# The structure of graphs with a vital linkage of order 2<sup>\*</sup>

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### Abstract

A linkage of order k of a graph G is a subgraph with k components, each of which is a path. A linkage is *vital* if it spans all vertices, and no other linkage connects the same pairs of end vertices. We give a characterization of the graphs with a vital linkage of order 2: they are certain minors of a family of highly structured graphs.

## 1 Introduction

Robertson and Seymour [4] defined a *linkage* in a graph *G* as a subgraph in which each component is a path. The *order* of a linkage is the number of components. A linkage *L* of order *k* is *unique* if no other collection of paths connects the same pairs of vertices, it is *spanning* if V(L) = V(G), and it is *vital* if it is both unique and spanning. Graphs with a vital linkage are well-behaved. For instance, Robertson and Seymour proved the following:

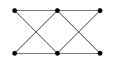
**Theorem 1.1** (Robertson and Seymour [4, Theorem 1.1]). There exists an integer w, depending only on k, such that every graph with a vital linkage of order k has tree width at most w.

Note that Robertson and Seymour use the term p-linkage to denote a linkage with p terminals. Robertson and Seymour's proof of this theorem is surprisingly elaborate, and uses their structural description of graphs with no large clique-minor. Recently Kawarabayashi and Wollan

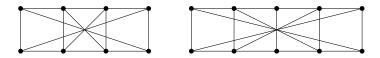
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**Figure 1:** The graph  $K_{2,4}$ .



**Figure 2:** The graphs  $\ddot{U}_4$  and  $\ddot{U}_5$ .

[2] proved a strengthening of this result. Their shorter proof avoids using the structure theorem.

Our interest in linkages, in particular those of order 2, stems from quite a different area of research: matroid theory. Truemper [5] studied a class of binary matroids that he calls *almost regular*. His proofs lean heavily on a class of matroids that are single-element extensions of the cycle matroids of graphs with a vital linkage of order 2. These matroids turned up again in the excluded-minor characterization of matroids that are either binary or ternary, by Mayhew et al. [3].

Truemper proves that an almost regular matroid can be built from one of two specific matroids by certain  $\Delta - Y$  operations. This is a deep result, but it does not yield bounds on the branch width of these matroids. In a forthcoming paper the authors of this paper, together with Chun, will give an explicit structural description of the class of almost regular matroids [1]. The main result of this paper will be of use in that project.

To state our main result we need a few more definitions. Fix a graph G and a spanning linkage L of order k. A *path edge* is a member of E(L); an edge in E(G) - E(L) is called a *chord* if its endpoints lie in a single path, and a *rung edge* otherwise. If L is vital, then G cannot have any chords.

A *linkage minor* of *G* with respect to a (chordless) linkage *L* is a minor *H* of *G* such that all path edges in E(G) - E(H) have been contracted, and all rung edges in E(G) - E(H) have been deleted. If the linkage *L* is clear from the context we simply say that *H* is a linkage minor of *G*. Moreover, let *G* be a graph with a chordless 2-linkage *L*. If *G* has a linkage minor isomorphic to  $K_{2,4}$ , such that the terminals of *L* are mapped to the degree-2 vertices of  $K_{2,4}$ , we say that *G* has an *XX* linkage minor (cf. Figure 1).

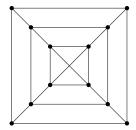
For each integer *n*, the graph  $\ddot{U}_n$  is the graph with  $V(\ddot{U}_n) = \{v_1, \dots, v_n\} \cup \{u_1, \dots, u_n\}$ , and

$$E(\ddot{U}_n) = \{v_i v_{i+1} \mid i = 1, \dots, n-1\} \cup \{u_i u_{i+1} \mid i = 1, \dots, n-1\} \cup \{u_i v_i \mid i = 1, \dots, n\} \cup \{u_i v_{n+1-i} \mid i = 1, \dots, n\}.$$
 (1)

We denote by  $L_n$  the linkage of  $\ddot{U}_n$  consisting of all edges  $v_i v_{i+1}$  and  $u_i u_{i+1}$  for i = 1, ..., n - 1. In Figure 2 the graphs  $\ddot{U}_4$  and  $\ddot{U}_5$  are depicted.

Finally, we say that *G* is a *Truemper graph* if *G* is a linkage minor of  $U_n$  for some *n*. The main result of this paper is the following:

**Theorem 1.2.** Let G be a graph. The following statements are equivalent:



**Figure 3:** The graph  $\ddot{U}_6$ . The linkage is formed by the two diagonally drawn paths.

- (i) G has a vital linkage of order 2;
- (ii) G has a chordless spanning linkage of order 2 with no XX linkage minor;
- (iii) G is a Truemper graph.

