# TILING 3-UNIFORM HYPERGRAPHS WITH $K_4^3 - 2e$

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ABSTRACT. Let  $K_4^3 - 2e$  denote the hypergraph consisting of two triples on four points. For an integer n, let  $t(n, K_4^3 - 2e)$  denote the smallest integer d so that every 3-uniform hypergraph G of order n with minimum pair-degree  $\delta_2(G) \ge d$  contains  $\lfloor n/4 \rfloor$  vertex-disjoint copies of  $K_4^3 - 2e$ . Kühn and Osthus [4] proved that  $t(n, K_4^3 - 2e) = \frac{n}{4}(1 + o(1))$  holds for large integers n. Here, we prove the exact counterpart, that for all sufficiently large integers n divisible by 4,

$$t(n, K_4^3 - 2e) = \begin{cases} \frac{n}{4} & \text{when } \frac{n}{4} \text{ is odd,} \\ \frac{n}{4} + 1 & \text{when } \frac{n}{4} \text{ is even.} \end{cases}$$

A main ingredient in our proof is the recent 'absorption technique' of Rödl, Ruciński and Szemerédi.

### 1. INTRODUCTION

For a fixed k-graph  $H_0$  of order m, we say that a given k-graph G of order n is  $H_0$ -tileable if G contains, as subhypergraphs,  $\lfloor n/m \rfloor$  vertex-disjoint copies of  $H_0$ . Now, suppose G has vertex set V, and for an integer  $1 \le \ell \le k$ , let  $U \in \binom{V}{\ell}$  be given. As is customary, let

$$N(U) = N_G(U) = \left\{ W \in \binom{V}{k-\ell} : U \cup W \in E(G) \right\} \text{ and } \delta_\ell(G) = \min\left\{ |N(U)| : U \in \binom{V}{\ell} \right\}$$

denote, respectively, the neighborhood of U in G, and the  $\ell$ -degree of G. Define  $t_{\ell}^k(n, H_0)$  to be the smallest integer d so that every k-graph G of order n for which  $\delta_{\ell}(G) \ge d$  holds is  $H_0$ -tileable.

In the case of graphs (k = 2),  $t_1^2(n, H_0)$  is known, up to an additive constant, for every fixed graph  $H_0$  (see [5]). Furthermore, there are some graphs  $H_0$  for which  $t_1^2(n, H_0)$  is known exactly. The most celebrated such result is the Hajnal-Szemerédi theorem [2], which says that for the *r*-clique  $H_0 = K_r$  and for *n* divisible by *r*,

$$t_1^2(n, K_r) = \left(1 - \frac{1}{r}\right)n.$$

A recent result of Wang [10] shows that for all integers n divisible by 4,  $t_1^2(n, C_4) = \frac{n}{2}$ . This result is a special case of the well-known El-Zahar conjecture, and had been independently conjectured by Erdős and Faudree.

In the case of hypergraphs  $(k \ge 3)$ , much less is known about tiling problems. For only the k-edge  $H_0 = K_k^k$  (the tiling of which is a perfect matching) is  $t_{k-1}^k(n, H_0)$  known for all  $k \ge 3$ . This significant result is due to Rödl, Ruciński and Szemerédi [9], and asserts that for all sufficiently large integers n divisible by k,

$$t_{k-1}^k(n, K_k^k) = \frac{n}{2} - k + \varepsilon_{k,n}, \text{ where } \varepsilon_{k,n} \in \left\{\frac{3}{2}, 2, \frac{5}{2}, 3\right\}$$

is determined by explicit divisibility conditions on n and k.

We are interested in tilings when k = 3 and  $\ell = 2$ , where some interesting results have recently developed. (In what follows, we abbreviate  $t_2^3(n, H_0)$  to  $t(n, H_0)$ .) As usual, let  $K_4^3$  denote the complete 3-graph on 4 vertices. Let  $K_4^3 - e$  denote its subhypergraph consisting of 3 edges, and

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let  $K_4^3 - 2e$  denote its subhypergraph consisting of 2 edges. Kühn and Osthus [4] proved that  $t(n, K_4^3 - 2e) = (1 + o(1))n/4$ . Recently, Lo and Markström [6, 7] have shown that  $t(n, K_4^3 - e) = (1 + o(1))n/2$  and that  $t(n, K_4^3) = (1 + o(1))3n/4$ . Keevash and Mycroft [3] showed the exact counterpart that, for sufficiently large integers n divisible by 4,  $t(n, K_4^3) = (3n/4) - \varepsilon_n$ , where  $\varepsilon_n = 2$  if 8|n and  $\varepsilon_n = 1$  otherwise. We shall prove the following exact result for  $K_4^3 - 2e$ .

**Theorem 1.1.** For all sufficiently large integers n divisible by 4,

$$t(n, K_4^3 - 2e) = \begin{cases} \frac{n}{4} & \text{when } \frac{n}{4} \text{ is odd,} \\ \frac{n}{4} + 1 & \text{when } \frac{n}{4} \text{ is even.} \end{cases}$$

The proof of Theorem 1.1 spans Sections 2 and 3. We mention that an essential ingredient in our proof is the 'absorption technique' (see Section 3) of Rödl, Ruciński and Szemerédi.

In the remainder of this paper, we shall make the abbreviation  $D = K_4^3 - 2e$ . (In the papers [4, 9],  $D = K_4^3 - 2e$  was abbreviated by C and  $C_4^{3,1}$ , respectively, since for those authors,  $K_4^3 - 2e$  was viewed as a type of cycle.) In the remainder of this introduction, we discuss the main concept used in the proof of Theorem 1.1, that of an ' $\varepsilon$ -extremal' 3-graph (for  $D = K_4^3 - 2e$ ).

1.1. Theorem 1.1 and  $\varepsilon$ -extremal 3-graphs. To motivate the concept of an  $\varepsilon$ -extremal 3-graph (stated in the upcoming Definition 1.2), we first observe the following constructions for the lower bounds of Theorem 1.1.

