Finite 2-geodesic transitive graphs of prime valency

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Abstract

We classify non-complete prime valency graphs satisfying the property that their automorphism group is transitive on both the set of arcs and the set of 2-geodesics. We prove that either Γ is 2-arc transitive or the valency p satisfies $p \equiv 1 \pmod 4$, and for each such prime there is a unique graph with this property: it is a non-bipartite antipodal double cover of the complete graph K_{p+1} with automorphism group $PSL(2,p) \times Z_2$ and diameter 3.

Keywords: 2-geodesic transitive graph; 2-arc transitive graph; cover

1 Introduction

In this paper, graphs are finite, simple and undirected. For a graph Γ, a vertex triple (u, v, w) with v adjacent to both u and w is called a 2-*arc* if $u \neq w$, and a 2-*geodesic* if in addition u, w are not adjacent. An *arc* is an ordered pair of adjacent vertices. A non-complete graph Γ is said to be 2-*arc transitive* or 2-*geodesic transitive* if its automorphism group is transitive on arcs, and also on 2-arcs or 2-geodesics, respectively. Clearly, every 2-geodesic is a 2-arc, but some 2-arcs may not be 2-geodesics. If Γ has girth 3 (length of the shortest cycle is 3), then the 2-arcs contained in 3-cycles are not 2-geodesics. Thus the family of non-complete 2-arc transitive graphs is properly contained in the family of 2-geodesic transitive graphs. The graph in Figure 1 is the icosahedron which is 2-geodesic transitive but not 2-arc transitive with valency 5.

The study of 2-arc transitive graphs goes back to Tutte [16, 17]. Since then, this family of graphs has been studied extensively, see [1, 9, 14, 18, 19]. In this paper, we are interested in 2-geodesic transitive graphs, in particular, which are not 2-arc transitive,

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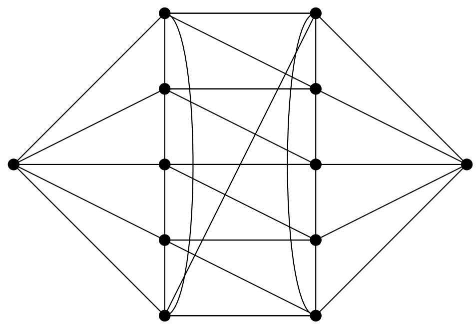

Figure 1: Icosahedron

that is, they have girth 3. We first construct a family of coset graphs, and prove that each of these graphs is 2-geodesic transitive but not 2-arc transitive of prime valency. We then prove that each graph with these properties belongs to the family.

For a finite group G, a core-free subgroup H (that is, $\bigcap_{g\in G} H^g = 1$), and an element $g \in G$ such that $G = \langle H, g\rangle$ and $HgH = Hg^{-1}H$, the *coset graph* $\mathrm{Cos}(G, H, HgH)$ is the graph with vertex set $\{Hx|x \in G\}$, such that two vertices Hx, Hy are adjacent if and only if $yx^{-1} \in HgH$. This graph is connected, undirected, and G-arc transitive of valency $|H : H \cap H^g|$, see [12].

Definition 1.1 Let $\mathcal{C}(5)$ be the singleton set containing the icosahedron, and for a prime $p > 5$ with $p \equiv 1 \pmod 4$, let $\mathcal{C}(p)$ consist of the coset graphs $\mathrm{Cos}(G, H, HgH)$ as follows. Let $G = PSL(2, p)$, choose $a \in G$ of order p, so $N_G(\langle a\rangle) = \langle a\rangle : \langle b\rangle \cong Z_p : Z_{\frac{p-1}{2}}$ for some $b \in G$ of order $\frac{p-1}{2}$. Then $N_G(\langle b^2\rangle) = \langle b\rangle : \langle c\rangle \cong D_{p-1}$ for some $c \in G$ of order 2. Let $H = \langle a\rangle : \langle b^2\rangle$ and $g = cb^{2i}$ for some i.

These graphs have appeared a number of times in the literature. They were constructed by D. Taylor [15] as a family of regular two-graphs (see [3, p.14]), they appeared in the classification of antipodal distance transitive covers of complete graphs in [6], and were also constructed explicitly as coset graphs and studied by the third author in [11]. (Antipodal covers of graphs are defined in Section 2.)

A path of shortest length from a vertex u to a vertex v is called a *geodesic* from u to v, or sometimes an *i-geodesic* if the distance between u and v is i. The graph Γ is said to be *geodesic transitive* if its automorphism group is transitive on the set of i-geodesics for all positive integers i less than or equal to the diameter of Γ.

