The structure of graphs with a vital linkage of order 2^{*}

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Abstract

A linkage of order k of a graph G is a subgraph with k components, each of which is a path. A linkage is *vital* if it spans all vertices, and no other linkage connects the same pairs of end vertices. We give a characterization of the graphs with a vital linkage of order 2: they are certain minors of a family of highly structured graphs.

1 Introduction

Robertson and Seymour [4] defined a *linkage* in a graph *G* as a subgraph in which each component is a path. The *order* of a linkage is the number of components. A linkage *L* of order *k* is *unique* if no other collection of paths connects the same pairs of vertices, it is *spanning* if V(L) = V(G), and it is *vital* if it is both unique and spanning. Graphs with a vital linkage are well-behaved. For instance, Robertson and Seymour proved the following:

Theorem 1.1 (Robertson and Seymour [4, Theorem 1.1]). There exists an integer w, depending only on k, such that every graph with a vital linkage of order k has tree width at most w.

Note that Robertson and Seymour use the term p-linkage to denote a linkage with p terminals. Robertson and Seymour's proof of this theorem is surprisingly elaborate, and uses their structural description of graphs with no large clique-minor. Recently Kawarabayashi and Wollan

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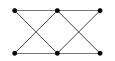


Figure 1: The graph $K_{2,4}$.

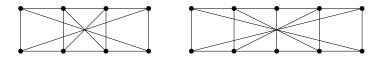


Figure 2: The graphs \ddot{U}_4 and \ddot{U}_5 .

[2] proved a strengthening of this result. Their shorter proof avoids using the structure theorem.

Our interest in linkages, in particular those of order 2, stems from quite a different area of research: matroid theory. Truemper [5] studied a class of binary matroids that he calls *almost regular*. His proofs lean heavily on a class of matroids that are single-element extensions of the cycle matroids of graphs with a vital linkage of order 2. These matroids turned up again in the excluded-minor characterization of matroids that are either binary or ternary, by Mayhew et al. [3].

Truemper proves that an almost regular matroid can be built from one of two specific matroids by certain $\Delta - Y$ operations. This is a deep result, but it does not yield bounds on the branch width of these matroids. In a forthcoming paper the authors of this paper, together with Chun, will give an explicit structural description of the class of almost regular matroids [1]. The main result of this paper will be of use in that project.

To state our main result we need a few more definitions. Fix a graph G and a spanning linkage L of order k. A *path edge* is a member of E(L); an edge in E(G) - E(L) is called a *chord* if its endpoints lie in a single path, and a *rung edge* otherwise. If L is vital, then G cannot have any chords.

A *linkage minor* of *G* with respect to a (chordless) linkage *L* is a minor *H* of *G* such that all path edges in E(G) - E(H) have been contracted, and all rung edges in E(G) - E(H) have been deleted. If the linkage *L* is clear from the context we simply say that *H* is a linkage minor of *G*. Moreover, let *G* be a graph with a chordless 2-linkage *L*. If *G* has a linkage minor isomorphic to $K_{2,4}$, such that the terminals of *L* are mapped to the degree-2 vertices of $K_{2,4}$, we say that *G* has an *XX* linkage minor (cf. Figure 1).

For each integer *n*, the graph \ddot{U}_n is the graph with $V(\ddot{U}_n) = \{v_1, \dots, v_n\} \cup \{u_1, \dots, u_n\}$, and

$$E(\ddot{U}_n) = \{v_i v_{i+1} \mid i = 1, \dots, n-1\} \cup \{u_i u_{i+1} \mid i = 1, \dots, n-1\} \cup \{u_i v_i \mid i = 1, \dots, n\} \cup \{u_i v_{n+1-i} \mid i = 1, \dots, n\}.$$
 (1)

We denote by L_n the linkage of \ddot{U}_n consisting of all edges $v_i v_{i+1}$ and $u_i u_{i+1}$ for i = 1, ..., n - 1. In Figure 2 the graphs \ddot{U}_4 and \ddot{U}_5 are depicted.

