set of $M$-complete vertices and $Z$ be the set of $M$ anticomplete vertices. In order to prove the lemma, we need only establish that $V(G)=M \cup T \cup Z$. Assume the contrary. Let $x$ be a vertex of $G$ that is not in $M \cup T \cup Z$. Let us fix some notation. Let $I=\{i \mid 0 \leq i \leq$ $q$ and $\left.A_{i} \neq \emptyset\right\}$. So $I$ is equal to either $\{0,1, \ldots, q\}$ or $\{1, \ldots, q\}$. Likewise, let $J=\left\{j \mid 0 \leq i \leq q\right.$ and $\left.B_{j} \neq \emptyset\right\}$. Note that we have $|I \cap J| \geq 3$ (even when $q=2$ ) by the definition of a boat. For each $i \in I$, let $a_{i}$ be an arbitrary vertex in $A_{i}$ and $u_{i}$ be a neighbor of $x$ in $A_{i}$ (if any). Likewise, for each $j \in J$, let $b_{j}$ be an arbitrary vertex in $B_{j}$ and $v_{j}$ be a neighbor of $x$ in $B_{j}$ (if any). We claim that:

If there is a pair of integers $i \in I, j \in J, i \neq j$ such that $x$ has a neighbor in each of $A_{i}, B_{j}$, then $x$ is complete to either $A \backslash\left(A_{i} \cup A_{j}\right)$ or $B \backslash\left(B_{i} \cup B_{j}\right)$.

Note that vertices $u_{i}$ and $v_{j}$ exist. If $x$ has non-neighbors $u^{\prime} \in A \backslash\left(A_{i} \cup A_{j}\right)$ and $v^{\prime} \in B \backslash\left(B_{i} \cup B_{j}\right)$, then $\left\{u^{\prime}, u_{i}, x, v_{j}, v^{\prime}\right\}$ induces an $F_{1}$ or a $C_{5}$, a contradiction. Thus (19) holds.

There is no pair of integers $i, j \in I \cap J, i \neq j$ such that $x$ has a neighbor in each of $A_{i}, A_{j}, B_{i}, B_{j}$.
For suppose the contrary; so vertices $u_{i}, u_{j}, v_{i}, v_{j}$ exist. Since $i, j$ play symmetric roles, we may assume that $i \neq 0$, so $u_{i}$ and $v_{i}$ are adjacent. By (19)
and up to symmetry, we may assume that $x$ is complete to $A \backslash\left(A_{i} \cup A_{j}\right)$. Consider any $h \in(I \cap J) \backslash\{i, j\}$, which is not empty. We know that $x$ is complete to $A_{h}$, and $B_{h} \neq \emptyset$. Then $x$ is adjacent to $b_{h}$, for otherwise $\left\{a_{h}, u_{i}, x, v_{i}, v_{j}, b_{h}\right\}$ induces an $F_{4}($ if $h=0)$ or $F_{10}($ if $h \neq 0)$. So $x$ is complete to $B_{h}$. We can repeat this argument for every pair of integers from $I \cap J$, and it follows that $x$ is complete to $A_{\ell} \cup B_{\ell}$ for every $\ell \in I \cap J$, in particular for $1 \leq \ell \leq q$. Thus, if $0 \in I \cap J$, we obtain $x \in T$, a contradiction. So $0 \notin I \cap J$, say $B_{0}=\emptyset$, and so $q \geq 3$. Since $x$ is not in $T$, it has a non-neighbor $w$ in $A \cup B$, and it must be that $w \in A_{0}$. But then $\left\{w, a_{1}, a_{2}, x, b_{1}, b_{3}\right\}$ induces an $F_{4}$. Thus (20) holds.

There do not exist two pairs of integers $g, h \in I, g \neq h$ and $i, j \in J, i \neq j$ such that $x$ has a neighbor in each of $A_{g}, A_{h}, B_{i}, B_{j}$.

For suppose the contrary; so vertices $u_{g}, u_{h}, v_{i}, v_{j}$ exist. First suppose that $\{g, h\} \cap\{i, j\} \neq \emptyset$, say $g=i$. By (20) we may assume that $h \neq j$. If $g=i=0$, then $A_{j}$ and $B_{h}$ are not empty, and by (20), $x$ has no neighbor in those two sets; but then $\left\{a_{j}, u_{0}, x, v_{0}, b_{h}\right\}$ induces an $F_{1}$. So $g=i \neq 0$, and $u_{g}$ sees $v_{i}$. One of $h, j$ is not equal to 0 , say $j \neq 0$. So $A_{j} \neq \emptyset$ and, by (20), $x$ has no neighbor in that set; but then $\left\{a_{j}, u_{g}, u_{h}, x, v_{i}, v_{j}\right\}$ induces an $F_{10}$. Now we may assume that the four integers $g, h, i, j$ are different. We may assume that none of $h, i, j$ is equal to 0 ; so $A_{j}$ and $B_{h}$ are not empty. If $x$ has no neighbor in those two sets, then $\left\{a_{j}, u_{g}, x, v_{i}, b_{h}\right\}$ induces an $F_{1}$. Now, by (20), $x$ has a neighbor in $A_{j}$, or in $B_{h}$, but not in both. If $x$ has a neighbor in $B_{h}$, then $\left\{a_{j}, u_{h}, u_{g}, x, v_{h}, v_{j}\right\}$ induces an $F_{10}$. If $x$ has a neighbor in $A_{j}$, then $\left\{b_{h}, v_{j}, v_{i}, x, u_{h}, u_{j}\right\}$ induces an $F_{10}$. Thus, (21) holds.

There is no pair of integers $i \in I, j \in J, i \neq j$ such that $x$ has a neighbor in each of $A_{i}, B_{j}$.

