Toughness and Vertex Degrees

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Abstract

We study theorems giving sufficient conditions on the vertex degrees of a graph G to guarantee G is t-tough. We first give a best monotone theorem when $t \geq 1$, but then show that for any integer $k \geq 1$, a best monotone theorem for $t = \frac{1}{k} \leq 1$ requires at least $f(k) \cdot |V(G)|$ nonredundant conditions, where $f(k)$ grows superpolynomially as $k \rightarrow \infty$. When $t < 1$, we give an additional, simple theorem for G to be t-tough, in terms of its vertex degrees.

1 Introduction

We consider only simple graphs without loops or multiple edges. Our terminology and notation will be standard except as indicated, and a good reference for any undefined terms or notation is [7]. For two graphs G, H on disjoint vertex sets, we

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denote their *union* by $G \cup H$. The *join* $G + H$ of G and H is the graph formed from $G \cup H$ by adding all edges between $V(G)$ and $V(H)$.

For a positive integer n, an *n-sequence* (or just a *sequence*) is an integer sequence $\pi = (d_1, d_2, \ldots, d_n)$, with $0 \le d_j \le n-1$ for all j. In contrast to [7], we will usually write the sequence in nondecreasing order (and may make this explicit by writing $\pi = (d_1 \le \cdots \le d_n)$). We will employ the standard abbreviated notation for sequences, e.g., $(4, 4, 4, 4, 4, 5, 5, 6)$ will be denoted $4^5\, 5^2\, 6^1$. If $\pi = (d_1, \ldots, d_n)$ and $\pi' = (d'_1, \ldots, d'_n)$ are two n-sequences, we say π' *majorizes* π, denoted $\pi' \ge \pi$, if $d'_j \ge d_j$ for all j.

A *degree sequence* of a graph is any sequence $\pi = (d_1, d_2, \ldots, d_n)$ consisting of the vertex degrees of the graph. A sequence π is *graphical* if there exists a graph G having π as one of its degree sequences, in which case we call G a *realization* of π. If P is a graph property (e.g., hamiltonian, k-connected, etc.), we call a graphical sequence π *forcibly* P if every realization of π has property P.

Historically, the degree sequence of a graph has been used to provide sufficient conditions for a graph to have certain properties, such as hamiltonicity or k-connectivity. In particular, sufficient conditions for π to be forcibly hamiltonian were given by several authors, culminating in the following theorem of Chvátal [4].

Theorem 1.1 ([4]). *Let $\pi = (d_1 \le \cdots \le d_n)$ be a graphical sequence, with $n \ge 3$. If $d_i \le i < \frac{1}{2}n$ implies $d_{n-i} \ge n - i$, then π is forcibly hamiltonian.*

Unlike its predecessors, Chvátal's theorem has the property that if it does not guarantee that π is forcibly hamiltonian because the condition fails for some $i < \frac{1}{2}n$, then π is majorized by $\pi' = i^i\, (n-i-1)^{n-2i}\, (n-1)^i$, which has a unique non-hamiltonian realization $K_i + (\overline{K_i} \cup K_{n-2i})$. As we will see below, this implies that Chvátal's theorem is the strongest of an entire class of theorems giving sufficient degree conditions for π to be forcibly hamiltonian.

Sufficient conditions for π to be forcibly k-connected were given by several authors, culminating in the following theorem of Bondy [3] (though the form in which we present it is due to Boesch [2]).

Theorem 1.2 ([2, 3]). *Let $\pi = (d_1 \le \cdots \le d_n)$ be a graphical sequence with $n \ge 2$, and let $1 \le k \le n-1$. If $d_i \le i+k-2$ implies $d_{n-k+1} \ge n-i$, for $1 \le i \le \frac{1}{2}(n-k+1)$, then π is forcibly k-connected.*

Boesch [2] also observed that Theorem 1.2 is the strongest theorem giving sufficient degree conditions for π to be forcibly k-connected, in exactly the same sense as Theorem 1.1.

Let $\omega(G)$ denote the number of components of a graph G. For $t \ge 0$, we call G *t-tough* if $t \cdot \omega(G - X) \le |X|$, for every $X \subseteq V(G)$ with $\omega(G - X) > 1$. The *toughness* of G, denoted $\tau(G)$, is the maximum $t \geq 0$ for which G is t-tough (taking $\tau(K_n) = n-1$, for all $n \geq 1$). So if G is not complete, then $\tau(G) = \min\left\{ \frac{|X|}{\omega(G-X)} \;\middle|\; X \subseteq V(G) \text{ is a cutset of } G \right\}$.