Let A be a set of  $\frac{n}{4} - 1$  vertices, and let B be a set of  $\frac{3n}{4} + 1$  additional vertices. Define  $G_0 = \binom{A \cup B}{3} \setminus \binom{B}{3}$ , and note that  $\delta_2(G_0) = \frac{n}{4} - 1$ . When  $\frac{n}{4}$  is even, add any Steiner triple system<sup>1</sup> on vertex set B to  $G_0$ , and call this hypergraph  $G_1$ , where we note that  $\delta_2(G_1) = \frac{n}{4}$ . Since  $G_i[B]$ , i = 0, 1, is D-free, every copy of D in  $G_i$  contains at least one vertex of A, and so  $G_i$  is not D-tileable.

**Definition 1.2** ( $\varepsilon$ -extremal). Let  $\varepsilon > 0$  be given, and suppose G is a 3-graph of order n. We say G is  $\varepsilon$ -extremal if there exists  $S \subset V(G)$  of size  $|S| \ge (1 - \varepsilon)\frac{3n}{4}$  for which G[S] is D-free.

While the lower bound constructions for Theorem 1.1 motivate the concept of Definition 1.2, the following fact indicates why we choose the terminology 'extremal'.

**Fact 1.3.** Let G be a 3-graph on n vertices, where n is divisible by 4, satisfying

$$\delta_2(G) \ge \begin{cases} \frac{n}{4} & \text{when } \frac{n}{4} \text{ is odd,} \\ \frac{n}{4} + 1 & \text{when } \frac{n}{4} \text{ is even.} \end{cases}$$
(1)

Then any  $S \subset V(G)$  for which G[S] is D-free satisfies  $|S| \leq \frac{3}{4}n$ .

Proof. Since G[S] is *D*-free, when  $\frac{n}{4}$  is even, we have  $\frac{n}{4} + 1 \leq \delta_2(G) \leq n - (|S| - 1)$ , and the result follows. When  $\frac{n}{4}$  is odd, suppose some  $S_0 \subset V(G)$  exists of size  $\frac{3n}{4} + 1$  for which  $G[S_0]$  is *D*-free. Since  $G[S_0]$  is not an STS (since  $\frac{3n}{4} + 1 \not\equiv 1, 3 \pmod{6}$ ), some pair  $s, s' \in S_0$  satisfies  $N(s,s') \cap S_0 = \emptyset$ , in which case  $\frac{n}{4} \leq |N(s,s')| \leq n - |S_0|$ , and the result follows.  $\Box$ 

Now, the upper bounds in Theorem 1.1 follow immediately from the following two statements.

**Theorem 1.4** (Theorem 1.1 – extremal case). There exists  $\varepsilon_0 > 0$  so that, for all sufficiently large integers n divisible by 4, the following holds. Whenever G is a 3-graph of order n satisfying (1) and which is  $\varepsilon_0$ -extremal, then G is D-tileable.

We prove Theorem 1.4 in Section 2.

<sup>&</sup>lt;sup>1</sup>A Steiner triple system (STS) is a 3-graph H where  $\delta_2(H) = \Delta_2(H) = 1$ . It is well-known that an STS of order m exists if, and only if,  $m \equiv 1, 3 \pmod{6}$ .

**Theorem 1.5** (Theorem 1.1 – non-extremal case). For every  $\varepsilon > 0$  and for all sufficiently large integers n divisible by 4, the following holds. Whenever G is a 3-graph of order n satisfying (1) (see Remark 1.6), which is not  $\varepsilon$ -extremal, then G is D-tileable.

We prove Theorem 1.5 in Section 3.

**Remark 1.6.** We mention that Theorem 1.5 can be proved, for the same money, under a slightly weaker hypothesis than (1). In particular, Theorem 1.5 remains valid if one only assumes that  $\delta_2(G) \ge (n/4)(1-\gamma)$ , for a constant  $\gamma > 0$  sufficiently smaller than  $\varepsilon$ .

## 2. Proof of Theorem 1.4

We shall use the following theorem of Pikhurko [8], stated here in a less general form.

**Theorem 2.1** ([8], Theorem 3). Let *H* be a 4-partite 4-graph with 4-partition  $V(H) = V_1 \cup V_2 \cup V_3 \cup V_4$ , where  $|V_1| = \cdots = |V_4| = m$ . Let  $\delta(V_1) = \min\{|N(v_1)| : v_1 \in V_1\}$  and

$$\delta(V_2, V_3, V_4) = \min\{|N(v_2, v_3, v_4)| : v_2 \in V_2, v_3 \in V_3, v_4 \in V_4\}.$$

For  $\gamma > 0$  and a sufficiently large integer m, if

$$m\delta(V_1) + m^3\delta(V_2, V_3, V_4) \ge (1+\gamma)m^4,$$

then H contains a perfect matching.

To prove Theorem 1.4, it suffices to take  $\varepsilon_0 = 10^{-18}$ , and we shall take *n* sufficiently large, whenever needed. We write n = 4k and  $\alpha^3 = \varepsilon_0$ . Let *G* be a 3-graph of order *n* satisfying (1) which is  $\varepsilon_0$ -extremal. We prove that *G* is *D*-tileable, and will construct a *D*-tiling in stages. In particular, we will build vertex-disjoint partial *D*-tilings  $\mathcal{Q}, \mathcal{R}, \mathcal{S}$  and  $\mathcal{T}$  whose union is a *D*-tiling of *G*. To build these partial tilings, we need a few initial considerations.

