A Characterization of Mixed Unit Interval Graphs

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Abstract

We give a complete characterization of mixed unit interval graphs, the intersection graphs of closed, open, and half-open unit intervals of the real line. This is a proper superclass of the well known unit interval graphs. Our result solves a problem posed by Dourado, Le, Protti, Rautenbach and Szwarcfiter (Mixed unit interval graphs, Discrete Math. **312**, 3357-3363 (2012)).

Keywords: unit interval graph; proper interval graph; intersection graph

1 Introduction

A graph G is an *interval graph*, if there is a function I from the vertex set of G to the set of intervals of the real line such that two vertices are adjacent if and only if their assigned intervals intersect. The function I is an *interval representation* of G. Interval graphs are well known and investigated [4, 6, 8]. There are several different algorithms that decide, if a given graph is an interval graph. See for example [2].

An important subclass of interval graphs are unit interval graphs. An interval graph G is a *unit interval graph*, if there is an interval representation I of G such that I assigns to every vertex a closed interval of unit length. This subclass is well understood and easy to characterize structurally [10] as well as algorithmically [1].

Frankl and Maehara [5] showed that it does not matter, if we assign the vertices of G only to closed intervals or only to open intervals of unit length. Rautenbach and Szwarcfiter [9] characterized, by a finite list of forbidden induced subgraphs, all interval graphs G such that there is an interval representation of G that uses only open and closed unit intervals.

Dourado et al. [3] gave a characterization of all diamond-free interval graphs that have an interval representation such that all vertices are assigned to unit intervals, where all kinds of unit intervals are allowed and a diamond is a complete graph on four vertices minus an edge. Furthermore, they made a conjecture concerning the general case. We prove that their conjecture is not completely correct and give a complete characterization of this class. Since the conjecture is rather technical and not given by a list of forbidden subgraphs, we refer the reader to [3] for a detailed formulation of the conjecture, but roughly speaking, they missed the class of forbidden subgraphs shown in Figure 6.

In Section 2 we introduce all definitions and relate our result to other work. In Section 3 we state and prove our results.

2 Preliminary Remarks

We only consider finite, undirected, and simple graphs. Let G be a graph. We denote by $V(G)$ and $E(G)$ the vertex and edge set of G, respectively. If C is a set of vertices, then we denote by $G[C]$ the subgraph of G induced by C. Let $\mathcal{M}$ be a set of graphs. We say G is $\mathcal{M}$-*free*, if for every $H \in \mathcal{M}$, the graph H is not an induced subgraph of G. For a vertex $v \in V(G)$, let the *neighborhood* $N_G(v)$ of v be the set of all vertices that are adjacent to v and let the *closed neighborhood* $N_G[v]$ be defined by $N_G(v) \cup \{v\}$. Two distinct vertices u and v are *twins* (in G) if $N_G[u] = N_G[v]$. If G contains no twins, then G is *twin-free*.

Let $\mathcal{N}$ be a family of sets. We say a graph G has an $\mathcal{N}$-*intersection representation*, if there is a function $f : V(G) \to \mathcal{N}$ such that for any two distinct vertices u and v there is an edge joining u and v if and only if $f(u) \cap f(v) \neq \emptyset$. If there is an $\mathcal{N}$-intersection representation for G, then G is an $\mathcal{N}$-*graph*. Let $x, y \in \mathbb{R}$. We denote by

$$[x, y] = \{z \in \mathbb{R} : x \leq z \leq y\}$$

the *closed interval*, by

$$(x, y) = \{z \in \mathbb{R} : x < z < y\}$$

the *open interval*, by

$$(x, y] = \{z \in \mathbb{R} : x < z \leq y\}$$

the *open-closed interval*, and by

$$[x, y) = \{z \in \mathbb{R} : x \leq z < y\}$$

the *closed-open interval* of x and y. For an interval A, let $\ell(A) = \inf\{x \in \mathbb{R} : x \in A\}$ and $r(A) = \sup\{x \in \mathbb{R} : x \in A\}$. If I is an interval representation of G and $v \in V(G)$, then we write $\ell(v)$ and $r(v)$ instead of $\ell(I(v))$ and $r(I(v))$, respectively, if there are no ambiguities. Let $\mathcal{I}^{++}$ be the set of all closed intervals, $\mathcal{I}^{--}$ be the set of all open intervals, $\mathcal{I}^{-+}$ be the set of all open-closed intervals, $\mathcal{I}^{+-}$ be the set of all closed-open intervals, and $\mathcal{I}$ be the set of all intervals. In addition, let $\mathcal{U}^{++}$ be the set of all closed unit intervals, $\mathcal{U}^{--}$ be the set of all open unit intervals, $\mathcal{U}^{-+}$ be the set of all open-closed unit intervals, $\mathcal{U}^{+-}$ be the set of all closed-open unit intervals, and $\mathcal{U}$ be the set of all unit intervals. We call a $\mathcal{U}$-graph a *mixed unit interval graph*.

By a result of [3] and [9], every interval graph is an $\mathcal{I}^{++}$-graph. With our notation unit interval graphs equals $\mathcal{U}^{++}$-graphs. An interval graph G is a *proper interval graph* if there is an interval representation of G such that $I(u) \not\subseteq I(v)$ for every distinct $u, v \in V(G)$.

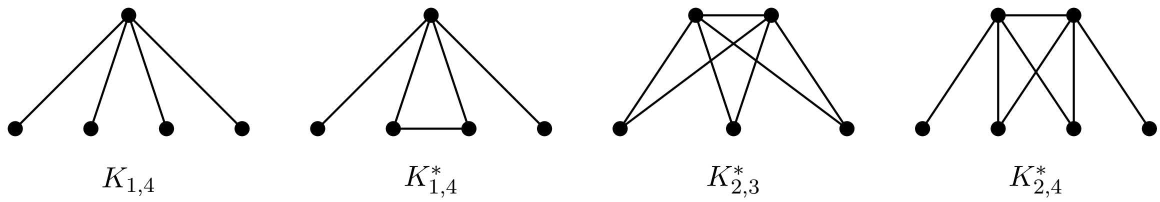


Figure 1: Forbidden induced subgraphs for twin-free $\mathcal{U}^{++} \cup \mathcal{U}^{--}$-graphs.

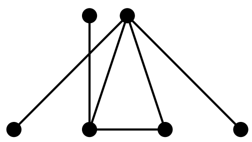


Figure 2: A graph, which is a $\mathcal{U}$-graph, but not a $\mathcal{U}^{++} \cup \mathcal{U}^{--}$-graph.

The next result due to Roberts characterizes unit interval graphs.

Theorem 1 (Roberts [10]). *The classes of unit interval graphs, proper interval graphs, and $K_{1,3}$-free interval graphs are the same.*

The second result shows that several natural subclasses of mixed unit interval graphs actually coincide with the class of unit interval graphs.

Theorem 2 (Dourado et al., Frankl and Maehara [3, 5]). *The classes of $\mathcal{U}^{++}$-graphs, $\mathcal{U}^{--}$-graphs, $\mathcal{U}^{+-}$-graphs, $\mathcal{U}^{-+}$-graphs, and $\mathcal{U}^{+-} \cup \mathcal{U}^{-+}$-graphs are the same.*

A graph G is a *mixed proper interval graph* (respectively an *almost proper interval graph*) if G has an interval representation $I : V(G) \to \mathcal{I}$ (respectively $I : V(G) \to \mathcal{I}^{++} \cup \mathcal{I}^{--}$) such that

- there are no two distinct vertices u and v of G with $I(u), I(v) \in \mathcal{I}^{++}$, $I(u) \subseteq I(v)$, and $I(u) \neq I(v)$, and
- for every vertex u of G with $I(u) \notin \mathcal{I}^{++}$, there is a vertex v of G with $I(v) \in \mathcal{I}^{++}$, $\ell(u) = \ell(v)$, and $r(u) = r(v)$.

A natural class extending the class of unit interval graphs are $\mathcal{U}^{++} \cup \mathcal{U}^{--}$-graphs. These were characterized by Rautenbach and Szwarcfiter.

Theorem 3 (Rautenbach and Szwarcfiter [9]). *For a twin-free graph G, the following statements are equivalent.*

- *G is a $\{K_{1,4}, K_{1,4}^*, K_{2,3}^*, K_{2,4}^*\}$-free graph. (See Figure 1 for an illustration.)*
- *G is an almost proper interval graph.*
- *G is a $\mathcal{U}^{++} \cup \mathcal{U}^{--}$-graph.*

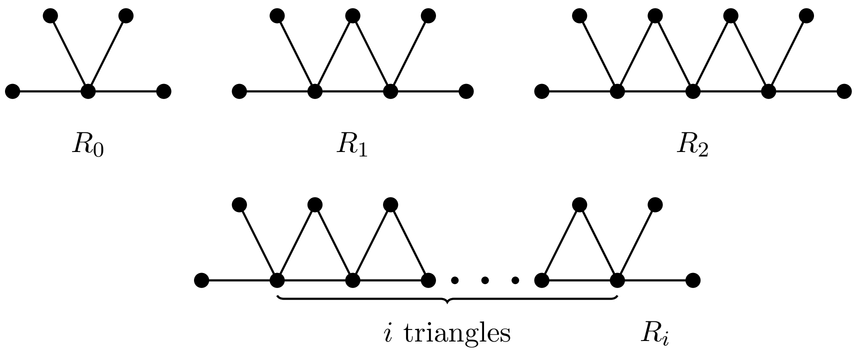


Figure 3: The class $\mathcal{R}$.