In this paper we consider forcibly t-tough theorems, for any $t \geq 0$. When trying to formulate and prove this type of theorem, we encountered very different behavior in the number of conditions required for a best possible theorem for the cases $t \geq 1$ and $t < 1$. In order to describe this behavior precisely, we need to say what we mean by a 'condition' and by a 'best possible theorem'.

First note that the conditions in Theorems 1.1 can be written in the form:

$$d_i \geq i+1 \;\text{ or }\; d_{n-i} \geq n-i, \;\text{ for } i = 1, \ldots, \left\lfloor \tfrac{1}{2}(n-1) \right\rfloor,$$

and the conditions in Theorem 1.2 can be written in a similar way. We will use the term 'Chvátal-type conditions' for such conditions. Formally, a *Chvátal-type condition* for n-sequences $(d_1 \leq d_2 \leq \cdots \leq d_n)$ is a condition of the form

$$d_{i_1} \geq k_{i_1} \;\vee\; d_{i_2} \geq k_{i_2} \;\vee\; \ldots \;\vee\; d_{i_r} \geq k_{i_r},$$

where all i_j and k_{i_j} are integers, with $1 \leq i_1 < i_2 < \cdots < i_r \leq n$ and $1 \leq k_{i_1} \leq k_{i_2} \leq \cdots \leq k_{i_r} \leq n$.

A graph property P is called *increasing* if whenever a graph G has P, so does every edge-augmented supergraph of G. In particular, "hamiltonian", "k-connected" and "t-tough" are all increasing graph properties. In this paper, the term "graph property" will always mean an increasing graph property.

Given a graph property P, consider a theorem T which declares certain degree sequences to be forcibly P, rendering no decision on the remaining degree sequences. We call such a theorem T a *forcibly P-theorem* (or just a *P-theorem*, for brevity). Thus Theorem 1.1 would be a forcibly hamiltonian theorem. We call a P-theorem T *monotone* if, for any two degree sequences π, π', whenever T declares π forcibly P and $\pi' \geq \pi$, then T declares π' forcibly P. We call a P-theorem T *optimal* if whenever T does not declare a degree sequence π forcibly P, then π is not forcibly P; T is *weakly optimal* if for any sequence π (not necessarily graphical) which T does not declare forcibly P, π is majorized by a degree sequence which is not forcibly P.

A P-theorem which is both monotone and weakly optimal is a best monotone P-theorem, in the following sense.

Theorem 1.3. *Let T, T_0 be monotone P-theorems, with T_0 weakly optimal. If T declares a degree sequence π to be forcibly P, then so does T_0.*

Proof of Theorem 1.3: Suppose to the contrary that there exists a degree sequence π so that T declares π forcibly P, but T_0 does not. Since T_0 is weakly optimal, there exists a degree sequence $\pi' \geq \pi$ which is not forcibly P. This means that also T will not declare π' forcibly P. But if T declares π forcibly P, $\pi' \geq \pi$, and T does not declare π' forcibly P, then T is not monotone, a contradiction. ■

If T_0 is Chvátal's hamiltonian theorem (Theorem 1.1), then T_0 is clearly monotone, and we noted above that T_0 is weakly optimal. So by Theorem 1.3, Chvátal's theorem is a best monotone hamiltonian theorem.

Our goal in this paper is to consider forcibly t-tough theorems, for any $t \geq 0$. In Section 2 we first give a best monotone t-tough theorem for n-sequences, requiring at most $\lfloor \frac{1}{2}n \rfloor$ Chvátal-type conditions, for any $t \geq 1$. In contrast to this, in Sections 3 and 4 we show that for any integer $k \geq 1$, a best monotone $1/k$-tough theorem contains at least $f(k) \cdot n$ nonredundant Chvátal-type conditions, where $f(k)$ grows superpolynomially as $k \to \infty$. A similar superpolynomial growth in the complexity of the best monotone k-edge-connected theorem in terms of k was previously noted by Kriesell [6].

