ARC-DISJOINT CYCLES AND FEEDBACK ARC SETS

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ABSTRACT. Isaak posed the following problem. Suppose T is a tournament having a minimum feedback arc set which induces an acyclic digraph with a hamiltonian path. Is it true that the maximum number of arc-disjoint cycles in T equals the cardinality of minimum feedback arc set of T? We prove that the answer to the problem is in the negative. Further, we study the number of arc-disjoint cycles through a vertex v of the minimum out-degree in an oriented graph D. We prove that if v is adjacent to all other vertices, then v belongs to $\delta^+(D)$ arc-disjoint cycles.

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1. INTRODUCTION

Let D = (V, A) be a digraph. A set of arcs $S \subseteq A$ is called a feedback arc set if D - S is acyclic. The minimum number of elements in a feedback arc set of D is denoted by $\tau(D)$. The maximum number of arc-disjoint cycles in D is denoted by $\nu(D)$. Let $\pi = v_1, v_2, \ldots, v_n$ be an ordering of the vertices of D. An arc $v_i v_j \in A$ is called backward with respect to π if i > j; π is τ -optimal if the number of backward arcs with respect to π is minimum among all orderings of vertices of D. For a vertex $v \in V$, we use the following notation:

$$N^{+}(v) = \{ u \in V - v : vu \in A \}, \quad N^{-}(v) = \{ u \in V - v : uv \in A \},$$
$$N^{+2}(v) = \bigcup_{u \in N^{+}(v)} N^{+}(u) - N^{+}(v) - \{v\}.$$

The out-degree of a vertex $v \in V$ is $d^+(v) = |N^+(v)|$, and $\delta^+(D) = \min\{d^+(v) : v \in V\}$. We use Bang-Jensen and Gutin [1] as reference for undefined terms.

It is well known that S is a minimum feedback arc set in a digraph D if and only if there exists a τ -optimal ordering π of vertices in D such that S is the set of backward arcs with respect to π (see Bang-Jensen and Gutin [1]). Hence it follows that for every digraph D we have $\tau(D) \geq \frac{1}{2}\delta^+(D)(\delta^+(D)+1)$ (see Remark 3). Erdős and Moon [4] proved that for every $n \geq 3$ there exists a tournament T_n of order n such that $\tau(T_n) \geq \frac{1}{4}n(n-1) - \frac{1}{2}\sqrt{n^3 \log_e n}$. A slightly better result was obtained by de la Vega in [14]. On the other hand, it follows from a result by Chatrand, Geller and Hedetniemi [2] that $\nu(T_n) \leq \lfloor \frac{1}{3}n\lfloor \frac{1}{2}(n-1) \rfloor \rfloor$. Even though not always $\tau(D) = \nu(D)$, Isaak [9] conjectured (Conjecture 15.4.1 of [1]) that if T is a tournament having a minimum feedback arc set which induces a transitive subtournament of T, then $\tau(T) = \nu(T)$. He posed also the following question (Problem 15.4.2 of [1]): Suppose T is a tournament having a minimum feedback arc set which induces an acyclic digraph with a hamiltonian path. Is it true that $\tau(T) = \nu(T)$? We prove that the answer to the Isaak question is in the negative.

Mathematical sociologist Landau [10] proved that in every tournament T, if a vertex v has the minimum out-degree, then it belongs to $\delta^+(T)$ different 3-cycles. If T is eulerian, then every vertex has the minimum out-degree. One may guess that for this case every vertex belongs to $\delta^+(T)$ arc-disjoint 3-cycles. However, it is not true (see Remark 4). We prove (Theorem 2.1) that in every oriented graph D, if v is a vertex of the minimum out-degree which is adjacent to all other vertices, then it belongs to $\delta^+(D)$ arc-disjoint cycles. Mader [11] consider linkages in digraphs with lower bounds on the out-degree. He proved a theorem which is related to our result: every digraph D with $\delta^+(D) \ge n$ contains a pair of distinct vertices x, y with n-1 arc-disjoint (x, y)-paths.