For suppose the contrary; so vertices $u_{i}$ and $v_{j}$ exist. By (19) and up to symmetry, we may assume that $x$ is complete to $A \backslash\left(A_{i} \cup A_{j}\right)$. Thus $x$ has neighbors in at least two of the sets $A_{0}, \ldots, A_{q}$, and so, by (21), it has no neighbor in $B \backslash B_{j}$. Consider any $h \in I \cap J \backslash\{i, j\}$. So $A_{h}$ and $B_{h}$ are not empty, and $x$ is complete to $A_{h}$ and anticomplete to $B_{h}$. We claim that there is a non-neighbor $w$ of $x$ in $B \backslash\left(B_{j} \cup B_{h}\right)$; indeed, if $B_{i} \neq \emptyset$, then we can take any $w \in B_{i}$; and if $B_{i}=\emptyset$, then $i=0, q \geq 3$, so there is an integer $g \in(I \cap J) \backslash\{h, j\}$, and we can take any $w \in B_{g}$. Now we can
apply (19) to $h$ and $j$, and the existence of $w$ implies that $x$ is complete to $A \backslash\left(A_{j} \cup A_{h}\right)$. In summary, we have established that $x$ is complete to $A \backslash A_{j}$ and anti-complete to $B \backslash B_{j}$. If $x$ is complete to $B_{j}$, then we can add $x$ to $A_{j}$ and obtain a boat (with sets $A_{0}, \ldots, A_{j} \cup\{x\}, \ldots, A_{q}, B_{0}, \ldots, B_{q}$ ) that contradicts the maximality of $M$. Therefore $x$ has a non-neighbor $w_{j}$ in $B_{j}$. Suppose that $j \neq 0$; so $A_{j} \neq \emptyset$ and $a_{j}$ sees both $v_{j}, w_{j}$. Then $w_{j}$ sees $v_{j}$, for otherwise $\left\{w_{j}, b_{h}, v_{j}, x, a_{i}\right\}$ induces an $F_{1}$; and $x$ sees $a_{j}$, for otherwise $\left\{b_{h}, w_{j}, a_{j}, a_{i}, x\right\}$ induces an $F_{1}$; but then $\left\{a_{h}, a_{j}, x, v_{j}, w_{j}, b_{h}\right\}$ induces an $F_{4}$ (if $h=0$ ) or $F_{10}($ if $h \neq 0)$, a contradiction. So $j=0$. If $x$ has a non-neighbor $y \in A_{0}$, then $\left\{y, a_{i}, x, v_{0}, b_{h}\right\}$ induces an $F_{1}$. Thus $x$ is complete to $A_{0}$. Set $A_{q+1}=\{x\}, B_{q+1}=B_{0} \cap N(x)$ and $B_{0}^{\prime}=B_{0} \backslash N(x)$. Note that $v_{j} \in B_{q+1}$ and $w_{j} \in B_{0}^{\prime}$. Moreover, every $v \in B_{q+1}$ sees every $w \in B_{0}^{\prime}$, for otherwise $\left\{a_{i}, x, v, b_{h}, w\right\}$ induces an $F_{1}$. But now we find a larger boat, with sets $A_{0}, \ldots, A_{q}, A_{q+1}, B_{0}^{\prime}, B_{1}, \ldots, B_{q}, B_{q+1}$, which contradicts the maximality of $M$. Thus (22) holds.

Since $x \notin Z$, up to symmetry we may assume that $x$ has a neighbor $u_{h} \in A_{h}$ for some $h$ with $0 \leq h \leq q$. By (22), $x$ has no neighbor in $B \backslash B_{h}$. Consider any $i \in(I \cap J) \backslash\{h\}, i \neq 0$, and suppose that $x$ has a non-neighbor $w_{i}$ in $A_{i}$. Pick any $j \in J \backslash\{h, i\}$. Then $\left\{x, u_{h}, w_{i}, b_{i}, b_{j}\right\}$ induces an $F_{1}$, a contradiction. Thus $x$ is complete to $A_{i}$. By repeating this argument with $i$ instead of $h$, we obtain that $x$ is complete to $A \backslash A_{0}$, and by (22) it is anti-complete to $B$. But now we find a larger boat, with sets $A_{0} \cup\{x\}, A_{1}, \ldots, A_{q}, B_{0}, B_{1}, \ldots, B_{q}$, which contradicts the maximality of $M$. This completes the proof of Lemma 4.1,

Theorem 4.2 Let $G$ be a minimal counterexample to Conjecture 1. Then $G$ contains no boat.

Proof. By Theorem 3.2, $G$ contains no $C_{5}$. Suppose that $G$ contains a boat. Then, by Lemma4.1, the vertex-set of $G$ can be partitioned into sets $M, T, Z$ such that $M$ induces a boat and every vertex of $M$ is $T$-complete and $Z$ anticomplete. Then $T$ and $Z$ are empty, for otherwise $M$ is a homogeneous set that contradicts Lemma 2.9. Thus $V(G)=M$. We use the same notation $A_{0}, A_{1}, \ldots, A_{q}, B_{0}, B_{1}, \ldots, B_{q}$ and properties as in the definition of a boat. As in Lemma 4.1, let $I=\left\{i \mid 0 \leq i \leq q\right.$ and $\left.A_{i} \neq \emptyset\right\}, J=\{j \mid 0 \leq i \leq$ $q$ and $\left.B_{j} \neq \emptyset\right\}$, and note that $|I \cap J| \geq 3$. For each $i \in I$, let $a_{i}$ be an arbitrary vertex in $A_{i}$, and for each $j \in J$, let $b_{j}$ be an arbitrary vertex in $B_{j}$.

Let $c$ be a b-coloring of $G$ with $k=b(G)>\chi(G)$ colors. For each color $\ell \in\{1, \ldots, k\}$ let $d_{\ell}$ be a b-vertex of color $\ell$, and let $D=\left\{d_{1}, \ldots, d_{k}\right\}$.

By the definition of a boat each of $A_{0}, A_{1}, \ldots, A_{q}, B_{0}, B_{1}, \ldots, B_{q}$ is a homogeneous set, so it satisfies the properties described in Lemma [2.9, It follows (recall Lemma (2.3) that:

Each $A_{i}(i \in I)$ is a clique or a stable set. All vertices of $A$ have different colors. Any two non-adjacent vertices of $A$ are not b-vertices. The same holds for $B$.

We will now prove
If two of the sets $A_{1}, \ldots, A_{q}$ contain a member of $D$, then $A$ contains vertices of all colors. The same holds for $B$.

For suppose up to symmetry that $d_{1} \in A_{1}$ and $d_{2} \in A_{2}$. Consider any color $\ell$ that appears in $B$. If $\ell$ appears in $B \backslash B_{1}$, then it does not appear in $B_{1}$, and since $d_{1}$ must have a neighbor of color $\ell$, such a neighbor must be in $A$. If $\ell$ appears in $B_{1}$, then it does not appear in $B_{2}$, and since $d_{2}$ must have a neighbor of color $\ell$, such a neighbor must be in $A$. So we have established that all colors that appear in $B$ also appear in $A$; and so, all colors appear in $A$. Thus (24) holds.

If $A_{i}$ is not a clique for some $i \in\{1, \ldots, q\}$, then every $B_{j}$ with $j \in\{1, \ldots, q\} \backslash\{i\}$ is a stable set. The same holds with
$A$ and $B$ interchanged.
For suppose on the contrary, and up to symmetry, that there are two nonadjacent vertices $u, v$ in $A_{1}$ and two adjacent vertices $x, y$ in $B_{2}$. Let $h=3$ if $q \geq 3$ and $h=0$ if $q=2$. Then $\left\{a_{h}, u, v, a_{2}, b_{1}, x, y\right\}$ induces an $F_{11}$, a contradiction. Thus (25) holds.

For each $i \in\{0, \ldots, q\}$, one of $A_{i}, B_{i}$ is a clique.
For suppose on the contrary that there exist non-adjacent vertices $u, v \in A_{i}$ and non-adjacent vertices $x, y \in B_{i}$. Pick two integers $h, j$ in $(I \cap J) \backslash\{i\}$. Then $\left\{u, v, x, y, a_{h}, a_{j}, b_{h}, b_{j}\right\}$ induces an $F_{12}($ if $i=0)$ of $F_{14}$ (if $h=0$ or $j=0$ ) or $F_{15}$ (if $h, i, j>0$ ). Thus (26) holds.

One of $A, B$ is a clique.