This superpolynomial complexity of a best monotone $1/k$-tough theorem suggests the desirability of finding more reasonable t-tough theorems, when $t < 1$. In Section 5 we give one such theorem. This theorem is a monotone, though not best monotone, t-tough theorem which is valid for any $t \leq 1$.

2 A Best Monotone t-Tough Theorem for $t \geq 1$

We first give a best monotone t-tough theorem for $t \geq 1$.

Theorem 2.1. *Let $t \geq 1$, $n \geq \lceil t \rceil + 2$, and let $\pi = (d_1 \leq \cdots \leq d_n)$ be a graphical sequence. If*

$$(*t) \qquad d_{\lfloor i/t \rfloor} \geq i+1 \quad \text{or} \quad d_{n-i} \geq n - \lfloor i/t \rfloor, \quad \text{for } t \leq i < \frac{tn}{(t+1)},$$

then π is forcibly t-tough.

Clearly, property $(*t)$ in Theorem 2.1 is monotone. Furthermore, if π does not satisfy $(*t)$ for some i with $t \leq i < tn/(t+1)$, then π is majorized by $\pi' = i^{\lfloor i/t \rfloor}\,(n - \lfloor i/t \rfloor - 1)^{n-i-\lfloor i/t \rfloor}\,(n-1)^i$, which has the non-$t$-tough realization $K_i + \big(\overline{K_{\lfloor i/t \rfloor}} \cup K_{n-i-\lfloor i/t \rfloor}\big)$. Thus $(*t)$ in Theorem 2.1 is also weakly optimal, and so Theorem 2.1 is best monotone by Theorem 1.3. Finally, note that when $t = 1$, $(*t)$ reduces to Chvátal's hamiltonian condition in Theorem 1.1.

Proof of Theorem 2.1: Suppose π satisfies $(*t)$ for some $t \geq 1$ and $n \geq \lceil t \rceil + 2$, but π has a realization G which is not t-tough. Then there exists a set $X \subseteq V(G)$ that is maximal with respect to $\omega(G - X) \geq 2$ and $\dfrac{|X|}{\omega(G-X)} < t$. Let $x \doteq |X|$

and $w \doteq \omega(G-X)$, so that $w \geq \lfloor x/t \rfloor + 1$. Also, let $H_1, H_2, \ldots, H_w$ denote the components of $G-X$, with $|H_1| \geq |H_2| \geq \cdots \geq |H_w|$, and let $h_j \doteq |H_j|$ for $j = 1, \ldots, w$. By adding edges (if needed) to G, we may assume $\langle X \rangle$ is complete, and each $\langle H_j \rangle$ is complete and completely joined to X.

Set $i \doteq x + h_2 - 1$.

Claim 1. $i \geq t$.

Proof: It is enough to show that $x \geq t$. Assume instead that $x < t$. Define $X' \doteq X \cup \{v\}$, with $v \in H_1$. If $h_1 \geq 2$, then

$$\frac{|X'|}{\omega(G-X')} = \frac{x+1}{\omega(G-X)} < \frac{t+1}{2} \leq t,$$

which contradicts the maximality of X. Similarly, if $h_1 = 1$ and $w \geq 3$, then

$$\frac{|X'|}{\omega(G-X')} = \frac{x+1}{\omega(G-X)-1} < \frac{t+1}{2} \leq t,$$

also a contradiction. Finally, if $h_1 = 1$ and $w = 2$, then G is the graph $K_{n-2} + \overline{K_2}$ with $n-2 = x < t$, contradicting $n \geq \lceil t \rceil + 2$. □

Claim 2. $i < \dfrac{tn}{t+1}$

Proof: Note that $n = x + h_1 + h_2 + \cdots + h_w \geq x + 2h_2 + w - 2$. Since $x < tw$, we obtain

$$\begin{aligned} i \;=\; x + h_2 - 1 \;&=\; \frac{tx + x + (t+1)(h_2-1)}{t+1} \\ &< \frac{t(x + w + (1+1/t)(h_2-1))}{t+1} \;\leq\; \frac{t(x+2h_2+w-2)}{t+1} \;\leq\; \frac{tn}{t+1}. \end{aligned}$$