Finally, we say that *G* is a *Truemper graph* if *G* is a linkage minor of U_n for some *n*. The main result of this paper is the following:

Theorem 1.2. Let G be a graph. The following statements are equivalent:

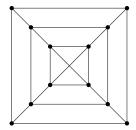


Figure 3: The graph \ddot{U}_6 . The linkage is formed by the two diagonally drawn paths.

- (i) G has a vital linkage of order 2;
- (ii) G has a chordless spanning linkage of order 2 with no XX linkage minor;
- (iii) G is a Truemper graph.

Robertson and Seymour [4] commented, without proof, that graphs with a vital linkage with $k \le 5$ terminal vertices have path width at most k. A weaker claim is the following:

Corollary 1.3. Let *G* be a graph with a vital linkage of order 2. Then *G* has path width at most 4.

Another consequence of our result is that graphs with a vital linkage of order 2 embed in the projective plane:

Corollary 1.4. Let G be a graph with a vital linkage of order 2. Then G can be embedded on a Möbius strip.

Both corollaries can be seen to be true by considering an alternative depiction of \ddot{U}_{2n} , analogous to Figure 3.

2 Proof of Theorem 1.2

We start with a few more definitions. Suppose *L* is a linkage of order 2 with components P_1 and P_2 , such that the terminal vertices of P_1 are s_1 and t_1 , and those of P_2 are s_2 and t_2 . We order the vertices on the paths in a natural way, as follows. If *v* and *w* are vertices of P_i , then we say that *v* is *(strictly) to the left* of *w* if the graph distance from s_i to *v* in the subgraph P_i is (strictly) smaller than the graph distance from s_i to *w*. The notion *to the right* is defined analogously.

We will frequently use the following elementary observation, whose proof we omit.

Lemma 2.1. Let *G* be a graph with a chordless spanning linkage *L* of order 2. Let P_1 and P_2 be the components of *L*, with terminal vertices respectively s_1, t_1 and s_2, t_2 . Let *H* be a linkage minor of *G*. If *v* and *w* are on P_i , and *v* is to the left of *w*, then the vertex corresponding to *v* in *H* is to the left of the vertex corresponding to *w* in *H*.

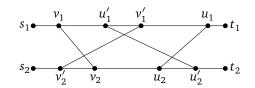


Figure 4: Detail of the proof of Lemma 2.2.

Without further ado we dive into the proof, which will consist of a sequence of lemmas. The first deals with the equivalence of the first two statements in the theorem.

Lemma 2.2. Let G be a graph with a chordless spanning linkage L of order 2. Then L is vital if and only if G has no XX linkage minor with respect to L.

Proof. First we suppose that there exists a graph *G* with a non-vital chordless spanning linkage *L* of order 2 such that *G* has no *XX* linkage minor. Let P_1 , P_2 be the paths of *L*, where P_1 runs from s_1 to t_1 , and P_2 runs from s_2 to t_2 . Let P'_1 , P'_2 be different paths connecting the same pairs of vertices. Without loss of generality, $P'_1 \neq P_1$. But then P'_1 must meet P_2 , so $P'_2 \neq P_2$. Let $e = v_1 v_2$ be an edge of P'_1 such that the subpath $s_1 - v_1$ of P'_1 is also a subpath of P_1 , but *e* is not an edge of P_1 . Let $f = u_2 u_1$ be an edge of P'_1 such that the subpath $u_1 - t_1$ of P'_1 is also a subpath of P_2 , but *f* is not an edge of P'_2 . Similarly, let $e' = v'_2 v'_1$ be an edge of P'_2 such that the subpath $s_2 - v'_2$ of P'_2 is also a subpath of P_2 , but *e'* is not an edge of P_2 . Let $f' = u'_1 u'_2$ be an edge of P'_2 such that the subpath $u'_2 - t_2$ of P'_2 is also a subpath of P_2 , but *f'* is not on P_2 . See Figure 4.