We conjecture that every tournament T has a vertex of the minimum out-degree which belongs to $\delta^+(T)$ arc-disjoint 3-cycles. It is connected with the following conjecture put forward by Hoang and Reed [8]: every digraph D with $\delta^+(D) = n$ contains a sequence C_1, C_2, \ldots, C_n of cycles such that $\bigcup_{i=1}^{j-1} C_i$ and C_j have at most one vertex in common. In the case of n = 2 the Hoang and Reed conjecture was proved by Thomassen [13]. The conjecture was verified for tournaments by Havet, Thomassé and Yeo [7]. They proved that every tournament T with $\delta^+(T) = n$ contains a sequence C_1, C_2, \ldots, C_n of 3-cycles such that $\bigcup_{i=1}^{j-1} C_i$ and C_j have exactly one vertex in common. Our conjecture is also connected with the following conjecture posed by Seymour [3] (see also Seymour's Second Neighbourhood Conjecture [1]): every oriented graph has a vertex v such that $|N^+(v)| \leq |N^{+2}(v)|$. In the case of tournaments this conjecture was proved by Fisher [5]. An elementary proof for the case of tournaments was found by Havet and Thomassé [6] (see also Bang-Jensen and Gutin [1]).

2. The Isaak problem

Let T be a tournament of order 13 in Figure 1 and let $\alpha = a, b, c, d, e, f, g, h, i, j, k, l, m$ be an ordering of vertices of T. We will prove that $\nu(T) = 11$. Notice that the set of all backward arcs with respect to α is a feedback arc set which induces an acyclic digraph with a hamiltonian path m, k, i, g, e, c, a. We will prove that the ordering α is τ -optimal. Hence, $\tau(T) = 12$

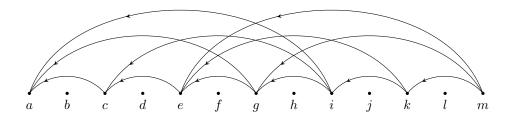


FIGURE 1. The tournament T. Only backward arcs with respect to the ordering α are shown.

Let \mathcal{C} be a family of the following arc-disjoint 3-cycles in T:

 $(a,b,c), (c,d,e), (e,f,g), (g,h,i), (i,j,k), (k,l,m), (a,d,g), (c,f,i), (e,h,k), (g,\ j,\ m), (a,e,i).$

Suppose that U is the union set of arcs of all cycles belonging to the family C. Notice that me is the only backward arc with respect to α which does not belong to U. If me is an arc of some cycle in the tournament, then this cycle has an arc belonging to U. Hence, C is a maximal family of arc-disjoint cycles in T. It is easy to see that every backward arc determines uniquely a family of eleven arc-disjoint cycles omitting this backward arc. By analogy, we can check that every such family is a maximal family of arc-disjoint cycles in T. Hence, C is a maximum family of arc-disjoint cycles in T. Thus $\nu(T) = 11$. Let S be a minimum feedback arc set in T. If S has eleven elements, then it satisfies the following conditions:

- (1) every cycle in \mathcal{C} has exactly one arc belonging to S,
- (2) $S \subset U$.

Since $im \notin U$ and $me \notin U$, by (2), the arc ei of a cycle (e, i, m) belongs to S. Hence, by (1), the arc ia of a cycle $(a, e, i) \in C$ does not belong to S. Since $ia \notin S$ and $ah \notin U$, by (2), the arc hi of a cycle (a, h, i) belongs to S. Hence, by (1), the arc gh of a cycle $(g, h, i) \in C$ does not belong to S. Since $gh \notin S$ and $hm \notin U$, by (2), the arc mg of a cycle (g, h, m) belongs to S. Hence, by (1), the arc mg of a cycle (g, h, m) belongs to S. Hence, by (1), the arc mg of a cycle (g, h, m) belongs to S. Hence, by (1), the arc jm of a cycle $(g, j, m) \in C$ does not belong to S. Thus we obtain a contradiction, a cycle (e, j, m) is arc-disjoint with the set S. Hence, S has at least twelve elements.