□

By the claims we have $t \leq i < \dfrac{tn}{t+1}$. Next note that

$$d_{\lfloor i/t \rfloor} \;=\; d_{\lfloor (x+h_2-1)/t \rfloor} \;\leq\; d_{\lfloor x/t \rfloor + h_2 - 1} \;\leq\; d_{w+h_2-2} \;\leq\; d_{(h_2+\cdots+h_w)} \;=\; x + h_2 - 1 \;=\; i.$$

However, we also have

$$\begin{aligned} d_{n-i} \;&\leq\; d_{n-x} \;=\; x + h_1 - 1 \;=\; n - h_2 - (h_3 + \cdots + h_w) - 1 \;\leq\; n - (w + h_2 - 1) \\ &< n - \left(\frac{x}{t} + h_2 - 1\right) \;\leq\; n - \frac{x+h_2-1}{t} \;=\; n - i/t \;\leq\; n - \lfloor i/t \rfloor, \end{aligned}$$

contradicting $(*t)$. ■

3 The Number of Chvátal-Type Conditions in Best Monotone Theorems

In this section we provide a theory that allows us to lower bound the number of degree sequence conditions required in a best monotone P-theorem.

Recall that a *Chvátal-type condition* for n-sequences $(d_1 \le d_2 \le \cdots \le d_n)$ is a condition of the form

$$d_{i_1} \ge k_{i_1} \ \vee \ d_{i_2} \ge k_{i_2} \ \vee \ \ldots \ \vee \ d_{i_r} \ge k_{i_r},$$

where all i_j and k_{i_j} are integers, with $1 \le i_1 < i_2 < \cdots < i_r \le n$ and $1 \le k_{i_1} \le k_{i_2} \le \cdots \le k_{i_r} \le n$. Given an n-sequence $\pi = (k_1 \le k_2 \le \cdots \le k_n)$, let $C(\pi)$ denote the Chvátal-type condition:

$$d_1 \ge k_1 + 1 \ \vee \ d_2 \ge k_2 + 1 \ \vee \ \ldots \ \vee \ d_n \ge k_n + 1.$$

Intuitively, $C(\pi)$ is the weakest condition that 'blocks' π. For instance, if $\pi = 2^2 3^3 5$, then $C(\pi)$ is

$$d_1 \ge 3 \ \vee \ d_2 \ge 3 \ \vee \ d_3 \ge 4 \ \vee \ d_4 \ge 4 \ \vee \ d_5 \ge 4 \ \vee \ d_6 \ge 6. \tag{1}$$

Since n-sequences are assumed to be nondecreasing, $d_1 \ge 3$ implies $d_2 \ge 3$, etc. Also, we cannot have $d_i \ge n$, so the condition $d_6 \ge 6$ is redundant. Hence (1) can be simplified to the equivalent Chvátal-type condition

$$d_2 \ge 3 \ \vee \ d_5 \ge 4, \tag{2}$$

and we use (1) $\cong$ (2) to denote this equivalence.

Conversely, given a Chvátal-type condition c, let $\Pi(c)$ denote the minimal n-sequence that majorizes all sequences which violate c ($\Pi(c)$ may not be graphical). So if c is the condition in (2) and $n = 6$, then $\Pi(c)$ is $2^2 3^3 5$. Of course, $\Pi(c)$ itself violates c. Note that C and Π are inverses: For any Chvátal-type condition c we have $C(\Pi(c)) \cong c$, and for any n-sequence π we have $\Pi(C(\pi)) = \pi$.

Given a graph property P, we call a Chvátal-type degree condition c *P-weakly-optimal* if any sequence π (not necessarily graphical) which does not satisfy c is majorized by a degree sequence which is not forcibly P. In particular, each of the $\left\lfloor \frac{1}{2}(n-1) \right\rfloor$ conditions in Chvátal's hamiltonian theorem is weakly optimal.

Next consider the poset whose elements are the graphical sequences of length n, with the majorization relation $\pi \le \pi'$ as the partial order relation. We call this poset the *n-degree-poset*. Posets of integer sequences with a different order relation were previously used by Aigner & Triesch [1] in their work on graphical sequences.

Given a graph property P, consider the set of n-vertex graphs without property P which are edge-maximal in this regard. The degree sequences of these edge-maximal, non-P graphs induce a subposet of the n-degree-poset, called the *P-subposet*. We refer to the maximal elements of this P-subposet as *sinks*, and denote their number by $s(n, P)$.

We first prove the following lemma.