For suppose the contrary. So one of the $A_{i}$ 's $(i \in I)$ is not a clique and one of the $B_{j}$ 's $(j \in J)$ is not a clique. By (26) and up to symmetry, we may assume that $A_{1}$ is not a clique and one of $B_{0}, B_{2}$ is not a clique. By (25) and (26), $B_{1}$ is a clique and each of $B_{2}, \ldots, B_{q}$ is a stable set. Note that this implies, by (23), that each of $B_{2}, \ldots, B_{q}$ contains at most one b-vertex (and if it contains one, then it has size one). Let $u, v$ be two non-adjacent vertices in $A_{1}$, and let $x, y$ be two non-adjacent vertices in $B_{0}$ or $B_{2}$. By (23), $u$ and $v$ are not b-vertices, and we may assume that they have color 1 and 2 . By (23) again, $d_{1}, d_{2}$ are in $B$, each of them is adjacent to all other vertices of $B$, and clearly they are not in $B_{1}$. Suppose that $B_{0}$ is not a clique, say $x, y$ are in $B_{0}$. If $q \geq 3$, then $\left\{u, v, a_{2}, a_{3}, b_{2}, b_{3}, x, y\right\}$ induces an $F_{12}$. So $q=2$, and so $A_{0} \neq \emptyset$. Note that $d_{1}, d_{2}$ are not both in $B_{2}$ (because $B_{2}$ can contain at most one b -vertex). So we may assume that $d_{1} \in B_{0}$. But $d_{1}$ is adjacent to $x$, a contradiction to (23). Therefore $B_{0}$ is a clique. So $x, y$ are in $B_{2}$, which restores the symmetry between $A$ and $B$. Thus $A_{2}$ is a clique, each of $A_{1}, A_{3}, \ldots, A_{q}$ is a stable set and contains at most one b-vertex, and $A_{0}$ is a clique. If $q \geq 4$, then $\left\{u, v, a_{3}, a_{4}, b_{3}, b_{4}, x, y\right\}$ induces an $F_{12}$. So $q \leq 3$. None of $d_{1}, d_{2}$ is in $B_{2}$ (because $B_{2}$ is now a stable set of size at least two, so it does not contain any b-vertex), so they are in $B_{0} \cup B_{3}$. Since $B_{3}$ is a stable set, it contains at most one of $d_{1}, d_{2}$, and so at least one of these is in $B_{0}$, say $d_{2} \in B_{0}$. By symmetry, we may assume that $x, y$ have color 3 and 4 respectively, vertices $d_{3}, d_{4}$ are in $A_{0} \cup A_{3}$, vertex $d_{4}$ is in $A_{0}$, and each of $d_{3}, d_{4}$ is adjacent to all other vertices of $A$. But then $\left\{u, v, d_{3}, d_{4}, x, y, d_{1}, d_{2}\right\}$ induces an $F_{7}$ (if $d_{3} \in A_{3}$ and $d_{1} \in B_{3}$ ) or an $F_{6}$ (else). Thus (27) holds.

Both $A, B$ are cliques.
For suppose the contrary. By (27), we may assume that $A$ is not a clique and $B$ is a clique. The vertices of $B_{0}$ are simplicial, so they are not b-vertices by Lemma 2.2. Moreover, if two of the sets $B_{1}, \ldots, B_{q}$ contain a b-vertex, then, by (24), $B$ contains vertices of all colors, and so $G$ has a clique of size $k$, a contradiction. Therefore we may assume up to symmetry that $B \cap D \subseteq B_{1}$. Let $u, v$ be two non-adjacent vertices in $A$. By (23), $u, v$ are not b -vertices and we may assume that they have color 1 and 2 . Then vertices $d_{1}, d_{2}$ are not in $A$, so they are in $B_{1}$. If $u, v \in A_{i}, i \neq 0, i \neq 1$, then by (24), $B_{1}$ is a stable set of size at least two, a contradiction to the assumption that $B$ is a clique. So, $u, v$ are not in $A \backslash A_{0}$, and this argument shows that $A \backslash A_{0}$ is a clique. Thus $u, v$ are in $A_{0}$. Now $A_{1} \cup B_{1}$ is a clique, so we may assume that it does not contain any vertex of color 3 . Since $d_{1}, d_{2}$ are b-vertices, they must have a neighbor $x_{3}$ of color 3 , which must be in $B \backslash B_{1}$. Vertex
$x_{3}$ is not a b-vertex (because $B \cap D \subseteq B_{1}$ ), so $d_{3}$ is in $A$, and $d_{3} \notin A_{1}$ by the choice of color 3. By (23), $d_{3}$ is adjacent to all of $A \backslash\left\{d_{3}\right\}$. But then $\left\{u, v, d_{3}, a_{1}, d_{1}, d_{2}, x_{3}\right\}$ induces an $F_{5}$. Thus (28) holds.

Since $A$ is a clique, there is a color, say color 1 , that does not appear in $A$. So $d_{1}$ is in $B$ and is the unique vertex of color 1 . Likewise, there is a color, say color 2 , that does not appear in $B$, and so $d_{2}$ is in $A$ and is the unique vertex of color 2 . Since $d_{1}$ must have a neighbor of color 2 , vertices $d_{1}, d_{2}$ are adjacent, and so we may assume that $d_{1} \in B_{1}$ and $d_{2} \in A_{1}$. By (24) we have $A \cap D \subseteq A_{1}$ and $B \cap D \subseteq B_{1}$. But then $A_{1} \cup B_{1}$ is a clique that contains vertices of all colors, a contradiction. This concludes the proof of the theorem.

## 5 Proof of the main result

Suppose that Theorem 1.1 fails. So there is a minimal counterexample $G$ to Conjecture 1. By Theorems 3.2 and 4.2, $G$ contains no $C_{5}$ and no boat. Since $G$ contains no $C_{5}$, no $F_{1}\left(=P_{5}\right)$ and no $F_{10}\left(=\bar{P}_{6}\right)$, it contains no odd hole and no odd antihole, so it is perfect [2], that is, $G$ satisfies $\chi(H)=\omega(H)$ for every induced subgraph $H$ of $G$. (Actually, since $G$ also contains no $\bar{C}_{6}$ (which is a boat), it is weakly chordal (i.e., it contains no hole and no antihole of length at least five), and reference [6] implies the perfectness of $G$ more simply than [2].)

Let $c$ be a b-coloring of $G$ with $k=b(G)>\chi(G)$ colors. For each color $i \in\{1, \ldots, k\}$, let $d_{i}$ be a b-vertex of color $i$, and let $D=\left\{d_{1}, \ldots, d_{k}\right\}$. Note that $G$ contains no clique of size $k$, for otherwise we would have $k \leq \omega(G) \leq$ $\chi(G)<b(G)=k$, which is impossible. In particular $D$ is not a clique.