To begin, let  $Z \subset V(G)$  be a maximal set for which G[Z] is D-free. Define

$$X = \left\{ x \in V(G) \setminus Z : \left| N(x) \cap \binom{Z}{2} \right| \ge (1 - \alpha) \binom{|Z|}{2} \right\},\tag{2}$$

and  $Y = V(G) \setminus (X \cup Z)$ . We estimate each of the quantities in |X| + |Y| + |Z| = 4k = n:

$$k(1 - 4\alpha^2) \le |X| \le k(1 + 3\varepsilon_0), \quad 0 \le |Y| \le 4\alpha^2 k, \quad 3k(1 - \varepsilon_0) \le |Z| \le 3k,$$
 (3)

i.e., |Y| is small, |X| is around n/4 and |Z| is around 3n/4. Indeed, the third estimate in (3) follows from our hypothesis and Fact 1.3. To see the second estimate, for  $W \subset X \cup Y$ , write G[Z, Z, W]for the collection of triples of G consisting of two vertices from Z and one vertex from W. Then,

$$(k-1)\binom{|Z|}{2} \le \left|G[Z,Z,X\cup Y]\right| \le (1-\alpha)\binom{|Z|}{2}|Y| + \binom{|Z|}{2}|X|,$$

so that  $k - 1 + \alpha |Y| \le |X| + |Y|$ . The estimate on |Z| implies that  $|X| + |Y| \le k + 3\varepsilon_0 k$ , and so we have the second estimate of (3). Finally, our bounds on |Y| and |Z| render the first estimate in (3).

Let us also check that (3) implies that

$$\forall z_1, z_2 \in Z, \ |N(z_1, z_2) \cap X| \ge (1 - \alpha)|X|.$$
 (4)

Indeed, since  $|N(z_1, z_2) \cap Z| \leq 1$ , we have

$$|N(z_1, z_2) \cap X| \ge k - 1 - |Y| \stackrel{(3)}{\ge} (1 - 5\alpha^2)k \stackrel{(3)}{\ge} \frac{1 - 5\alpha^2}{1 + 3\varepsilon_0}|X| \ge (1 - \alpha)|X|.$$

We now introduce the first of our partial D-tilings, namely, Q.

**The partial** *D***-tiling**  $\mathcal{Q}$ . Let  $\mathcal{Q}$  be a largest *D*-tiling in *G* for which each element  $D_0 \in \mathcal{Q}$  has three vertices in *Z* and one vertex in *Y*. Write  $q = |\mathcal{Q}|$ , write  $Y_{\mathcal{Q}} \subset Y$  for the set of vertices of *Y* covered by  $\mathcal{Q}$ , and write  $Z_{\mathcal{Q}} \subset Z$  for the set of vertices of *Z* covered by  $\mathcal{Q}$ . Clearly,  $|Y_{\mathcal{Q}}| = q$  and  $|Z_{\mathcal{Q}}| = 3q$ . Write  $\ell = k - |X|$ , where we note from (3) that

$$-3\varepsilon_0 k \le \ell = k - |X| \le 4\alpha^2 k.$$
(5)

For future reference, we make the following two claims.

Claim 2.2.  $q \ge \ell = k - |X|$ .

*Proof.* If  $\ell \leq 0$ , there is nothing to show. If  $\ell = 1$ , we have  $|Y \cup Z| = 3k + 1$ , and thus Fact 1.3 implies that  $G[Y \cup Z]$  contains a copy of D, which requires  $|Y| \geq 1$ . Now, if q = 0, then we could move a vertex from Y to Z, which contradicts the maximality of Z. Finally, suppose  $\ell \geq 2$ , and observe that the quantity  $|G[Z, Z, Y]| = |G[Z, Z, Y_Q]| + |G[Z, Z, Y \setminus Y_Q]|$  satisfies that

$$\begin{aligned} (\ell-1)\binom{|Z|}{2} &\leq |G[Z,Z,Y]| \leq |Y_{\mathcal{Q}}|(1-\alpha)\binom{|Z|}{2} + \left(\frac{|Z| - |Z_{\mathcal{Q}}|}{2} + |Z_{\mathcal{Q}}||Z|\right) \left|Y \setminus Y_{\mathcal{Q}}\right| \\ &= q(1-\alpha)\binom{|Z|}{2} + \left(\frac{|Z| - 3q}{2} + 3q|Z|\right) \left(|Y| - q\right) \stackrel{(3)}{\leq} q(1-\alpha)\binom{|Z|}{2} + 16\alpha^2 q|Z|k. \end{aligned}$$

Now, if  $q \leq \ell - 1$ , then

$$1 \le 1 - \alpha + 32 \frac{\alpha^2 k}{|Z| - 1} \stackrel{(3)}{\le} 1 - \alpha + 16\alpha^2,$$

a contradiction.

Note that, on account of the claim above,

$$0 \le q - \ell \stackrel{(5)}{\le} |Y| + 3\varepsilon_0 k \le |Y| + 4\alpha^2 k \stackrel{(3)}{\le} 8\alpha^2 k.$$

$$\tag{6}$$

Claim 2.3. For all  $y \in Y \setminus Y_Q$  and  $z \in Z \setminus Z_Q$ ,  $|N(y,z) \cap X| \ge (1-\alpha)|X|$ .

*Proof.* Fix  $y \in Y \setminus Y_Q$  and  $z \in Z \setminus Z_Q$ . By the maximality of Q, we have  $|N(y, z) \cap Z| \leq |Z_Q| + 1 = 3q + 1$ . As such, since  $|Y| \geq q$ , we have

$$|N(y,z) \cap X| \ge k - (3q+1) - (|Y|-1) \ge k - 4|Y| \stackrel{(3)}{\ge} (1 - 16\alpha^2)k \stackrel{(3)}{\ge} \frac{1 - 16\alpha^2}{1 + 3\varepsilon_0}|X| \ge (1 - \alpha)|X|.$$

