# ARC-DISJOINT CYCLES AND FEEDBACK ARC SETS 

JAN FLOREK


#### Abstract

Isaak posed the following problem. Suppose $T$ is a tournament having a minimum feedback arc set which induces an acyclic digraph with a hamiltonian path. Is it true that the maximum number of arc-disjoint cycles in $T$ equals the cardinality of minimum feedback arc set of $T$ ? We prove that the answer to the problem is in the negative. Further, we study the number of arc-disjoint cycles through a vertex $v$ of the minimum out-degree in an oriented graph $D$. We prove that if $v$ is adjacent to all other vertices, then $v$ belongs to $\delta^{+}(D)$ arc-disjoint cycles.

Mathematics Subject Classification: 05C20, 05C38. Key words and phrases: feedback arc set, $\tau$-optimal ordering, Isaak conjecture, arcdisjoint cycles, Menger's theorem, linkages in digraphs.


## 1. Introduction

Let $D=(V, A)$ be a digraph. A set of $\operatorname{arcs} S \subseteq A$ is called a feedback arc set if $D-S$ is acyclic. The minimum number of elements in a feedback arc set of $D$ is denoted by $\tau(D)$. The maximum number of arc-disjoint cycles in $D$ is denoted by $\nu(D)$. Let $\pi=v_{1}, v_{2}, \ldots, v_{n}$ be an ordering of the vertices of $D$. An arc $v_{i} v_{j} \in A$ is called backward with respect to $\pi$ if $i>j ; \pi$ is $\tau$-optimal if the number of backward arcs with respect to $\pi$ is minimum among all orderings of vertices of $D$. For a vertex $v \in V$, we use the following notation:

$$
\begin{gathered}
N^{+}(v)=\{u \in V-v: v u \in A\}, \quad N^{-}(v)=\{u \in V-v: u v \in A\}, \\
N^{+2}(v)=\bigcup_{u \in N^{+}(v)} N^{+}(u)-N^{+}(v)-\{v\}
\end{gathered}
$$

The out-degree of a vertex $v \in V$ is $d^{+}(v)=\left|N^{+}(v)\right|$, and $\delta^{+}(D)=\min \left\{d^{+}(v): v \in V\right\}$. We use Bang-Jensen and Gutin [1] as reference for undefined terms.

It is well known that $S$ is a minimum feedback arc set in a digraph $D$ if and only if there exists a $\tau$-optimal ordering $\pi$ of vertices in $D$ such that $S$ is the set of backward arcs with respect to $\pi$ (see Bang-Jensen and Gutin [1]). Hence it follows that for every digraph $D$ we have $\tau(D) \geq \frac{1}{2} \delta^{+}(D)\left(\delta^{+}(D)+1\right)$ (see Remark 3). Erdős and Moon [4] proved that for every $n \geq 3$ there exists a tournament $T_{n}$ of order $n$ such that $\tau\left(T_{n}\right) \geq \frac{1}{4} n(n-1)-\frac{1}{2} \sqrt{n^{3} \log _{e} n}$. A slightly better result was obtained by de la Vega in [14. On the other hand, it follows from a result by Chatrand, Geller and Hedetniemi [2] that $\nu\left(T_{n}\right) \leq\left\lfloor\frac{1}{3} n\left\lfloor\frac{1}{2}(n-1)\right\rfloor\right\rfloor$. Even though not always $\tau(D)=\nu(D)$, Isaak [9] conjectured (Conjecture 15.4.1 of [1]) that if $T$ is a tournament having a minimum feedback arc set which induces a transitive subtournament of $T$, then $\tau(T)=\nu(T)$. He posed also the following question (Problem 15.4.2 of [1]): Suppose $T$ is a tournament having a minimum feedback arc set which induces an acyclic digraph with a hamiltonian path. Is it true that $\tau(T)=\nu(T)$ ? We prove that the answer to the Isaak question is in the negative.

Mathematical sociologist Landau [10] proved that in every tournament $T$, if a vertex $v$ has the minimum out-degree, then it belongs to $\delta^{+}(T)$ different 3 -cycles. If $T$ is eulerian, then every vertex has the minimum out-degree. One may guess that for this case every vertex belongs to
$\delta^{+}(T)$ arc-disjoint 3-cycles. However, it is not true (see Remark 4). We prove (Theorem 2.1) that in every oriented graph $D$, if $v$ is a vertex of the minimum out-degree which is adjacent to all other vertices, then it belongs to $\delta^{+}(D)$ arc-disjoint cycles. Mader [11] consider linkages in digraphs with lower bounds on the out-degree. He proved a theorem which is related to our result: every digraph $D$ with $\delta^{+}(D) \geq n$ contains a pair of distinct vertices $x, y$ with $n-1$ arc-disjoint $(x, y)$-paths.