Remark 1. If we add three arcs mc, kc and ka to the tournament T we obtain a tournament T' having a minimum feedback arc set which induces an acyclic digraph with a hamiltonian path, such that $\nu(T') = 14$ and $\tau(T') = 15$.

Remark 2. We checked case by case that if $T_{\leq 6}$ is a tournament of order at most 6, then the maximum number of arc-disjoint cycles in $T_{\leq 6}$ is equal to the cardinality of a minimum feedback arc set of $T_{\leq 6}$. We show in Figure 2 a tournament T_7 of order 7 such that $\nu(T_7) = 4$ and $\tau(T_7) = 5$.

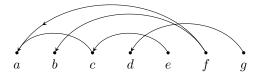


FIGURE 2. The tournament T_7 . Only backward arcs with respect to an ordering a, b, c, d, e, f, g are presented.

Remark 3. If D is a digraph, then $\tau(D) \geq \frac{1}{2}\delta^+(D)(\delta^+(D)+1)$. We proceed by induction on the order of D. Let S be a feedback arc set in D of minimum size, and suppose that $\pi = v_1, v_2, \ldots, v_n$ is a τ -optimal ordering of vertices in D such that S is the set of backward arcs with respect to π . Since $v_1, v_2, \ldots, v_{n-1}$ is a τ -optimal ordering of vertices in $D - v_n$, $S' = S - \{v_n v : v \in N^+(v_n)\}$ is a feedback arc set in $D - v_n$ of minimum size. Notice that $\delta^+(D - v_n) \geq \delta^+(D) - 1$. Hence, the size of S' is at least $\frac{1}{2}(\delta^+(D) - 1)\delta^+(D)$. Therefore, the size of S is at least

$$\frac{1}{2}(\delta^+(D) - 1)\delta^+(D) + \delta^+(D) = \frac{1}{2}\delta^+(D)(\delta^+(D) + 1).$$

3. Arc-disjoint cycles through a vertex of the minimum out-degree

For a pair X, Y of vertex sets of an oriented graph D = (V, A), we define

$$(X, Y)_D = \{(x, y) \in A : x \in X, y \in Y\}.$$

Theorem 3.1. Let D be an oriented graph, and suppose that v_0 is a vertex which is adjacent to all other vertices in D. Let $a = \min\{d^+(v) : v \in N^+(v_0)\}$ and $b = \min\{d^+(v) : v \in N^-(v_0)\}$. If $d^+(v_0) \le \min(a, \frac{1}{2}(a+b+1))$, then v_0 belongs to $d^+(v_0)$ arc-disjoint cycles.

Proof Let $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_m\}$ be a maximum family of arc-disjoin cycles through v_0 . Let $\gamma_i = v_0 v_1^i \ldots v_{n(i)}^i v_0$, for $i = 1, \ldots, m$. By Menger's theorem (see [12] and [1] there exists a set Δ of m arcs covering all cycles containing the vertex v_0 . Suppose that k is the number of arcs in Δ with the head v_0 . If k > 0, we can assume that $(v_{n(i)}^i, v_0) \in \Delta$ if and only if $1 \le i \le k$.

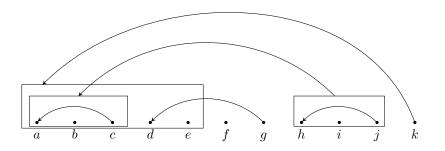


FIGURE 3. The eulerian tournament T_{11} . $(\{h, i, j\}, \{a, b, c\})_{T_{11}} \cup (\{k\}, \{a, b, c, d, e\})_{T_{11}} \cup \{(c, a), (g, d), (j, h)\}$ is the set of all backward arcs with respect to the ordering β .