Lemma 3.1. *Let P be a graph property. If a sink π of the P-subposet violates a P-weakly-optimal Chvátal-type condition c, then $c \cong C(\pi)$.*

Proof: Since π violates c, $\pi \leq \Pi(c)$. Since $\Pi(c)$ violates c, and c is P-weakly-optimal, there is a sequence $\pi' \geq \Pi(c)$ such that π' has a non-P realization. But $\pi' \leq \pi''$ for some sink π'', giving $\pi \leq \Pi(c) \leq \pi' \leq \pi''$. Since distinct sinks are incomparable, $\pi = \pi''$. This implies $\Pi(c) = \pi$, and thus $c \cong C(\Pi(c)) \cong C(\pi)$. ■

Theorem 3.2. *Let P be a graph property. Then any P-theorem for n-sequences whose hypothesis consists solely of P-weakly-optimal Chvátal-type conditions must contain at least $s(n, P)$ such conditions.*

Proof: Consider a P-theorem whose hypothesis consists solely of P-weakly-optimal Chvátal-type conditions. By Lemma 3.1, a sink π satisfies every Chvátal-type condition besides $C(\pi)$. So the theorem must include all the Chvátal-type conditions $C(\pi)$, as π ranges over the $s(n, P)$ sinks. ■

On the other hand, it is easy to see that if we take the collection of Chvátal-type conditions $C(\pi)$ for all sinks π in the P-subposet, then this gives a best monotone P-theorem.

We do not have a comparable result for P-theorems if we do not require the conditions to be P-weakly-optimal, let alone if we consider conditions that are not of Chvátal-type. On the other hand, all results we have discussed so far, and most of the forcibly P-theorems we know in the literature, involve only P-weakly-optimal Chvátal-type degree conditions.

4 Best Monotone t-Tough Theorems for $t \leq 1$

Using the terminology from Section 3, it follows that Theorem 2.1 gives, for $t \geq 1$, a best monotone t-tough theorem using a linear number (in n) of weakly optimal Chvátal-type conditions. On the other hand, we now show that for any integer $k \geq 1$, a best monotone $1/k$-tough theorem for n-sequences requires at least $f(k) \cdot n$ weakly optimal Chvátal-type conditions, where $f(k)$ grows superpolynomially as $k \to \infty$. In view of Theorem 3.2, to prove this assertion it suffices to prove the following lemma.

Lemma 4.1. *Let $k \geq 2$ be an integer, and let $n = m(k+1)$ for some integer $m \geq 9$. Then the number of (1/k-tough)-subposet sinks in the n-degree-subposet is at least* $\dfrac{p(k-1)}{5(k+1)}n$*, where p denotes the integer partition function.*

Recall that the integer partition function $p(r)$ counts the number of ways a positive integer r can be written as a sum of positive integers. Since $p(r) \sim \dfrac{1}{4r\sqrt{3}} e^{\pi\sqrt{2r/3}}$ as $r \to \infty$ [5], $f(k) = \dfrac{p(k-1)}{5(k+1)}$ grows superpolynomially as $k \to \infty$.

Proof of Lemma 4.1: Consider the collection $\mathcal{C}$ of all connected graphs on n vertices which are edge-maximally not-($1/k$-tough). Each $G \in \mathcal{C}$ has the form $G = K_j + (K_{c_1} \cup \cdots \cup K_{c_{kj+1}})$, where $j < n/(k+1) = m$, so that $1 \leq j \leq m-1$, and $c_1 + \cdots + c_{kj+1}$ is a partition of $n-j$. Assuming $c_1 \leq \cdots \leq c_{kj+1}$, the degree sequence of G becomes $\pi \doteq (c_1+j-1)^{c_1} \ldots (c_{kj+1}+j-1)^{c_{kj+1}} (n-1)^j$. Note that π cannot be majorized by the degrees of any disconnected graph on n vertices, since a disconnected graph has no vertex of degree $n-1$. By a *complete degree* of a degree sequence we mean an entry in the sequence equal to $n-1$.

Partition the degree sequences of the graphs in $\mathcal{C}$ into $m-1$ groups, where the sequences in the j^{th} group, $1 \leq j \leq m-1$, are precisely those containing j complete degrees. We establish two basic properties of the j^{th} group.