Note that an interval representation can assign the same interval to twins and hence the restriction to twin-free graphs does not weaken the statement but simplifies the description.

The next step is to allow all different types of unit intervals. The class of $\mathcal{U}$-graphs is a proper superclass of the $\mathcal{U}^{++} \cup \mathcal{U}^{--}$-graphs, because the graph illustrated in Figure 2 is a $\mathcal{U}$-graph, but not a $\mathcal{U}^{++} \cup \mathcal{U}^{--}$-graph (it contains a $K_{1,4}^*$). Dourado et al. already made some progress in characterizing this class.

Theorem 4 (Dourado et al. [3])**.** *For a graph G, the following two statements are equivalent.*

- *G is a mixed proper interval graph.*
- *G is a mixed unit interval graph.*

They also characterized diamond-free mixed unit interval graphs. There is another approach by Le and Rautenbach [7] to understand the class of $\mathcal{U}$-graphs by restricting the ends of the unit intervals to integers. They found a infinite list of forbidden induced subgraphs, which characterize these so-called *integral $\mathcal{U}$-graphs.*

3 Results

In this section we state and prove our main results. We start by introducing a list of forbidden induced subgraphs. See Figures 3, 4, 5, and 6 for illustration. Let $\mathcal{R} = \bigcup_{i=0}^{\infty}\{R_i\}$, $\mathcal{S} = \bigcup_{i=1}^{\infty}\{S_i\}$, $\mathcal{S}' = \bigcup_{i=1}^{\infty}\{S_i'\}$, and $\mathcal{T} = \bigcup_{i\geq j\geq 0}\{T_{i,j}\}$. For $k \in \mathbb{N}$ let the graph Q_k arise from the graph R_k by deleting two vertices of degree 1 that have a common neighbor. We call the common neighbor of the two deleted vertices and its neighbor of degree 2 *special vertices* of Q_k. Note that if a graph G is twin-free, then the interval representation of G is injective.

Lemma 5 (Dourado et al.[3])**.** *Let $k \in \mathbb{N}$.*

(a) Every $\mathcal{U}$-representation of the claw $K_{1,3}$ arises by translation (replacing I by $I + x$ for some $x \in \mathbb{R}$; that is, shifting all intervals by x) of the following $\mathcal{U}$-representation $I : V(K_{1,3}) \to \mathcal{U}$ of $K_{1,3}$, where $I(V(K_{1,3}))$ consists of the following intervals

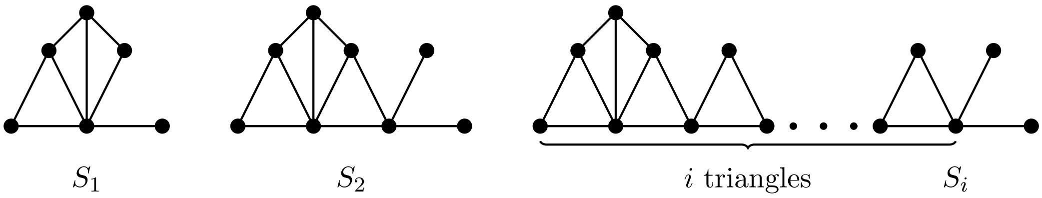


Figure 4: The class $\mathcal{S}$.

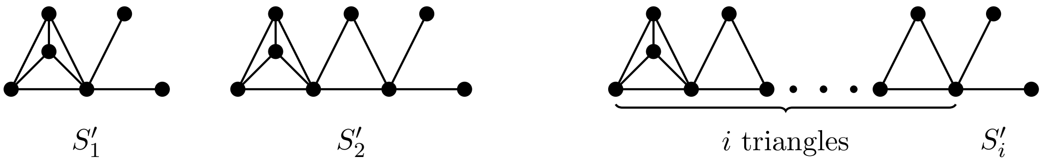


Figure 5: The class $\mathcal{S}'$.

- *either* $[0,1]$ *or* $(0,1]$,
- $[1,2]$ *and* $(1,2)$, *and*
- *either* $[2,3]$ *or* $[2,3)$.

(b) Every injective $\mathcal{U}$-representation of Q_k arises by translation and inversion (replacing I by $-I$; that is, multiplying all endpoints of the intervals by -1) of one of the two injective $\mathcal{U}$-representations $I : V(Q_k) \to \mathcal{U}$ of Q_k, where $I(V(Q_k))$ consists of the following intervals

- *either* $[0,1]$ *or* $(0,1]$,
- $[1,2]$ *and* $(1,2)$, *and*
- $[i,i+1]$ *and* $[i,i+1)$ *for* $2 \leq i \leq k+1$.

(c) The graphs in $\{T_{0,0}\} \cup \mathcal{R}$ are minimal forbidden subgraphs for the class of $\mathcal{U}$-graphs with respect to induced subgraphs.

(d) If G is a $\mathcal{U}$-graph, then every induced subgraph H in G that is isomorphic to Q_k and every vertex $u^* \in V(G) \setminus V(H)$ such that u^* is adjacent to exactly one of the two special vertices x of H, the vertex u^* has exactly one neighbor in $V(H)$, namely x.

Lemma 6. *If a graph G is a twin-free mixed unit interval graph, then G is $\{K^*_{2,3}\} \cup \mathcal{R} \cup \mathcal{S} \cup \mathcal{S}' \cup \mathcal{T}$-free.*

Proof of Lemma 6: It is easy to see that G is $\{K^*_{2,3}\}$-free. Lemma 5 (c) shows that G is $\mathcal{R}$-free and Lemma 5 (d) shows that G is $\mathcal{S}$-free.

Let $k \in \mathbb{N}$. Note that the graph S'_k arises from the graph Q_k by adding a vertex z and joining it to the two special vertices of Q_k and the unique common neighbor of these two vertices. For contradiction, we assume that S'_k has a $\mathcal{U}$-representation I. By Lemma 5

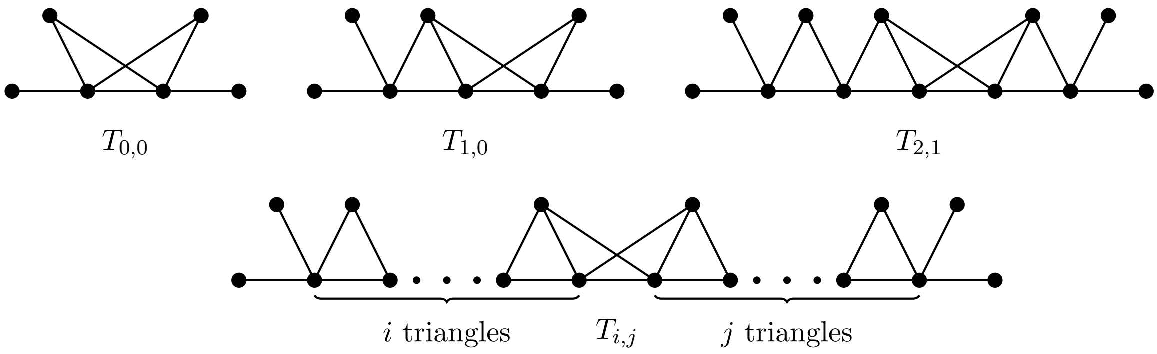


Figure 6: The class $\mathcal{T}$.

(b) there are only two possibilities for the $\mathcal{U}$-representation of Q_k. Thus we assume that the subgraph Q_k of S'_k has the interval representation as described in Lemma 5 (b). In both cases we conclude $\ell(z) = k+1$ and $k+1 \in I(z)$. Thus $r(z) = k+2$ and hence $I(z) \in \{[k+1, k+2], [k+1, k+2)\}$. Therefore, G is not twin-free, which is a contradiction. This implies that G is $\mathcal{S}'$-free.

By Lemma 5 (c), G is $T_{0,0}$-free. Let C be a claw with vertex set $\{c, a_1, a_2, a_3\}$, where c is the center vertex. Denote by v_k and w_k the special vertices of Q_k. Note that $T_{k,0}$ arises by the disjoint union of the graph Q_k and C, identifying v_k and a_1, and adding the edges $w_k c$ and $v_k a_2$. For contradiction, we assume that $T_{k,0}$ has a $\mathcal{U}$-representation I. By Lemma 5 (b), we assume without loss of generality that the induced subgraph Q_k of $T_{k,0}$ is represented by exactly the intervals described in Lemma 5 (b). Thus $I(v_k) = [k+1, k+2]$ and $I(w_k) = [k+1, k+2)$, because $v_k a_2 \in E(T_{k,0})$ but $w_k a_2 \notin E(T_{k,0})$. Since $I(v_k)$ is not an open interval and by Lemma 5 (a), we obtain $I(c) = [k+2, k+3]$ and hence $I(w_k) \cap I(c) = \emptyset$. This is a contradiction, which implies that G is $\bigcup_{i \geq 0}\{T_{i,0}\}$-free.