We conjecture that every tournament $T$ has a vertex of the minimum out-degree which belongs to $\delta^{+}(T)$ arc-disjoint 3-cycles. It is connected with the following conjecture put forward by Hoang and Reed [8]: every digraph $D$ with $\delta^{+}(D)=n$ contains a sequence $C_{1}, C_{2}, \ldots$, $C_{n}$ of cycles such that $\bigcup_{i=1}^{j-1} C_{i}$ and $C_{j}$ have at most one vertex in common. In the case of $n=2$ the Hoang and Reed conjecture was proved by Thomassen [13]. The conjecture was verified for tournaments by Havet, Thomassé and Yeo [7. They proved that every tournament $T$ with $\delta^{+}(T)=n$ contains a sequence $C_{1}, C_{2}, \ldots, C_{n}$ of 3 -cycles such that $\bigcup_{i=1}^{j-1} C_{i}$ and $C_{j}$ have exactly one vertex in common. Our conjecture is also connected with the following conjecture posed by Seymour [3] (see also Seymour's Second Neighbourhood Conjecture [1]): every oriented graph has a vertex $v$ such that $\left|N^{+}(v)\right| \leq\left|N^{+2}(v)\right|$. In the case of tournaments this conjecture was proved by Fisher [5]. An elementary proof for the case of tournaments was found by Havet and Thomassé [6] (see also Bang-Jensen and Gutin [1]).

## 2. The Isaak problem

Let $T$ be a tournament of order 13 in Figure 1 and let $\alpha=a, b, c, d, e, f, g, h, i, j, k, l, m$ be an ordering of vertices of $T$. We will prove that $\nu(T)=11$. Notice that the set of all backward arcs with respect to $\alpha$ is a feedback arc set which induces an acyclic digraph with a hamiltonian path $m, k, i, g, e, c, a$. We will prove that the ordering $\alpha$ is $\tau$-optimal. Hence, $\tau(T)=12$


Figure 1. The tournament $T$. Only backward arcs with respect to the ordering $\alpha$ are shown.

Let $\mathcal{C}$ be a family of the following arc-disjoint 3 -cycles in $T$ :

$$
(a, b, c),(c, d, e),(e, f, g),(g, h, i),(i, j, k),(k, l, m),(a, d, g),(c, f, i),(e, h, k),(g, j, m),(a, e, i)
$$

Suppose that $U$ is the union set of arcs of all cycles belonging to the family $\mathcal{C}$. Notice that me is the only backward arc with respect to $\alpha$ which does not belong to $U$. If $m e$ is an arc of some cycle in the tournament, then this cycle has an arc belonging to $U$. Hence, $\mathcal{C}$ is a maximal family of arc-disjoint cycles in $T$. It is easy to see that every backward arc determines uniquely a family of eleven arc-disjoint cycles omitting this backward arc. By analogy, we can check that every such family is a maximal family of arc-disjoint cycles in $T$. Hence, $\mathcal{C}$ is a maximum family of arc-disjoint cycles in $T$. Thus $\nu(T)=11$. Let $S$ be a minimum feedback arc set in $T$. If $S$ has eleven elements, then it satisfies the following conditions:
(1) every cycle in $\mathcal{C}$ has exactly one arc belonging to $S$,
(2) $S \subset U$.

Since $i m \notin U$ and $m e \notin U$, by (2), the arc $e i$ of a cycle $(e, i, m)$ belongs to $S$. Hence, by (1), the arc $i a$ of a cycle $(a, e, i) \in \mathcal{C}$ does not belong to $S$. Since $i a \notin S$ and $a h \notin U$, by (2), the arc $h i$ of a cycle $(a, h, i)$ belongs to $S$. Hence, by (1), the arc $g h$ of a cycle $(g, h, i) \in \mathcal{C}$ does not belong to $S$. Since $g h \notin S$ and $h m \notin U$, by (2), the arc $m g$ of a cycle $(g, h, m)$ belongs to $S$. Hence, by (1), the arc $j m$ of a cycle $(g, j, m) \in \mathcal{C}$ does not belong to $S$. Thus we obtain a contradiction, a cycle $(e, j, m)$ is arc-disjoint with the set $S$. Hence, $S$ has at least twelve elements.
Remark 1. If we add three arcs $m c, k c$ and $k a$ to the tournament $T$ we obtain a tournament $T^{\prime}$ having a minimum feedback arc set which induces an acyclic digraph with a hamiltonian path, such that $\nu\left(T^{\prime}\right)=14$ and $\tau\left(T^{\prime}\right)=15$.