The partial *D*-tiling  $\mathcal{R}$ . We now use (4) and Claim 2.3 to build a collection  $\mathcal{R}$  of  $|Y \setminus Y_Q|$  vertex-disjoint copies of *D*, each with 1 vertex in  $Y \setminus Y_Q$ , 1 vertex in *X*, and two vertices in  $Z \setminus Z_Q$ . For sake of argument, assume  $|Y \setminus Y_Q| \ge 1$ . Inductively, assume we have obtained  $0 \le i < |Y \setminus Y_Q|$  vertex-disjoint copies of *D*, each with 1 vertex in  $Y \setminus Y_Q$ , 1 vertex in *X*, and two vertices in  $Z \setminus Z_Q$ . Arbitrarily select an uncovered  $y' \in Y \setminus Y_Q$  and uncovered  $z'_1, z'_2 \in Z \setminus Z_Q$ , noting that the latter is possible since at most  $|Z_Q| + 2i \le 5|Y| \le |Z| - 2$  (cf. (3)) vertices in *Z* are unavailable for selection. Since  $|N(y', z'_1) \cap N(z'_1, z'_2) \cap X| \ge (1-2\alpha)|X|$ , we have at least  $(1-2\alpha)|X| - i \ge (1-2\alpha)|X| - |Y| > 0$  (cf. (3)) choices for an uncovered vertex  $x' \in X$ , to complete the  $(i + 1)^{\text{st}}$  copy of *D*.

Note that all vertices of Y are covered by  $\mathcal{Q}$  or  $\mathcal{R}$ . Let  $Z_{\mathcal{Q},\mathcal{R}} \supset Z_{\mathcal{Q}}$  denote the set of vertices of Z covered by  $\mathcal{Q}$  or  $\mathcal{R}$ , and let  $X_{\mathcal{R}}$  denote the set of vertices of X covered by  $\mathcal{R}$  (no vertices of X were covered by  $\mathcal{Q}$ ). Observe that

$$|X \setminus X_{\mathcal{R}}| = |X| - (|Y| - |Y_{\mathcal{Q}}|) = k - |Y| + (q - \ell), \text{ and} |Z \setminus Z_{\mathcal{Q},\mathcal{R}}| = |Z| - |Z_{\mathcal{Q}}| - 2(|Y| - |Y_{\mathcal{Q}}|) = 3(k - |Y|) - (q - \ell),$$
(7)

where we used that  $|Z| = 4k - |X| - |Y| = 3k + \ell - |Y|$ .

**The partial** *D*-tiling *S*. We now obtain a collection *S* of  $q - \ell$  vertex-disjoint copies of *D*, each with 2 vertices in  $X \setminus X_{\mathcal{R}}$  and 2 vertices in  $Z \setminus Z_{\mathcal{Q},\mathcal{R}}$ . Indeed, arbitrarily pick vertices  $z_1, z'_1, \ldots, z_{q-\ell}, z'_{q-\ell} \in Z \setminus Z_{\mathcal{Q},\mathcal{R}}$ , which is possible since

$$|Z \setminus Z_{\mathcal{Q},\mathcal{R}}| - 2(q-\ell) \stackrel{(7)}{=} 3(k-|Y| - (q-\ell)) \stackrel{(3), \ (6)}{\geq} 3k(1-12\alpha^2) \ge 2.$$

Inductively, assume we have covered  $0 \le i < q - \ell$  pairs  $z_1, z'_1, \ldots, z_i, z'_i$  by vertex-disjoint copies  $D_1, \ldots, D_i$  of D, where each  $D_j, 0 \le j \le i$ , has vertices  $\{z_j, z'_j, x_j, x'_j\}$ , where  $x_j, x'_j \in X \setminus X_{\mathcal{R}}$ . We infer from (4) that

$$|N(z_1, z_1') \cap (X \setminus (X_{\mathcal{R}} \cup \{x_1, x_1', \dots, x_i, x_i'\}))| \ge (1 - \alpha)|X| - |X_{\mathcal{R}}| - 2i \ge (1 - \alpha)|X| - |Y| - 2(q - \ell) \ge 2,$$

where the last inequality holds on account of (3) and (6). We thus obtain the  $(i + 1)^{st}$  copy of D.

Let  $Z_{\mathcal{Q},\mathcal{R},\mathcal{S}} \supset Z_{\mathcal{Q},\mathcal{R}}$  denote the set of vertices of Z covered by  $\mathcal{Q}$ ,  $\mathcal{R}$  or  $\mathcal{S}$ , and let  $X_{\mathcal{R},\mathcal{S}} \supset X_{\mathcal{R}}$  denote the set of vertices of X covered by  $\mathcal{R}$  or  $\mathcal{S}$ . Set  $m := |X \setminus X_{\mathcal{R},\mathcal{S}}|$  and note that

$$m = |X \setminus X_{\mathcal{R},\mathcal{S}}| \stackrel{(7)}{=} k - |Y| - (q - \ell) \quad \text{and} \quad |Z \setminus Z_{\mathcal{Q},\mathcal{R},\mathcal{S}}| \stackrel{(7)}{=} 3(k - |Y| - (q - \ell)) = 3m.$$
(8)

We conclude the proof of Theorem 1.4 by building the remaining partial D-tiling  $\mathcal{T}$ .

The partial *D*-tiling  $\mathcal{T}$ . Arbitrarily partition  $Z \setminus Z_{\mathcal{Q},\mathcal{R},\mathcal{S}} = Z_1 \cup Z_2 \cup Z_3$  into three sets of size m, and for simplicity of notation, write  $X_0 = X \setminus X_{\mathcal{R},\mathcal{S}}$ . Define the following auxiliary 4-partite 4-graph H with 4-partition  $V(H) = X_0 \cup Z_1 \cup Z_2 \cup Z_3$ , obtained by including each  $\{x, z_1, z_2, z_3\} \in H$ ,  $x \in X_0, z_i \in Z_i$  for i = 1, 2, 3, if  $\{x, z_1, z_2, z_3\}$  spans a copy of D in G. We claim that H satisfies the hypothesis of Theorem 2.1 with  $\gamma = 1/2$ , and hence contains a perfect matching, which will then define  $\mathcal{T}$  and finish our proof of Theorem 1.4.

