Linear-Time Algorithms for Scattering Number and Hamilton-Connectivity of Interval Graphs *

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Abstract. We prove that for all $k \leq -1$ an interval graph is -(k + 1)-Hamilton-connected if and only if its scattering number is at most k. This complements a previously known fact that an interval graph has a nonnegative scattering number if and only if it contains a Hamilton cycle, as well as a characterization of interval graphs with positive scattering numbers in terms of the minimum size of a path cover. We also give an O(n + m) time algorithm for computing the scattering number of an interval graph with n vertices and m edges, which improves the previously best-known $O(n^3)$ time bound for solving this problem. As a consequence of our two results, the maximum k for which an interval graph is k-Hamilton-connected can be computed in O(n + m) time.

1 Introduction

The HAMILTON CYCLE problem is that of testing whether a given graph has a Hamilton cycle, i.e., a cycle passing through all the vertices. This problem is a notorious NP-complete problem, which remains NP-complete on many graph classes such as the classes of planar cubic 3-connected graphs [23], chordal bipartite graphs [37], and strongly chordal split graphs [37]. Bertossi and Bonucelli [6] proved that HAMILTON CYCLE is NP-complete for undirected path graphs, double interval graphs and rectangle graphs, all three of which are classes of intersection graphs that contain the class of interval graphs. A graph G is an *interval graph* if it is the intersection graph of a set of closed intervals on the real line, i.e., the vertices of G correspond to the intervals and two vertices are adjacent in G if and only if their intervals have at least one point in common. For interval graphs, Keil [29] showed in 1985 that HAMILTON CYCLE can be solved in O(n+m) time, thereby strengthening an earlier result of Bertossi [5] for proper interval graphs, which are interval graphs that have a closed interval representation, in which no interval is properly contained in another one. The approach of Keil was later extended by Damaschke [18] to an $O(n^5)$ -time algorithm for circular-arc graphs. By using an algorithm that computes the so-called bump number of a graph, Deogun and Steiner [21] proved that the HAMILTON CYCLE is solvable in polynomial time on cocomparability graphs.

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Recently, Corneil, Dalton and Habib [17] proved and further extended this result by using an LexBFS search algorithm.

In this paper we examine whether the linear-time result of Keil [29] can be extended on interval graphs to hold for other connectivity properties, which are NP-complete or coNP-complete to verify in general. This line of research is well embedded in the literature. Before surveying existing work for interval graphs and presenting our new results, we first give necessary terminology, and in addition, we formally introduce the decision problems considered (we also state their computational complexity for general graphs).

1.1 Connectivity Properties and Corresponding Decision Problems

We only consider undirected finite graphs with no self-loops and no multiple edges. We refer to the textbook of Bondy and Murty [8] for any undefined graph terminology. Throughout the paper we let n and m denote the number of vertices and edges, respectively, of the input graph.

Let G = (V, E) be a graph. If G has a Hamilton cycle, i.e., a cycle containing all the vertices of G, then G is hamiltonian. Recall that the corresponding NP-complete decision problem is called HAMILTON CYCLE. If G contains a Hamilton path, i.e., a path containing all the vertices of G, then G is traceable. In this case, the corresponding decision problem is called the HAMILTON PATH problem, which is also well known to be NP-complete (cf. [22]).

The problems 1-HAMILTON PATH and 2-HAMILTON PATH are those of testing whether a given graph has a Hamilton path that starts in some given vertex u or that is between two given vertices u and v, respectively. The LONGEST PATH problem is to compute the maximum length of a path in a given graph. All three problems are NP-hard by a straightforward reduction from HAMILTON PATH, the former two are indeed NP-complete.

Let G = (V, E) be a graph. If for each two distinct vertices $s, t \in V$ there exists a Hamilton path with end-vertices s and t, then G is *Hamilton-connected*. If G-S is Hamilton-connected for every set $S \subset V$ with $|S| \leq k$ for some integer $k \geq 0$, then G is k-Hamilton-connected. Note that a graph is Hamilton-connected if and only if it is 0-Hamilton-connected. The HAMILTON CONNECTIVITY problem is that of computing the maximum value of k for which a given graph is k-Hamilton-connected. Dean [19] showed that already deciding whether k = 0 is NP-complete. Kužel, Ryjáček and Vrána [31] proved this for k = 1. A straightforward generalization of the latter result yields the same for any integer $k \geq 1$. As an aside, the HAMILTON CONNECTIVITY problem has recently been studied by Kužel, Ryjáček and Vrána [31], who showed that NP-completeness of the case k = 1 for line graphs would disprove the conjecture of Thomassen that every 4-connected line graph is hamiltonian, unless P = NP.

A path cover of a graph G is a set of mutually vertex-disjoint paths P_1, \ldots, P_k with $V(P_1) \cup \cdots \cup V(P_k) = V(G)$. The size of a smallest path cover is denoted by $\pi(G)$. The PATH COVER problem is to compute this number, whereas the 1-PATH COVER problem is to compute the size of a smallest path cover that contains a path in which some given vertex u is an end-vertex. Because a Hamilton path of a graph is a path cover of size 1, PATH COVER and 1-PATH COVER are NP-hard via a reduction from HAMILTON PATH and 1-HAMILTON PATH, respectively.

We denote the number of connected components of a graph G = (V, E) by c(G). A subset $S \subset V$ is a vertex cut of G if $c(G-S) \geq 2$, and G is called k-connected if the size of a smallest vertex cut of G is at least k. We say that G is t-tough if $|S| \geq t \cdot c(G-S)$ for every vertex

cut S of G. The toughness $\tau(G)$ of a graph G = (V, E) was defined by Chvátal [16] as

$$au(G) = \min \left\{ \frac{|S|}{c(G-S)} : S \subset V \text{ and } c(G-S) \ge 2 \right\},$$

where $\tau(G) = \infty$ if G is a complete graph. Note that $\tau(G) \ge 1$ if G is hamiltonian; the reverse statement does not hold in general (see [8]). The TOUGHNESS problem is to compute $\tau(G)$ for a graph G. Bauer, Hakimi and Schmeichel [4] showed that already deciding whether $\tau(G) = 1$ is coNP-complete.