We observe that:

$$
\begin{equation*}
G \text { contains a } 2 K_{2} . \tag{29}
\end{equation*}
$$

For suppose that $G$ contains no $2 K_{2}$. Since $D$ is not a clique, we may assume without loss of generality that $d_{1}, d_{2}$ are not adjacent. Since $d_{1}$ is a b-vertex, it has a neighbor $x_{2}$ of color 2. Since $d_{2}$ is a b-vertex, it has a neighbor $x_{1}$ of color 1 . Then $x_{1}, x_{2}$ are adjacent, for otherwise $\left\{d_{1}, d_{2}, x_{1}, x_{2}\right\}$ induces a $2 K_{2}$. Since $d_{1}$ is a b-vertex, by Lemma 2.3 we cannot have $N\left(d_{1}\right) \subseteq N\left(x_{1}\right)$; so there exists a vertex $u$ that is adjacent to $d_{1}$ and not to $x_{1}$. Likewise, there exists a vertex $v$ that is adjacent to $d_{2}$ and not to $x_{2}$. Then $u$ is adjacent to $d_{2}$, for otherwise $\left\{u, d_{1}, d_{2}, x_{1}\right\}$ induces a $2 K_{2}$; and $u$ is adjacent to $x_{2}$, for
otherwise $\left\{u, d_{1}, d_{2}, x_{1}, x_{2}\right\}$ induces a $C_{5}$. Likewise, $v$ is adjacent to $d_{1}$ and $x_{1}$. But now $\left\{u, v, d_{1}, d_{2}, x_{1}, x_{2}\right\}$ induces a boat (if $u, v$ are not adjacent) or an $F_{10}$ (if $u, v$ are adjacent), a contradiction. Thus (29) holds.

Since $G$ contains a $2 K_{2}$, there is a subset $B$ of $V(G)$ such that the subgraph induced by $B$ has at least two components, each component of $B$ has at least two vertices, and $B$ is maximal with this property. Let $B_{1}, \ldots, B_{r}$ be the components of $B$, with $r \geq 2$. Let $S$ be the set of vertices of $V \backslash B$ that are $B$-anticomplete. Note that $S$ is a stable set, for otherwise two adjacent vertices of $S$ could be added to $B$, which would contradict the maximality of $B$. Let $A=V \backslash(B \cup S)$. We claim that:
$r=2$ and there is a component of $B$, say $B_{2}$, such that $B_{2}$ is a clique and every vertex of $A$ is $B_{2}$-complete and has a neighbor in $B_{1}$.
Consider any vertex $a \in A$. By the definition of $S, a$ has a neighbor in $B$. If $a$ has no neighbor in some component $B_{j}$ of $B$, then $B \cup\{a\}$ contradicts the maximality of $B$ (as every component of $B \cup\{a\}$ has size at least two and $B$ has at least two components, one that contains $a$ and one that includes $B_{j}$ ). So $a$ has a neighbor in each component of $B$. If every vertex of $A$ is $B$-complete, then $B$ is a homogeneous set, which contradicts Lemma 2.9, So there is a vertex $a_{0}$ of $A$ that is not $B$-complete, say $a_{0}$ has a non-neighbor in $B_{1}$. Since $B_{1}$ is connected, there are adjacent vertices $u_{1}, v_{1}$ in $B_{1}$ such that $a_{0}$ is adjacent to $u_{1}$ and not to $v_{1}$. If $a_{0}$ also has a non-neighbor in another component $B_{i}$ of $B(i \neq 1)$, then there are adjacent vertices $u_{i}, v_{i}$ in $B_{i}$ such that $a_{0}$ is adjacent to $u_{i}$ and not to $v_{i}$, and $\left\{a_{0}, u_{1}, v_{1}, u_{i}, v_{i}\right\}$ induces an $F_{1}$. Therefore only component $B_{1}$ of $B$ contains a non-neighbor of $a_{0}$, that is, $a_{0}$ is $B \backslash B_{1}$-complete. Consider any other vertex $a_{0}^{\prime}$ in $A$ that is not $B$-complete. Just like for $a_{0}$, there is a component $B_{i}$ and adjacent vertices $u_{i}, v_{i}$ in $B_{i}$ such that $a_{0}^{\prime}$ is adjacent to $u_{i}$ and not to $v_{i}$ and $a_{0}^{\prime}$ is $B \backslash B_{i}$-complete. If $i \neq 1$, then $\left\{a_{0}, a_{0}^{\prime}, u_{1}, v_{1}, u_{i}, v_{i}\right\}$ induces an $F_{4}$ or a boat, a contradiction. So $i=1$ for each $a_{0}^{\prime}$. This implies that every vertex of $A$ is $B \backslash B_{1}$-complete, so $B \backslash B_{1}$ is homogeneous, and Lemma 2.9 implies that $r=2$ and $B_{2}$ is a clique. Thus (30) holds.

For each $a \in A$ and each component $C$ of $B_{1} \backslash N(a)$, every vertex in $B_{1} \backslash C$ is either $C$-complete or $C$-anticomplete. Furthermore, there is a vertex in $B_{1} \backslash C$ that is $(C \cup\{a\})$-complete.
Suppose that some vertex $z$ in $B_{1} \backslash C$ has a neighbour and a non-neighbour in $C$. Then $z$ is adjacent to $a$ by the definition of $C$. Since $C$ is connected,
there are adjacent vertices $y, y^{\prime} \in C$ such that $z$ sees $y$ and misses $y^{\prime}$. Let $t$ be any vertex in $B_{2}$. Then $\left\{t, a, z, y, y^{\prime}\right\}$ induces an $F_{1}$, a contradiction. Furthermore, since $B_{1}$ is connected, some vertex in $B_{1} \cap N(a)$ must have a neighbour in $C$, and so is $(C \cup\{a\})$-complete. Thus (31) holds.

For every $a \in A$, each component of $B_{1} \backslash N(a)$ is a homogeneous set and a clique.

Let $C$ be any component of $B_{1} \backslash N(a)$. Suppose that $C$ is not homogeneous. So there are adjacent vertices $x, y$ in $C$ and a vertex $u$ not in $C$ that sees $x$ and misses $y$. By (31), we have $u \notin B_{1}$ and so $u \in A$. Let $t$ be any vertex in $B_{2}$. Then $u$ sees $a$, for otherwise $\{a, t, u, x, y\}$ induces an $F_{1}$. By (31), there is a vertex $z$ in $B_{1} \cap N(a)$ that is $(C \cup\{a\})$-complete. But now $\{a, t, u, x, y, z\}$ induces a boat or an $F_{4}$, a contradiction. So $C$ is homogeneous. Then Lemma 2.9 implies that $C$ is a clique. Thus (32) holds.

$$
\begin{equation*}
S \text { contains no b-vertex. } \tag{33}
\end{equation*}
$$

Indeed, if $x$ is any vertex in $S$ and $t$ is any vertex in $B_{2}$, then $N(x) \subset N(t)$ and Lemma 2.3 implies that $x$ cannot be a b-vertex. Thus (33) holds.