To bound  $\delta_H(Z_1, Z_2, Z_3)$ , fix  $z_1 \in Z_1, z_2 \in Z_2, z_3 \in Z_3$ . We infer from (4) that

$$|N_{H}(z_{1}, z_{2}, z_{3})| \ge |N_{G}(z_{1}, z_{2}) \cap N_{G}(z_{1}, z_{3}) \cap X_{0}| \ge (1 - 2\alpha)|X| - |X_{\mathcal{R}, \mathcal{S}}|$$
  

$$\ge (1 - 2\alpha)|X| - |Y| - 2(q - \ell) \stackrel{(3), (6)}{\ge} (1 - 2\alpha)|X| - 20\alpha^{2}k \stackrel{(3)}{\ge} ((1 - 2\alpha)((1 - 4\alpha^{2}) - 20\alpha^{2}))k$$
  

$$\stackrel{(3)}{\ge} \frac{1 - 26\alpha}{1 + 3\varepsilon_{0}}|X| \ge (1 - 27\alpha)|X| \ge (1 - 27\alpha)|X_{0}| = (1 - 27\alpha)m.$$

Thus,  $\delta_H(Z_1, Z_2, Z_3) \ge (1 - 27\alpha)m$ .

To bound  $\delta_H(X_0)$ , fix  $x \in X_0$ , and for clarity of notation in what follows, write  $N_G(x) = G_x$ . By the definition of X, we have that  $|G_x[Z]| \ge (1 - \alpha) {|Z| \choose 2}$ , and so all but at most  $\sqrt{\alpha} |Z|$  vertices  $z \in Z$  satisfy that  $\deg_{G_x[Z]}(z) \ge (1 - \sqrt{\alpha}) |Z|$ . For each such  $z \in Z$  and i = 1, 2, 3,

$$|N_{G_x}(z) \cap Z_i| \ge (1 - \sqrt{\alpha})|Z| - |Z_{\mathcal{Q},\mathcal{R},\mathcal{S}}| - 2m \stackrel{(8)}{=} m - \sqrt{\alpha}|Z| = \left(1 - \sqrt{\alpha}\frac{|Z|}{m}\right)m$$

Since, by (3) and (8), we have

$$3m = |Z| - |Z_{\mathcal{Q},\mathcal{R},\mathcal{S}}| = |Z| - \left(3q + 2(|Y| - q) + 2(q - \ell)\right) \ge |Z| - 5|Y| + 2\ell \stackrel{(3), (5)}{\ge} |Z| - 26\alpha^2 k \stackrel{(3)}{\ge} \frac{|Z|}{2}, \quad (9)$$

we conclude that

$$|N_{G_x}(z) \cap Z_i| \ge (1 - 6\sqrt{\alpha})m.$$

As such,

$$|N_H(x)| \ge \sum_{z_1 \in Z_1} |N_{G_x}(z_1) \cap Z_2| |N_{G_x}(z_1) \cap Z_3| \ge (m - \sqrt{\alpha}|Z|) \left( (1 - 6\sqrt{\alpha}) m \right)^2 \stackrel{(9)}{\ge} (1 - 6\sqrt{\alpha})^3 m^3,$$

and so  $\delta_H(X_0) \ge (1 - 234\sqrt{\alpha})m^3$ .

The obtained bounds on  $\delta_H(Z_1, Z_2, Z_3)$  and  $\delta_H(X_0)$  then implies

$$m\delta_H(X_0) + m^3\delta_H(Z_1, Z_3, Z_3) \ge (2 - 234\sqrt{\alpha} - 27\alpha) m^4 \ge (2 - 261\sqrt{\alpha}) m^4 > \frac{3}{2}m^4$$

so that, as claimed, H satisfies the hypothesis of Theorem 2.1 with  $\gamma = 1/2$ .

## 3. Proof of Theorem 1.5

Our proof of Theorem 1.5 is based on the following two lemmas, the second of which mirrors an 'absorption' lemma of Rödl, Ruciński and Szemerédi [9].

**Lemma 3.1.** For all  $\gamma > 0$  and sufficiently large integers m divisible by 4, the following holds. Let H be a 3-graph of order m. If  $\delta_2(H) \ge (\frac{1}{4} - \gamma) m$  and H is not  $(8\gamma)$ -extremal, then H admits a D-tiling covering all but  $50/\gamma$  vertices.

**Lemma 3.2.** For all  $\alpha > 0$  and sufficiently large integers n divisible by 4, the following holds. Let G be a 3-graph of order n. If  $\delta_2(G) \ge n/4$ , then there exists  $A \subset V(G)$  of size  $|A| \le \alpha n$  so that, for every  $W \subset V \setminus A$  of size  $|W| \le 50/\alpha$  for which  $|A \cup W|$  is divisible by 4, the hypergraph  $G[A \cup W]$  is D-tileable.

We defer the proofs of Lemmas 3.1 and 3.2 to Sections 3.1 and 3.2 respectively in favor of first proving Theorem 1.5.

Proof of Theorem 1.5. Let  $\varepsilon > 0$  be given, together with a sufficiently large integer n which is divisible by 4. Let G be a 3-graph of order n satisfying (1) which is not  $\varepsilon$ -extremal. For  $\alpha = \varepsilon/9$ , let  $A \subset V(G)$  be the set given by Lemma 3.2. Set  $H = G[V \setminus A]$ , and write m = n - |A|.

We claim that H satisfies the hypothesis of Lemma 3.1 with  $\gamma = \alpha$ . Indeed, note that

$$\delta_2(H) \ge \frac{n}{4} - |A| \ge \frac{n}{4} - \alpha n = \left(\frac{1}{4} - \alpha\right) n \ge \left(\frac{1}{4} - \alpha\right) m.$$

Observe, moreover, that H is not  $(8\alpha)$ -extremal. Indeed, if  $S \subset V(H)$  satisfies that H[S] is D-free, then G[S] is also D-free, and if

$$|S| \ge (1 - 8\alpha)\frac{3m}{4} = (1 - 8\alpha)\frac{3}{4}(n - |A|) \ge (1 - 8\alpha)(1 - \alpha)\frac{3n}{4} \ge (1 - 9\alpha)\frac{3n}{4} = (1 - \varepsilon)\frac{3n}{4},$$

then G would be  $\varepsilon$ -extremal, a contradiction.

