TREEWIDTH OF CARTESIAN PRODUCTS OF HIGHLY CONNECTED GRAPHS

DAVID R. WOOD

ABSTRACT. The following theorem is proved: For all k-connected graphs G and H each with at least n vertices, the treewidth of the cartesian product of G and H is at least $k(n-2k+2)-1$. For $n \gg k$ this lower bound is asymptotically tight for particular graphs G and H. This theorem generalises a well known result about the treewidth of planar grid graphs.

Treewidth is a graph parameter of fundamental importance in graph minor theory, with numerous applications in algorithmic theory and practical computing. The planar grid graph is a key example for treewidth, in that the $n \times n$ planar grid has treewidth n, and every graph with sufficiently large treewidth contains the $n \times n$ planar grid as a minor.

Motivated by the fact that the planar grid can be defined to be the cartesian product of two paths, in this note we consider the treewidth of cartesian products of general graphs. Our main result is a lower bound on the treewidth of the cartesian product of two highly connected graphs; see [1, 5–12, 14, 15] for related results. Before stating the theorem, we introduce the necessary definitions.

The *cartesian product* of graphs G and H, denoted by $G \square H$, is the graph with vertex set $V(G \square H) := V(G) \times V(H)$, where $(v,x)(w,y)$ is an edge of $G \square H$ if and only if $vw \in E(G)$ and $x = y$, or $v = w$ and $xy \in E(H)$. For each vertex $v \in V(G)$ the subgraph of $G \square H$ induced by $\{(v,w) : w \in V(H)\}$ is isomorphic to H; we call it the *v-copy* of H, denoted by H_v. Similarly, for each vertex $w \in V(H)$ the subgraph of $G \square H$ induced by $\{(v,w) : v \in V(G)\}$ is isomorphic to G; we call it the *w-copy* of G, denoted by G_w.

A *tree decomposition* of a graph G consists of a tree T and a set $\{T_x \subseteq V(G) : x \in V(T)\}$ of 'bags' of vertices of G indexed by T, such that

- for each edge $vw \in E(G)$, some bag T_x contains both v and w, and
- for each vertex $v \in V(G)$, the set $\{x \in V(T) : v \in T_x\}$ induces a non-empty (connected) subtree of T.

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Department of Mathematics and Statistics, The University of Melbourne, Australia (woodd@unimelb.edu.au). Supported by QEII Research Fellowship from the Australian Research Council.

The *width* of the tree decomposition is $\max\{|T_x| : x \in V(T)\} - 1$. The *treewidth* of G, denoted by $\mathsf{tw}(G)$, is the minimum width of a tree decomposition of G. For example, G has treewidth 1 if and only if G is a forest.

Let G be a graph. Two subgraphs X and Y of G *touch* if $X \cap Y \neq \emptyset$ or there is an edge of G between X and Y. A *bramble* in G is a set of pairwise touching connected subgraphs. A set S of vertices in G is a *hitting set* of a bramble $\mathcal{B}$ if S intersects every element of $\mathcal{B}$. The *order* of $\mathcal{B}$ is the minimum size of a hitting set. The canonical example of a bramble of order ℓ is the set of crosses (union of a row and column) in the $\ell \times \ell$ grid. The following 'Treewidth Duality Theorem' shows the intimate relationship between treewidth and brambles; see [2] for an alternative proof.

Theorem 1 ([13]). *A graph G has treewidth at least ℓ if and only if G contains a bramble of order at least $\ell + 1$.*

This paper proves the following general lower bound on the treewidth of cartesian products of highly connected graphs.

Theorem 2. *For all k-connected graphs G and H each with at least n vertices,*

$$\mathsf{tw}(G \,\square\, H) \geq k(n - 2k + 2) - 1 \ .$$

Proof. To prove this theorem, we construct a bramble $\mathcal{B}$ in $G \,\square\, H$ and then apply Theorem 1. If $n \leq 2k - 2$ then the claim is vacuously true. Now assume that $n \geq 2k - 1$.

Let $\mathcal{B}$ be the set of all subgraphs X of $G \,\square\, H$ formed in the following way. Let S be a set of $2k - 1$ vertices in G. Let T be a set of $2k - 1$ vertices in H. Initialise X to be the union of $\cup\{H_v : v \in S\}$ and $\cup\{G_w : w \in T\}$. Now delete vertices from X such that at most $k - 1$ vertices are deleted from H_v for each $v \in S$, and at most $k - 1$ vertices are deleted from G_w for each $w \in T$. We claim that $\mathcal{B}$ is a bramble of $G \,\square\, H$.