Theorem 1.2 (a) *A graph $\Gamma \in \mathcal{C}(p)$ if and only if Γ is a connected non-bipartite antipodal double cover of K_{p+1} with $p \equiv 1 \pmod 4$, and* $\mathrm{Aut}\Gamma \cong PSL(2, p) \times Z_2$.

(b) *For a given p, all graphs in $\mathcal{C}(p)$ are isomorphic, geodesic transitive and have diameter* 3.

Our second result shows that the graphs in Definition 1.1 are the only 2-geodesic transitive graphs of prime valency that are not 2-arc transitive.

Theorem 1.3 *Let Γ be a connected non-complete graph of prime valency p. Then Γ is 2-geodesic transitive if and only if Γ is 2-arc transitive, or $p \equiv 1 \pmod 4$ and $\Gamma \in \mathcal{C}(p)$.*

These two theorems show that up to isomorphism, there is a unique connected 2-geodesic transitive but not 2-arc transitive graph of prime valency p and $p \equiv 1 \pmod 4$. The family of 2-geodesic transitive but not 2-arc transitive graphs of valency 4 has been determined in [4]. It would be interesting to know if a similar classification is possible for non-prime valencies at least 6. This is the subject of further research by the second author, see [10].

2 Preliminaries

In this section, we give some definitions and prove some results which will be used in the following discussion. Let Γ be a graph. We use $V\Gamma$, $E\Gamma$ and $\mathrm{Aut}\Gamma$ to denote its *vertex set*, *edge set* and *automorphism group*, respectively. The size of $V\Gamma$ is called the *order* of the graph. The graph Γ is said to be *vertex transitive* if the action of $\mathrm{Aut}\Gamma$ on $V\Gamma$ is transitive.

For two distinct vertices u, v of Γ, the smallest value for n such that there is a path of length n from u to v is called the *distance* from u to v and is denoted by $d_\Gamma(u, v)$. The *diameter* $\mathrm{diam}(\Gamma)$ of a connected graph Γ is the maximum of $d_\Gamma(u, v)$ over all $u, v \in V\Gamma$. We set $\Gamma_2(v) = \{u \in V\Gamma | d_\Gamma(v, u) = 2\}$ for every vertex v.

Quotient graphs play an important role in this paper. Let G be a group of permutations acting on a set Ω. A *G-invariant partition* of Ω is a partition $\mathcal{B} = \{B_1, B_2, \ldots, B_n\}$ such that for each $g \in G$, and each $B_i \in \mathcal{B}$, the image $B_i^g \in \mathcal{B}$. The parts of Ω are often called *blocks* of G on Ω. For a G-invariant partition $\mathcal{B}$ of Ω, we have two smaller transitive permutation groups, namely the group $G^{\mathcal{B}}$ of permutations of $\mathcal{B}$ induced by G; and the group $G_{B_i}^{B_i}$ induced on B_i by G_{B_i} (the setwise stabiliser of B_i in G) where $B_i \in \mathcal{B}$. Let Γ be a graph, and let $G \le \mathrm{Aut}\Gamma$. Suppose $\mathcal{B} = \{B_1, B_2, \ldots, B_n\}$ is a G-invariant partition of $V\Gamma$. The *quotient graph* $\Gamma_{\mathcal{B}}$ of Γ relative to $\mathcal{B}$ is defined to be the graph with vertex set $\mathcal{B}$ such that $\{B_i, B_j\}$ $(i \ne j)$ is an edge of $\Gamma_{\mathcal{B}}$ if and only if there exist $x \in B_i, y \in B_j$ such that $\{x, y\} \in E\Gamma$. We say that $\Gamma_{\mathcal{B}}$ is *nontrivial* if $1 < |\mathcal{B}| < |V\Gamma|$. The graph Γ is said to be a *cover* of $\Gamma_{\mathcal{B}}$ if for each edge $\{B_i, B_j\}$ of $\Gamma_{\mathcal{B}}$ and $v \in B_i$, we have $|\Gamma(v) \cap B_j| = 1$.