Lemma 3.1 implies that H admits a D-tiling covering all but  $50/\alpha$  vertices. Set  $W \subset V(H)$  to be the set of vertices (if any) uncovered by this D-tiling. Since  $|V(H) \setminus W|$  is divisible by 4, it must be that  $|A \cup W|$  is divisible by 4, and so Lemma 3.2 guarantees that  $G[A \cup W]$  is D-tileable. Thus, G is D-tileable.

3.1. **Proof of Lemma 3.1.** Let  $\gamma > 0$  be given, and let m be a sufficiently large integer which is divisible by 4. Let H be a 3-graph of order m, which is not  $(8\gamma)$ -extremal, and for which  $\delta_2(H) \ge (\frac{1}{4} - \gamma) m$ . We prove that H contains a D-tiling covering all but  $50/\gamma$  vertices. To that end, let  $\mathcal{M}$  be a maximum D-tiling in H, but assume, on the contrary, that  $\mathcal{M}$  leaves more than  $50/\gamma$  vertices uncovered.

We use the following notation and terminology. Let  $V_{\mathcal{M}}$  denote the set of vertices of H covered by  $\mathcal{M}$ , and let  $W = V(H) \setminus V_{\mathcal{M}}$ . For a vertex  $v \in V_{\mathcal{M}}$ , write  $H_v[W]$  for  $N_H(v) \cap {\binom{W}{2}}$ , and say that  $v \in V_{\mathcal{M}}$  is W-big if  $|H_v[W]| \ge 10|W|$ , and W-small otherwise. Observe that every element  $D_0 \in \mathcal{M}$  contains at most one W-big vertex. Indeed, assuming otherwise, let  $u, v \in V(D_0)$  both be W-big vertices. Since  $|H_u[W]| \ge 10|W| > |W|/2$ , the graph  $H_u[W]$  contains a path of length 2, with vertices denoted by  $w_1, w_2, w_3$ . The graph  $H_v[W \setminus \{w_1, w_2, w_3\}$ ] then has size

$$|H_v[W \setminus \{w_1, w_2, w_3\}]| \ge |H_v[W]| - 3|W| \ge 7|W| > |W|/2,$$
(10)

and so  $H_v[W \setminus \{w_1, w_2, w_3\}]$  contains a path of length 2, with vertices denoted by  $w'_1, w'_2, w'_3$ . Then,  $\{u, w_1, w_2, w_3\}$  and  $\{v, w'_1, w'_2, w'_3\}$  span vertex-disjoint copies of D, which can replace  $D_0$  in  $\mathcal{M}$  to contradict that  $\mathcal{M}$  was a maximum D-tiling in H.

Now, write B for the set of W-big vertices  $v \in V_{\mathcal{M}}$ , and write |B| = b. We now observe that  $b \geq (\frac{1}{4} - 2\gamma) m$ . Indeed, write  $H[W, W, V_{\mathcal{M}}]$  for the set of triples from H containing exactly two vertices from W. From our definitions above, note that

$$|H[W, W, V_{\mathcal{M}}]| \le b \left( 30|W| + \binom{|W|}{2} \right) + 40(|\mathcal{M}| - b)|W| \le b \binom{|W|}{2} + 40|\mathcal{M}||W| \le b \binom{|W|}{2} + 10m|W|.$$

On the other hand, the maximality of  $\mathcal{M}$  implies that H[W] is D-free, and so

$$|H[W, W, V_{\mathcal{M}}]| \ge \left(\left(\frac{1}{4} - \gamma\right)m - 1\right)\binom{|W|}{2}.$$

The inequalities above imply that

$$b \ge \left(\frac{1}{4} - \gamma\right)m - 1 - \frac{20m}{|W| - 1} \ge \left(\frac{1}{4} - \gamma\right)m - 1 - \frac{40m}{|W|},$$

and by our assumption that  $|W| > 50/\gamma$ , we infer that  $b \ge (\frac{1}{4} - 2\gamma) m$ , as claimed.

Now, write  $\mathcal{M}_B \subset \mathcal{M}$  for elements of  $\mathcal{M}$  which contain a W-big vertex, and let  $V_{\mathcal{M}_B}$  denote the set of vertices of H covered by  $\mathcal{M}_B$ . Then,  $S_B = V_{\mathcal{M}_B} \setminus B$  consists of W-small vertices and we have  $|S_B| = 3|B| \ge (1 - 8\gamma)3m/4$ . Since H is not  $(8\gamma)$ -extremal,  $H[S_B]$  contains a copy  $D_0$  of D, say with vertices  $v_1, v_2, v_3, v_4$ . Let  $u_1, u_2, u_3, u_4$  denote the W-big vertices corresponding to  $v_1, v_2, v_3, v_4$ , respectively, in  $\mathcal{M}_B$ . Among  $u_1, \ldots, u_4$ , at least two and at most 4 are distinct, and so w.l.o.g., let  $u_1, \ldots, u_j$ , for some  $j \in \{2, 3, 4\}$ , denote the distinct vertices of  $u_1, \ldots, u_4$ . For  $1 \le i \le j$ , let  $D_i \in \mathcal{M}_B$  be the element containing  $u_i$ .