Let $i, j \in \mathbb{N}$. Note that the graph $T_{i,j}$ arises by the disjoint union of Q_i and Q_j and adding three edges between the special vertices of Q_i and Q_j. We may assume that the intervals of the subgraph Q_i are exactly the intervals as described in Lemma 5 (b). Let w_i (respectively v_i) be the vertex of Q_i that has one (two) neighbor(s) in the subgraph Q_j; that is, $I(v_i) = [i+1, i+2]$ and $I(w_i) = [i+1, i+2)$ because $N_{T_{i,j}}(w_i) \subset N_{T_{i,j}}(v_i)$. Let w_j (respectively v_j) be the vertex of Q_j that has one (two) neighbor(s) in the subgraph Q_i. Since the subgraph Q_j has also an interval representation as described in Lemma 5 (b) and the vertices of $Q_i \setminus \{v_i, w_i\}$ and not joined by an edge to the vertices of $Q_j \setminus \{v_j, w_j\}$, we conclude that the intervals of the vertices of Q_j arise by an inversion and a translation of the interval representation as described in Lemma 5 (b). This implies that $I(v_j) = [x, x+1]$ and $I(w_j) = (x, x+1]$ for some $x \in \mathbb{R}$. Obviously, $x \in [i+1, i+2]$. If $x = i+2$, then neither v_i is adjacent to w_j nor w_i is adjacent to v_i. If $x \in [i+1, i+2)$, then the intervals of w_i and w_j intersect, which is not possible. Therefore, G is $\mathcal{T}$-free and this completes the proof. $\square$

We proceed to our main result.

Theorem 7. *A twin-free graph G is a mixed unit interval graph if and only if G is a $\{K^*_{2,3}\} \cup \mathcal{R} \cup \mathcal{S} \cup \mathcal{S}' \cup \mathcal{T}$-free interval graph.*

Proof of Theorem 7: We use a similar approach as in [9]. By Lemma 6, we know if G is a twin-free mixed unit interval graph, then G is a $\{K^*_{2,3}\} \cup \mathcal{R} \cup \mathcal{S} \cup \mathcal{S}' \cup \mathcal{T}$-free interval graph. Let G be a twin-free $\{K^*_{2,3}\} \cup \mathcal{R} \cup \mathcal{S} \cup \mathcal{S}' \cup \mathcal{T}$-free interval graph. We show that G is a mixed proper interval graph. By Theorem 4, this proves Theorem 7. Since G is an interval graph, G has an $\mathcal{I}^{++}$-representation I. As in [9] we call a pair (u, v) of distinct vertices a *bad pair* if $I(u) \subseteq I(v)$. Let I be such that the number of bad pairs is as small as possible. If I has no bad pair, then we are done by Theorem 1. Hence we assume that there is at least one bad pair. The strategy of the proof is as follows. Claim 1 to Claim 6 collect properties of G and I, before we modify our interval representation of G to show that G is a mixed proper interval graph. In Claim 7 to Claim 10 we prove that our modification of the interval representation preserves all intersections and non-intersections. Claim 1 to Claim 3 are similar to Claim 1 to Claim 3 in [9], respectively. For the sake of completeness we state the proofs here.

Claim 1. *If (u, v) is a bad pair, then there are vertices x and y such that $\ell(v) \leq r(x) < \ell(u)$ and $r(u) < \ell(y) \leq r(v)$.*

Proof of Claim 1: For contradiction, we assume the existence of a bad pair (w, v) such that there is no vertex x with $\ell(v) \leq r(x) < \ell(w)$. A symmetric argument implies the existence of y. Let u be a vertex such that $\ell(u)$ is as small as possible with respect to $I(u) \subseteq I(v)$. By our assumption there is no vertex x such that $\ell(v) \leq r(x) < \ell(u)$. Let ϵ be the smallest distance between two distinct endpoints of intervals of I. Let $I' : V(G) \to \mathcal{I}^{++}$ be such that $I'(u) = [\ell(v) - \epsilon/2, r(u)]$, $I'(v) = [\ell(v), r(v) + \epsilon/2]$, and $I'(z) = I(z)$ for $z \in V(G) \setminus \{u, v\}$. By the choice of u and ϵ, we conclude that I' is an interval representation of G, but I' has less bad pairs than I, which is a contradiction to our choice of I. This completes the proof. □

Let a_1 and a_2 be two distinct vertices. Claim 1 implies that $\ell(a_1) \neq \ell(a_2)$ and $r(a_1) \neq r(a_2)$. Suppose $\ell(a_1) < \ell(a_2)$. If $r(a_1) = \ell(a_2)$, then let ϵ be as in the proof of Claim 1 and $I' : V(G) \to \mathcal{I}^{++}$ be such that $I'(a_1) = [\ell(a_1), r(a_1) + \epsilon/2]$, and $I'(z) = I(z)$ for $z \in V(G) \setminus \{a_1\}$. By the choice of ϵ, we conclude that I' is an interval representation of G with as many bad pairs as I. Therefore, we assume without loss of generality that we chose I such that all endpoints of the intervals of I are distinct. Hence the inequalities in Claim 1 are strict inequalities.

Claim 2. *If (u, w) and (v, w) are bad pairs, then $u = v$, that is, no interval contains two distinct intervals.*

Proof of Claim 2: For contradiction, we assume that there are distinct vertices u', v' and w such that (u', w) and (v', w) are bad pairs. Let u be a vertex such that (u, w) is a bad pair and $\ell(u)$ is as small as possible. Let v be a vertex such that (v, w) is a bad pair and $r(v)$ is as large as possible. Claim 1 ensures two distinct vertices x and y such that $\ell(w) < r(x) < \ell(u)$ and $r(v) < \ell(y) < r(w)$.

If $u \neq v$ and $I(u) \cap I(v) = \emptyset$, then $G[\{w, x, u, v, y\}]$ is isomorphic to R_0, which is a contradiction. If $u \neq v$ and $I(u) \cap I(v) \neq \emptyset$, then in the graph $G[\{w, x, u, v, y\}]$ the vertices u and v are twins. Since G is twin-free, u and v do not have the same closed neighborhood in G and hence there is a vertex z, which is adjacent to say u (by symmetry) and not to v. Since $I(u) \subset I(w)$, z is adjacent to w. If z is not adjacent to x, then $G[\{w, x, z, v, y\}]$ is isomorphic to R_0 and if z is adjacent to x, then $G[\{w, x, z, u, v, y\}]$ is isomorphic to S_1, which is a contradiction.

If $u = v$, then there is a vertex z such that (z, u) is a bad pair because u' or v' is a suitable choice. We choose z such that $\ell(z)$ is minimal. Claim 1 ensures the existence of a vertex x' such that $\ell(u) < r(x') < \ell(z)$. Note that the choice of u and z guarantees $\ell(x') < \ell(w)$, so $xx' \in E(G)$. Therefore, $G[\{w, x, x', u, z, y\}]$ is isomorphic to S_1, which is a contradiction. This completes the proof of Claim 2. $\square$

Claim 3. *If (u, v) and (u, w) are bad pairs, then $v = w$, that is, no interval is contained in two distinct intervals.*

Proof of Claim 3: Claim 2 implies that neither (v, w) nor (w, v) is a bad pair. Thus we may assume $\ell(w) < \ell(v) < \ell(u)$ and $r(u) < r(w) < r(v)$. By Claim 1, there are vertices x and y such that $\ell(v) < r(x) < \ell(u)$ and $r(u) < \ell(y) < r(w)$. Now, $G[\{v, w, x, u, y\}]$ is isomorphic to $K^*_{2,3}$, which is a contradiction and completes the proof of Claim 3. $\square$

A vertex x is to the *left* (respectively *right*) of a vertex y (in I), if $r(x) < \ell(y)$ (respectively $r(y) < \ell(x)$). Two adjacent vertices x and y are *distinguishable* by vertices to the left (respectively right) of them, if there is a vertex z, which is adjacent to exactly one of them and to the left (respectively right) of one of them. The vertex z *distinguishes* x and y. Next, we show that for a bad pair (u, v) there is the structure as shown in Figure 7 in G. We introduce a positive integer $\ell^{\max}_{u,v}$ that, roughly speaking, indicates how large this structure is.

For a bad pair (u, v) let $v = X^0_{u,v}$ and let $X^1_{u,v}$ be the set of vertices that are adjacent to v and to the left of u. Let $y_{u,v}$ be a vertex to the right of u and adjacent to v. Claim 1 guarantees $|X^1_{u,v}| \geq 1$ and the existence of $y_{u,v}$. If $|X^1_{u,v}| = 1$, then let $\ell^{\max}_{u,v} = 1$ and we stop here. Suppose $|X^1_{u,v}| \geq 2$. Since G is R_0-free, $X^1_{u,v}$ is a clique and since G is S'_1-free, we conclude $|X^1_{u,v}| = 2$. Let $\{x, x'\} = X^1_{u,v}$ such that $r(x) < r(x')$. For contradiction, we assume that there is a vertex z to the right of x that distinguishes x and x'. We conclude $\ell(v) < \ell(z)$. By Claim 2, $r(v) < r(z)$. This implies that (u, z) is a bad pair, which contradicts Claim 3. Thus z does not exist. In addition (x, x') is not a bad pair, otherwise Claim 1 guarantees a vertex z such that $r(x) < \ell(z) < r(x')$, which is a contradiction. Thus $\ell(x) < \ell(x') < r(x) < r(x')$. Let $x^1_{u,v} = x$ and ${x^1_{u,v}}' = x'$. Note that $N_G({x^1_{u,v}}') \subset N_G(x^1_{u,v})$.