The scattering number of a graph G = (V, E) was defined by Jung [28] as

 $sc(G) = max\{c(G-S) - |S| : S \subset V \text{ and } c(G-S) \ge 2\},\$

where $\operatorname{sc}(G) = -\infty$ if G is a complete graph. We call a set S on which $\operatorname{sc}(G)$ is attained a scattering set. Note that $\operatorname{sc}(G) \leq 0$ if G is hamiltonian. Shih, Chern and Hsu [38] showed that $\operatorname{sc}(G) \leq \pi(G)$ for all graphs G. Hence, $\operatorname{sc}(G) \leq 1$ if G is traceable. The SCATTERING NUMBER problem is to compute $\operatorname{sc}(G)$ for a graph G. The observation that $\operatorname{sc}(G) = 0$ if and only if $\tau(G) = 1$ combined with the aforementioned result of Bauer, Hakimi and Schmeichel [4] implies that already deciding whether $\operatorname{sc}(G) = 0$ is coNP-complete.

1.2 Known Results for Interval Graphs

We first briefly discuss the results on testing hamiltonicity properties for proper interval graphs. Besides giving a linear-time algorithm for solving HAMILTON CYCLE on proper interval graphs, Bertossi [5] also showed that a proper interval graph is traceable if and only if it is connected. His work was extended by Chen, Chang and Chang [13] who showed that a proper interval graph is hamiltonian if and only if it is 2-connected, and that a proper interval graph is Hamilton-connected if and only if it is 3-connected. In addition, Chen and Chang [12] showed that a proper interval graph has scattering number at most 2 - k if and only if it is k-connected.

Below we survey the results on testing hamiltonicity properties for interval graphs that appeared after Keil [29] solved the HAMILTON CYCLE problem on interval graphs.

Testing for Hamilton cycles and Hamilton paths. The O(n + m) time algorithm of Keil [29] makes use of an interval representation. One can find such a representation by executing the O(n + m) time interval recognition algorithm of Booth and Lueker [9]. If an interval representation is already given, Manacher, Mankus and Smith [36] showed that HAMILTON CYCLE and HAMILTON PATH can be solved in $O(n \log n)$ time. In the same paper, they ask whether the time bound for these two problems can be improved to O(n) time if a so-called sorted interval representation is given. Chang, Peng and Liaw [11] answered this question in the affirmative. They showed that this even holds for PATH COVER.

When no Hamilton path exists. In this case, LONGEST PATH and PATH COVER are natural problems to consider. Ioannidou, Mertzios and Nikolopoulos [26] gave an $O(n^4)$ algorithm for solving LONGEST PATH on interval graphs. Arikati and Pandu Rangan [1] and also Damaschke [18] showed that PATH COVER can be solved in O(n+m) time on interval graphs.

Damaschke [18] posed the complexity status of 1-HAMILTON PATH and 2-HAMILTON PATH on interval graphs as open questions. The latter question is still open, but Asdre and Nikolopoulos [3] answered the former question by presenting an $O(n^3)$ time algorithm that solves 1-PATH COVER, and hence 1-HAMILTON PATH, on interval graphs. Li and Wu [32] announced an O(n + m) time algorithm for 1-PATH COVER on interval graphs. Deogun, Kratsch and Steiner [20] showed that for all $k \ge 1$ any cocomparability graph has a path cover of size at most k if and only if its scattering number is at most k.⁷ Because every interval graph is cocomparability (see e.g. [10]), this result holds for interval graphs as well. Deogun, Kratsch and Steiner also proved that a cocomparability graph G is hamiltonian if and only if $sc(G) \le 0$. Recall that the latter condition is equivalent to $\tau(G) \ge 1$. As such, this result restricted to interval graphs is known (see e.g. [15]) to be implicit already in Keil's algorithm [29]. Hung and Chang [25] gave an O(n+m) time algorithm that finds a scattering set of an interval graph G if $sc(G) \ge 0$.

1.3 Our Results

When a Hamilton path does exist. In this case, HAMILTON CONNECTIVITY is a natural problem to consider. Isaak [27] used a closely related variant of toughness called k-path toughness to characterize interval graphs that contain the kth power of a Hamiltonian path. However, the aforementioned results of Deogun, Kratsch and Steiner [20] suggest that trying to characterize k-Hamilton-connectivity in terms of the scattering number of an interval graph may be more appropriate than doing this in terms of its toughness. We confirm this by showing that for all $k \ge 0$ an interval graph is k-Hamilton-connected if and only if its scattering number is at most -(k + 1). Together with the results of Deogun, Kratsch and Steiner [20], this leads to the following theorem.

Theorem 1. Let G be an interval graph. Then $sc(G) \leq k$ if and only if

- (i) G has a path cover of size at most k when $k \ge 1$
- (ii) G has a Hamilton cycle when k = 0
- (iii) G is -(k+1)-Hamilton-connected when $k \leq -1$.

Moreover, we give an O(n+m) time algorithm for solving SCATTERING NUMBER on interval graphs that also produces a scattering set. This improves the $O(n^3)$ time bound of a previous algorithm due to Kratsch, Kloks and Müller [30]. Combining our result with Theorem 1 yields that HAMILTON CONNECTIVITY can be solved in O(n+m) time on interval graphs.

For proper interval graphs we can express k-Hamilton-connectivity also in the following way. Recall that a proper interval graph has scattering number at most 2 - k if and only if it is k-connected [12]. Combining this result with Theorem 1 yields that for all $k \ge 0$, a proper interval graph is k-Hamilton-connected if and only if it is (k + 3)-connected.