Robertson and Seymour [4] commented, without proof, that graphs with a vital linkage with  $k \le 5$  terminal vertices have path width at most k. A weaker claim is the following:

**Corollary 1.3.** Let *G* be a graph with a vital linkage of order 2. Then *G* has path width at most 4.

Another consequence of our result is that graphs with a vital linkage of order 2 embed in the projective plane:

**Corollary 1.4.** Let G be a graph with a vital linkage of order 2. Then G can be embedded on a Möbius strip.

Both corollaries can be seen to be true by considering an alternative depiction of  $\ddot{U}_{2n}$ , analogous to Figure 3.

# 2 Proof of Theorem 1.2

We start with a few more definitions. Suppose *L* is a linkage of order 2 with components  $P_1$  and  $P_2$ , such that the terminal vertices of  $P_1$  are  $s_1$  and  $t_1$ , and those of  $P_2$  are  $s_2$  and  $t_2$ . We order the vertices on the paths in a natural way, as follows. If *v* and *w* are vertices of  $P_i$ , then we say that *v* is *(strictly) to the left* of *w* if the graph distance from  $s_i$  to *v* in the subgraph  $P_i$  is (strictly) smaller than the graph distance from  $s_i$  to *w*. The notion *to the right* is defined analogously.

We will frequently use the following elementary observation, whose proof we omit.

**Lemma 2.1.** Let *G* be a graph with a chordless spanning linkage *L* of order 2. Let  $P_1$  and  $P_2$  be the components of *L*, with terminal vertices respectively  $s_1, t_1$  and  $s_2, t_2$ . Let *H* be a linkage minor of *G*. If *v* and *w* are on  $P_i$ , and *v* is to the left of *w*, then the vertex corresponding to *v* in *H* is to the left of the vertex corresponding to *w* in *H*.

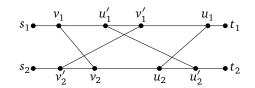


Figure 4: Detail of the proof of Lemma 2.2.

Without further ado we dive into the proof, which will consist of a sequence of lemmas. The first deals with the equivalence of the first two statements in the theorem.

**Lemma 2.2.** Let G be a graph with a chordless spanning linkage L of order 2. Then L is vital if and only if G has no XX linkage minor with respect to L.

*Proof.* First we suppose that there exists a graph *G* with a non-vital chordless spanning linkage *L* of order 2 such that *G* has no *XX* linkage minor. Let  $P_1$ ,  $P_2$  be the paths of *L*, where  $P_1$  runs from  $s_1$  to  $t_1$ , and  $P_2$  runs from  $s_2$  to  $t_2$ . Let  $P'_1$ ,  $P'_2$  be different paths connecting the same pairs of vertices. Without loss of generality,  $P'_1 \neq P_1$ . But then  $P'_1$  must meet  $P_2$ , so  $P'_2 \neq P_2$ . Let  $e = v_1 v_2$  be an edge of  $P'_1$  such that the subpath  $s_1 - v_1$  of  $P'_1$  is also a subpath of  $P_1$ , but *e* is not an edge of  $P_1$ . Let  $f = u_2 u_1$  be an edge of  $P'_1$  such that the subpath  $u_1 - t_1$  of  $P'_1$  is also a subpath of  $P_2$ , but *f* is not an edge of  $P'_2$ . Similarly, let  $e' = v'_2 v'_1$  be an edge of  $P'_2$  such that the subpath  $s_2 - v'_2$  of  $P'_2$  is also a subpath of  $P_2$ , but *e'* is not an edge of  $P_2$ . Let  $f' = u'_1 u'_2$  be an edge of  $P'_2$  such that the subpath  $u'_2 - t_2$  of  $P'_2$  is also a subpath of  $P_2$ , but *f'* is not on  $P_2$ . See Figure 4.

Since  $P'_1$  and  $P'_2$  are vertex-disjoint,  $v'_2$  must be strictly to the left of  $v_2$  and  $u_2$ . For the same reason,  $v'_1$  must be strictly between  $v_1$  and  $u_1$ . Likewise,  $u'_2$  must be strictly to the right of  $v_2$  and  $u_2$ , and  $u'_1$  must be strictly between  $v_1$  and  $u_1$ . Now construct a linkage minor H of G, as follows. Contract all edges on the subpaths  $s_1 - v_1$ ,  $v'_1 - u'_1$ , and  $u_1 - t_1$  of  $P_1$ , contract all edges on the subpaths  $s_2 - v'_2$ ,  $v_2 - u_2$ , and  $u'_2 - t_2$  of  $P_2$ , delete all rung edges but  $\{e, f, e', f'\}$ , and contract all but one of the edges of each series class in the resulting graph. Clearly H is isomorphic to XX, a contradiction.