Let $X^2_{u,v} = N_G(x^1_{u,v}) \setminus N_G({x^1_{u,v}}')$. Note that all vertices in $X^2_{u,v}$ are to the left of ${x^1_{u,v}}'$. Since G is twin-free, $|X^2_{u,v}| \geq 1$. If $|X^2_{u,v}| = 1$, then let $\ell^{\max}_{u,v} = 2$ and we stop here.

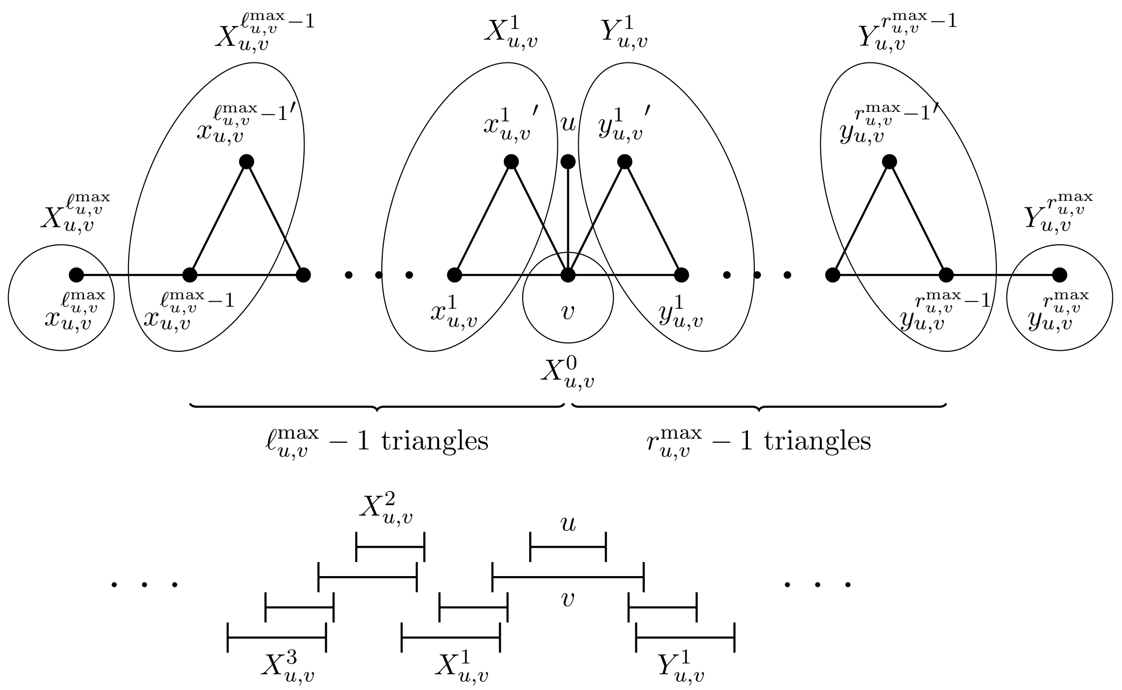


Figure 7: The structure in G forced by a bad pair (u,v).

Suppose $|X_{u,v}^2| \geq 2$. Since G is R_1-free, $X_{u,v}^2$ is a clique and since G is S_2'-free, we conclude $|X_{u,v}^2| = 2$. Let $\{x,x'\} = X_{u,v}^2$ such that $r(x) < r(x')$. For contradiction, we assume that there is a vertex z to the right of x that distinguishes x and x'. Since $z \notin X_{u,v}^2$, we conclude $\ell(x_{u,v}^1{}') < r(z)$. If $r(z) < \ell(v)$, then $G[\{z,x,x',x_{u,v}^1,x_{u,v}^1{}',v,u,y_{u,v}\}]$ is isomorphic to S_2, which is a contradiction. Thus $\ell(v) < r(z)$. If $r(z) < \ell(u)$, then $|X_{u,v}^1| = 3$, which is a contradiction. Thus $\ell(u) < r(z)$. If $r(u) < r(z)$, then (u,v) and (u,z) are bad pairs, which is a contradiction to Claim 3. Thus $\ell(u) < r(z) < r(u)$. Now $G[\{z,x',x_{u,v}^1{}',v,u,y_{u,v}\}]$ is isomorphic to $T_{0,0}$, which is the final contradiction.

Note that (x,x') is not a bad pair, otherwise Claim 1 guarantees a vertex z such that $r(x) < \ell(z) < r(x')$, which is a contradiction. Thus $\ell(x) < \ell(x') < r(x) < r(x')$. Let $x_{u,v}^2 = x$ and $x_{u,v}^2{}' = x'$. Note that $N_G(x_{u,v}^2{}') \subset N_G(x_{u,v}^2)$. Let $X_{u,v}^3 = N_G(x_{u,v}^2) \setminus N_G(x_{u,v}^2{}')$. Note that all vertices in $X_{u,v}^3$ are to the left of $x_{u,v}^2{}'$.
We assume that for $k \geq 3$, $i \in [k-1]$ and $j \in [k]$

- we defined $X_{u,v}^j$,
- $|X_{u,v}^i| = 2$ holds,
- we defined $x_{u,v}^i$ and $x_{u,v}^i{}'$,
- $\ell(x_{u,v}^i) < \ell(x_{u,v}^i{}') < r(x_{u,v}^i) < r(x_{u,v}^i{}')$ holds,
- the vertices in $X_{u,v}^{i+1}$ are to the left of $x_{u,v}^i{}'$, and
- the vertices in $X_{u,v}^i$ are not distinguishable to the right.

If $|X^k_{u,v}| = 1$, then let $\ell^{\max}_{u,v} = k$ and we stop here. Suppose $|X^k_{u,v}| \geq 2$. Since G is R_{k-1}-free, $X^k_{u,v}$ is a clique and since G is S'_k-free, we obtain $|X^k_{u,v}| = 2$. Let $\{x, x'\} = X^k_{u,v}$ such that $r(x) < r(x')$. For contradiction, we assume that there is a vertex z to the right of x that distinguishes x and x'. Since $z \notin X^k_{u,v}$, we conclude $\ell(x^{k-1\prime}_{u,v}) < r(z)$. If $r(z) < \ell(x^{k-2}_{u,v})$, then $G[\{z, x, x', v, u, y_{u,v}\} \cup \bigcup_{i=1}^{k-1} X^i_{u,v}]$ is isomorphic to S_k, which is a contradiction. Thus $\ell(x^{k-2}_{u,v}) < r(z)$. If $r(z) < \ell(x^{k-2\prime}_{u,v})$, then $|X^{k-1}_{u,v}| = 3$, which is a contradiction. Thus $\ell(x^{k-2\prime}_{u,v}) < r(z)$. If $r(z) < \ell(x^{k-3}_{u,v})$, then $G[\{z, x', x^{k-1\prime}_{u,v}, v, u, y_{u,v}\} \cup \bigcup_{i=1}^{k-2} X^i_{u,v}]$ is isomorphic to $T_{k-3,0}$, which is a contradiction. Thus $\ell(x^{k-3}_{u,v}) < r(z)$. If $r(z) < r(x^{k-2}_{u,v})$, then $|X^{k-2}_{u,v}| = 3$, which is a contradiction. Thus $r(x^{k-2}_{u,v}) < r(z)$ and hence $(x^{k-1\prime}_{u,v}, z)$ and $(x^{k-2}_{u,v}, z)$ are bad pairs, which is a contradiction to Claim 2. Thus x, x' are not distinguishable to the right. We obtain that (x, x') is not a bad pair, otherwise Claim 1 guarantees a vertex z such that $r(x) < \ell(z) < r(x')$, which is a contradiction. Thus $\ell(x) < \ell(x') < r(x) < r(x')$. Let $x^k_{u,v} = x$ and ${x^k_{u,v}}' = x'$. Note that $N_G({x^k_{u,v}}') \subset N_G(x^k_{u,v})$. Let $X^{k+1}_{u,v} = N_G(x^k_{u,v}) \setminus N_G({x^k_{u,v}}')$. Note that all vertices in $X^{k+1}_{u,v}$ are to the left of ${x^k_{u,v}}'$. By induction this leads to the following properties.

Claim 4. *If (u, v) is a bad pair, $k \in [\ell^{\max}_{u,v} - 1]$, then the following holds:*

(a) $|X^k_{u,v}| = 2$.

(b) The vertices in $X^k_{u,v}$ are not distinguishable by vertices to the right of them.