1.4 Our Proof Method

In order to explain our approach we first need to introduce some additional terminology. A set of p internally vertex-disjoint paths P_1, \ldots, P_p , all of which have the same end-vertices u and vof a graph G, is called a *stave* or p-stave of G, which is *spanning* if $V(P_1) \cup \cdots \cup V(P_p) = V(G)$. A spanning p-stave between two vertices u and v is also called a spanning (p; u, v)-path-system [14], a p^* -container between u and v [24,34] or a spanning p-trail [32]. We call a spanning p-stave between two vertices u and v of a graph an *optimal spanning stave* between u and vif there does not exist a spanning (p+1)-stave between u and v.

⁷ This has also been shown by Lehel in an unpublished manuscript [35].

By Menger's Theorem (Theorem 9.1 in [8]), a graph G is p-connected if and only if there exists a p-stave between any pair of vertices of G. It is also well-known that the existence of a p-stave between two given vertices can be decided in polynomial time (cf. [8]). However, given an integer $p \ge 1$ and two vertices u and v of a general input graph G, deciding whether there exists a spanning p-stave between u and v is an NP-complete problem: for p = 1 the problem is equivalent to the NP-complete problem 2-HAMILTON PATH; for p = 2 the problem is equivalent to the NP-complete problem HAMILTON CYCLE; for $p \ge 3$ NP-completeness follows by induction and by considering the graph obtained after adding one vertex adjacent to u and v.

Damaschke's algorithm [18] for solving PATH COVER on interval graphs, which is based on the approach of Keil [29], actually solves the following problem in O(n + m) time: given an interval graph G and an integer p, does G have a spanning p-stave between the vertex u_1 corresponding to the leftmost interval of an interval model of G and the vertex u_n corresponding to the rightmost one? Here, the *leftmost* interval is the interval with the smallest right end-point, and the *rightmost* interval is the interval with the largest left end-point. Note that we may assume without loss of generality that these intervals are unique.

In Section 2 we present our O(n+m) time algorithm that solves SCATTERING NUMBER on interval graphs. Our approach is as follows. We extend Damaschke's algorithm to an O(n+m)time algorithm that takes as input only an interval graph G and finds an optimal spanning stave of G between u_1 and u_n , unless it detects that there does not exist a spanning stave between u_1 and u_n . In the latter case G it is not hamiltonian, and hence, $\operatorname{sc}(G) \geq 1$ as shown by Deogun, Kratsch and Steiner [20]. Therefore, the O(n+m) time algorithm of Hung and Chang [25] for computing a scattering set may be applied. In the case that there is an optimal spanning stave between u_1 and u_n , we show how this enables us to compute a scattering set S of G in O(n+m) time. In fact we show a stronger relationship, namely that $|S| \geq 2 - p^*$ if p^* is the size of an optimal spanning stave between u_1 and u_n . We consequently obtain a structural result, proved also in Section 2, which states that G contains a spanning p-stave between u_1 and u_n if and only if $\operatorname{sc}(G) \leq 2 - p$.

In Section 3 we prove our contribution to Theorem 1 (iii), that is, the case when $k \leq -1$. In particular, for proving the subcase k = -1, we show that an interval graph G is Hamiltonconnected if it contains a spanning 3-stave between the vertex corresponding to the leftmost interval of an interval model and the vertex corresponding to the rightmost one. We then combine this claim with the structural result obtained in Section 2.

2 Spanning Staves and the Scattering Number

In order to present our algorithm we start by giving the necessary terminology and notations.

Additional Terminology. A set $D \subseteq V$ dominates a graph G = (V, E) if each vertex of G belongs to D or has a neighbor in D. We will usually denote a path in a graph by its sequence of distinct vertices such that consecutive vertices are adjacent. If $P = u_1 \ldots u_n$ is a path, then we denote its reverse by $P^{-1} = u_n \ldots u_1$. We may concatenate two paths P and P' whenever they are vertex-disjoint except for the last vertex of P coinciding with the first vertex of P'. The resulting path is then denoted by $P \circ P'$.

A clique path of an interval graph G with vertices u_1, \ldots, u_n is a sequence C_1, \ldots, C_s of all maximal cliques of G, such that each edge of G is present in some clique C_i and each vertex of G appears in consecutive cliques only. It is well known that a graph is interval if and only

if it has a clique path. We use the O(n + m) time recognition algorithm of interval graphs due to Booth and Lueker [9], which produces a clique path C_1, \ldots, C_s when the input graph is interval. Hence, in the remainder of this section, we let G denote an interval graph with a clique path C_1, \ldots, C_s . The latter yields a specific interval model for G that we will also use throughout the remainder of this paper: a vertex u_i of G is represented by the interval $I_{u_i} = [\ell_i, r_i]$, where $\ell_i = \min\{j : u_i \in C_j\}$ and $r_i = \max\{j : u_i \in C_j\}$, which are referred to as the *start point* and the *end point* of u_i , respectively. By definition, C_1 and C_s are maximal cliques. Hence both C_1 and C_s contain at least one vertex that does not occur in any other clique. We assume that u_1 is such a vertex in C_1 , and hence it corresponds to the leftmost interval defined earlier. Analogously, u_n , corresponding to the rightmost one, is such a vertex in C_s . Note that $I_{u_1} = [1, 1]$ and $I_{u_n} = [s, s]$ are single points.

Keil [29] made the useful observation that any Hamilton path in an interval graph can be reordered into a path with a special property that allows to build a greedy-like algorithm. A path in an interval graph is *monotone* if every edge (u, v) can be assigned a point from $I_u \cap I_v$ such that these points ordered in the appearance of the edges in the path form a nondecreasing sequence (it is called "straight" in previous works [29,18]).

Lemma 1 ([29]). If the interval graph G contains a Hamilton path, then it contains a monotone Hamilton path from u_1 to u_n .