Let $G^{\prime}$ be the graph obtained from $G$ by removing all edges whose two endvertices are in $B_{2}$.

$$
\begin{equation*}
G^{\prime} \text { does not contain any } C_{5} \text {, boat, or member of } \mathcal{F} \text {. } \tag{34}
\end{equation*}
$$

For suppose that $G^{\prime}$ has an induced subgraph $F$ that is either a $C_{5}$, a boat, or a member of $\mathcal{F}$. If $F$ is a boat, we may assume that it is a small boat, since every boat contains a small boat. If $F$ contains at most one vertex of $B_{2}$, then $F$ is an induced subgraph of $G$, a contradiction. So $F$ must contain at least two vertices of $B_{2}$. Then these vertices are pairwise non-adjacent twins in $F$, which implies that $F$ is not a $C_{5}$ or a small boat (since such graphs do not have any pair of non-adjacent twins); more precisely $F$ is one of $F_{2}, F_{3}, F_{5}, F_{6}, F_{7}, F_{9}, F_{11}, F_{12}, F_{13}, F_{14}, F_{15}, F_{19}, F_{21}$ and it has exactly two vertices of $B_{2}$. In fact $F$ cannot be any of $F_{19}, F_{21}$, for that would imply that $G^{\prime}$ contains a $C_{5}$, which we have already excluded. Likewise, $F$ cannot be any of $F_{12}, F_{14}, F_{15}$, since that would imply that $G^{\prime}$ contains a boat, which is also excluded. Therefore $F$ is one $F_{2}, F_{3}, F_{5}, F_{6}, F_{7}, F_{9}, F_{11}, F_{13}$.
Suppose that $F$ is either $F_{2}$ or $F_{3}$. So $F$ has vertices $x, y, a, z_{1}, \ldots, z_{p}$, with $x, y \in B_{2}$, and edges $x a, y a$, and either (if $F$ is $\left.F_{2}\right) p=4$ and $\left\{z_{1}, \ldots, z_{4}\right\}$ induces a $P_{4}$, or (if $F$ is $F_{3}$ ) $p=6$ and $\left\{z_{1}, \ldots, z_{6}\right\}$ induces a $2 P_{3}$. Since $x, y$
are in $B_{2}$, vertices $z_{1}, \ldots, z_{p}$ must be in $B_{1}$ and $a$ must be in $A$. But then $a, z_{1}, \ldots, z_{p}$ contradict Claim (32).
Suppose that $F$ is either $F_{5}$ or $F_{9}$. So $F$ has vertices $x, y, a, b, z_{1}, \ldots, z_{p}$, with $x, y \in B_{2}$, and edges $x a, x b, y a, y b, a b, a z_{1}, z_{1} z_{2}, z_{1} z_{3}, z_{2} z_{3}$ and either (if $F$ is $F_{5}$ ) $p=3$ and $a z_{2}$ is an edge, or (if $F$ is $F_{9}$ ) $p=6$ and vertices $z_{4}, z_{5}, z_{6}$ induce a $P_{3}$ and are adjacent to $a$. Since $x, y$ are in $B_{2}$, vertices $z_{1}, \ldots, z_{p}$ must be in $B_{1}$ and $a, b$ must be in $A$. But then $b, z_{1}, z_{2}, z_{3}$ contradict Claim (32).

Suppose that $F$ is either $F_{6}$ or $F_{7}$. So $F$ has vertices $x, y, a, b, z_{1}, \ldots, z_{4}$, with $x, y \in B_{2}$, and edges $x a, x b, y a, y b, a b, z_{1} z_{2}, z_{1} z_{3}, z_{1} z_{4}, z_{2} z_{3}, z_{2} z_{4}$ and (if $F$ is $F_{7}$ ) the edge $a z_{1}$. Since $x, y$ are in $B_{2}$, vertices $z_{1}, \ldots, z_{4}$ must be in $B_{1}$ and $a, b$ must be in $A$. But then $b, z_{2}, z_{3}, z_{4}$ contradict Claim (32).
Suppose that $F$ is $F_{11}$. So $F$ has vertices $x, y, a, b, z_{1}, z_{2}, w$, with $x, y \in B_{2}$, and edges $x a, x b, y a, y b, a b, a z_{1}, a z_{2}, z_{1} z_{2}, z_{1} w, z_{2} w, x w, y w$. Since $x, y$ are in $B_{2}$, vertices $z_{1}, z_{2}$ must be in $B_{1}$ and $a, b, w$ must be in $A$. By Claim (31), there is a vertex $z$ in $B_{1}$ that is adjacent to $b, z_{1}, z_{2}$. Then $z$ sees $w$, for otherwise $\left\{z, b, x, w, z_{1}\right\}$ induces a $C_{5}$. But then $\left\{a, b, x, w, z, z_{1}\right\}$ induces a small boat $\left(\bar{C}_{6}\right)$ or $F_{10}$ in $G$, a contradiction.
Finally suppose that $F$ is an $F_{13}$. So $F$ has vertices $x, y, a, b, u, v, z_{1}, z_{2}$, with $x, y \in B_{2}$, and edges $a b, a x, a y, b x, b y, x u, x v, y u, y v, u z_{1}, u z_{2}, v z_{1}, v z_{2}, z_{1} z_{2}$. Since $x, y$ are in $B_{2}$, vertices $z_{1}, z_{2}$ must be in $B_{1}$ and $a, b, u, v$ must be in A. By Claim (31), there is a vertex $z$ in $B_{1}$ that is adjacent to $b, z_{1}, z_{2}$. Vertex $z$ sees $u$, for otherwise $\left\{z, b, x, u, z_{1}\right\}$ induces a $C_{5}$. Similarly $z$ sees $v$. Then $z$ sees $a$, for otherwise $\left\{a, b, x, u, z, z_{1}\right\}$ induces a boat. But then $\left\{a, b, x, z, u, v, z_{1}\right\}$ induces an $F_{11}$ in $G$, a contradiction. Thus (34) holds.

Some vertex $d$ in $B_{2}$ is the unique b-vertex of $G$ of color $c(d)$.
Suppose for each b-vertex $d$ in $B_{2}$, there is a b-vertex $d^{\prime}$ of the same color. Then $d^{\prime} \notin B_{2}$ ( $B_{2}$ is a clique); so, $c$ is a b-coloring of $G^{\prime}$ with $k$ colors, and $G^{\prime}$ is a smaller counterexample than $G$, a contradiction. Thus, (35) holds.

Since the vertices of $B_{2}$ are pairwise adjacent twins, (35) implies that they are all b-vertices, and we may assume that $B_{2}=\left\{d_{1}, \ldots, d_{\ell}\right\}$, with $\ell \geq 2$.

Next, we will prove
$B_{1}$ contains a vertex $x_{i}$ of color $i, 1 \leq i \leq \ell$, that is not a b-vertex of $G$.