(c) We have $\ell(x^i_{u,v}) < \ell({x^i_{u,v}}') < r(x^i_{u,v}) < r({x^i_{u,v}}')$, that is $(x^k_{u,v}, {x^k_{u,v}}')$ and $({x^k_{u,v}}', x^k_{u,v})$ are not bad pairs.

Note that $\ell^{\max}_{u,v}$ is the smallest integer k such that $|X^{k-1}_{u,v}| = 2$ and $|X^k_{u,v}| = 1$.

Claim 5. *If (u, v) is a bad pair and $k \in [\ell^{\max}_{u,v} - 1]$, then the following holds.*

(a) ${x^k_{u,v}}'$ is not contained in a bad pair.

(b) There is no vertex $z \in V(G)$ such that $(x^k_{u,v}, z)$ is a bad pair.

Proof of Claim 5: (a): For contradiction, we assume that there is a vertex $z \in V(G)$ such that $({x^k_{u,v}}', z)$ is a bad pair. Trivially $z \notin \{\{v, y, u\} \cup \bigcup_{i=1}^{\ell^{\max}_{u,v}} X^i_{u,v}\}$. We have $r({x^k_{u,v}}') < r(z)$ and $\ell(z) < \ell({x^k_{u,v}}')$. In addition $\ell(x^k_{u,v}) < \ell(z)$, otherwise $(x^k_{u,v}, z)$ is also a bad pair, which contradicts Claim 2. Claim 1 implies the existence of a vertex a, such that $\ell(z) < r(a) < \ell({x^k_{u,v}}')$.

Let $k = 1$. If $r(z) < \ell(u)$, then $z \in X^1_{u,v}$, which is a contradiction to $|X^1_{u,v}| = 2$. Thus $\ell(u) < r(z)$. If $r(z) < r(u)$, then $G[\{a, z, {x^k_{u,v}}', u, v, y\}]$ is isomorphic to $T_{0,0}$, which is a contradiction. Thus $r(u) < r(z)$ and now (u, z) is a bad pair, which is a contradiction to Claim 2.

Let $k \geq 2$. If $r(z) < \ell(x^{k-1\prime}_{u,v})$, then $z \in X^k_{u,v}$, which is a contradiction to $|X^k_{u,v}| = 2$. Thus $\ell(x^{k-1\prime}_{u,v}) < r(z)$. If $r(z) < \ell(x^{k-2}_{u,v})$, then $G[\{a, z, {x^k_{u,v}}', v, u, y\} \cup \bigcup_{i=1}^{k-1} X^i_{u,v}]$ is isomorphic to $T_{k-1,0}$. Thus $\ell(x^{k-2}_{u,v}) < r(z)$. If $r(z) < r(x^{k-1}_{u,v})$, then $z \in X^{k-1}_{u,v}$, which is a contradiction to $|X^{k-1}_{u,v}| = 2$. Thus $r(x^{k-1}_{u,v}) < r(z)$, but now $(x^{k-1}_{u,v}, z)$ is also a bad pair, which is a contradiction to Claim 2 and completes this part of the proof.

For contradiction, we assume that there is a vertex $z \in V(G)$ such that $(z, {x^k_{u,v}}')$ is a bad pair. By Claim 1, $\ell({x^k_{u,v}}') < \ell(z)$ and $r(z) < r({x^k_{u,v}}')$. By Claim 3, $r(x^k_{u,v}) < r(z)$. Let y_z be the vertex guaranteed by Claim 1 such that $r(z) < \ell(y_z)$, but this contradicts Claim 4 (b).

(b): For contradiction, we assume the existence of a vertex $z \in V(G)$ such that $(x^k_{u,v}, z)$ is a bad pair. Trivially $z \neq {x^k_{u,v}}'$. If $r(z) < r({x^k_{u,v}}')$, then this contradicts Claim 4 (a), that is $|X^k_{u,v}| = 2$ and if $r({x^k_{u,v}}') < r(z)$, then $({x^k_{u,v}}', z)$ is also a bad pair and this contradicts Claim 2. This completes the proof of Claim 5. $\square$

For a bad pair (u, v) define $Y^k_{u,v}$ as $X^k_{u,v}$ by interchanging in the definition right by left. Let $r^{\max}_{u,v}$ be the smallest integer k such that $|Y^{k-1}_{u,v}| = 2$ and $|Y^k_{u,v}| = 1$. By symmetry, one can prove a "y"-version of Claim 4, Claim 5 and Claim 6 (a) and (b). Let $\{y^k_{u,v}, {y^k_{u,v}}'\} = Y^k_{u,v}$ such that $N_G({y^k_{u,v}}') \subset N_G(y^k_{u,v})$ for $k \leq r^{\max}_{u,v} - 1$.

Claim 6. *Let (u, v) and (w, z) be bad pairs and $k \in [\ell^{\max}_{u,v}]$.*

(a) If $X^k_{u,v} \cap X^{\tilde{k}}_{w,z} \neq \emptyset$, then $x^{k-1}_{u,v} = x^{\tilde{k}-1}_{w,z}$ for $\tilde{k} \in [\ell^{\max}_{w,z}]$.

(b) If $X^k_{u,v} \cap X^{\tilde{k}}_{w,z} \neq \emptyset$, then $X^k_{u,v} = X^{\tilde{k}}_{w,z}$ for $\tilde{k} \in [\ell^{\max}_{w,z}]$.

(c) If $X^k_{u,v} \cap Y^{\tilde{k}}_{w,z} \neq \emptyset$, then $X^k_{u,v} \cap Y^{\tilde{k}}_{w,z} = x^k_{u,v} = y^{\tilde{k}}_{w,z}$ for $\tilde{k} \in [r^{\max}_{w,z}]$

Proof of Claim 6: (a): For contradiction we assume $x^{k-1}_{u,v} \neq x^{\tilde{k}-1}_{w,z}$. Without loss of generality we assume $\ell(x^{k-1}_{u,v}) < \ell(x^{\tilde{k}-1}_{w,z})$. Note that $x^{\tilde{k}-1}_{w,z}$ is adjacent to the vertices in $X^k_{u,v} \cap X^{\tilde{k}}_{w,z}$. Since the vertices in $X^k_{u,v}$ are not distinguishable to the right, we conclude $\ell(x^{\tilde{k}-1}_{w,z}) < r(x^k_{u,v})$.

First, we suppose $k = 1$. Thus $v = x^{k-1}_{u,v}$. If $r(x^{\tilde{k}-1}_{w,z}) < r(v)$, then $(x^{\tilde{k}-1}_{w,z}, v)$ is a bad pair and this contradicts Claim 2 and if $r(x^{\tilde{k}-1}_{w,z}) > r(v)$, then $(u, x^{\tilde{k}-1}_{w,z})$ is a bad pair and this contradicts Claim 3. Now we suppose $k \geq 2$. If $r({x^{k-1}_{u,v}}') < r(x^{\tilde{k}-1}_{w,z})$, then $({x^{k-1}_{u,v}}', x^{\tilde{k}-1}_{w,z})$ is a bad pair, which contradicts Claim 5 (a). Thus $r(x^{\tilde{k}-1}_{w,z}) < r({x^{k-1}_{u,v}}')$. If $r(x^{k-1}_{u,v}) < r(x^{\tilde{k}-1}_{w,z})$, then $x^{\tilde{k}-1}_{w,z} \in X^{k-1}_{u,v}$, which implies $|X^{k-1}_{u,v}| = 3$ and hence contradicts Claim 4 (a). Thus $r(x^{\tilde{k}-1}_{w,z}) < r(x^{k-1}_{u,v})$. Therefore, $(x^{\tilde{k}-1}_{w,z}, x^{k-1}_{u,v})$ is a bad pair. Claim 1 implies the existence of a vertex a which is to the left of $x^{\tilde{k}-1}_{w,z}$ and adjacent to $x^{k-1}_{u,v}$. Thus $a \in X^k_{u,v}$. However, $r(a) < r(x^k_{u,v})$, which contradicts Claim 4 (c). This is the final contradiction and this completes the proof of Claim 6 (a).

(b): If $|X^k_{u,v}| = |X^{\tilde{k}}_{w,z}| = 1$, then there is nothing to show. Thus we assume, $|X^k_{u,v}| = 2$. Note that by Claim 6 (a), $x^{k-1}_{u,v} = x^{\tilde{k}-1}_{w,z}$. If ${x^k_{u,v}}' \in X^{\tilde{k}}_{w,z}$, then $x^k_{u,v} \in X^{\tilde{k}}_{w,z}$ and we are done. Thus we assume ${x^k_{u,v}}' \notin X^{\tilde{k}}_{w,z}$. Since $X^k_{u,v} \cap X^{\tilde{k}}_{w,z} \neq \emptyset$, we conclude $x^k_{u,v} \in X^{\tilde{k}}_{w,z}$. Hence w or ${x^{\tilde{k}-1}_{w,z}}'$ distinguishes the vertices in $X^k_{u,v}$ to the right of them, which is a contradiction to Claim 4 (b). This completes the proof.

