# The structure of graphs with a vital linkage of order $2^{*}$ 

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#### Abstract

A linkage of order $k$ of a graph $G$ is a subgraph with $k$ components, each of which is a path. A linkage is vital if it spans all vertices, and no other linkage connects the same pairs of end vertices. We give a characterization of the graphs with a vital linkage of order 2: they are certain minors of a family of highly structured graphs.


## 1 Introduction

Robertson and Seymour [4] defined a linkage in a graph $G$ as a subgraph in which each component is a path. The order of a linkage is the number of components. A linkage $L$ of order $k$ is unique if no other collection of paths connects the same pairs of vertices, it is spanning if $V(L)=V(G)$, and it is vital if it is both unique and spanning. Graphs with a vital linkage are well-behaved. For instance, Robertson and Seymour proved the following:

Theorem 1.1 (Robertson and Seymour [4, Theorem 1.1]). There exists an integer $w$, depending only on $k$, such that every graph with a vital linkage of order $k$ has tree width at most $w$.

Note that Robertson and Seymour use the term $p$-linkage to denote a linkage with $p$ terminals. Robertson and Seymour's proof of this theorem is surprisingly elaborate, and uses their structural description of graphs with no large clique-minor. Recently Kawarabayashi and Wollan

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Figure 1: The graph $K_{2,4}$.


Figure 2: The graphs $\ddot{U}_{4}$ and $\ddot{U}_{5}$.
[2] proved a strengthening of this result. Their shorter proof avoids using the structure theorem.

Our interest in linkages, in particular those of order 2, stems from quite a different area of research: matroid theory. Truemper [5] studied a class of binary matroids that he calls almost regular. His proofs lean heavily on a class of matroids that are single-element extensions of the cycle matroids of graphs with a vital linkage of order 2 . These matroids turned up again in the excluded-minor characterization of matroids that are either binary or ternary, by Mayhew et al. [3].

Truemper proves that an almost regular matroid can be built from one of two specific matroids by certain $\Delta-Y$ operations. This is a deep result, but it does not yield bounds on the branch width of these matroids. In a forthcoming paper the authors of this paper, together with Chun, will give an explicit structural description of the class of almost regular matroids [1]. The main result of this paper will be of use in that project.

To state our main result we need a few more definitions. Fix a graph $G$ and a spanning linkage $L$ of order $k$. A path edge is a member of $E(L)$; an edge in $E(G)-E(L)$ is called a chord if its endpoints lie in a single path, and a rung edge otherwise. If $L$ is vital, then $G$ cannot have any chords.

A linkage minor of $G$ with respect to a (chordless) linkage $L$ is a minor $H$ of $G$ such that all path edges in $E(G)-E(H)$ have been contracted, and all rung edges in $E(G)-E(H)$ have been deleted. If the linkage $L$ is clear from the context we simply say that $H$ is a linkage minor of $G$. Moreover, let $G$ be a graph with a chordless 2-linkage $L$. If $G$ has a linkage minor isomorphic to $K_{2,4}$, such that the terminals of $L$ are mapped to the degree2 vertices of $K_{2,4}$, we say that $G$ has an $X X$ linkage minor (cf. Figure 1).

For each integer $n$, the graph $\ddot{U}_{n}$ is the graph with $V\left(\ddot{U}_{n}\right)=\left\{v_{1}, \ldots, v_{n}\right\} \cup$ $\left\{u_{1}, \ldots, u_{n}\right\}$, and

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\begin{align*}
E\left(\ddot{U}_{n}\right)= & \left\{v_{i} v_{i+1} \mid i=1, \ldots, n-1\right\} \cup\left\{u_{i} u_{i+1} \mid i=1, \ldots, n-1\right\} \cup \\
& \left\{u_{i} v_{i} \mid i=1, \ldots, n\right\} \cup\left\{u_{i} v_{n+1-i} \mid i=1, \ldots, n\right\} . \tag{1}
\end{align*}
$$

We denote by $L_{n}$ the linkage of $\ddot{U}_{n}$ consisting of all edges $v_{i} v_{i+1}$ and $u_{i} u_{i+1}$ for $i=1, \ldots, n-1$. In Figure 2 the graphs $\ddot{U}_{4}$ and $\ddot{U}_{5}$ are depicted.