Let $S_{1}, \ldots, S_{k}$ be the color classes of the b-coloring $c$, with $d_{i} \in S_{i}$ for
each $i=1, \ldots, \ell$. We have $k=b(G)>\chi(G)=\omega(G)$ since $G$ is perfect. So $k-1 \geq \omega(G)$. Note that $S_{2}, \ldots, S_{k}$ form a b-coloring of $G \backslash S_{1}$; so $b\left(G \backslash S_{1}\right) \geq k-1$. Since $G$ is minimally b-imperfect, we have $b\left(G \backslash S_{1}\right)=$ $\chi\left(G \backslash S_{1}\right)=\omega\left(G \backslash S_{1}\right)$. Combining the above inequalities, we get $k-1 \geq$ $\omega(G) \geq \omega\left(G \backslash S_{1}\right)=\chi\left(G \backslash S_{1}\right)=b\left(G \backslash S_{1}\right) \geq k-1$. So, equality must hold throughout, in particular we have $\omega\left(G \backslash S_{1}\right)=\omega(G)=k-1$. So $G \backslash d_{1}$ contains a clique $K$ of size $\omega(G)=k-1$. If $K$ contains a vertex $x$ of $S$, then $(K \backslash x) \cup\left\{d_{1}, d_{2}\right\}$ is a clique of size $k$ in $G$, a contradiction. If $K$ contains no vertex of $B_{1}$, then we have $K \subseteq A \cup B_{2} \backslash d_{1}$, and then $K \cup\left\{d_{1}\right\}$ is a clique of size $k$ in $G$, again a contradiction. So $K$ contains a vertex of $B_{1}$ and $K \subseteq B_{1} \cup A$. Then we have $|K \cap A| \leq k-1-\ell$, for otherwise $(K \cap A) \cup B_{2}$ would be a clique of size at least $k$. Consequently, $K \cap B_{1}$ has size at least $\ell$ and at least $\ell-1$ of the colors $1, \ldots, \ell$ appear in $K \cap B_{1}$. So we may assume up to symmetry that $B_{1}$ contains a vertex $x_{i}$ of color $i, 1 \leq i \leq \ell$. If $x_{i}$ is not a b-vertex, then we are done. Suppose $x_{i}$ is a b-vertex. So, it must have neighbors $x_{j}$ of color $j$, for all $j \in\{1, \ldots, \ell\} \backslash\{i\}$. The vertices $x_{j}$ are in $B_{1}$ necessarily. By (35), some such $x_{j}$ is not a b-vertex. So, (36) holds.

For simplicity, let $x_{1}$ be the vertex of color 1 in $B_{1}$ that is not a bvertex. There must be a color $m$ such that $x_{1}$ is the only neighbor of color 1 of every b-vertex of color $m$. Thus $d_{m}$ is not adjacent to $d_{1}$, so it is not in $A$; therefore, it is in $B_{1}$. Moreover $m>\ell$. Vertices $d_{1}, \ldots, d_{\ell}$ must have a neighbor $u_{m}$ of color $m$, and clearly $u_{m}$ is in $A$. Moreover $u_{m}$ is not a b-vertex (because it is adjacent to $d_{1}$ and by the property of $x_{1}$ ), so there is a color $n \neq 1$ such that $u_{m}$ has no neighbor of color $n$. Thus $n>\ell$. Vertices $d_{1}, \ldots, d_{\ell}$ must have a neighbor $u_{n}$ of color $n$, and clearly $u_{n}$ is in $A$. Let $C$ be the component of $B_{1} \backslash N\left(u_{m}\right)$ that contains $d_{m}$. By (32), $C$ is a homogeneous set and a clique.

$$
\begin{equation*}
N\left(d_{m}\right) \cap B_{1} \text { is a clique. } \tag{37}
\end{equation*}
$$

Suppose on the contrary that $d_{m}$ has two neighbors $x, y$ in $B_{1}$ that are not adjacent. Since $C$ is a homogeneous set and a clique, vertices $x, y$ are in $B_{1} \backslash C$ and so they are both adjacent to $u_{m}$. Then $u_{n}$ sees $x$, for otherwise $\left\{u_{n}, d_{1}, u_{m}, x, d_{m}\right\}$ induces an $F_{1}$ or $C_{5}$. Likewise $u_{n}$ sees $y$. Then $u_{n}$ misses $d_{m}$, for otherwise $\left\{d_{1}, d_{2}, u_{n}, u_{m}, d_{m}, x, y\right\}$ induces an $F_{11}$. Since this holds for every vertex $u_{n}$ of color $n$ in $A$, and $d_{m}$ must have a neighbor $z_{n}$ of color $n$, it must be that such a vertex $z_{n}$ is in $B_{1}$. Then $z_{n}$ sees $x$ since $C$ is a homogeneous set. Likewise $z_{n}$ sees $y$. But then $\left\{d_{m}, z_{n}, x, y, u_{n}, u_{m}, d_{1}, d_{2}\right\}$ induces an $F_{13}$. So Claim (37) holds.

Every neighbor of $d_{m}$ in $A$ is adjacent to all of $N\left(d_{m}\right) \cap B_{1}$.
For suppose that some neighbor $a$ of $d_{m}$ in $A$ is not adjacent to some vertex $y$ in $N\left(d_{m}\right) \cap B_{1}$. Vertex $y$ is not in $C$ since $C$ is homogeneous. So $y$ is adjacent to $u_{m}$. Then $a$ sees $u_{m}$, for otherwise $\left\{d_{1}, a, u_{m}, d_{m}, y\right\}$ induces a $C_{5}$. Thus $a \neq u_{n}$. Then $u_{n}$ sees $y$, for otherwise $\left\{u_{n}, d_{1}, u_{m}, y, d_{m}\right\}$ induces an $F_{1}$ or $C_{5}$. Then $u_{n}$ misses $d_{m}$, for otherwise $\left\{d_{1}, u_{m}, u_{n}, d_{m}, y, a\right\}$ induces a boat $\left(\bar{C}_{6}\right)$ or an $F_{10}$. Since this holds for every vertex $u_{n}$ of color $n$ in $A$, and $d_{m}$ must have a neighbor $z_{n}$ of color $n$, it must be that such a vertex $z_{n}$ is in $B_{1}$. Since $u_{m}$ has no neighbor of color $n$, we have $z_{n} \in C$. Since $C$ is homogeneous, $a$ sees $z_{n}$. Note that $a$ sees $u_{n}$, for otherwise $\left\{d_{1}, a, u_{n}, d_{m}, y\right\}$ induces a $C_{5}$. Recall that $z_{n}$ sees $y$ by (37) and misses $u_{m}$ by the definition of color $n$. But then $\left\{d_{1}, a, u_{m}, u_{n}, d_{m}, y, z_{n}\right\}$ induces an $F_{11}$, a contradiction. So (38) holds.

Vertex $d_{m}$ must have a neighbor $z_{i}$ of color $i$ for each $i \in\{1, \ldots, \ell\}$, and clearly $z_{i}$ is in $B_{1}$ since it is not adjacent to $d_{i}$. By (37) and (38), every neighbor of $d_{m}$ (other than $z_{i}$ ) is adjacent to $z_{i}$. It follows that $z_{1}, \ldots, z_{\ell}$ are b-vertices, a contradiction to (35). This completes the proof of the main theorem.

## 6 Optimizing b-perfect graphs

In this section we describe polynomial time algorithms to find an optimal coloring and a largest clique of a b-perfect graph.

Suppose a graph $G$ is assigned an arbitrary coloring. We want to find a way to reduce the number of colors used to hopefully obtain a better coloring of $G$. The notion of b-vertices can be used for this purpose. If there is a color $c$ with no b-vertex then we can eliminate $c$ from our coloring as follows. For each vertex $x$ of color $c$, give $x$ a color that is missing in the neighborhood of $x$. We may repeat this process until every color has a bvertex, thus obtaining a b-coloring of $G$. We will call the above algorithm the $b$-greedy (coloring) algorithm. It is easy to see that the b-greedy algorithm can be implemented in polynomial time. If $G$ is a b-perfect graph, then the b-greedy algorithm will deliver an optimal coloring since $b(G)=\chi(G)$.