(c): If $|X^k_{u,v}| = |Y^{\tilde{k}}_{w,z}| = 1$, then there is nothing to show. Thus we assume by symmetry $|Y^{\tilde{k}}_{w,z}| = 2$. First, we assume for contradiction ${y^{\tilde{k}}_{w,z}}' = x^k_{u,v}$. Note that $\ell(y^{\tilde{k}-1}_{w,z}) < \ell({y^{\tilde{k}}_{w,z}}')$ and $r(y^{\tilde{k}-1}_{w,z}) < r({y^{\tilde{k}}_{w,z}}')$.

Suppose $|X^k_{u,v}| = 1$. If $\ell(x^{k-1}_{u,v}) < r(y^{\tilde{k}-1}_{w,z})$, then $y^{\tilde{k}-1}_{w,z} \in X^k_{u,v}$, which is a contradiction to $|X^k_{u,v}| = 1$. Thus $r(y^{\tilde{k}-1}_{w,z}) < \ell(x^{k-1}_{u,v})$. Note that $\ell({y^{\tilde{k}}_{w,z}}') < \ell(y^{\tilde{k}}_{w,z}) < r(y^{\tilde{k}-1}_{w,z})$ and $r({y^{\tilde{k}}_{w,z}}') < r(y^{\tilde{k}}_{w,z})$. Suppose $k = 1$. If $r(y^{\tilde{k}}_{w,z}) < \ell(u)$, then $y^{\tilde{k}}_{w,z} \in X^k_{u,v}$, which is a contradiction to $|X^k_{u,v}| = 1$. If $\ell(u) < r(y^{\tilde{k}}_{w,z}) < r(u)$, then $G[\{x^1_{w,z}, w, z, u, v, y^1_{u,v}\} \cup \bigcup_{i=1}^{\tilde{k}} Y^i_{w,z}]$ is isomorphic to $T_{\tilde{k},0}$, which is a contradiction. If $r(u) < r(y^{\tilde{k}}_{w,z})$, then $(u, y^{\tilde{k}}_{w,z})$ is a bad pair, which is a contradiction to Claim 3. Now we suppose $k \geq 2$. If $r(y^{\tilde{k}}_{w,z}) < \ell({x^{k-1}_{u,v}}')$, then $y^{\tilde{k}}_{w,z} \in X^k_{u,v}$, which is a contradiction to $|X^k_{u,v}| = 1$. If $\ell({x^{k-1}_{u,v}}') < r(y^{\tilde{k}}_{w,z}) < \ell(x^{k-2}_{u,v})$, then $G[\{x^1_{w,z}, w, z, u, v, y^1_{u,v}\} \cup \bigcup_{i=1}^{\tilde{k}} Y^i_{w,z} \cup \bigcup_{i=1}^{k-1} X^i_{u,v}]$ is isomorphic to $T_{\tilde{k},k-1}$, which is a contradiction. If $\ell(x^{k-2}_{u,v}) < r(y^{\tilde{k}}_{w,z}) < \ell({x^{k-1}_{u,v}}')$, then $y^{\tilde{k}}_{w,z} \in X^{k-1}_{u,v}$ and hence $|X^{k-1}_{u,v}| = 3$, which is a contradiction to Claim 4 (a). If $\ell({x^{k-1}_{u,v}}') < r(y^{\tilde{k}}_{w,z})$, then $({x^{k-1}_{u,v}}', y^{\tilde{k}}_{w,z})$ is a bad pair, which is a contradiction to Claim 5 (a).

This shows $|X^k_{u,v}| \neq 1$ and thus we suppose $|X^k_{u,v}| = 2$. If $\ell(x^{k-1}_{u,v}) < r(y^{\tilde{k}-1}_{w,z})$, then $y^{\tilde{k}-1}_{w,z} \in X^k_{u,v}$, which is a contradiction to $|X^k_{u,v}| = 2$. Thus $r(y^{\tilde{k}-1}_{w,z}) < \ell(x^{k-1}_{u,v})$. Note that $\ell({y^{\tilde{k}}_{w,z}}') < \ell(y^{\tilde{k}}_{w,z}) < r(y^{\tilde{k}-1}_{w,z})$ and $r({y^{\tilde{k}}_{w,z}}') < r(y^{\tilde{k}}_{w,z})$. If $\ell({x^k_{u,v}}') < r(y^{\tilde{k}-1}_{w,z})$, then ${x^k_{u,v}}' = y^{\tilde{k}}_{w,z}$. Thus $\{{x^k_{u,v}}', x^k_{u,v}\} = Y^{\tilde{k}}_{w,z}$. By Claim 4 (b), these vertices are not distinguishable to the right and to the left. Thus they are twins, which is a contradiction. Thus $r(y^{\tilde{k}-1}_{w,z}) < \ell({x^k_{u,v}}')$. Note that $\ell({y^{\tilde{k}}_{w,z}}') < \ell(y^{\tilde{k}}_{w,z}) < r(y^{\tilde{k}-1}_{w,z})$. If $r(y^{\tilde{k}}_{w,z}) < r({x^k_{u,v}}')$, then $y^{\tilde{k}}_{w,z} \in X^k_{u,v}$, which is a contradiction to $|X^k_{u,v}| = 2$ and if $r({x^k_{u,v}}') < r(y^{\tilde{k}}_{w,z})$, then $({x^k_{u,v}}', y^{\tilde{k}}_{w,z})$ is a bad pair, which is a contradiction to Claim 5 (a). This shows ${y^{\tilde{k}}_{w,z}}' \neq x^k_{u,v}$. A totally symmetric argumentation shows $y^{\tilde{k}}_{w,z} \neq {x^k_{u,v}}'$.

To complete the proof, we show that ${y^{\tilde{k}}_{w,z}}' \neq {x^k_{u,v}}'$. For contradiction, we assume ${y^{\tilde{k}}_{w,z}}' = {x^k_{u,v}}'$. If $\ell(x^{k-1}_{u,v}) < r(y^{\tilde{k}-1}_{w,z})$, then $x^{k-1}_{u,v} = y^{\tilde{k}}_{w,z}$. Thus $G[\{x^1_{w,z}, w, z, u, v, y^1_{u,v}\} \cup \bigcup_{i=1}^{\tilde{k}} Y^i_{w,z} \cup \bigcup_{i=1}^{k-1} X^i_{u,v}]$ is isomorphic to $R_{k+\tilde{k}-1}$, which is a contradiction. Hence we assume $r(y^{\tilde{k}-1}_{w,z}) < \ell(x^{k-1}_{u,v})$. If $\ell(x^k_{u,v}) < \ell(y^{\tilde{k}-1}_{w,z})$, then $(y^{\tilde{k}-1}_{w,z}, x^k_{u,v})$ is a bad pair, which is a contradiction to the "y"-version of Claim 5 (b). Hence we assume $\ell(y^{\tilde{k}-1}_{w,z}) < \ell(x^k_{u,v})$. If $x^k_{u,v} \in Y^{\tilde{k}}_{w,z}$, then this is a contradiction to the "y"-version of Claim 4 (a), because $\ell(x^k_{u,v}) < \ell({y^{\tilde{k}}_{w,z}}')$. Suppose $\tilde{k} = 1$. Since $x^k_{u,v} \notin Y^{\tilde{k}}_{w,z}$, we conclude $x^k_{u,v}w \in E(G)$. If $\ell(w) < \ell(x^k_{u,v})$, then $G[\{x^1_{w,z}, w, z, u, v, y^1_{u,v}\} \cup \bigcup_{i=1}^{k} X^i_{u,v}]$ is isomorphic to $T_{k,0}$, which is a contradiction. If $\ell(x^k_{u,v}) < \ell(w)$, then $(w, x^k_{u,v})$ is a bad pair, which is a contradiction to Claim 3. Hence we suppose $\tilde{k} \geq 2$. Note that $\ell(x^k_{u,v}) < r({y^{\tilde{k}-1}_{w,z}}')$. If $r(y^{\tilde{k}-2}_{w,z}) < \ell(x^k_{u,v})$, then $G[\{x^1_{w,z}, w, z, u, v, y^1_{u,v}\} \cup \bigcup_{i=1}^{\tilde{k}-1} Y^i_{w,z} \cup \bigcup_{i=1}^{k} X^i_{u,v}]$ is isomorphic to $T_{\tilde{k}-1,k}$. If $\ell({y^{\tilde{k}-1}_{w,z}}') < \ell(x^k_{u,v}) < r(y^{\tilde{k}-2}_{w,z})$, then $x^k_{u,v} \in Y^{\tilde{k}-1}_{w,z}$, which is a contradiction to the "y"-version of Claim 4 (a). If $\ell(x^k_{u,v}) < \ell({y^{\tilde{k}-1}_{w,z}}')$, then $({y^{\tilde{k}-1}_{w,z}}', x^k_{u,v})$ is a bad pair, which is a contradiction to the "y"-version of Claim 5 (a). This completes the proof of Claim 6.
□

Next, we define step by step new interval representations of G as follows. First we shorten the intervals of $X_{u,v}^k$ for every bad pair (u,v) and $k \in [\ell_{u,v}^{\max}]$. Let $I' : V(G) \to \mathcal{I}^{++}$ be such that $I'(x) = [\ell(x), \ell(x_{u,v}^{k-1})]$ if $x \in X_{u,v}^k$ for some bad pair (u,v) and $I'(x) = I(x)$ otherwise. By Claim 6 (a), I' is well-defined; that is, if $x \in X_{u,v}^k \cap X_{w,z}^{\tilde{k}}$, then $\ell(x_{u,v}^{k-1}) = \ell(x_{w,z}^{\tilde{k}-1})$. Let $\ell'(x)$ and $r'(x)$ be the left and right endpoint of the interval $I'(x)$ for $x \in V(G)$, respectively.