Remark 2. We checked case by case that if $T_{\leq 6}$ is a tournament of order at most 6 , then the maximum number of arc-disjoint cycles in $T_{\leq 6}$ is equal to the cardinality of a minimum feedback arc set of $T_{\leq 6}$. We show in Figure 2 a tournament $T_{7}$ of order 7 such that $\nu\left(T_{7}\right)=4$ and $\tau\left(T_{7}\right)=5$.


Figure 2. The tournament $T_{7}$. Only backward arcs with respect to an ordering $a, b, c, d, e, f, g$ are presented.

Remark 3. If $D$ is a digraph, then $\tau(D) \geq \frac{1}{2} \delta^{+}(D)\left(\delta^{+}(D)+1\right)$. We proceed by induction on the order of $D$. Let $S$ be a feedback arc set in $D$ of minimum size, and suppose that $\pi=v_{1}, v_{2}, \ldots, v_{n}$ is a $\tau$-optimal ordering of vertices in $D$ such that $S$ is the set of backward arcs with respect to $\pi$. Since $v_{1}, v_{2}, \ldots, v_{n-1}$ is a $\tau$-optimal ordering of vertices in $D-v_{n}$, $S^{\prime}=S-\left\{v_{n} v: v \in N^{+}\left(v_{n}\right)\right\}$ is a feedback arc set in $D-v_{n}$ of minimum size. Notice that $\delta^{+}\left(D-v_{n}\right) \geq \delta^{+}(D)-1$. Hence, the size of $S^{\prime}$ is at least $\frac{1}{2}\left(\delta^{+}(D)-1\right) \delta^{+}(D)$. Therefore, the size of $S$ is at least

$$
\frac{1}{2}\left(\delta^{+}(D)-1\right) \delta^{+}(D)+\delta^{+}(D)=\frac{1}{2} \delta^{+}(D)\left(\delta^{+}(D)+1\right)
$$

## 3. Arc-Disjoint cycles through a vertex of the minimum out-Degree

For a pair $X, Y$ of vertex sets of an oriented graph $D=(V, A)$, we define

$$
(X, Y)_{D}=\{(x, y) \in A: x \in X, y \in Y\}
$$

Theorem 3.1. Let $D$ be an oriented graph, and suppose that $v_{0}$ is a vertex which is adjacent to all other vertices in $D$. Let $a=\min \left\{d^{+}(v): v \in N^{+}\left(v_{0}\right)\right\}$ and $b=\min \left\{d^{+}(v): v \in N^{-}\left(v_{0}\right)\right\}$. If $d^{+}\left(v_{0}\right) \leq \min \left(a, \frac{1}{2}(a+b+1)\right)$, then $v_{0}$ belongs to $d^{+}\left(v_{0}\right)$ arc-disjoint cycles.

Proof Let $\Gamma=\left\{\gamma_{1}, \gamma_{2}, \ldots, \gamma_{m}\right\}$ be a maximum family of arc-disjoin cycles through $v_{0}$. Let $\gamma_{i}=v_{0} v_{1}^{i} \ldots v_{n(i)}^{i} v_{0}$, for $i=1, \ldots, m$. By Menger's theorem (see [12] and [1] there exists a set $\Delta$ of $m$ arcs covering all cycles containing the vertex $v_{0}$. Suppose that $k$ is the number of arcs in $\Delta$ with the head $v_{0}$. If $k>0$, we can assume that $\left(v_{n(i)}^{i}, v_{0}\right) \in \Delta$ if and only if $1 \leq i \leq k$.


Figure 3. The eulerian tournament $T_{11}$.
$(\{h, i, j\},\{a, b, c\})_{T_{11}} \cup(\{k\},\{a, b, c, d, e\})_{T_{11}} \cup\{(c, a),(g, d),(j, h)\}$ is the set of all backward arcs with respect to the ordering $\beta$.

Let us denote $K=\left\{v_{1}^{1}, v_{1}^{2}, \ldots, v_{1}^{k}\right\}, X=\left\{v_{n(1)}^{1}, v_{n(2)}^{2}, \ldots, v_{n(k)}^{k}\right\}$ (if $k=0$ we set $K=X=\emptyset$ ), $L=\left\{v_{1}^{k+1}, v_{1}^{k+2}, \ldots, v_{1}^{m}\right\}, Y=N^{-}\left(v_{0}\right)-X$, and $M=N^{+}\left(v_{0}\right)-K-L$. First we prove the following inequality:
(1) $\left|(K \cup X \cup M, Y)_{D}\right| \leq\left|(L, K \cup X \cup M)_{D}\right|$.