Claim 1. *There are exactly $p_{kj+1}\big((k+1)(m-j)-1\big)$ sequences in the j^{th} group.*

Here $p_\ell(r)$ denotes the number of partitions of integer r into at most ℓ parts, or equivalently the number of partitions of r with largest part at most ℓ.

Proof of Claim 1: Each sequence in the j^{th} group corresponds uniquely to a set of $kj+1$ component sizes which sum to $n-j$. If we subtract 1 from each of those component sizes, we obtain a corresponding collection of $kj+1$ integers (some possibly 0) which sum to $n-j-(kj+1) = (k+1)(m-j)-1$, and which therefore form a partition of $(k+1)(m-j)-1$ into at most $kj+1$ parts. $\square$

Claim 2. *No sequence in the j^{th} group majorizes another sequence in the j^{th} group.*

Proof: Suppose the sequences $\pi \doteq (c_1+j-1)^{c_1} \ldots (c_{kj+1}+j-1)^{c_{kj+1}} (n-1)^j$ and $\pi' \doteq (c'_1+j-1)^{c'_1} \ldots (c'_{kj+1}+j-1)^{c'_{kj+1}} (n-1)^j$ are in the j^{th} group, with $\pi \geq \pi'$. Deleting the j complete degrees from each sequence gives sequences $\sigma \doteq (c_1-1)^{c_1} \ldots (c_{kj+1}-1)^{c_{kj+1}}$ and $\sigma' \doteq (c'_1-1)^{c'_1} \ldots (c'_{kj+1}-1)^{c'_{kj+1}}$, with $\sigma \geq \sigma'$.

Let m be the smallest index with $c_m \neq c'_m$; since $\sigma \geq \sigma'$, we have $c_m > c'_m$. In particular, $c_1+\cdots+c_m > c'_1+\cdots+c'_m$. But $c_1+\cdots+c_{kj+1} = c'_1+\cdots+c'_{kj+1} = n-j$, and so there exists a smallest index $\ell > m$ with $c_1+\cdots+c_\ell \leq c'_1+\cdots+c'_\ell$. In particular, $c_\ell < c'_\ell$. Since $c'_1+\cdots+c'_{\ell-1} < c_1+\cdots+c_{\ell-1} < c_1+\cdots+c_\ell \leq c_1+\cdots+c'_\ell$, we have $d_{c_1+\cdots+c_\ell} = c_\ell - 1 < c'_\ell - 1 = d'_{c_1+\cdots+c_\ell}$, and thus $\sigma \not\geq \sigma'$, a contradiction. $\square$

Since $K_j + (K_{c_1} \cup \cdots \cup K_{c_{kj+1}})$ has n vertices, $K_{c_{kj+1}}$ has at most $n-j-kj$ vertices. This means the largest possible noncomplete degree in a sequence in the j^{th} group is $j + (n-j-kj-1) = n-kj-1$. Using this observation we can prove the following.

Claim 3. *If a sequence $\pi = \cdots d^{d-j+1}\,(n-1)^j$ in the j^{th} group has largest noncomplete degree $d \geq n-k(j+1)$, then π is not majorized by any sequence in the i^{th} group, for $i \geq j+1$.*

In particular, such a π is a sink, since π is certainly not majorized by another sequence in the j^{th} group by Claim 2, nor by a sequence in groups $1, 2, \ldots, j-1$, since any such sequence has fewer than j complete degrees.

Proof of Claim 3: If $d \geq n-k(j+1)$, then the $d+1$ largest degrees $d^{d-j+1}\,(n-1)^j$ in π could be majorized only by complete degrees in a sequence in group $i \geq j+1$, since the largest noncomplete degree in any sequence in group i is at most $n-ki-1 < n-k(j+1)$. There are only $i \leq m-1$ complete degrees in a sequence in group i. On the other hand, since $j+1 \leq i < m$, we have $d+1 \geq n-k(j+1)+1 > m(k+1)-km+1 = m+1 > m-1$, a contradiction. $\square$