First we prove that each $X \in \mathcal{B}$ is connected. Say X is defined with respect to $S \subseteq V(G)$ and $T \subseteq V(H)$. First note that for each $v \in S$ and $w \in T$, since H_v and G_w are k-connected, $H_v \cap X$ and $G_w \cap X$ are connected. Let Q be the bipartite graph with $V(Q) = S \cup T$, where for all $v \in S$ and $w \in T$, the edge vw is in Q whenever the vertex (v, w) is in X (i.e., it was not deleted). The degree in Q of each vertex $v \in S$ is at least $(2k - 1) - (k - 1) = k$ since at most $k - 1$ vertices were deleted from H_v. Similarly, each vertex in T has degree at least k in Q. So Q has $2k - 1$ vertices in each colour class, and minimum degree k. If Q is disconnected then some component H of Q contains at most $k - 1$ vertices in S, implying that the vertices in $H \cap T$ have degree at most $k - 1$. Thus Q is connected. Now consider two vertices (v_1, w_1) and (v_2, w_2) in X. Thus v_1w_1 and v_2w_2 are edges of Q. Since Q is connected, there is a path P in Q between one endpoint of v_1w_1 and one endpoint of v_2w_2. For each 2-edge path vwv' of P, since $G_w \cap X$ is connected, there is a path in X between the vertices (v, w) and (v', w). Similarly, for each 2-edge path wvw' of P, there is a path in X between the vertices (v, w) and (v, w'). The union of these paths is a walk between (v_1, w_1) and (v_2, w_2) in X. Therefore X is connected, as claimed.

Now we prove that X and X' touch for all $X, X' \in \mathcal{B}$. Say X is defined with respect to S and T, and X' is defined with respect to S' and T'. At most $(2k-1)(k-1)$ vertices in $S \times T'$ were deleted in the construction of X, and at most $(2k-1)(k-1)$ vertices in $S \times T'$ were deleted in the construction of X'. Since $|S \times T'| = (2k-1)^2 > 2(2k-1)(k-1)$, some vertex $(v, w) \in S \times T'$ was deleted in neither the construction of X nor the construction of X'. Hence (v, w) is in both $H_v \cap X$ and $G_w \cap X$. Thus X and X' have a common vertex.

Therefore $\mathcal{B}$ is a bramble. Let J be a hitting set of $\mathcal{B}$. We claim that $|J| \geq k(n-2k+2)$. Let $S_0 := \{v \in V(G) : |V(H_v) \cap J| \leq k-1\}$ and $T_0 := \{w \in V(H) : |V(G_w) \cap J| \leq k-1\}$. If $|S_0| \leq 2k-2$ then at least $n-(2k-2)$ pairwise-disjoint copies of H contain at least k vertices in J, implying $|J| \geq k(n-2k+2)$, as claimed. Otherwise, $|S_0| \geq 2k-1$. Similarly, $|T_0| \geq 2k-1$. Let $S \subseteq S_0$ and $T \subseteq T_0$ such that $|S| = |T| = 2k-1$. Let X be the union of $\cup\{H_v - J : v \in S\}$ and $\cup\{G_w - J : w \in T\}$. Thus $X \in \mathcal{B}$ (since $|V(H_v \cap J)| \leq k-1$ and $|V(G_w \cap J)| \leq k-1$ for each $v \in S$ and $w \in T$). However, $X \cap J = \emptyset$. Thus J is not a hitting set for $\mathcal{B}$. Hence the order of $\mathcal{B}$ is at least $k(n-2k+2)$. The result follows from Theorem 1. $\square$

We now show that the bound in Theorem 2 is tight (ignoring lower order terms and assuming $n \gg k$). The *bandwidth* of a graph G, denoted by $\mathsf{bw}(G)$, is the minimum, taken over all bijections $\phi : V(G) \to \{1, 2, \dots, |V(G)|\}$, of the maximum, taken over all edges $vw \in E(G)$, of $|\phi(v) - |\phi(w)|$. It is well known that $\mathsf{tw}(G) \leq \mathsf{bw}(G)$; see [3]. Let P_n^k be the k-th power of a path, which has vertex set $\{1, 2, \dots, n\}$, where ij is an edge if and only if $|i-j| \leq k$. Clearly P_n^k is k-connected. Let ϕ be the vertex ordering of $P_n^k \,\square\, P_n^k$ defined by $\phi((x, y)) = (x-1)n+y$. Each edge $(x, y)(x, y')$ has width $|y'-y| \leq k$, and each edge $(x, y)(x', y)$ has width $|(x'-x)n| \leq kn$. Hence $\mathsf{tw}(P_n^k \,\square\, P_n^k) \leq \mathsf{bw}(P_n^k \,\square\, P_n^k) \leq kn$. (This upper bound can be slightly improved by ordering the vertices with respect to the function $x(n+1) + yn$.)

In fact, there is a much broader class of graphs that provide an upper bound only slightly weaker than kn. Let G and H be chordal graphs with n vertices and connectivity k. It is well known that G and H have clique-number $k+1$ and treewidth k. A tree decomposition of G with width k can be easily turned into a tree decomposition of $G \,\square\, H$ with width $(k+1)n-1$; see [7, 14]. Thus $\mathsf{tw}(G \,\square\, H) \leq (k+1)n-1$.

We expect that the dependence on k in Theorem 2 can be slightly improved (although it is not obvious how to do so). For example, Theorem 2 with $k = 1$ implies that the $n \times n$ grid has treewidth at least $n-1$, whereas it actually has treewidth n; see [4] for a proof. Another interesting example is the toroidal grid graph $C_n \,\square\, C_n$. By Theorem 2 with $k = 2$ and since C_n is a subgraph of P_n^2,

$$2n - 5 \leq \mathsf{tw}(C_n \,\square\, C_n) \leq \mathsf{tw}(P_n^2 \,\square\, P_n^2) \leq 2n \enspace .$$

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