Assume that $(v, y) \in(K \cup X \cup M, Y)_{D}$. Notice that $\left(y, v_{0}\right) \notin \Delta$. If $v \in K \cup M$, then $\left(v_{0}, v\right) \notin \Delta$. Hence, the arc $(v, y)$ of a cycle $v_{0} v y v_{0}$ belongs to $\Delta$. If $v \in X$, then $v=v_{n(i)}^{i}$, for some $i \leq k$. Hence, the $\operatorname{arc}\left(v_{n(i)}^{i}, y\right)$ of a cycle $v_{0} v_{1}^{i} \ldots v_{n(i)}^{i} y v_{0}$ belongs to $\Delta$. Thus, $(v, y)$ is an arc of some cycle $\gamma_{i}$, for $i>k$. Accordingly, to every arc $(v, y) \in(K \cup X \cup M, Y)_{D} \subseteq \Delta$ we can assign an $\operatorname{arc}\left(l, v^{\prime}\right) \in(L, K \cup X \cup M)_{D}$ such that $\left(l, v^{\prime}\right)$ and $(v, y)$ belong to the same cycle in $\Gamma$. The above assignment is injective, because different arcs in $\Delta$ belong to arc-disjoint cycles in Г. Hence, (1) holds.

Let us complete the oriented graph $D$ to a tournament $T$ with the same vertex set $V$. Hence, $\mid K \cup X \cup M, V-Y)_{T}\left|\geq\left|(K \cup X \cup M, V-Y)_{D}\right|\right.$. Therefore, by (1) we obtain:

$$
\begin{aligned}
\left|(L, K \cup X \cup M)_{D}\right| & \left.+\left|(K \cup X \cup M, V)_{T}\right| \geq \mid K \cup X \cup M, Y\right)_{D}\left|+\left|(K \cup X \cup M, V)_{T}\right|\right. \\
& \geq\left|(K \cup X \cup M, Y)_{T}\right|+\left|(K \cup X \cup M, V)_{D}\right| .
\end{aligned}
$$

Accordingly,

$$
\begin{aligned}
|K \cup X \cup M|(|V|-1)= & \left|(V, K \cup X \cup M)_{T}\right|+\left|(K \cup X \cup M, V)_{T}\right| \\
= & \left|(V-L, K \cup X \cup M)_{T}\right|+\left|(L, K \cup X \cup M)_{T}\right|+\left|(K \cup X \cup M, V)_{T}\right| \\
\geq \geq & \left|(V-L, K \cup X \cup M)_{T}\right|+\left|(K \cup X \cup M, Y)_{T}\right|+\left|(K \cup X \cup M, V)_{D}\right| \\
= & \left|(K \cup X \cup M, K \cup X \cup M)_{T}\right|+\left|\left(\left\{v_{0}\right\}, K \cup X \cup M\right)_{T}\right| \\
& +\left|(Y, K \cup X \cup M)_{T}\right|+\left|(K \cup X \cup M, Y)_{T}\right|+\left|(K \cup X \cup M, V)_{D}\right| \\
\geq & \frac{1}{2}|K \cup X \cup M|(|K \cup X \cup M|-1)+|K|+|M| \\
& +|K \cup X \cup M||Y|+a|K|+b|X|+a|M| .
\end{aligned}
$$

Since $|V|-1=d^{+}\left(v_{0}\right)+|X|+|Y|$ and $|K|=|X|$, we have

$$
2|K| d^{+}\left(v_{0}\right)+|M| d^{+}\left(v_{0}\right) \geq\left(|K|+\frac{1}{2}|M|\right)(|M|-1)+|M|+(a+b+1)|K|+a|M| .
$$

Thus, $|M|=0$, because $(a+b+1) \geq 2 d^{+}\left(v_{0}\right)$ and $a \geq d^{+}\left(v_{0}\right)$.

Remark 4. Let $T_{11}$ be the eulerian tournament in Fig. 3, and $\beta=a, b, c, d, e, f, g, h, i, j, k$ an ordering of its vertices. Suppose that

$$
(\{h, i, j\},\{a, b, c\})_{T_{11}} \cup(\{k\},\{a, b, c, d, e\})_{T_{11}} \cup\{(c, a),(g, d),(j, h)\}
$$

is the set of all backward arcs with respect to the ordering $\beta$. Let $k v_{1} v_{2}$ be a 3 -cycle through the vertex $k$. Notice that, if $v_{1} \in\{a, b, c\}$, then $v_{2} \in\{f, g\}$. Hence, the vertex $k$ does not belong to $\delta^{+}\left(T_{11}\right)$ arc-disjoint 3-cycles.

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E-mail address: jan.florek@ue.wroc.pl
Institute of Mathematics, University of Economics, ul. Komandorska 118/120, 53-345 Wroceaw, Poland