We use Lemma 1 to rearrange certain path systems in G into a single path as follows. Let P be a path between u_1 and u_n and let $\mathcal{Q} = (Q_1, \ldots, Q_k)$ be a collection of paths, each of which contains u_1 or u_n as an end-vertex. Furthermore, P and all the paths of \mathcal{Q} are assumed to be vertex-disjoint except for possible intersections at u_1 or u_n . Consider the path Q_1 . By symmetry, it may be assumed to contain u_1 . We apply Lemma 1 to the subgraph induced by the path $P \circ (Q_1 - u_n)$ and obtain a path P' between u_1 and u_n containing all the vertices of $P \cup Q_1$. Proceeding in a similar way for the paths Q_2, \ldots, Q_k , we obtain a path between u_1 and u_n on the same vertex set as $P \cup \bigcup_{j=1}^k Q_j$. We denote the resulting path by $\operatorname{merge}(P, Q_1, \ldots, Q_k)$ or simply by $\operatorname{merge}(P, Q)$.

Algorithm 1 is our O(n+m) time algorithm for finding an optimal spanning stave between u_1 and u_n if it exists. Similarly to the algorithm of Damaschke [18], it gradually builds up a set \mathcal{P} of internally disjoint monotone paths starting at u_1 and passing through vertices of $C_t \setminus C_{t+1}$ before moving to $C_t \cap C_{t+1}$ for $t = 1, \ldots, s-1$. In contrast to Damaschke's algorithm, where the number of paths is fixed, our algorithm starts with the maximum possible number of paths, i.e. the degree of the leftmost vertex u_1 . Intuitively, if some path cannot be further extended, it is abandoned and the set of possible paths is reduced. At the final phase we clear all abandoned paths by merging them with any path that reaches u_n .

It is convenient to consider all these paths ordered from u_1 to their (temporary) endvertices that we call *terminals*, and to use the terms *predecessor*, *successor*, and *descendant* of a fixed vertex v in one of the paths with the usual meaning of a vertex immediately before, immediately after, and somewhere after v in one of these paths, respectively. For a path Pwith end-vertex u and a vertex $v \notin V(P)$, we say that P has been *extended by attaching* v, if uv becomes the last edge of the resulting path, while all edges of P are preserved in the resulting path too. By extending a path by attaching a set of vertices we mean attaching vertices of the set one by one, in an arbitrary order.

We note that the path system \mathcal{P} provided by Algorithm 1 is a valid stave. A routine check confirms that the following loop invariant holds at line 6: the last vertices of paths from \mathcal{P}

Input: A clique-path C_1, \ldots, C_s in an interval graph G. **Output**: An optimal spanning stave \mathcal{P} between u_1 and u_n , if it exists. 1 begin let $p = \deg(u_1)$; $\mathbf{2}$ let $R_i = u_1$ for all $i = 1, \ldots, p$; 3 let $\mathcal{P} = \{R_1, \ldots, R_p\};$ $\mathbf{4}$ let $\mathcal{Q} = \emptyset$; $\mathbf{5}$ for t := 1 to s - 1 do 6 choose a $P \in \mathcal{P}$ whose terminal has the smallest end point among all terminals; $\mathbf{7}$ if $C_t \setminus (C_{t+1} \cup \bigcup (\mathcal{P} \cup \mathcal{Q})) \neq \emptyset$ then extend P by attaching vertices of $C_t \setminus (C_{t+1} \cup \bigcup (\mathcal{P} \cup \mathcal{Q}))$; 8 9 for every path $R \in \mathcal{P}$ do if the terminal of R is not in C_{t+1} then 10 try to extend R by attaching a new vertex u from $(C_t \cap C_{t+1}) \setminus \bigcup (\mathcal{P} \cup \mathcal{Q})$ with the 11 smallest end point; if such u does not exist then $\mathbf{12}$ remove R from \mathcal{P} ; 13 insert R into Q; 14 decrement p; 15 if p = 0 then report that G has no spanning 1-stave between u_1 and u_n and quit 16 end 17 \mathbf{end} 18 19 end end 20 choose any $P \in \mathcal{P}$; 21 extend P by attaching vertices of $C_s \setminus \bigcup (\mathcal{P} \cup \mathcal{Q})$; 22 let P = merge(P, Q);23 for every path $R \in \mathcal{P} \setminus P$ do extend R by attaching u_n ; $\mathbf{24}$ report the optimal spanning *p*-stave \mathcal{P} . $\mathbf{25}$ $_{26}$ end

Algorithm 1: Finding an optimal spanning stave.

all belong to the clique C_t . This is guaranteed by the computations at lines 10–18. At line 20 it also holds that all vertices of $C_t \setminus C_{t+1}$ appear in the current $\mathcal{P} \cup \mathcal{Q}$, as they have been included at line 8. When the loop terminates, the remaining vertices are incorporated at line 22. Thus the resulting path system \mathcal{P} is a spanning stave.

In Theorem 2 we show that no spanning stave may consist of more than $2 - \operatorname{sc}(G)$ paths. On the other hand, we will also show that the *p*-stave found by Algorithm 1 can be supplied with a scattering set witnessing that $p \ge 2 - \operatorname{sc}(G)$. In other words this is an optimal scattering set whose existence also proves the optimality of the spanning stave. For this goal, we first develop some auxiliary terminology related to our algorithm.

If vertex u_i has been processed by the algorithm and attached to a path at lines 8 or 11 of Algorithm 1, we say that u_i has been *activated* at time a_i , and we assign a_i the current value of the variable t. Thus, we think of time steps $t = 1, \ldots, t = s$ during the execution of the algorithm. When at the same or at a later stage a vertex u_j has been attached as a successor of u_i to a path, we say that u_i has been *deactivated* at time d_i , and assign $d_i = a_j$. Hence, as soon as a_i and d_i have been assigned values, we have $\ell_i \leq a_i \leq d_i \leq r_i$. Furthermore, any of the implied inequalities holds whenever both of its sides are defined. Note that any of these inequalities may be an equality; in particular, a vertex can be activated and deactivated at the same time.