Similarly to (10), the definition of a W-big vertex will guarantee, for each  $1 \le i \le j$ , the existence of a 2-path  $P_2(u_i) \subset H_{u_i}[W]$  so that  $P_2(u_1), \ldots, P_2(u_j)$  are each pair-wise vertex-disjoint. Indeed, if we already have the desired 2-paths  $P_2(u_1), \ldots, P_2(u_{i-1})$ , where  $2 \le i \le j \le 4$ , then

$$\left| H_{u_i} \Big[ W \setminus \big( V(P_2(u_1)) \cup \dots \cup V(P_2(u_{i-1})) \big) \Big] \right| \ge |H_{u_i}[W]| - 3(i-1)|W| \ge |H_{u_i}[W]| - 9|W| \ge |W| > |W|/2,$$

and so there exists a 2-path  $P_2(u_i) \subset H_{u_i}[W]$  which is vertex-disjoint from each of  $P_2(u_1), \ldots, P_2(u_{i-1})$ .

Clearly, for each  $1 \leq i \leq j$ ,  $\{u_i\} \cup V(P_2(u_i))$  spans a copy of D, which we shall denote as  $D^{u_i}$ . Then,  $D^{u_1}, \ldots, D^{u_j}$  are pair-wise vertex-disjoint copies of D, and so, deleting from  $\mathcal{M}$  the elements  $D_1, \ldots, D_j$  and adding  $D_0, D^{u_1}, \ldots, D^{u_j}$  contradicts that  $\mathcal{M}$  was a maximum D-tiling. This concludes the proof of Lemma 3.1.

3.2. **Proof of Lemma 3.2** – **Absorption.** We shall prove the following stronger form of Lemma 3.2, which allows for a smaller co-degree and larger choices of subset W.

**Lemma 3.3** (Lemma 3.2 - strong form). For all  $\alpha, \delta > 0$ , there exists  $\omega > 0$  so that for all sufficiently large integers n divisible by 4, the following holds. Let G be a 3-graph of order n. If  $\delta_2(G) \geq \delta n$ , then there exists  $A \subset V(G)$  of size  $|A| \leq \alpha n$  so that, for every  $W \subset V \setminus A$  of size  $|W| \leq \omega n$  for which  $|A \cup W|$  is divisible by 4, the hypergraph  $G[A \cup W]$  is D-tileable.

Our proof of Lemma 3.3 will be based on Proposition 3.5, for which we need the following definition.

**Definition 3.4.** Suppose G is a 3-graph with vertex set V, and let  $U \in {\binom{V}{4}}$ . We say that a set  $S \in {\binom{V \setminus U}{8}}$  absorbs U if G[S] is D-tileable and  $G[S \cup U]$  is D-tileable.

**Proposition 3.5.** For all  $\delta > 0$ , there exists  $\sigma > 0$  so that for all sufficiently large integers n, the following holds. Suppose G is a 3-graph with vertex set V of order |V| = n for which  $\delta_2(G) \ge \delta n$ . For each  $U \in \binom{V}{4}$ , there are  $\sigma n^8$  sets  $S \in \binom{V}{8}$  which absorb U.

To prove Proposition 3.5, we require the following well-known 'supersaturation' result of Erdős [1] (stated here only in special case form).

**Theorem 3.6** (Erdős [1]). For all  $c_1 > 0$  there exists  $c_2 > 0$  so that for all sufficiently large integers n, the following holds. If H is a 3-graph of order n and size  $|H| \ge c_1 n^3$ , then H contains at least  $c_2 n^9$  copies of  $K_{3,3,3}^3$  (the balanced complete 3-partite 3-graph of order 9).

Proof of Proposition 3.5. Let  $\delta > 0$  be given. Let  $c_1 = \delta^3/36$ , and let  $c_2 > 0$  be the constant guaranteed by Theorem 3.6. We define  $\sigma = c_2$ , and in all that follows, we take *n* to be a sufficiently large integer. Let *G* be a 3-graph with vertex set *V* of order |V| = n for which  $\delta_2(G) \ge \delta n$ . Fix  $U = \{u_1, u_2, u_3, u_4\} \subset V$ . We prove there are  $\sigma n^8$  sets  $S \in {V \choose 8}$  which absorb *U*.

To that end, define  $V_1 = N(u_1, u_2), V_2 = N(u_3, u_4)$  and

$$V_3 = \bigcup \{ N(v_1, v_2) : (v_1, v_2) \in V_1 \times V_2 \}.$$

Note that  $V_1 \cup V_2 \cup V_3$  is not necessarily a partition, but it will not be difficult to find pairwise disjoint subsets  $W_i \subset V_i$ , i = 1, 2, 3, for which  $|G[W_1, W_2, W_3]| \ge c_1 n^3$ . To that end, let  $W_1 \subset V_1 \setminus \{u_3, u_4\}$  be any set of size (exactly)  $\lceil \delta n/3 \rceil$  (recall  $|V_1| \ge \delta n$ ). Let  $W_2 \subset V_2 \setminus (W_1 \cup \{u_1, u_2\})$  be any set of size (exactly)  $\lceil \delta n/3 \rceil$  (recall  $|V_2| \ge \delta n$ ). Now, set  $W_3 = V_3 \setminus (W_1 \cup W_2 \cup \{u_1, u_2, u_3, u_4\})$ . Then,

$$|G[W_1, W_2, W_3]| = \sum_{(w_1, w_2) \in W_1 \times W_2} |N(w_1, w_2) \cap W_3| \ge \left\lceil \frac{\delta n}{3} \right\rceil^2 \left(\delta n - 2\left\lceil \frac{\delta n}{3} \right\rceil - 4\right) \ge \frac{\delta^3 n^3}{36} = c_1 n^3.$$