Conversely, suppose that *G* has an *XX* linkage minor, but that *L* is unique. Clearly having a vital linkage is preserved under taking linkage minors. But *XX* has two linkages, a contradiction.

Next we show that the third statement of Theorem 1.2 implies the second.

#### **Lemma 2.3.** For all n, $\ddot{U}_n$ has no XX linkage minor with respect to $L_n$ .

*Proof.* The result holds for  $n \le 2$ , because then  $|V(\ddot{U}_n)| < |V(XX)|$ . Suppose the lemma fails for some  $n \ge 3$ , but is valid for all smaller n. Every edge of *XX* is incident with exactly one of the four end vertices of the paths. Hence all rung edges incident with at least two of the four end vertices are not in any *XX* linkage minor. But after deleting those edges

from  $\ddot{U}_n$  the end vertices have degree one, and hence the edges incident with them will not be in any *XX* linkage minor. Contracting these four edges produces  $\ddot{U}_{n-2}$ , a contradiction.

*Reversing a path*  $P_i$  means exchanging the labels of vertices  $s_i$  and  $t_i$ , thereby reversing the order on the vertices of the path.

**Lemma 2.4.** Let *G* be a graph, and *L* a chordless spanning linkage of order 2 of *G* consisting of paths  $P_1$ , running from  $s_1$  to  $t_1$ , and  $P_2$ , running from  $s_2$  to  $t_2$ . If *G* has no XX linkage minor, then *G* is a linkage minor of  $U_n$  with respect to  $L_n$  for some integer *n*, such that *L* is a contraction of  $L_n$ .

*Proof.* Suppose the statement is false. Let *G* be a counterexample with as few edges as possible. If some end vertex of a path, say  $s_1$ , has degree one (with  $e = s_1v$  the only edge), then we can embed G/e in  $\ddot{U}_n$  for some *n*. Let *G'* be obtained from  $\ddot{U}_n$  by adding four vertices  $s'_1, t'_1, s'_2, t'_2$ , and edges  $s'_1v_1, s'_1s'_2, s'_1t'_2, s'_2u_1, s'_2t'_1, v_nt'_1, u_nt'_2, t'_1t'_2$ . Then *G'* is isomorphic to  $\ddot{U}_{n+2}$ , and *G'* certainly has *G* as linkage minor.

Hence we may assume that each end vertex of  $P_1$  and  $P_2$  has degree at least two. Suppose no rung edge runs between two of these end vertices. Then it is not hard to see that *G* has an *XX* minor, a contradiction. Therefore some two end vertices must be connected. By reversing paths as necessary, we may assume there is an edge  $e = s_1s_2$ .

By our assumption,  $G \setminus e$  can be embedded in  $\ddot{U}_n$  for some n. Again, let G' be obtained from  $\ddot{U}_n$  by adding four vertices  $s'_1, t'_1, s'_2, t'_2$ , and edges  $s'_1v_1, s'_1s'_2, s'_1t'_2, s'_2u_1, s'_2t'_1, v_nt'_1, u_nt'_2, t'_1t'_2$ . Then G' is isomorphic to  $\ddot{U}_{n+2}$ , and G' certainly has G as linkage minor, a contradiction.

As an aside, it is possible to prove a stronger version of the previous lemma. We say a partition (A, B) of the rung edges is *valid* if the edges in *A* are pairwise non-crossing, and the edges in *B* are pairwise non-crossing after reversing one of the paths. One can show:

- Each Truemper graph has a valid partition.
- For every valid partition (A, B) of a Truemper graph G, some  $\ddot{U}_n$  has G as linkage minor in such a way that (A, B) extends to a valid partition of  $\ddot{U}_n$ .

Now we have all ingredients of our main result.

*Proof of Theorem* **1.2**. From Lemma **2.2** we learn that  $(i) \Leftrightarrow (ii)$ . From Lemma **2.3** we learn that  $(iii) \Rightarrow (ii)$ , and from Lemma **2.4** we conclude that  $(ii) \Rightarrow (iii)$ .

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