For a graph Γ, the *k-distance graph* Γ_k of Γ is the graph with vertex set $V\Gamma$, such that two vertices are adjacent if and only if they are at distance k in Γ. If $d = \mathrm{diam}(\Gamma) \ge 2$ and Γ_d is a disjoint union of complete graphs, then Γ is said to be an *antipodal graph.* In other words, the vertex set of an antipodal graph Γ of diameter d, may be partitioned into so-called *fibres*, such that any two distinct vertices in the same fibre are at distance d and two vertices in different fibres are at distance less than d. For an antipodal graph Γ of diameter d, its *antipodal quotient graph* Σ is the quotient graph of Γ where $\mathcal{B}$ is the set of fibres. If further, Γ is a cover of Σ, then Γ is called an *antipodal cover* of Σ.

Paley graphs were first defined by Paley in 1933, see [13]. These graphs are vertex transitive, self-complementary, and have many nice properties. Let $q = p^e$ be a prime power such that $q \equiv 1 \pmod 4$. Let F_q be the finite field of order q. The *Paley graph* $P(q)$ is the graph with vertex set F_q, where two distinct vertices u, v are adjacent if and only if $u - v$ is a nonzero square in F_q. The congruence condition on q implies that -1 is a square in F_q, and hence $P(q)$ is an undirected graph.

Lemma 2.2 is used in the proof of Theorem 1.3, and its proof uses the following famous result of Burnside.

Lemma 2.1 ([5, Theorem 3.5B]) *A primitive permutation group G of prime degree p is either 2-transitive, or solvable and $G \leq AGL(1,p)$.*

For a finite group G, and a subset S of G such that $1 \notin S$ and $S = S^{-1}$, the *Cayley graph* $\mathrm{Cay}(G,S)$ of G with respect to S is the graph with vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. The Paley graph $P(q)$ is a Cayley graph for the additive group $G = F_q^+$ with $S = \{w^2, w^4, \ldots, w^{q-1} = 1\}$, where w is a primitive element of F_q.

Lemma 2.2 *Let Γ be an arc transitive graph of prime order p and valency $\frac{p-1}{2}$. Then $p \equiv 1 \pmod 4$, $\mathrm{Aut}\Gamma \cong Z_p : Z_{\frac{p-1}{2}}$, and $\Gamma \cong P(p)$.*

Proof. Since Γ has valency $\frac{p-1}{2}$, p is an odd prime. Since Γ has the given order and valency, it follows that Γ has $p(\frac{p-1}{2})/2$ edges. This implies that $p \equiv 1 \pmod 4$.

Let $A = \mathrm{Aut}\Gamma$. Since A is transitive on $V\Gamma$ and p is a prime, A is primitive on $V\Gamma$, and since Γ is arc transitive, $|A|$ is divisible by $\frac{p(p-1)}{2}$. Since Γ is neither complete nor empty, it follows by Lemma 2.1 that $A < AGL(1,p) = Z_p : Z_{p-1}$. Thus $|A|$ is a proper divisor of $p(p-1)$, and at least $\frac{p(p-1)}{2}$, and so $|A| = \frac{p(p-1)}{2}$. Hence $A \cong Z_p : Z_{\frac{p-1}{2}}$.

Since Z_p is regular on $V\Gamma$, it follows from [2, Lemma 16.3] that Γ is a Cayley graph for Z_p. Thus $\Gamma = \mathrm{Cay}(G,S)$ where $G \cong Z_p$, $S \subseteq G \setminus \{0\}$, $S = S^{-1}$ and $|S| = \frac{p-1}{2}$. Now we may identify G with F_p^+ where F_p is a finite field of order p. Let $v \in V\Gamma$ be the vertex corresponding to $0 \in G$. Then A_v is the unique subgroup of order $\frac{p-1}{2}$ of $F_p^* = \langle w \rangle$, that is, $A_v = \langle w^2 \rangle$. The A_v-orbits in F_p are $\{0\}$, $S_1 = \{w^2, w^4, \ldots, w^{p-1}\}$ and $S_2 = \{w, w^3, \ldots, w^{p-2}\}$, and so $S = S_1$ or S_2, and $\Gamma = P(p)$ or its complement respectively. In either case, $\Gamma \cong P(p)$. $\square$

To end the section, we cite a property of Paley graphs which will be used in the next section.

Lemma 2.3 ([7, p.221]) *Let $\Gamma = P(q)$, where q is a prime power such that $q \equiv 1 \pmod 4$. Let u, v be distinct vertices of Γ. If u, v are adjacent, then $|\Gamma(u) \cap \Gamma(v)| = \frac{q-5}{4}$; if u, v are not adjacent, then $|\Gamma(u) \cap \Gamma(v)| = \frac{q-1}{4}$.*

3 Proof of Theorem 1.2

We study graphs in the family $\mathcal{C}(p)$ for each prime $p \equiv 1 \pmod 4$. We first collect some properties of graphs in $\mathcal{C}(p)$ for $p > 5$, which can be found in [11, Theorem 1.1] and its proof.