Now, set  $H = G[W_1, W_2, W_3]$ , which we view as a hypergraph of order n. Since H has size  $|H| \ge c_1 n^3$ , Theorem 3.6 guarantees that H has at least  $c_2 n^9 = \sigma n^9$  copies of  $K_{3,3,3}^3$ . Note that each such copy has exactly 3 vertices in each of  $W_1, W_2, W_3$  and that, for some fixed  $w_3 \in W_3$  (it doesn't matter which), at least  $\sigma n^8$  such copies contain the vertex  $w_3$ . Let  $\{w_1, w'_1, w''_1, w_2, w'_2, w''_2, w''_3, w''_3\}$  denote the vertex set of such a copy, where  $w_i, w'_i, w''_i \in W_i$ , i = 1, 2, 3. We claim that

$$S_U = S_U(w_3) = \{w_1, w_1', w_1'', w_2, w_2', w_2'', w_3', w_3''\}$$

absorbs the set U (see Figure 1). Indeed,

$$S_1 := \left\{ \{w_1, w_2, w_3'\}, \{w_1', w_2, w_3'\} \right\}, \quad S_2 := \left\{ \{w_1'', w_2', w_3''\}, \{w_1'', w_2'', w_3''\} \right\}$$

is a *D*-tiling of  $G[S_U]$  and

$$T_1 := \left\{ \{u_1, u_2, w_1\}, \{u_1, u_2, w_1'\} \right\}, T_2 := \left\{ \{u_3, u_4, w_2\}, \{u_3, u_4, w_2'\} \right\}, T_3 := \left\{ \{w_1'', w_2'', w_3'\}, \{w_1'', w_2'', w_3''\} \right\}$$



FIGURE 1. Absorbing structure.

is a *D*-tiling of  $G[S_U \cup U]$ .

Finally we use Proposition 3.5 to prove Lemma 3.3.

*Proof of Lemma 3.3.* Let  $\alpha, \delta > 0$  be given. Let  $\sigma = \sigma(\delta) > 0$  be the constant guaranteed by Proposition 3.5. We define

$$\omega = \frac{\alpha \sigma^2}{128}.\tag{11}$$

In all that follows, we take n to be a sufficiently large integer divisible by 4. Let G be a given 3-graph with vertex set V of order |V| = n for which  $\delta_2(G) \ge \delta n$ . We prove that G admits a set  $A \subset V$  described in the conclusion of Lemma 3.3. To produce the desired set A, we employ the well-known deletion method in probabilistic combinatorics.

To begin, set  $p = (1/16)\alpha\sigma n^{-7}$ , and let  $\mathbb{H} = \mathbb{H}^{(8)}(n,p)$  be the binomial random 8-uniform hypergraph with *n*-element vertex set *V*. We note several basic properties of  $\mathbb{H}$  (due to the Chernoff inequality, unless otherwise indicated):

(i) With probability  $1 - \exp\{-n/\log n\}$ ,

$$\mathbb{H}| \le 2p\binom{n}{8} \le \frac{1}{8}\alpha n;$$

(*ii*) Let  $\mathbb{H} \otimes \mathbb{H} = \{(S_1, S_2) \in \mathbb{H} \times \mathbb{H} : S_1 \cap S_2 \neq \emptyset\}$ . Then,

$$\mathbb{E}\left[|\mathbb{H} \otimes \mathbb{H}|\right] \le 8 \binom{n}{8} \binom{n}{7} p^2 \le \frac{1}{256} \alpha^2 \sigma^2 n.$$

As such, by the Markov inequality,

$$\Pr\left[|\mathbb{H} \otimes \mathbb{H}| \ge \frac{1}{128} \alpha^2 \sigma^2 n\right] \le \frac{1}{2};$$

(iii) For  $U \in {\binom{V}{4}}$ , let  $\mathcal{A}(U)$  be the collection of sets  $S \in {\binom{V}{8}}$  which absorb U. By Proposition 3.5,  $|\mathcal{A}(U)| \geq \sigma n^8$ , and so with probability  $1 - \exp\{-n/\log n\}$ ,  $\mathbb{H}$  satisfies that for every  $U \in {\binom{V}{4}}$ ,

$$|\mathcal{A}(U) \cap \mathbb{H}| \ge \frac{1}{2}p|\mathcal{A}(U)| \ge \frac{1}{32}\alpha\sigma^2 n.$$

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Let H be an instance of  $\mathbb{H}$  for which properties (i)–(iii) hold (and specifically, where  $|H \otimes H| < \alpha^2 \sigma^2 n/128$ ). Now,

(a) delete any  $S \in H$  for which there exists  $S' \in H$  for which  $S \cap S' \neq \emptyset$ . This deletes at most

$$2 \times \frac{\alpha^2 \sigma^2 n}{128} = \frac{\alpha^2 \sigma^2 n}{64}$$

elements  $S \in H$ ;

(b) delete any  $S \in H$  for which no  $U \in \binom{V}{4}$  has  $S \in \mathcal{A}(U)$ .

The resulting hypergraph is then, importantly, a (partial) matching M in V. Let m := |M|,  $\{S_1, \ldots, S_m\} = M$ , and  $A := \bigcup_{i=1}^m S_i$  (the set of vertices covered by M). We now confirm that A satisfies its claimed properties.

Observe from (i) that  $|A| = 8|M| \le \alpha n$ , as promised. Now, let  $W \subset V \setminus A$  have size  $4t := |W| \le \omega n$  (cf. (11)) and then arbitrarily partition W into 4-sets  $\{W_1, W_2, \ldots, W_t\} =: W$ .

Note that by (iii), (a), and (11) we have that for all  $W_i \in \mathcal{W}$ ,

$$|\mathcal{A}(W_i) \cap M| \ge \frac{1}{32}\alpha\sigma^2 n - \frac{1}{64}\alpha^2\sigma^2 n \ge \frac{1}{64}\alpha\sigma^2 n \ge \frac{\omega n}{4} \ge t.$$

So for each  $W_i \in \mathcal{W}$  we can greedily choose some unique  $S'_i \in \mathcal{A}(W_i) \subseteq M$ , which guarantees that each of  $G[S'_1 \cup W_1], \ldots, G[S'_t \cup W_t]$  are *D*-tileable. Finally, since G[S] is *D*-tilable for all  $S \in M$  (by (b) and Definition 3.4), and since  $\{S_1, \ldots, S_m, W_1, \ldots, W_t\}$  is a partition of  $A \cup W$ , we infer that  $G[A \cup W]$  is *D*-tileable as desired.

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