Since P'_1 and P'_2 are vertex-disjoint, v'_2 must be strictly to the left of v_2 and u_2 . For the same reason, v'_1 must be strictly between v_1 and u_1 . Likewise, u'_2 must be strictly to the right of v_2 and u_2 , and u'_1 must be strictly between v_1 and u_1 . Now construct a linkage minor H of G, as follows. Contract all edges on the subpaths $s_1 - v_1$, $v'_1 - u'_1$, and $u_1 - t_1$ of P_1 , contract all edges on the subpaths $s_2 - v'_2$, $v_2 - u_2$, and $u'_2 - t_2$ of P_2 , delete all rung edges but $\{e, f, e', f'\}$, and contract all but one of the edges of each series class in the resulting graph. Clearly H is isomorphic to XX, a contradiction.

Conversely, suppose that *G* has an *XX* linkage minor, but that *L* is unique. Clearly having a vital linkage is preserved under taking linkage minors. But *XX* has two linkages, a contradiction.

Next we show that the third statement of Theorem 1.2 implies the second.

Lemma 2.3. For all n, \ddot{U}_n has no XX linkage minor with respect to L_n .

Proof. The result holds for $n \le 2$, because then $|V(\ddot{U}_n)| < |V(XX)|$. Suppose the lemma fails for some $n \ge 3$, but is valid for all smaller n. Every edge of *XX* is incident with exactly one of the four end vertices of the paths. Hence all rung edges incident with at least two of the four end vertices are not in any *XX* linkage minor. But after deleting those edges

from \ddot{U}_n the end vertices have degree one, and hence the edges incident with them will not be in any *XX* linkage minor. Contracting these four edges produces \ddot{U}_{n-2} , a contradiction.

Reversing a path P_i means exchanging the labels of vertices s_i and t_i , thereby reversing the order on the vertices of the path.

Lemma 2.4. Let *G* be a graph, and *L* a chordless spanning linkage of order 2 of *G* consisting of paths P_1 , running from s_1 to t_1 , and P_2 , running from s_2 to t_2 . If *G* has no XX linkage minor, then *G* is a linkage minor of U_n with respect to L_n for some integer *n*, such that *L* is a contraction of L_n .

Proof. Suppose the statement is false. Let *G* be a counterexample with as few edges as possible. If some end vertex of a path, say s_1 , has degree one (with $e = s_1v$ the only edge), then we can embed G/e in \ddot{U}_n for some *n*. Let *G'* be obtained from \ddot{U}_n by adding four vertices s'_1, t'_1, s'_2, t'_2 , and edges $s'_1v_1, s'_1s'_2, s'_1t'_2, s'_2u_1, s'_2t'_1, v_nt'_1, u_nt'_2, t'_1t'_2$. Then *G'* is isomorphic to \ddot{U}_{n+2} , and *G'* certainly has *G* as linkage minor.

Hence we may assume that each end vertex of P_1 and P_2 has degree at least two. Suppose no rung edge runs between two of these end vertices. Then it is not hard to see that *G* has an *XX* minor, a contradiction. Therefore some two end vertices must be connected. By reversing paths as necessary, we may assume there is an edge $e = s_1s_2$.

By our assumption, $G \setminus e$ can be embedded in \ddot{U}_n for some n. Again, let G' be obtained from \ddot{U}_n by adding four vertices s'_1, t'_1, s'_2, t'_2 , and edges $s'_1v_1, s'_1s'_2, s'_1t'_2, s'_2u_1, s'_2t'_1, v_nt'_1, u_nt'_2, t'_1t'_2$. Then G' is isomorphic to \ddot{U}_{n+2} , and G' certainly has G as linkage minor, a contradiction.

As an aside, it is possible to prove a stronger version of the previous lemma. We say a partition (A, B) of the rung edges is *valid* if the edges in *A* are pairwise non-crossing, and the edges in *B* are pairwise non-crossing after reversing one of the paths. One can show:

- Each Truemper graph has a valid partition.
- For every valid partition (A, B) of a Truemper graph G, some \ddot{U}_n has G as linkage minor in such a way that (A, B) extends to a valid partition of \ddot{U}_n .

Now we have all ingredients of our main result.

Proof of Theorem **1.2**. From Lemma **2.2** we learn that $(i) \Leftrightarrow (ii)$. From Lemma **2.3** we learn that $(iii) \Rightarrow (ii)$, and from Lemma **2.4** we conclude that $(ii) \Rightarrow (iii)$.

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