Our notion of b-perfect graph is thus analogous to Chvátal's notion of perfectly orderable graph [3]. On a perfectly ordered graph, the greedy
algorithm delivers an optimal coloring. The recognition of perfectly orderable graphs is NP-complete [15]; in comparison, the recognition of b-perfect graphs can be done in polynomial time, since our main result above is that bperfect graphs are characterized by forbidding as induced subgraphs twentytwo graphs, which have at most eight vertices.

Now, we consider the problem of finding a largest clique of a b-perfect graph. First, we need establish some preliminary results.

Lemma 6.1 Let $G$ be a b-perfect graph that contains a $C_{5}$. Then either (i) $G$ has two non-adjacent comparable vertices, or
(ii) $X_{6}=\emptyset$, and $\left|X_{i}\right|=1$ for $i=1,2,3,4,5$ (with the notation of Lemma (3.1).

Proof of Lemma 6.1. We know that $G$ has the structure described in Lemma [3.1, and we use the same notation. Write $X_{1}^{\prime \prime}=X_{1} \backslash X_{1}^{\prime}, X_{2}^{\prime \prime}=X_{2} \backslash X_{2}^{\prime}$. Let $a_{1}$ be any vertex in $X_{1}^{\prime}$. Suppose that $X_{1}^{\prime \prime}$ contains a vertex $x_{1}$. By items 6 and 8 of Lemma 3.1, $x_{1}$ dominates $a_{1}$ and we obtain conclusion (i). Now let us assume that $X_{1}^{\prime \prime}$ is empty, and similarly $X_{4}^{\prime \prime}$ is empty. Any two vertices of $X_{1}^{\prime}$ are non-adjacent (by item 6 of Lemma 3.1) and comparable (by item 8 ); so if $\left|X_{1}\right| \geq 2$ or $\left|X_{4}\right| \geq 2$ we obtain (i) again. So let $\left|X_{1}\right|=1$ and $\left|X_{4}\right|=1$. Suppose that $X_{6}$ contains a vertex $x_{6}$. Since $\left|X_{4}\right|=1, a_{1}$ has no neighbor in $X_{4}$ and so is dominated by $x_{6}$ and we obtain (i). Thus, $X_{6}=\emptyset$. Now, if $\left|X_{i}\right| \geq 2(i=2,3,5)$, then $X_{i}$ contains two non-adjacent comparable vertices, and we have (i) again. Thus the lemma holds.

A special boat is a boat (with the same notation as in Section (4) such that all $A_{i}$ 's and all $B_{i}$ 's are cliques.

Lemma 6.2 Let $G$ be a $C_{5}$-free b-perfect graph that contains a boat. Then either
(i) $G$ has a proper homogeneous set that is not a clique, or
(ii) $G$ is a special boat.

Proof of Lemma 6.2. Consider a set $M$ that induces a largest boat in $G$. By Lemma 4.1, $M$ is a homogeneous set of $G$. If $M \neq V(G)$, we obtain conclusion (i). So let $M=V(G)$. Since each $A_{i}$ and $B_{i}$ with at least two vertices is a homogeneous set of $G$, either one of them is not a clique, and we obtain (i), or all are cliques, and we obtain (ii). Thus the Lemma holds.

Lemma 6.3 Let $G$ be a b-perfect graph. Then one of the following holds:

1. G has two non-adjacent comparable vertices.
2. G has a proper homogeneous set that is not a clique.
3. $G$ is a $C_{5}$.
4. $G$ is weakly chordal.
5. $G$ is a special boat.

Proof of Lemma 6.3. Let $G$ be a b-perfect graph, and suppose that it does not have two non-adjacent comparable vertices. First suppose that $G$ contains a $C_{5}$, and let us use the same notation as in Lemma 3.1. By Lemma [6.1, we have $X_{6}=\emptyset$, and $\left|X_{i}\right|=1$ for $i=1,2,3,4,5$. Thus the set $X=X_{1} \cup X_{2} \cup X_{3} \cup X_{4} \cup X_{5}$ induces a $C_{5}$. By Lemma 3.1, $X$ is a homogeneous set. If $X \neq V(G)$, we obtain conclusion 2. If $X=V(G)$, we obtain conclusion 3. Now let $G$ be $C_{5}$-free. If $G$ contains a $\bar{C}_{6}$ (which is a boat), then, by Lemma 6.2, we obtain conclusion 2 or 5 . If $G$ does not contain a $\bar{C}_{6}$, then it does not contains any hole or antihole of length at least five (because $G$ contains no $F_{1}=P_{5}$ and $F_{10}=\bar{P}_{6}$ ), and we obtain conclusion 4.

We are now in position to describe a polynomial-time algorithm to find a largest clique in a b-perfect graph $G$. First, if $G$ has two non-adjacent comparable vertices $x, y$, where $x$ dominates $y$, then $\omega(G)=\omega(G-y)$; thus, we can remove $y$ from consideration and recursively find a largest clique in $G-y$. We apply this reduction as long as possible. Second, if $G$ has a proper homogeneous set $H$ that is not a clique, then we recursively find a largest clique $K$ of $H$. Let $G^{\prime}$ be the graph $G-(H-K)$. Clearly, every maximal clique of $G$ either is disjoint from $H$ or contains a maximal clique of $H$; so $\omega(G)=\omega\left(G^{\prime}\right)$. Note that $G^{\prime}$ does not have two non-adjacent comparable vertices $x, y$ (for otherwise, it is easy to see that $x, y$ would have the same property in $G$, a situation which we have already dealt with); so we do not need to go back to the first step with $G^{\prime}$. Finally, if $G$ does not have the above two properties, then by Lemma 6.3, $G$ is either a $C_{5}$, a special boat, or weakly chordal. It is easy to determine $\omega(G)$ when $G$ is a special boat (which is the complement of a bipartite graph), and there are efficient algorithms to find a largest clique of a weakly chordal graph [7]. We can formalize our algorithm as follows.

```
Algorithm CLIQUE(G)
Input: A b-perfect graph \(G\)
Output: A largest clique of \(G\)
```

1. If $G$ has two non-adjacent comparable vertices $x, y$ with $x$ dominating $y$, then return the clique produced by $\operatorname{CLIQUE}(G-y)$;
2. If $G$ has a proper homogeneous set $H$ that is not a clique then

Let $K$ be the clique returned by CLIQUE $(H)$
Return the clique produced by CLIQUE $(G-(H-K))$;
3. If $G$ is a $C_{5}$ then return a clique of size two of $G$;
4. If $G$ is weakly chordal then return a largest clique of $G$ produced by the algorithm in [7];
5. /* Now is $G$ is a special boat */

Return (as described below) a largest clique of $G$.
In Step 5, the special boat is the complement of a bipartite graph. There are well-known algorithms (see, for example, [4) for finding a largest clique of the complement of a bipartite graph. However, we can directly find a largest clique in a special boat given the sets $A_{i}, B_{i}$ (with the same notation as in Section (4). Indeed, the largest clique of $G$ is, clearly, one of the sets $A, B, A_{1} \cup B_{1}, \ldots, A_{q} \cup B_{q}$. So the question is how to find the sets $A_{i}, B_{i}$. When the algorithm reaches Step 5 , we know that $G$ must contain a $\bar{C}_{6}$ because $G$ contains no $P_{5}, C_{5}, \bar{P}_{6}$ and is not weakly chordal. By Lemmas 4.1 and 6.2, every boat extends into a special boat containing all vertices of $G$. Thus, starting with the $\bar{C}_{6}$, we can extend it into a special boat with sets $A_{0}, A_{1}, \ldots, A_{q}, B_{0}, B_{1}, \ldots, B_{q}$ as desired. Clearly, this can be done in polynomial time.