Finally, we say that $G$ is a Truemper graph if $G$ is a linkage minor of $\ddot{U}_{n}$ for some $n$. The main result of this paper is the following:
Theorem 1.2. Let $G$ be a graph. The following statements are equivalent:


Figure 3: The graph $\ddot{U}_{6}$. The linkage is formed by the two diagonally drawn paths.
(i) G has a vital linkage of order 2;
(ii) $G$ has a chordless spanning linkage of order 2 with no XX linkage minor;
(iii) $G$ is a Truemper graph.

Robertson and Seymour [4] commented, without proof, that graphs with a vital linkage with $k \leq 5$ terminal vertices have path width at most $k$. A weaker claim is the following:

Corollary 1.3. Let $G$ be a graph with a vital linkage of order 2 . Then $G$ has path width at most 4.

Another consequence of our result is that graphs with a vital linkage of order 2 embed in the projective plane:

Corollary 1.4. Let $G$ be a graph with a vital linkage of order 2. Then $G$ can be embedded on a Möbius strip.

Both corollaries can be seen to be true by considering an alternative depiction of $\ddot{U}_{2 n}$, analogous to Figure 3.

## 2 Proof of Theorem 1.2

We start with a few more definitions. Suppose $L$ is a linkage of order 2 with components $P_{1}$ and $P_{2}$, such that the terminal vertices of $P_{1}$ are $s_{1}$ and $t_{1}$, and those of $P_{2}$ are $s_{2}$ and $t_{2}$. We order the vertices on the paths in a natural way, as follows. If $v$ and $w$ are vertices of $P_{i}$, then we say that $v$ is (strictly) to the left of $w$ if the graph distance from $s_{i}$ to $v$ in the subgraph $P_{i}$ is (strictly) smaller than the graph distance from $s_{i}$ to $w$. The notion to the right is defined analogously.

We will frequently use the following elementary observation, whose proof we omit.
Lemma 2.1. Let $G$ be a graph with a chordless spanning linkage $L$ of order 2. Let $P_{1}$ and $P_{2}$ be the components of $L$, with terminal vertices respectively $s_{1}, t_{1}$ and $s_{2}, t_{2}$. Let $H$ be a linkage minor of $G$. If $v$ and $w$ are on $P_{i}$, and $v$ is to the left of $w$, then the vertex corresponding to $v$ in $H$ is to the left of the vertex corresponding to $w$ in $H$.


Figure 4: Detail of the proof of Lemma 2.2.

Without further ado we dive into the proof, which will consist of a sequence of lemmas. The first deals with the equivalence of the first two statements in the theorem.
Lemma 2.2. Let $G$ be a graph with a chordless spanning linkage $L$ of order 2. Then $L$ is vital if and only if $G$ has no XX linkage minor with respect to $L$.