Remark 3.1 Let $\Gamma \in \mathcal{C}(p)$ and $p > 5$. Then $G = \langle H, g \rangle$, Γ is connected and G-arc transitive of valency p, $\mathrm{Aut}\Gamma \cong G \times Z_2$, $|V\Gamma| = |G : H| = 2p + 2$. Further, $\mathrm{diam}(\Gamma) = \mathrm{girth}(\Gamma) = 3$, so Γ is not 2-arc transitive.

The orbit set $\mathcal{B} = \{\Delta_1, \Delta_2, \ldots, \Delta_{p+1}\}$ of the normal subgroup $K \cong Z_2$ of $\mathrm{Aut}\Gamma$ forms a system of imprimitivity for $\mathrm{Aut}\Gamma$ in $V\Gamma$, and it follows from the proof of [11, Theorem 1.1] that this is the unique nontrivial system of imprimitivity and the kernel of the action of $\mathrm{Aut}\Gamma$ on $\mathcal{B}$ is the normal subgroup K. For $i = 1, \ldots, p+1$, let $\Delta_i = \{v_i, v_i'\}$. Then v_i is not adjacent to v_i', and for each $j \neq i$, v_i is adjacent to exactly one point of Δ_j and v_i' is adjacent to the other. Thus, $\Gamma(v_1) \cap \Gamma(v_1') = \emptyset$, $V\Gamma = \{v_1\} \cup \Gamma(v_1) \cup \{v_1'\} \cup \Gamma(v_1')$, and Γ is a non-bipartite double cover of K_{p+1}.

The next lemma shows that graphs in $\mathcal{C}(p)$ are geodesic transitive.

Lemma 3.2 *Let p be a prime and $p \equiv 1 \pmod 4$. Then each graph in $\mathcal{C}(p)$ is geodesic transitive of girth 3 and diameter 3.*

Proof. Let $\Gamma \in \mathcal{C}(p)$. If $p = 5$, then Γ is the icosahedron of girth 3 and diameter 3. Its automorphism group is $PSL(2,5) \times Z_2$ and it is geodesic transitive. Now suppose that $p > 5$. Let $\mathcal{B}$ be as in Remark 3.1, $A := \mathrm{Aut}\Gamma$, $v_1 \in V\Gamma$ and $u \in \Gamma(v_1)$. Let K be the kernel of the A-action on $\mathcal{B}$ so that the induced group $A^{\mathcal{B}} = A/K$. Then by the proof of [11, Theorem 1.1], $K \cong Z_2 \lhd A$, $A = G \times K$, $A^{\mathcal{B}} \cong G = PSL(2,p)$ and $(A^{\mathcal{B}})_{\Delta_1} \cong A_{v_1}$. Since $A \cong G \times Z_2$, it follows that $|A_{v_1}| = \frac{p(p-1)}{2}$, and by Lemma 2.4 of [11], $A_{v_1} \cong Z_p : Z_{\frac{p-1}{2}}$, which has a unique permutation action of degree p, up to permutational isomorphism. Since Γ is A-arc transitive, A_{v_1} is transitive on $\Gamma(v_1)$ and hence on $\mathcal{B} \setminus \{\Delta_1\}$, and therefore also on $\Gamma(v_1')$, all of degree p. Thus the A_{v_1}-orbits in $V\Gamma$ are $\{v_1\}, \Gamma(v_1), \Gamma(v_1')$ and $\{v_1'\}$, and it follows that $\Gamma(v_1') = \Gamma_2(v_1)$. Moreover, $A_{v_1,u} \cong Z_{\frac{p-1}{2}}$ has orbit lengths $1, \frac{p-1}{2}, \frac{p-1}{2}$ in $\Gamma(v_1)$, and hence has the same orbit lengths in $\Gamma_2(v_1)$, and also in $\Gamma(u)$ (since $A_{v_1,u}$ is the point stabiliser of A_u acting on $\Gamma(u)$). Since $\Gamma(v_1) \cap \Gamma(u) \neq \emptyset$, it follows that the $A_{v_1,u}$-orbits in $\Gamma(u)$ are $\{v_1\}, \Gamma(v_1) \cap \Gamma(u)$, and $\Gamma_2(v_1) \cap \Gamma(u)$. Thus Γ is $(A,2)$-geodesic transitive and $\mathrm{girth}(\Gamma) = 3$. Further, as $\Gamma_3(v_1) = \{v_1'\}$, it follows that Γ is geodesic transitive and has diameter 3. $\Box$

In the proof of the second part of Theorem 1.2, we repeatedly use the fact that each $\sigma \in \mathrm{Aut}G$ induces an isomorphism from $\mathrm{Cos}(G,H,HgH)$ to $\mathrm{Cos}(G,H^\sigma,H^\sigma g^\sigma H^\sigma)$, and in particular, we use this fact for the conjugation action by elements of G. For a subset Δ of the vertex set of a graph Γ, we use $[\Delta]$ to denote the subgraph of Γ induced by Δ.