Claim 7. *I' is an interval representation of G.*

Proof of Claim 7: Trivially, if two intervals do not intersect in I, then they do not intersect in I'. For contradiction, we assume that there are two vertices $a, b \in V(G)$ such that $I(a) \cap I(b) \neq \emptyset$ and $I'(a) \cap I'(b) = \emptyset$. At least one interval is shortened by changing the interval representation. Say $a \in X_{u,v}^k$ for some bad pair (u,v) and $k \in [\ell_{u,v}^{\max}]$. Hence $b \neq x_{u,v}^{k-1}$ and $\ell(x_{u,v}^{k-1}) < \ell(b)$ and by Claim 4 (b), $\ell(b) < r(x_{u,v}^k)$. We conclude that $(b, x_{u,v}^{k-1})$ is not a bad pair, otherwise Claim 1 implies the existence of a vertex $z \in X_{u,v}^k$ to the left of b, but $z \notin \{x_{u,v}^k, {x_{u,v}^k}'\}$, which is a contradiction to Claim 4 (a). Thus $r(x_{u,v}^{k-1}) < r(b)$. If $k = 1$, then (u,b) is also a bad pair, which is a contradiction to Claim 3. Thus $k \geq 2$. Since $\ell(b) < r(x_{u,v}^k)$, we obtain $\ell(b) < \ell({x_{u,v}^{k-1}}')$. Since $({x_{u,v}^{k-1}}', b)$ is not a bad pair by Claim 5 (a), $r(b) < r({x_{u,v}^{k-1}}')$. Thus $b \in X_{u,v}^{k-1}$, which is a contradiction to $|X_{u,v}^{k-1}| = 2$. □

Claim 8. *The change of the interval representation of G from I to I' creates no new bad pair (a,b) such that $\{a,b\} \neq X_{u,v}^k$ for some $k \in [\ell_{u,v}^{\max}]$ and some bad pair (u,v).*

Proof of Claim 8: For contradiction, we assume that (a,b) is a new bad pair and $\{a,b\} \neq X_{u,v}^k$. Since (a,b) is a new bad pair, $I'(a)$ is a proper subset of $I(a)$. Thus let $a \in X_{u,v}^k$ and $b \notin X_{u,v}^k$. If $a \in X_{u,v}^k$ and $|X_{u,v}^k| = 2$, then $\ell(b) < \ell({x_{u,v}^k}')$ and $r'(a) = \ell(x_{u,v}^{k-1}) < r(b) < r({x_{u,v}^k}')$, because of Claim 5 (a). Thus $b \in X_{u,v}^k$, which is a contradiction. If $a \in X_{u,v}^k$ and $|X_{u,v}^k| = 1$, then $\ell(b) < \ell(x_{u,v}^k)$ and $r'(a) = \ell(x_{u,v}^{k-1}) < r(b) < r(x_{u,v}^k)$. Thus $b \in X_{u,v}^k$, which is the final contradiction. □

In a second step, we shorten the intervals of $Y_{u,v}^i$ for every bad pair (u,v) and $i \in [r_{u,v}^{\max}]$. Let $I'' : V(G) \to \mathcal{I}^{++}$ be such that $I''(y) = [r'(y_{u,v}^{k-1}), r'(y)]$ if $y \in Y_{u,v}^k$ for some bad pair (u,v) and $I''(y) = I'(y)$ otherwise. Note that bad pairs are only referred to the interval representation I. Let $\ell''(x)$ and $r''(x)$ be the left and right endpoints of the interval $I''(x)$ for $x \in V(G)$, respectively.

Claim 9. *I'' is an interval representation of G.*

Proof of Claim 9: Again, two intervals do not intersect in I'' if they do not intersect in I' (and in I). For contradiction, we assume that there are two vertices $a, b \in V(G)$ such that $I(a) \cap I(b) \neq \emptyset$ and $I''(a) \cap I''(b) = \emptyset$. Again, at least one interval is shortened by the change of the interval representation. Say $a \in Y_{u,v}^k$ for some bad pair (u,v) and $k \in [r_{u,v}^{\max}]$.

Suppose $a \in X_{w,z}^{\tilde{k}}$ for some bad pair (w,z) and $\tilde{k} \in [\ell_{w,z}^{\max} - 1]$. By Claim 6 (c), we have $a = x_{w,z}^{\tilde{k}} = y_{u,v}^k$. If $y_{u,v}^{k-1} = x_{w,z}^{\tilde{k}+1}$, then we did not change the interval of a. Thus we assume $y_{u,v}^{k-1} \neq x_{w,z}^{\tilde{k}+1}$. Now $\ell(y_{u,v}^k) < r(b) < r(y_{u,v}^{k-1})$. The rest of the proof is similar to a symmetric version of the proof of Claim 7.

If $a \notin X_{\tilde{u},\tilde{v}}^{\tilde{k}}$, then $r(b) < r(y_{u,v}^{k-1})$ and $\ell({y_{u,v}^k}') < r(b)$, if ${y_{u,v}^k}'$ exists, otherwise $\ell(y_{u,v}^k) < r(b)$. If $\ell(y_{u,v}^{k-1}) < \ell(b)$, then by Claim 8, $(b, y_{u,v}^{k-1})$ is a bad pair and by Claim 5, $I(b) = I'(b)$. Thus Claim 1 implies the existence of a vertex, which contradicts the "y"-version of Claim 4 (a) and (b) and hence we suppose $\ell(b) \leq \ell(y_{u,v}^{k-1})$. Thus $k \geq 2$, otherwise (u', b) is a bad pair, which contradicts Claim 3. If $\ell(b) \leq \ell({y_{u,v}^{k-1}}')$, then $({y_{u,v}^{k-1}}', b)$ is a bad pair, which contradicts the "y"-version of Claim 5 (a). Therefore, $\ell({y_{u,v}^{k-1}}') < \ell(b)$, which implies $b \in Y_{u,v}^{k-1}$, but $b \notin \{y_{u,v}^{k-1}, {y_{u,v}^{k-1}}'\}$, which contradicts the "y"-version of Claim 4 (a). □

Claim 10. *The change of the interval representation of G from I to I'' creates no new bad pair (a, b) such that $\{a, b\} \neq X_{u,v}^k$ for some $k \in [\ell_{u,v}^{\max}]$ or $\{a, b\} \neq Y_{u,v}^i$ for some $i \in [r_{u,v}^{\max}]$ and some bad pair (u, v).*

Proof of Claim 10: For contradiction, we assume that (a, b) is a new bad pair and $Y_{u,v}^i \neq \{a, b\} \neq X_{u,v}^k$. Thus $a \in X_{u,v}^k$ or $a \in Y_{u,v}^i$ and $b \notin X_{u,v}^k$ or $b \notin Y_{u,v}^i$, respectively. If $a \in X_{u,v}^k$ and $|X_{u,v}^k| = 2$, then $\ell(b) < \ell({x_{u,v}^k}')$ and $\ell(x_{u,v}^{k-1}) < r(b) < r({x_{u,v}^k}')$. Thus $b \in X_{u,v}^k$, which is a contradiction. If $a \in X_{u,v}^k$ and $|X_{u,v}^k| = 1$, then $\ell(b) < \ell(x_{u,v}^k)$ and $\ell(x_{u,v}^{k-1}) < r(b) < r(x_{u,v}^k)$. Thus $b \in X_{u,v}^k$, which is a contradiction. If $a \in Y_{u,v}^i$ the proof is almost exactly the same. □

Now we are in a position to blow up some intervals to open or half-open intervals to get a mixed proper interval graph. Let $I^* : V(G) \to \mathcal{I}$ be such that

$$I^*(x) = \begin{cases} (\ell(v), r(v)), & \text{if } (x, v) \text{ is a bad pair,} \\ (\ell''(x_{u,v}^k), r''(x_{u,v}^k)], & \text{if } x = {x_{u,v}^k}' \text{ for some bad pair } (u, v) \text{ and } k \in [\ell_{u,v}^{\max} - 1], \\ \left[\ell''(y_{u,v}^i), r''(y_{u,v}^i)\right), & \text{if } x = {y_{u,v}^i}' \text{ for some bad pair } (u, v) \text{ and } i \in [r_{u,v}^{\max} - 1], \text{ and} \\ \left[\ell''(x), r''(x)\right], & \text{else.} \end{cases}$$

Note that I^* is well-defined by Claim 5 and Claim 6; that is, the four cases in the definition of I^* induces a partition of the vertex set of G. Moreover, the interval representation I^* defines a mixed proper interval graph. As a final step, we prove that I'' and I^* define the same graph. Since we make every interval bigger, we show that for every two vertices a, b such that $I''(a) \cap I''(b) = \emptyset$, we still have $I^*(a) \cap I^*(b) = \emptyset$. For contradiction, we assume the opposite. Let a, b be two vertices such that $I''(a) \cap I''(b) = \emptyset$ and $I^*(a) \cap I^*(b) \neq \emptyset$. It follows by our approach and definition of our interval representation I'', that both a and b are blown up intervals.