Proof. First we suppose that there exists a graph $G$ with a non-vital chordless spanning linkage $L$ of order 2 such that $G$ has no $X X$ linkage minor. Let $P_{1}, P_{2}$ be the paths of $L$, where $P_{1}$ runs from $s_{1}$ to $t_{1}$, and $P_{2}$ runs from $s_{2}$ to $t_{2}$. Let $P_{1}^{\prime}, P_{2}^{\prime}$ be different paths connecting the same pairs of vertices. Without loss of generality, $P_{1}^{\prime} \neq P_{1}$. But then $P_{1}^{\prime}$ must meet $P_{2}$, so $P_{2}^{\prime} \neq P_{2}$. Let $e=v_{1} v_{2}$ be an edge of $P_{1}^{\prime}$ such that the subpath $s_{1}-v_{1}$ of $P_{1}^{\prime}$ is also a subpath of $P_{1}$, but $e$ is not an edge of $P_{1}$. Let $f=u_{2} u_{1}$ be an edge of $P_{1}^{\prime}$ such that the subpath $u_{1}-t_{1}$ of $P_{1}^{\prime}$ is also a subpath of $P_{2}$, but $f$ is not an edge of $P_{2}$. Similarly, let $e^{\prime}=v_{2}^{\prime} \nu_{1}^{\prime}$ be an edge of $P_{2}^{\prime}$ such that the subpath $s_{2}-v_{2}^{\prime}$ of $P_{2}^{\prime}$ is also a subpath of $P_{2}$, but $e^{\prime}$ is not an edge of $P_{2}$. Let $f^{\prime}=u_{1}^{\prime} u_{2}^{\prime}$ be an edge of $P_{2}^{\prime}$ such that the subpath $u_{2}^{\prime}-t_{2}$ of $P_{2}^{\prime}$ is also a subpath of $P_{2}$, but $f^{\prime}$ is not on $P_{2}$. See Figure 4.

Since $P_{1}^{\prime}$ and $P_{2}^{\prime}$ are vertex-disjoint, $v_{2}^{\prime}$ must be strictly to the left of $v_{2}$ and $u_{2}$. For the same reason, $v_{1}^{\prime}$ must be strictly between $v_{1}$ and $u_{1}$. Likewise, $u_{2}^{\prime}$ must be strictly to the right of $v_{2}$ and $u_{2}$, and $u_{1}^{\prime}$ must be strictly between $v_{1}$ and $u_{1}$. Now construct a linkage minor $H$ of $G$, as follows. Contract all edges on the subpaths $s_{1}-v_{1}, v_{1}^{\prime}-u_{1}^{\prime}$, and $u_{1}-t_{1}$ of $P_{1}$, contract all edges on the subpaths $s_{2}-v_{2}^{\prime}, v_{2}-u_{2}$, and $u_{2}^{\prime}-t_{2}$ of $P_{2}$, delete all rung edges but $\left\{e, f, e^{\prime}, f^{\prime}\right\}$, and contract all but one of the edges of each series class in the resulting graph. Clearly $H$ is isomorphic to $X X$, a contradiction.

Conversely, suppose that $G$ has an $X X$ linkage minor, but that $L$ is unique. Clearly having a vital linkage is preserved under taking linkage minors. But $X X$ has two linkages, a contradiction.

Next we show that the third statement of Theorem 1.2 implies the second.

Lemma 2.3. For all $n, \ddot{U}_{n}$ has no $X X$ linkage minor with respect to $L_{n}$.
Proof. The result holds for $n \leq 2$, because then $\left|V\left(\ddot{U}_{n}\right)\right|<|V(X X)|$. Suppose the lemma fails for some $n \geq 3$, but is valid for all smaller $n$. Every edge of $X X$ is incident with exactly one of the four end vertices of the paths. Hence all rung edges incident with at least two of the four end vertices are not in any $X X$ linkage minor. But after deleting those edges
from $\ddot{U}_{n}$ the end vertices have degree one, and hence the edges incident with them will not be in any $X X$ linkage minor. Contracting these four edges produces $\ddot{U}_{n-2}$, a contradiction.

Reversing a path $P_{i}$ means exchanging the labels of vertices $s_{i}$ and $t_{i}$, thereby reversing the order on the vertices of the path.

Lemma 2.4. Let $G$ be a graph, and $L$ a chordless spanning linkage of order 2 of $G$ consisting of paths $P_{1}$, running from $s_{1}$ to $t_{1}$, and $P_{2}$, running from $s_{2}$ to $t_{2}$. If $G$ has no $X X$ linkage minor, then $G$ is a linkage minor of $\ddot{U}_{n}$ with respect to $L_{n}$ for some integer $n$, such that $L$ is a contraction of $L_{n}$.