If the involved parameters have been assigned values, we consider the open (time) intervals $(\ell_i, a_i), (a_i, d_i)$ and (d_i, r_i) , and we say that u_i is *free* during (ℓ_i, a_i) if this interval is nonempty,

active during (a_i, d_i) if this interval is nonempty, and depleted during (d_i, r_i) if this interval is nonempty. In particular, note that the vertices that are attached to a path at line 8 (if any) are from $C_t \setminus C_{t+1}$, so they satisfy $r_i = t$ and $a_i = t$. Such vertices will not be active or depleted during any (nonempty) time interval, but they are free during the time interval (ℓ_i, r_i) if this interval is nonempty.

For $1 \leq j \leq k \leq s$, we define $C_{j,k} = \bigcup_{i=j}^{k} C_i$. The following lemma is crucial

The following lemma is crucial.

Lemma 2. Suppose that Algorithm 1 terminates at line 16 or finishes an iteration of the loop at lines 6–20. Let the current value of the variable t be also denoted by t. If there is at least one depleted vertex during the interval (t, t + 1), then there exists an integer t' < t with the following properties (see Fig. 1a for an illustration):

- (i) $C_{t'+1,t} \setminus (C_{t'} \cup C_{t+1}) \neq \emptyset$,
- (ii) a unique vertex $u_i \in C_{t'} \cap C_{t+1}$ is active during (t', t'+1) and is depleted during (t, t+1),
- (iii) all vertices that are active during (t, t + 1) are also active during (t', t' + 1), with the only possible exception of the last descendant of u_i (which we denote by v) that can be free during (t', t' + 1),
- (iv) all vertices that are depleted during (t, t + 1) and distinct from u_i are also depleted during (t', t' + 1),
- (v) all vertices that are active during (t', t' + 1) are also active during (t, t + 1), with the only exception of u_i , and
- (vi) all vertices that are free during (t', t' + 1) are also free during (t, t + 1), with the only possible exception of v if it is active during (t, t + 1).



Fig. 1. A path system as described in Lemma 2. The vertical arrows indicate successors in the paths and the time of activation and deactivation.

Proof. Assume that there is at least one depleted vertex during the interval (t, t + 1), and let u_i be a vertex with the latest deactivation time among those that are depleted during (t, t+1). To prove that this vertex is unique, we note that all but at most one of the vertices deactivated during a given iteration of the loop on lines 6–20 (say, at time t) have end point equal to t and hence cannot be depleted during a nonempty interval. The only possible exception is the terminal of the path P chosen at line 7 (and only if it is deactivated due to attaching a vertex to P at line 8).

We define Q to be the subpath of P formed by all descendants of u_i , except that if the last descendant v of u_i is active during (t, t + 1), we do not include v in Q. Observe that the successor of u_i has the same deactivation time as u_i , hence it is distinct from v, and therefore Q is nonempty. Let ℓ_Q be the smallest start point among intervals corresponding to vertices of Q, and let r_Q be the largest such end point.

If P has a vertex that is active during (t, t + 1), this vertex is v and it is not a vertex of Q. Thus all vertices of Q are either depleted during (t, t + 1) or their end point is less than or equal to t. By the choice of u_i , none of them belongs to C_{t+1} , and hence $r_Q \leq t$. We choose $t' = \ell_Q - 1$. Notice that for $u_j \in V(Q)$, $r_j \geq d_i$. Thus if we let u_q be the vertex of u such that $\ell_q = \ell_Q$, then u_q is free during $(t' + 1, d_i)$.

Observe that all vertices of Q are in $C_{t'+1,t} \setminus (C_{t'} \cup C_{t+1})$. Hence, this set is not empty and property (i) is proved.

We prove (ii). Since the deactivation of u_i happened when its successor u_j was free, we have $d_i \ge \ell_j > t'$. Hence, u_i cannot be depleted during (t', t' + 1). Observe that $u_i \ne u_1$, as u_1 is not depleted during (t - 1, t). Therefore, u_i has a predecessor. Denote it by u'. If u' were adjacent to the vertex u_q of Q, then the algorithm would choose u_q as the successor of u', since $r_i > r_Q \ge r_q$. Consequently, the start point of u' is less than or equal to t', so u_i is active during (t', t' + 1). The uniqueness of u_i will follow easily once we establish property (iv).

To show property (iii), assume that u_m is a vertex different from v that is active during (t, t+1) but has been activated after t'. Since u_1 is not active during (t, t+1), $u_m \neq u_1$ and u_m has a predecessor u'. We first suppose that u_m is active during $(d_i - 1, d_i)$. The vertex u' is deactivated at some time t'' such that $t' + 1 \leq t'' \leq d_i - 1$. Hence, it is adjacent to the previously defined vertex u_q of Q that is free during $(t' + 1, d_i)$. Since $r_q \leq r_Q < t + 1 \leq r_m$, the successor of u' should be u_q rather than u_m , a contradiction.

It follows that u_m is not active during $(d_i - 1, d_i)$. The vertex u_m is included in some path $R \in \mathcal{P}, R \neq P$. This path contains a vertex w' that is active during $(d_i - 1, d_i)$ (see Fig. 1b), where u_m is a descendant of w'. Observe that w' is not active during (t, t + 1) because u_m is. Suppose that the end point of w' is at least t + 1. Then w' is depleted during (t, t + 1), so by the choice of u_i, w' is deactivated before time d_i and cannot be active during $(d_i - 1, d_i)$, a contradiction.

Thus, the end point of w' is not larger than t. But then w' should have been chosen at line 7 of the algorithm instead of u_i .