So by Claim 3, the sequences π in the j^{th} group which could possibly be nonsinks (i.e., majorized by a sequence in group i, for some $i \geq j+1$), must have largest noncomplete degree at most $n-k(j+1)-1$. So in a graph $G \in \mathcal{C}$, $G = K_j + (K_{c_1} \cup \cdots \cup K_{c_{kj+1}})$, which realizes a nonsink π, each of the K_c's must have order at most $(n-k(j+1)-1)-j+1 = (k+1)(m-j)-k$. Subtracting 1 from the order of each of these components gives a sequence of $kj+1$ integers (some possibly 0) which sum to $(n-j)-(kj+1) = (k+1)(m-j)-1$, and which have largest part at most $(k+1)(m-j)-k-1 = (k+1)(m-j-1)$. Thus there are exactly $p_{(k+1)(m-j-1)}\big((k+1)(m-j)-1\big)$ such sequences, and so there are at most this many nonsinks in the j^{th} group. Setting $N(j) \doteq (k+1)(m-j)-1$, so that $(k+1)(m-j-1) = N(j)-k$, this becomes at most $p_{N(j)-k}\big(N(j)\big)$ nonsinks in the j^{th} group of sequences.

But by Claim 1, there are exactly $p_{kj+1}\big(N(j)\big)$ sequences in group j, and so the number of sinks in the j^{th} group is at least $p_{kj+1}\big(N(j)\big) - p_{N(j)-k}\big(N(j)\big)$.

Note that $p_{kj+1}(N(j))$ reduces to $p(N(j))$ if $kj+1 \geq N(j)$. However, $kj+1 \geq N(j)$ is equivalent to $j \geq \dfrac{(k+1)m-2}{2k+1}$. Since $k \geq 2$, the inequality $j \geq \dfrac{(k+1)m-2}{2k+1}$ holds if $j \geq \frac{3}{5}m$. Thus $p_{kj+1}(N(j)) = p(N(j))$ holds for $j \geq \frac{3}{5}m$.

On the other hand, for $j \leq m-2$ we can show the following.

Claim 4. *If $j \le m-2$, then*

$$p\big(N(j)\big) - p_{N(j)-k}\big(N(j)\big) \;=\; 1 + p(1) + \cdots + p(k-1) \;\ge\; p(k-1).$$

Proof: Note that if $j \le m-2$, then $k < \frac{1}{2}N(j)$. The left side of the equality in the claim counts partitions of $N(j)$ with largest part at least $N(j)-(k-1)$. The right side counts the same according to the exact order $N(j)-\ell$, $0 \le \ell \le k-1$, of the largest part in the partition, using that the largest part is unique since $N(j)-\ell \ge N(j)-(k-1) > \frac{1}{2}N(j)$. □

Completing the proof of Lemma 4.1, we find that the number of sinks in the ($1/k$-tough)-subposet of the n-degree-poset is at least

$$\begin{aligned}\sum_{j=\lceil 3m/5\rceil}^{m-2} \big[p_{kj+1}\big(N(j)\big) - p_{N(j)-k}\big(N(j)\big)\big] \;&=\; \sum_{j=\lceil 3m/5\rceil}^{m-2} \big[p\big(N(j)\big) - p_{N(j)-k}\big(N(j)\big)\big] \\ &\ge\; \sum_{j=\lceil 3m/5\rceil}^{m-2} p(k-1) \;\ge\; \big(\tfrac{2}{5}m - \tfrac{9}{5}\big)p(k-1) \\ &=\; \Big(\frac{2n}{5(k+1)} - \frac{9}{5}\Big)p(k-1) \;\ge\; \frac{n}{5(k+1)}p(k-1),\end{aligned}$$

as asserted, since $n = m(k+1) \ge 9(k+1)$ implies $\dfrac{2n}{5(k+1)} - \dfrac{9}{5} \ge \dfrac{n}{5(k+1)}$. ■

Combining Lemma 4.1 with Theorem 3.2 gives the promised superpolynomial growth in the number of weakly optimal Chvátal-type conditions for $1/k$-toughness.

Theorem 4.2. *Let $k \ge 2$ be an integer, and let $n = m(k+1)$ for some integer $m \ge 9$. Then a best monotone $1/k$-tough theorem for n-sequences whose degree conditions consist solely of weakly optimal Chvátal-type conditions requires at least $\dfrac{p(k-1)n}{5(k+1)}$ such conditions, where $p(r)$ is the integer partition function.*

5 A Simple t-Tough Theorem

The superpolynomial complexity as $k \to \infty$ of a best monotone $1/k$-tough theorem suggests the desirability of finding simple t-tough theorems, when $t < 1$. We give such a theorem below. It will again be convenient to assume at first that $t = 1/k$, for some integer $k \ge 1$. Note that the conditions in the theorem are still Chvátal-type conditions.