Proof of Theorem 1.2 (a) Suppose first that Γ is a connected non-bipartite antipodal double cover of K_{p+1} with $p \equiv 1 \pmod 4$, and $A := \mathrm{Aut}\Gamma \cong PSL(2,p) \times Z_2$. Then $|V\Gamma| = 2p+2$, and for each $u \in V\Gamma$, let $u' \in V\Gamma$ be its unique vertex at maximum distance. Then $|\Gamma(u)| = p = |\Gamma(u')|$, and $\Gamma(u) \cap \Gamma(u') = \emptyset$. Since Γ is connected, it follows that $V\Gamma = \{u\} \cup \Gamma(u) \cup \Gamma(u') \cup \{u'\}$, and the diameter of Γ is 3.

Let $\mathcal{B} = \{B_1, B_2, \ldots, B_{p+1}\}$ be the invariant partition of $V\Gamma$ such that $\Gamma_{\mathcal{B}} \cong K_{p+1}$ and Γ is a non-bipartite antipodal double cover of $\Gamma_{\mathcal{B}}$. Let K be the kernel of the A-action on $\mathcal{B}$. As each $|B_i| = 2$, it follows that K is a 2-group. Further, as K is a normal subgroup of A and $PSL(2,p)$ is a simple group, it follows that $K \cong Z_2$. Thus $G := PSL(2,p)$ acts faithfully on $\mathcal{B}$. Since the G-action on $p+1$ points is unique and this action is 2-transitive, it follows that G is 2-transitive on $\mathcal{B}$, and so $\Gamma_{\mathcal{B}}$ is G-arc transitive. Thus either G is transitive on $V\Gamma$ or G has two orbits Δ_1, Δ_2 in $V\Gamma$ of size $p+1$. Suppose the latter holds. If the induced subgraph $[\Delta_i]$ contains an edge, then $[\Delta_i] \cong K_{p+1}$, as the G-action on $p+1$ points is 2-transitive. It follows that $\Gamma = 2 \cdot K_{p+1}$ contradicting the fact that Γ is connected. Hence $[\Delta_i]$ does not contain edges of Γ, and so Γ is a bipartite graph, again a contradiction. Thus G is transitive on $V\Gamma$.

Let B_1 be a block and $u \in B_1$. Then $G_{B_1} \cong Z_p : Z_{\frac{p-1}{2}}$ and $G_u \cong Z_p : Z_{\frac{p-1}{4}}$. As G_u has an element of order p, G_u is transitive on $\Gamma(u)$, and hence Γ is G-arc transitive.

Let $p = 5$. Suppose $B_1 = \{u, u'\}$. Since Γ is G-arc transitive, it follows that G_u is transitive on $\Gamma(u)$ and $G_{u'}$ is transitive on $\Gamma(u')$. As $G_u = G_{u,u'} = G_{u'} \cong Z_5$ and

$\Gamma_3(u) = \{u'\}$, it follows that Γ is G-distance transitive. Thus by [3, p.222, Theorem 7.5.3 (ii)], Γ is the icosahedron, so $\Gamma \in \mathcal{C}(5)$.

Now assume that $p > 5$. As Γ is connected and G-arc transitive, $\Gamma \cong Cos(G, H, HgH)$ for the subgroup $H = G_u$ and some element $g \in G \setminus H$, such that $\langle H, g\rangle = G$ and $g^2 \in H$. Let $a \in H$ and $o(a) = p$. Then $\langle a\rangle$ is a Sylow p-subgroup of G. Thus $H = \langle a\rangle : \langle b^2\rangle$ where $N_G(\langle a\rangle) = \langle a\rangle : \langle b\rangle$.