Let us denote $K = \{v_1^1, v_1^2, \dots, v_1^k\}$, $X = \{v_{n(1)}^1, v_{n(2)}^2, \dots, v_{n(k)}^k\}$ (if k = 0 we set $K = X = \emptyset$), $L = \{v_1^{k+1}, v_1^{k+2}, \dots, v_1^m\}$, $Y = N^-(v_0) - X$, and $M = N^+(v_0) - K - L$. First we prove the following inequality:

(1) $|(K \cup X \cup M, Y)_D| \le |(L, K \cup X \cup M)_D|.$

Assume that $(v, y) \in (K \cup X \cup M, Y)_D$. Notice that $(y, v_0) \notin \Delta$. If $v \in K \cup M$, then $(v_0, v) \notin \Delta$. Hence, the arc (v, y) of a cycle $v_0 v y v_0$ belongs to Δ . If $v \in X$, then $v = v_{n(i)}^i$, for some $i \leq k$. Hence, the arc $(v_{n(i)}^i, y)$ of a cycle $v_0 v_1^i \dots v_{n(i)}^i y v_0$ belongs to Δ . Thus, (v, y) is an arc of some cycle γ_i , for i > k. Accordingly, to every arc $(v, y) \in (K \cup X \cup M, Y)_D \subseteq \Delta$ we can assign an arc $(l, v') \in (L, K \cup X \cup M)_D$ such that (l, v') and (v, y) belong to the same cycle in Γ . The above assignment is injective, because different arcs in Δ belong to arc-disjoint cycles in Γ . Hence, (1) holds.

Let us complete the oriented graph D to a tournament T with the same vertex set V. Hence, $|K \cup X \cup M, V - Y)_T| \ge |(K \cup X \cup M, V - Y)_D|$. Therefore, by (1) we obtain:

$$|(L, K \cup X \cup M)_D| + |(K \cup X \cup M, V)_T| \ge |K \cup X \cup M, Y)_D| + |(K \cup X \cup M, V)_T|$$
$$\ge |(K \cup X \cup M, Y)_T| + |(K \cup X \cup M, V)_D|.$$

Accordingly,

$$\begin{split} |K \cup X \cup M|(|V| - 1) &= |(V, K \cup X \cup M)_T| + |(K \cup X \cup M, V)_T| \\ &= |(V - L, K \cup X \cup M)_T| + |(L, K \cup X \cup M)_T| + |(K \cup X \cup M, V)_T| \\ &\geq |(V - L, K \cup X \cup M)_T| + |(K \cup X \cup M, Y)_T| + |(K \cup X \cup M, V)_D| \\ &= |(K \cup X \cup M, K \cup X \cup M)_T| + |(\{v_0\}, K \cup X \cup M)_T| \\ &+ |(Y, K \cup X \cup M)_T| + |(K \cup X \cup M, Y)_T| + |(K \cup X \cup M, V)_D| \\ &\geq \frac{1}{2} |K \cup X \cup M|(|K \cup X \cup M| - 1) + |K| + |M| \\ &+ |K \cup X \cup M||Y| + a|K| + b|X| + a|M|. \end{split}$$

Since $|V| - 1 = d^+(v_0) + |X| + |Y|$ and |K| = |X|, we have

$$2|K|d^{+}(v_{0}) + |M|d^{+}(v_{0}) \ge \left(|K| + \frac{1}{2}|M|\right)\left(|M| - 1\right) + |M| + (a + b + 1)|K| + a|M|.$$

Thus, |M| = 0, because $(a + b + 1) \ge 2d^+(v_0)$ and $a \ge d^+(v_0)$.

Remark 4. Let T_{11} be the eulerian tournament in Fig. 3, and $\beta = a, b, c, d, e, f, g, h, i, j, k$ an ordering of its vertices. Suppose that

$$\{\{h, i, j\}, \{a, b, c\}\}_{T_{11}} \cup (\{k\}, \{a, b, c, d, e\})_{T_{11}} \cup \{(c, a), (g, d), (j, h)\}$$

is the set of all backward arcs with respect to the ordering β . Let kv_1v_2 be a 3-cycle through the vertex k. Notice that, if $v_1 \in \{a, b, c\}$, then $v_2 \in \{f, g\}$. Hence, the vertex k does not belong to $\delta^+(T_{11})$ arc-disjoint 3-cycles.

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