For (iv), assume that some $u_h \neq u_i$ is depleted during (t, t+1), but $d_h \geq t'+1$. By the choice of u_i , we have $d_h < d_i$. Without loss of generality, assume that u_h was chosen such that d_h is maximal. Let R be the path in $\mathcal{P} \cup \mathcal{Q}$ containing u_h . Note that $R \neq P$. If R contains a vertex w that is active during (t, t+1), then by (iii), w is active during (t', t'+1) and we conclude that u_h cannot be included in R; a contradiction.

It follows that no vertex of R is active during (t, t + 1) (see Fig. 1c). Moreover, by the choice of u_h , the end points of all its descendants are less than or equal to t, because if there is a descendant u_j of u_h with $r_j \ge t + 1$, then w is depleted during (t, t + 1) and $d_j > d_h$,

a contradiction. Recall that the vertex u_q is free during $(t'+1, d_i)$. Since the path R cannot be terminated while a free vertex is available, it must contain a vertex that is active during $(d_i - 1, d_i)$. However, this vertex has a smaller end point than u_i , contradicting the correct execution of the algorithm at line 7.

To obtain (v), assume that $w \neq u_i$ is active during (t', t'+1) but not active during (t, t+1). The vertex w is included in some path $R \in \mathcal{P} \cup \mathcal{Q}$, $R \neq P$. If one of the descendants of w is active during (t, t+1), then by (iii), this vertex is active during (t', t'+1) contradicting the activeness of w at the same time. Similarly, if w or one of its descendants is depleted during (t, t+1), then by (iv), this vertex is depleted during (t', t'+1) and w cannot be active. It follows that the end points of w and its descendants are less than or equal to t. If $d_i = t'+1$, then R has a vertex that is active during $(d_i - 1, d_i)$. If $d_i > t'+1$, then we use the observation that the vertex u_q is free during $(t' + 1, d_i)$, and again conclude that R has an active vertex during $(d_i - 1, d_i)$. Then this vertex should be selected by the algorithm in line 7 instead of u_i ; a contradiction.

It remains to prove (vi). Let w be a vertex that is free during (t', t'+1) and not free during (t, t+1). Moreover, we assume that $w \neq v$ if v is active during (t, t+1). Our algorithm does not terminate until time t. Therefore, w is included in some path $R \in \mathcal{P} \cup \mathcal{Q}, R \neq P$. This path has a vertex that is active during (t', t'+1). By (v), this vertex remains active until t+1, but it means that w is not included in R.

Now we are ready to state and prove the main structural result.

Theorem 2. A non-complete interval graph G contains a spanning p-stave between u_1 and u_n if and only if $sc(G) \leq 2 - p$.

Proof. Let us first assume that $\mathcal{P} = (R_1 \dots, R_p)$ is a spanning *p*-stave between u_1 and u_n . If G is complete, then the claim is trivial. Otherwise, let $S \subset V(G)$ be a scattering set of G. We claim that $u_1, u_n \notin S$. Suppose the contrary. Since u_1 is simplicial, i.e., its neighborhood induces a clique, we get that $c(G - S) \leq c(G - (S - \{u_1\}))$ and therefore $c(G - S) - |S| < c(G - (S - \{u_1\})) - |S - \{u_1\}|$, a contradiction with the choice of S. The argument for u_n is symmetric.

Since each path in \mathcal{P} connects u_1 and u_n , the union of intervals corresponding to the internal vertices of such a path is the interval [1, s]. In other words, the internal vertices of each path in \mathcal{P} dominate G. Hence the set S, which is a vertex cut by definition, contains an internal vertex from each path of \mathcal{P} . From each path R_i of \mathcal{P} , we choose a vertex $s_i \in S$. We let $S' = \{s_1, \ldots, s_p\}$.

Consider the spanning subgraph G' of G induced by the edges of \mathcal{P} . Observe that G' - S' has two components. If we remove the remaining vertices of $S \setminus S'$ one by one, then with each vertex we remove, the number of components of the remaining graph can increase by at most one, as $u_1, u_n \notin S$. Hence $c(G-S) \leq c(G'-S) \leq 2+|S|-p$, which means that $\mathrm{sc}(G) \leq 2-p$, as S is a scattering set of G. This proves the forward implication of the statement.

For the other direction, let us assume that G does not have a spanning p-stave between u_1 and u_n . If deg $(u_1) < p$, then let S be the set of neighbors of u_1 . Because G is not a complete graph, $u_n \notin S$, i.e., S is a vertex cut and $c(G - S) \ge 2$. Then $sc(G) \ge c(G - S) - |S| \ge$ 2 - |S| > 2 - p. Otherwise, if deg $(u_1) \ge p$, then during the execution of Algorithm 1, at some stage the value set at line 15 becomes smaller than p. Suppose t_1 is the value of the variable t at this moment. We will complete the proof by constructing a scattering set S, for which we show that c(G - S) - |S| > 2 - p. We repeatedly use Lemma 2 and find a finite sequence t_1, t_2, \ldots, t_k , such that $t_{i+1} = (t_i)'$ as long as there are depleted vertices during $(t_i, t_i + 1)$ for i < k. Note that this sequence is decreasing, as we analyze the execution of the algorithm backwards, that is, we first analyze what happened before t_1 , then what happened before t_2 , and so on. As the sequence is limited to positive integers, this process stops at some moment. In particular, we have no depleted vertices during $(t_k, t_k + 1)$. In the utmost case this might be the time interval (1, 2), where no vertex is depleted.

We choose $S = \bigcup_{i=1}^{k} (C_{t_i} \cap C_{t_{i+1}})$ and prove that G - S has at least |S| - p + 3 components. The subgraphs $G[C_{1,t_k}] - S$ and $G[C_{t_1+1,s}] - S$ contain u_1 and u_n , respectively; in particular, they have at least one component each. By property (i) in Lemma 2, $G[C_{t_{i+1}+1,t_i}] - S$ has at least one component for each $i \in \{1, \ldots, k-1\}$. Since all these components are distinct components of G - S, the graph G - S has at least k + 1 components.