First we suppose a and b are intervals that are blown up to open intervals, that is, there are distinct vertices $\tilde{a}$ and $\tilde{b}$ such that $(a, \tilde{a})$ and $(b, \tilde{b})$ are bad pairs. Furthermore, the intervals of $\tilde{a}$ and $\tilde{b}$ intersect not only in one point. By Claim 2 and 3, we assume without loss of generality, that $\ell''(\tilde{a}) < \ell''(\tilde{b}) < r''(\tilde{a}) < r''(\tilde{b})$. Therefore, by the construction of I'', we obtain a is adjacent to $\tilde{b}$ and $\tilde{a}$ is adjacent to b, and in addition they intersect

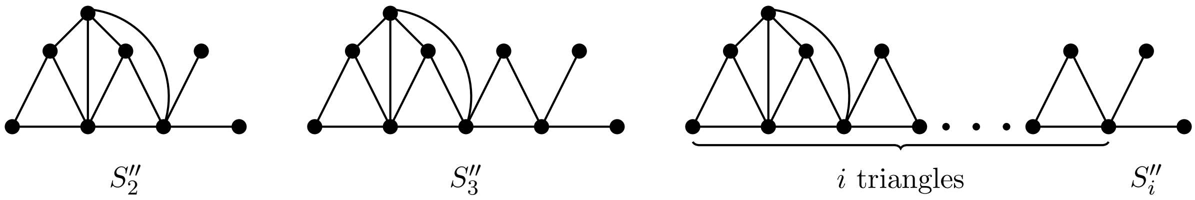
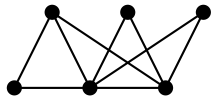


Figure 8: The class $\mathcal{S}_i''$.



G_1

Figure 9: The graph G_1.

in one point, respectively. Now, $G[\{x^1_{a,\tilde{a}}, a, \tilde{a}, b, \tilde{b}, y^1_{b,\tilde{b}}\}]$ is isomorphic to $T_{0,0}$, which is a contradiction.

Now we suppose a is blown up to an open interval and b is blown up to an open-closed interval (the case closed-open is exactly symmetric). Let $\tilde{a}$ be the vertex such that $(a, \tilde{a})$ is a bad pair. Let $\tilde{b}, u, v \in V(G)$ and $k \in \mathbb{N}$ such that $\{b, \tilde{b}\} = X^k_{u,v}$. We suppose $\tilde{a} \neq \tilde{b}$. We conclude $\ell''(\tilde{a}) < \ell''(\tilde{b}) < r''(\tilde{a}) < r''(\tilde{b})$. As above, we conclude a is adjacent to $\tilde{b}$ and $\tilde{a}$ is adjacent to b, and in addition they intersect in one point, respectively. Thus $G[\{x^1_{a,\tilde{a}}, a, \tilde{a}, v, u, y^1_{u,v}\} \cup \bigcup_{i=1}^{k} X^i_{u,v}]$ induces a $T_{k,0}$, which is a contradiction. Now we suppose $\tilde{a} = \tilde{b}$. We conclude that $G[\{x^1_{a,\tilde{a}}, a, v, u, y^1_{u,v}\} \cup \bigcup_{i=1}^{k} X^i_{u,v}]$ is isomorphic to R_k, which is a contradiction.

It is easy to see that a and b cannot be both blown up to closed-open or both open-closed intervals, because G is R_k-free for $k \geq 0$ and the definition of I''.

Therefore, we consider finally the case that a is blown up to a closed-open and b to an open-closed interval. Let $\tilde{a}, \tilde{b}, u, v, w, z \in V(G)$ and $k, \tilde{k} \in \mathbb{N}$ such that $\{a, \tilde{a}\} = Y^k_{u,v}$ and $\{b, \tilde{b}\} = X^{\tilde{k}}_{w,z}$. First we suppose $\tilde{a} \neq \tilde{b}$. Again, we obtain $\ell''(\tilde{a}) < \ell''(\tilde{b}) < r''(\tilde{a}) < r''(\tilde{b})$ and a is adjacent to $\tilde{b}$ and $\tilde{a}$ is adjacent to b, and furthermore they intersect in one point, respectively. Thus $G[\{x^1_{u,v}, u, v, w, z, y^1_{w,z}\} \cup \bigcup_{i=1}^{k} Y^i_{u,v} \cup \bigcup_{i=1}^{\tilde{k}} X^i_{w,z}]$ is isomorphic to $T_{k,\tilde{k}}$. Next we suppose $\tilde{a} = \tilde{b}$ and hence $G[\{x^1_{u,v}, u, v, w, z, y^1_{w,z}\} \cup \bigcup_{i=1}^{k} Y^i_{u,v} \cup \bigcup_{i=1}^{\tilde{k}} X^i_{w,z}]$ is isomorphic to $R_{k+\tilde{k}}$. This is the final contradiction and completes the proof of Theorem 7. $\square$

In Theorem 7 we only consider twin-free $\mathcal{U}$-graphs to reduce the number of case distinctions in the proof. In Corollary 8 we resolve this technical condition. See Figure 8 and 9 for illustration. Let $\mathcal{S}'' = \bigcup_{i=2}^{\infty}\{S_i''\}$.

Corollary 8. *A graph G is a mixed unit interval graph if and only if G is a $\{G_1\} \cup \mathcal{R} \cup \mathcal{S} \cup \mathcal{S}'' \cup \mathcal{T}$-free interval graph.*

Proof of Corollary 8: We first show that $\{G_1\} \cup \mathcal{R} \cup \mathcal{S} \cup \mathcal{S}'' \cup \mathcal{T}$ is the set of all twin-free graphs that contain all graphs of $\{K_{2,3}^*\} \cup \mathcal{R} \cup \mathcal{S} \cup \mathcal{S}' \cup \mathcal{T}$ and are minimal with subject to induced subgraphs. We leave it as an exercise to show that G_1 is the only minimal twin-free and R_0-free graph that contains $K_{2,3}^*$. Since all graphs in $\mathcal{R} \cup \mathcal{S} \cup \mathcal{T}$ are twin-free graphs, there is nothing to show.

Let now $G \in \mathcal{S}'$, that is $G = S_k'$ for some $k \in \mathbb{N}$. With the notation as in the proof of Theorem 7, G can be interpreted as a bad pair (u, v) together with $\{y_{u,v}^1\} \cup \bigcup_{i=1}^k X_{u,v}^i$ such that $|X_{u,v}^i| = 2$ if $i < k$ and $|X_{u,v}^k| = 3$. Note that Claim 4 (b) of Theorem 7 is still true even if G is not $\mathcal{S}'$-free. Therefore, we know that the vertices in $X_{u,v}^i$ cannot be distinguished by vertices from the right. Thus the vertices that distinguish the vertices in $X_{u,v}^k$ are only adjacent to $X_{u,v}^k$. Clearly, there are at least two of them, say a, b. Without loss of generality a and b they do not have the same neighborhood on $X_{u,v}^k$. We conclude either $N_{G[X_{u,v}^k]}(a) \subset N_{G[X_{u,v}^k]}(b)$ or $N_{G[X_{u,v}^k]}(b) \subset N_{G[X_{u,v}^k]}(a)$. We assume the first possibility. Since $0 < |N_{G[X_{u,v}^k]}(x) \cap X_{u,v}^k| < 3$ for $x \in \{a, b\}$, it follows $|N_{G[X_{u,v}^k]}(a) \cap X_{u,v}^k| = 1$ and $|N_{G[X_{u,v}^k]}(b) \cap X_{u,v}^k| = 2$. Since G is R_k-free, a and b are adjacent. Now $G[\bigcup_{i=1}^k X_{u,v}^i \cup \{a, b, u, v, y_{u,v}^1\}]$ is isomorphic to S_{k+1}''. This completes this part of the proof.

Let G be an interval graph. The relation $\sim$, where $u \sim v$ if and only if u and v are twins, defines an equivalence relation on $V(G)$. Let $U \subseteq V(G)$ such that there is exactly one vertex of every equivalence class in U. Therefore, $G[U]$ is a twin-free graph. Furthermore, G contains an induced subgraph in $\{G_1\} \cup \mathcal{R} \cup \mathcal{S} \cup \mathcal{S}'' \cup \mathcal{T}$ if and only if $G[U]$ contains an induced subgraph in $\{K_{2,3}^*\} \cup \mathcal{R} \cup \mathcal{S} \cup \mathcal{S}' \cup \mathcal{T}$. In addition, $G[U]$ is a twin-free $\mathcal{U}$-graph if and only if G is a $\mathcal{U}$-graph. By Theorem 7 this completes the proof. $\square$

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