Proof. Suppose the statement is false. Let $G$ be a counterexample with as few edges as possible. If some end vertex of a path, say $s_{1}$, has degree one (with $e=s_{1} v$ the only edge), then we can embed $G / e$ in $\ddot{U}_{n}$ for some $n$. Let $G^{\prime}$ be obtained from $\ddot{U}_{n}$ by adding four vertices $s_{1}^{\prime}, t_{1}^{\prime}, s_{2}^{\prime}, t_{2}^{\prime}$, and edges $s_{1}^{\prime} v_{1}, s_{1}^{\prime} s_{2}^{\prime}, s_{1}^{\prime} t_{2}^{\prime}, s_{2}^{\prime} u_{1}, s_{2}^{\prime} t_{1}^{\prime}, v_{n} t_{1}^{\prime}, u_{n} t_{2}^{\prime}, t_{1}^{\prime} t_{2}^{\prime}$. Then $G^{\prime}$ is isomorphic to $\ddot{U}_{n+2}$, and $G^{\prime}$ certainly has $G$ as linkage minor.

Hence we may assume that each end vertex of $P_{1}$ and $P_{2}$ has degree at least two. Suppose no rung edge runs between two of these end vertices. Then it is not hard to see that $G$ has an $X X$ minor, a contradiction. Therefore some two end vertices must be connected. By reversing paths as necessary, we may assume there is an edge $e=s_{1} s_{2}$.

By our assumption, $G \backslash e$ can be embedded in $\ddot{U}_{n}$ for some $n$. Again, let $G^{\prime}$ be obtained from $\ddot{U}_{n}$ by adding four vertices $s_{1}^{\prime}, t_{1}^{\prime}, s_{2}^{\prime}, t_{2}^{\prime}$, and edges $s_{1}^{\prime} v_{1}, s_{1}^{\prime} s_{2}^{\prime}, s_{1}^{\prime} t_{2}^{\prime}, s_{2}^{\prime} u_{1}, s_{2}^{\prime} t_{1}^{\prime}, v_{n} t_{1}^{\prime}, u_{n} t_{2}^{\prime}, t_{1}^{\prime} t_{2}^{\prime}$. Then $G^{\prime}$ is isomorphic to $\ddot{U}_{n+2}$, and $G^{\prime}$ certainly has $G$ as linkage minor, a contradiction.

As an aside, it is possible to prove a stronger version of the previous lemma. We say a partition $(A, B)$ of the rung edges is valid if the edges in $A$ are pairwise non-crossing, and the edges in $B$ are pairwise non-crossing after reversing one of the paths. One can show:

- Each Truemper graph has a valid partition.
- For every valid partition $(A, B)$ of a Truemper graph $G$, some $\ddot{U}_{n}$ has $G$ as linkage minor in such a way that $(A, B)$ extends to a valid partition of $\ddot{U}_{n}$.
Now we have all ingredients of our main result.
Proof of Theorem 1.2. From Lemma 2.2 we learn that $(i) \Leftrightarrow(i i)$. From Lemma 2.3 we learn that (iii) $\Rightarrow$ (ii), and from Lemma 2.4 we conclude that $(i i) \Rightarrow(i i i)$.


## References

[1] Carolyn Chun, Dillon Mayhew, Geoff Whittle, and Stefan H. M. van Zwam. The structure of binary Fano-fragile matroids. In preparation.
[2] Ken-ichi Kawarabayashi and Paul Wollan. A shorter proof of the graph minor algorithm: the unique linkage theorem. In STOC, pages 687-694, 2010.
[3] Dillon Mayhew, Bogdan Oporowski, James Oxley, and Geoff Whittle. The excluded minors for the class of matroids that are binary or ternary. European J. Combin., 32(6):891-930, 2011.
[4] Neil Robertson and P. D. Seymour. Graph minors. XXI. graphs with unique linkages. J. Combin. Theory Ser. B, 99(3):583-616, 2009.
[5] K. Truemper. A decomposition theory of matroids. VI. Almost regular matroids. J. Combin. Theory Ser. B, 55(2):235-301, 1992. ISSN 0095-8956.


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