By properties (ii), (v) and (vi) in Lemma 2, $(C_{t_{i+1}} \cap C_{t_{i+1}+1}) \setminus (C_{t_i} \cap C_{t_i+1})$ contains only vertices that are depleted during $(t_{i+1}, t_{i+1} + 1)$ for each $i \in \{1, \ldots, k-1\}$. Further, $C_{t_1} \cap C_{t_1+1}$ has no vertices that are free during (t, t + 1), because at least one path is not extendable at time t_1 . Also this set has at most p-1 vertices that are active during (t, t+1). Hence, the remaining vertices are depleted. By properties (ii) and (iv) in Lemma 2, for each $i \in \{1, \ldots, k-1\}$, exactly one vertex that is depleted during (t_i, t_{i+1}) has a different status during $(t_{i+1}, t_{i+1} + 1)$ and is active. It follows that $|S| \leq (p-1) + (k-1) = k + p - 2$ as required.

We will now discuss how to compute a scattering set of G in O(n + m) time. We apply Algorithm 1. This takes O(n + m) time; the only operation whose time complexity has not been discussed is merge(P, Q) at line 23, and we refer to Damaschke's proof of Lemma 1 to verify that this line can be implemented in O(n + m) time. If Algorithm 1 outputs that there is no spanning stave between u_1 and u_n , then G it is not hamiltonian. Recall that in that case $sc(G) \ge 1$ [20] and that we then may apply the O(n + m) time algorithm of Hung and Chang [25] for computing a scattering set. Otherwise, Algorithm 1 finds an optimal spanning stave. Our proof of Theorem 2 provides a construction of a scattering set of G that can be straightforwardly implemented in O(n + m) time.

Corollary 1. A scattering set of an interval graph can be computed in O(n+m) time.

3 Hamilton-connectivity

In this section we prove our contribution to Theorem 1, which is the following.

Theorem 3. For all $k \ge 0$, an interval graph G is k-Hamilton-connected if and only if $sc(G) \le -(k+1)$.

Proof. Let $k \ge 0$ and G be an interval graph with leftmost and rightmost vertices u_1 and u_n as defined before. The statement of Theorem 3 is readily seen to hold when G is a complete graph. Hence we may assume without loss of generality that G is not complete.

First suppose that G is k-Hamilton-connected. Then G has at least k + 3 vertices. We claim that G - R is traceable for every subset $R \subset V(G)$ with $|R| \leq k + 2$. In order to see this, suppose that $R \subseteq V(G)$ with $|R| \leq k + 2$. We may assume without loss of generality that |R| = k + 2. Let s and t be two vertices of R. By definition, $G^* = G - (R \setminus \{s, t\})$ has a

Hamilton path with end-vertices s and t. Hence $G - R = G^* - \{s, t\}$ is traceable. Below we apply this claim twice.

Because G is not complete, G has a scattering set S. By definition, S is a vertex cut. Hence $S = \{s_1, \ldots, s_\ell\}$ for some $\ell \ge k + 3$, as otherwise G - S would be traceable, and thus connected, due to our claim. Let $T = \{s_1, \ldots, s_{k+2}\}$ and let $U = \{s_{k+3}, \ldots, s_\ell\}$. By our claim, G' = G - T is traceable implying that $sc(G') \le 1$ [38]. Because $c(G' - U) = c(G - S) \ge 2$, we find that U is a vertex cut of G'. We use these two facts to derive that

 $1 \ge \operatorname{sc}(G') \\ \ge c(G' - U) - |U| \\ = c(G - T - U) - |T| - |U| + |T| \\ = c(G - S) - |S| + |T| \\ = \operatorname{sc}(G) + |T| \\ = \operatorname{sc}(G) + k + 2,$

implying that $sc(G) \le 1 - (k+2) = -(k+1)$, as required.



Fig. 2. The essential cases in the proof of Theorem 3 for k = 0.

Now suppose that $sc(G) \leq -(k+1)$. First let k = 0. By Theorem 2, there exists a spanning 3-stave $\mathcal{P} = (P, Q, R)$ between u_1 and u_n . Let v, w be an arbitrary pair of vertices of G. We distinguish four cases in order to find a Hamilton path between v and w; see Fig. 2 for an illustration.

Case 1: $v = u_1$ and $w = u_n$. In this case, merge(P, Q, R) is the desired Hamilton path.

Case 2: $v = u_1$ and $w \neq u_n$. Assume without loss of generality that $w \in R$. We split R before w into the subpaths R_1 and R_2 , i.e., w becomes the first vertex of R_2 and it does not belong to R_1 . Then merge $(P, Q, R_1) \circ R_2^{-1}$ is the desired path. The case with $v \neq u_1$ and $w = u_n$ is symmetric.

Case 3: $v \neq u_1$ and $w \neq u_n$ belong to different paths, say $v \in Q$ and $w \in R$. We split Q after v into Q_1 and Q_2 , and we also split R before w, as above. Then $Q_1^{-1} \circ \operatorname{merge}(P, Q_2, R_1) \circ R_2^{-1}$ is the desired path.

Case 4: $v \neq u_1$ and $w \neq u_n$ belong to the same path, say Q. Without loss of generality, assume that both $v \neq u_1$ and $w \neq u_n$ appear in this order on Q. We split Q after v and before w into three subpaths Q_1, Q_2, Q_3 . If v and w are consecutive on Q, i.e., when Q_2 is empty, then $Q_1^{-1} \circ \operatorname{merge}(P, R) \circ Q_3^{-1}$ is the desired path. Otherwise, let z be any vertex on R that is a neighbor of the first vertex of Q_2 . Such z exists since the path R dominates G. We split R after z into R_1 and R_2 . By the choice of z, R_1 and Q_2 can be combined through z into a valid path R' containing exactly the same vertices as R_1 and Q_2 and starting at u_1 . Then we choose $Q_1^{-1} \circ \operatorname{merge}(P, R', R_2) \circ Q_3^{-1}$.