Now, we show that our algorithm can be implemented in polynomial time. Clearly, Step 1 can be performed in polynomial time. Considering Step 2, there are many efficient algorithms to find a homogeneous set in a graph $G$. The most efficient ones are based on the theory of modular decomposition. This theory is rich and complex, and we recall here the relevant facts only. A module is defined to be any homogeneous set $M$ such that every homogeneous set $H$ satisfies either $H \subseteq M, M \subseteq H$, or $M \cap H=\emptyset$. Note that $V(G)$ and each singleton $\{v\} \subseteq V(G)$ is a module. For every module $M$ of size at least two, let $M_{1}, \ldots, M_{h}$ be those modules of $G$ that are properly included in $M$ and are maximal with that property;
then $M_{1}, \ldots, M_{h}$ form a partition of $M$; they are called the children of $M$. The child relation defines a tree, which is called the modular decomposition tree of $G$. Note that the root of the tree is the module $V(G)$ and the leaves of the tree are the singleton modules. Here is an important property of every module $M$. (Let $G[M]$ denotes the subgraph of $G$ induced by $M$.)
Property (*): If $G[M]$ is not connected, then the children of $M$ are the vertex-sets of the components of $G[M]$; if $\bar{G}[M]$ is not connected, then the children of $M$ are the vertex-sets of the components of $\bar{G}[M]$; if $G[M]$ and $\bar{G}[M]$ are connected, and $H$ is any homogeneous set of $G$ that is properly included in $M$, then $H$ is included in a child of $M$.
Now Algorithm Clique can be implemented so as to return a maximum clique $K_{M}$ of every module $M$ of $G$, starting from the leaves up to the root, as follows. If $M$ is a leaf, let the algorithm return $K_{M}=M$. Now suppose that $M$ is not a leaf, that its children are $M_{1}, \ldots, M_{h}$, and that the algorithm has already produced cliques $K_{M_{1}}, \ldots, K_{M_{h}}$. If $G[M]$ is not connected, then let $K_{M}$ be the largest of $K_{M_{1}}, \ldots, K_{M_{h}}$. If $\bar{G}[M]$ is not connected, then let $K_{M}$ be the union of $K_{M_{1}}, \ldots, K_{M_{h}}$. If $G[M]$ and $\bar{G}[M]$ are connected, then consider the graph $G_{M}^{\prime}=G\left[K_{M_{1}} \cup \cdots \cup K_{M_{h}}\right]$. By Property $\left(^{*}\right)$, graph $G_{M}^{\prime}$ does not have a homogeneous set that is not a clique. Moreover, as observed earlier, $G_{M}^{\prime}$ does not have two non-adjacent comparable vertices. Thus, by Lemma 6.3, $G_{M}^{\prime}$ is either a $C_{5}$, a boat or a weakly chordal graph. So we can compute a maximum clique of $G_{M}^{\prime}$ in polynomial time, and the algorithm returns such a clique as $K_{M}$.

There are efficient (but conceptually complex) algorithms that compute the modular decomposition tree of any graph with $n$ vertices and $m$ edges in time $O(n+m)$, see [14, 17]. Moreover, the modular decomposition tree has at most $2 n$ nodes (including the leaf nodes). This ensures that Step 2 of the Algorithm is performed at most $n$ times. Consequently, Steps 3, 4 and 5 are performed at most $n$ times and Algorithm Clique runs in polynomial time.

## References

[1] C. Berge. Graphs. North Holland, 1985.
[2] M. Chudnovsky, N. Robertson, P. Seymour, R. Thomas. The strong perfect graph theorem. Annals of Mathematics 164 (2006) 51-229.
[3] V. Chvátal, Perfectly ordered graphs. In: Topics on perfect graphs, North-Holland Math. Stud., 88 (1984), North-Holland, Amsterdam-New York, pp. 63-65.
[4] V. Chvátal. Linear Programming. W.H. Freeman, 1983.
[5] T. Faik. La b-continuité des b-colorations: complexité, propriétés structurelles et algorithmes. PhD thesis, Univ. Orsay, France, 2005.
[6] R. Hayward. Weakly triangulated graphs. J. Comb. Theory B 39 (1985) 200-209.
[7] R. Hayward, J. P. Spinrad, R. Sritharan. Improved algorithms for weakly chordal graphs. ACM Transactions on Algorithms (TALG) 3 (2) (2007), article 14.
[8] C.T. Hoàng, M. Kouider. On the b-dominating coloring of graphs. Disc. Appl. Math. 152 (2005) 176-186.
[9] C.T. Hoàng, C. Linhares Sales, F. Maffray. On minimally b-imperfect graphs. To appear in Discrete Applied Mathematics.
[10] J. Kratochvíl, Zs. Tuza, M. Voigt. On the b-chromatic number of graphs. Lecture Notes in Computer Science 2573 (2002) 310-320.
[11] R.W. Irving, D.F. Manlove. The b-chromatic number of graphs. Discrete Appl. Math. 91 (1999) 127-141.
[12] F. Maffray, M. Mechebbek. On b-perfect chordal graphs. Graphs and Combinatorics 25 (2009) 365-375.
[13] D.F. Manlove. Minimaximal and maximinimal optimisation problems: a partial order-based approach. PhD thesis. Tech. Rep. 27, Comp. Sci. Dept., Univ. Glasgow, Scotland, 1998.
[14] R.M. McConnell, J.P. Spinrad. Linear-time modular decomposition and efficient transitive orientation of comparability graphs. Proc. 5th ACMSIAM Symposium on Discrete Algorithms (1994) 536-545.
[15] M. Middendorf, F. Pfeiffer. On the complexity of recognizing perfectly orderable graphs. Discrete Math. 80 (1990) 327-333.
[16] S. Seinsche. On a property of the class of n-colorable graphs. J. Combin. Th. B 16 (1974) 191-193.
[17] M. Tedder, D. Corneil, M. Habib, C. Paul. Simple, linear-time modular decomposition. Lecture Notes in Computer Science 5125 (2008) 634-645.


[^0]:    *Supported by Centre Jacques Cartier.
    ${ }^{\dagger}$ Dept. of Physics and Computer Science, Wilfrid Laurier University, 75 University Avenue West, Waterloo, Ontario, Canada N2L 3C5. Supported by NSERC.
    ${ }^{\ddagger}$ C.N.R.S, Laboratoire G-SCOP, 46 Avenue Félix Viallet, 38031 Grenoble Cedex, France.
    ${ }^{\S}$ USTHB, Laboratoire LAID3, BP32 El Alia, Bab Ezzouar 16111, Alger, Algeria.