Lemma 5.1. *Let $k \geq 1$ be an integer, $n \geq k+2$, and $\pi = (d_1 \leq \cdots \leq d_n)$ a graphical sequence. If*

(i) $d_i \geq i-k+2$ *or* $d_{n-i+k-1} \geq n-i$, *for* $k \leq i < \frac{1}{2}(n+k-1)$, *and*

(ii) $d_i \geq i$ *or* $d_n \geq n-i$, *for* $1 \leq i \leq \frac{1}{2}n$,

then π is forcibly $1/k$-tough.

Proof of Lemma 5.1: Suppose π has a realization G which is not $1/k$-tough. By (ii) and Theorem 1.2, G is connected. So we may assume (by adding edges if necessary) that there exists $X \subseteq V(G)$, with $x \doteq |X| \geq 1$, such that $G = K_x + (K_{a_1} \cup K_{a_2} \cup \cdots \cup K_{a_{kx+1}})$, where $1 \leq a_1 \leq a_2 \leq \cdots \leq a_{kx+1}$.

Set $i \doteq x + k - 2 + a_{kx}$.

Claim 1. $k \leq i < \frac{1}{2}(n+k-1)$

Proof: The fact that $i \geq k$ follows immediately from the definition of i. Since $kx - x - k + 1 = (k-1)(x-1) \geq 0$, we have

$$kx - 1 \;\geq\; x + k - 2. \tag{3}$$

This leads to

$$\begin{aligned} n \;=\; x + \sum_{j=1}^{kx-1} a_j + a_{kx} + a_{kx+1} \;&\geq\; x + kx - 1 + 2a_{kx} \\ &\geq\; 2x + k - 2 + 2a_{kx} \;=\; 2i - k + 2, \end{aligned}$$

which is equivalent to $i < \frac{1}{2}(n+k-1)$. $\square$

Claim 2. $d_i \leq i - k + 1$.

Proof: From (3) we get

$$i \;=\; x + k - 2 + a_{kx} \;\leq\; kx - 1 + a_{kx} \;\leq\; \sum_{j=1}^{kx} a_j. \tag{4}$$

This gives $d_i \leq x + (a_{kx} - 1) \;=\; i - k + 1$. $\square$

Claim 3. $d_{n-i+k-1} < n - i$.

Proof: We have $n - i + k - 1 = n - x - a_{kx} + 1 \leq \sum_{j=1}^{kx+1} a_j$. Thus, using the bound (4) for i,

$$d_{n-i+k-1} \;\leq\; x + a_{kx+1} - 1 \;<\; n - \sum_{j=1}^{kx} a_j \;\leq\; n - i.$$

$\square$

Claims 1, 2 and 3 together contradict condition (i), completing the proof of the lemma ■

We can extend Lemma 5.1 to arbitrary $t \leq 1$ by letting $k = \lfloor 1/t \rfloor$.

Theorem 5.2. *Let $t \leq 1$, $n \geq \lfloor 1/t \rfloor + 2$, and $\pi = (d_1 \leq \cdots \leq d_n)$ a graphical sequence. If*

(i) $d_i \geq i - \lfloor 1/t \rfloor + 2$ *or* $d_{n-i+\lfloor 1/t \rfloor -1} \geq n - i$, *for* $\lfloor 1/t \rfloor \leq i < \frac{1}{2}\big(n + \lfloor 1/t \rfloor - 1\big)$, *and*

(ii) $d_i \geq i$ *or* $d_n \geq n - i$, *for* $1 \leq i \leq \frac{1}{2}n$,

then π is forcibly t-tough.

Proof: Set $k = \lfloor 1/t \rfloor \geq 1$. If π satisfies conditions (i), (ii) in Theorem 5.2, then π satisfies conditions (i), (ii) in Lemma 5.1, and so is forcibly $1/k$-tough. But $k = \lfloor 1/t \rfloor \leq 1/t$ means $1/k \geq t$, and so π is forcibly t-tough. ■

In summary, if $\dfrac{1}{k+1} < t \leq \dfrac{1}{k}$ for some integer $k \geq 1$, then Theorem 5.2 declares π forcibly t-tough precisely if Lemma 5.1 declares π forcibly $1/k$-tough.

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