Now let $k \ge 1$. Let S be a set of vertices with $|S| \le k$. We need to show that G - S is Hamilton-connected. Let T be a scattering set of G - S and let $S^* = S \cup T$. Because T is a scattering set of G - S, we find that S^* is a vertex cut of G. We use this to derive that

$$sc(G - S) = c(G - S - T) - |T|$$

= $c(G - S^*) - |S^*| + |S^*| - |T|$
 $\leq sc(G) + k - 0$
 $\leq -1.$

Then, by returning to the case k = 0 with G - S instead of G, we find that G - S is Hamiltonconnected, as required. This completes the proof of Theorem 3.

4 Future Work

We conclude our paper by posing a number of open problems. We start with recalling two open problems posed in the literature, the first of which is the aforementioned question of Damaschke [18]:

1. What is the complexity of 2-HAMILTON PATH for interval graphs?

Our results imply that we may restrict ourselves to interval graphs with scattering number equal to 0 or 1. This can be seen as follows. Let G be an interval graph that together with two of its vertices u and v forms an instance of 2-HAMILTON PATH. We apply Corollary 1 to compute sc(G) in O(n+m) time. If sc(G) < 0, then G is Hamilton-connected by Theorem 1. Then, by definition, there exists a Hamilton path between u and v. If sc(G) > 1, then G is not traceable, also due to Theorem 1. Hence, there exists no Hamilton path between u and v.

The second open problem the literature is due to Asdre and Nikolopoulos [3], who consider the ℓ -PATH COVER problem. This problem generalizes the 1-PATH COVER problem and is that of determining the size of a smallest path cover of a graph G subject to the additional condition that every vertex of a given set T of size ℓ is an end-vertex of a path in the path cover. Note that this problem generalizes 2-HAMILTON PATH. Asdre and Nikolopoulos ask the following question:

2. What is the complexity of ℓ -PATH COVER for interval graphs?

In another paper [2], Asdre and Nikolopoulos proved that ℓ -PATH COVER, and hence 2-HAMILTON PATH, can be solved in O(n+m) time on proper interval graphs.

The SPANNING STAVE problem is that of computing the minimum value of p for which a given graph has a spanning p-stave. Because a Hamilton path of a graph is a spanning 1-stave and HAMILTON PATH is NP-complete, this problem is NP-hard in general.

3. What is the complexity of SPANNING STAVE for interval graphs?

The following example shows that we cannot generalize Lemma 1 and apply Algorithm 1 as an attempt to solve this problem. Take the graph $K_4 - e$, which has four vertices a, b, c, d and five edges, say (a, b), (a, c), (b, c), (b, d), (c, d). This graph is interval. However, we only have a spanning 2-stave between a and d (as their degrees are 2) but there is a spanning 3-stave between b and c, namely paths $\{b, a, c; b, c; b, d, c\}$.

Chen et al. [14] define the spanning connectivity of a Hamilton-connected graph G as the largest integer q such that G has a spanning p-stave between any two vertices of G for all integers $1 \le p \le q$. So, for instance, the complete graph on n vertices has spanning connectivity n-1, and a graph has spanning connectivity at least 1 if and only if it is Hamilton-connected. By the latter statement, the corresponding optimization problem SPANNING CONNECTIVITY is NP-hard. We posed as an open problem to determine the complexity of this problem for proper interval graphs and interval graphs [7]. In response, very recently, Li and Wu [33] announced an O(n + m) time algorithm for solving SPANNING CONNECTIVITY on interval graphs.

Kratsch, Kloks and Müller [30] gave an $O(n^3)$ time algorithm for solving TOUGHNESS on interval graphs. We showed that SCATTERING NUMBER can be solved in linear time on interval graphs.

4. Can Toughness be solved in linear time on interval graphs?

Finally, we ask whether our algorithmic results can be generalized to superclasses of interval graphs, such as circular-arc graphs or cocomparability graphs. The complexity status of HAMILTON CONNECTIVITY is still open for both circular-arc graphs and cocomparability graphs, although HAMILTON CYCLE can be solved in $O(n^2 \log n)$ time on circular-arc graphs [38] and in $O(n^3)$ time on cocomparability graphs [21]. It is known that SCATTER-ING NUMBER can be solved in $O(n^4)$ time on circular-arc graphs and in polynomial time on cocomparability graphs of bounded dimension [30].

- 5. Can HAMILTON CONNECTIVITY be solved in polynomial time for circular-arc graphs or cocomparability graphs?
- 6. Can SCATTERING NUMBER be solved in linear time for circular-arc graphs or cocomparability graphs?

It seems that our approach by relating the Hamilton connectivity of a graph to its scattering number via the existence of a k-stave has been tailored just for the class of interval graphs. The method fails for any graph class that contains all complete bipartite graphs $K_{n,n}$, so for example, for the class of cocomparability graphs and many of its subclasses, such as the classes of permutation graphs and convex graphs. For graphs from these classes, it is no longer clear which two vertices must be chosen as the "leftmost" and "rightmost" vertex, respectively. It can be seen from the previous example, which considered the graph $K_4 - e$, that this choice is important even for interval graphs. For all $n \geq 2$, the complete bipartite graph $K_{n,n}$ is not Hamilton-connected. However, there exists a spanning *n*-stave between any pair of adjacent vertices, but only a spanning 2-stave between the remaining pairs. Acknowledgement. We thank referees for valuable references to related work and for useful comments that helped us to improve the readability of our paper.

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