Now we determine the element g. Let $u = H$ and $v = Hg$ in $V\Gamma$. Then $G_u = H$ and $G_{u,v} = \langle b^2\rangle$. Further, $G^g_{u,v} = (G_u \cap G_v)^g = G^g_u \cap G^g_v = G_v \cap G_u = G_{u,v}$, and hence $\langle b^2\rangle^g = \langle b^2\rangle$. Thus $g \in N_G(\langle b^2\rangle) \cong D_{p-1} = \langle b\rangle : \langle x\rangle$ for some involution x. If $g = b^i$ for some $i \geq 1$, then $\langle H, g\rangle \leq N_X(\langle a\rangle) = \langle a\rangle : \langle y\rangle$ where $X = PGL(2, p)$ and $y^2 = b$, contradicting the fact that $\langle H, g\rangle = G$. Thus $g = b^i x$ for some i, and so $N_G(\langle b^2\rangle) \cong D_{p-1} = \langle b\rangle : \langle g\rangle$. Thus $\Gamma \cong Cos(G, H, HgH) \in \mathcal{C}(p)$.

Conversely, assume that $\Gamma \in \mathcal{C}(p)$. If Γ is the icosahedron, then we easily see that Γ is a connected non-bipartite antipodal double cover of K_6 and its automorphism group is $PSL(2, 5) \times Z_2$. If $p > 5$, then by Remark 3.1, Γ is a connected non-bipartite antipodal double cover of K_{p+1} and $\mathrm{Aut}\Gamma \cong PSL(2, p) \times Z_2$.

(b) The claims in part (b) hold for the icosahedron, so assume that $p > 5$ and $p \equiv 1 \pmod 4$, and let $G = PSL(2, p)$. Let elements a_i, b_i, g_i and subgroups H_i be chosen as in Definition 1.1 for $i \in \{1, 2\}$. Let $X = PGL(2, p) \cong \mathrm{Aut}G$.

Since all subgroups of G of order p are conjugate there exists $x \in G$ such that $\langle a_2\rangle^x = \langle a_1\rangle$, so we may assume that $\langle a_1\rangle = \langle a_2\rangle = M$, say. Let $Y = N_X(M)$. Then $Y = M : \langle y\rangle$ where $o(y) = p - 1$, and $H_1 = M : \langle b_1^2\rangle$ and $H_2 = M : \langle b_2^2\rangle$ are equal to the unique subgroup of Y of order $\frac{p(p-1)}{4}$, that is, $H_1 = H_2 = M : \langle y^4\rangle = H$, say. Next, since all subgroups of Y of order $\frac{p-1}{4}$ are conjugate, there exist $x_1, x_2 \in Y$ such that $\langle b_1^2\rangle^{x_1} = \langle b_2^2\rangle^{x_2} = \langle y^4\rangle$. Since each x_i normalises H we may assume in addition that $\langle b_1^2\rangle = \langle b_2^2\rangle = \langle y^4\rangle < \langle y\rangle$. Thus g_1, g_2 are non-central involutions in $N_G(\langle y^4\rangle) \cong D_{p-1}$, an index 2 subgroup of $N_X(\langle y^4\rangle) = \langle y\rangle : \langle z\rangle \cong D_{2(p-1)}$. The set of non-central involutions in $N_G(\langle y^4\rangle)$ forms a conjugacy class of $N_X(\langle y^4\rangle)$ of size $\frac{p-1}{2}$ and consists of the elements $y^{2i}z$, for $0 \leq i < \frac{p-1}{2}$. The group $\langle y\rangle$ acts transitively on this set of involutions by conjugation (and normalises H). Hence, for some $u \in \langle y\rangle$, $H^u = H$ and $g_2^u = g_1$. Thus all graphs in $\mathcal{C}(p)$ are isomorphic. Finally, by Lemma 3.2, these graphs are geodesic transitive of diameter 3. $\Box$

4 Proof of Theorem 1.3

In this section, we will prove Theorem 1.3 in a series of lemmas. For all lemmas of this section, we assume that Γ is a connected 2-geodesic transitive graph of prime valency p and we denote $\mathrm{Aut}\Gamma$ by A. Note that the assumption of 2-geodesic transitivity implies that the graph is not complete. If Γ is 2-arc transitive, there is nothing to prove, so we assume further that this is not the case, that is to say, we assume that Γ has girth 3. The first lemma determines some intersection parameters.

Lemma 4.1 *Let (v, u, w) be a 2-geodesic of Γ. Then $p \equiv 1 \pmod 4$, $|\Gamma(v) \cap \Gamma(u)| = |\Gamma_2(v) \cap \Gamma(u)| = \frac{p-1}{2}$ and $|\Gamma(v) \cap \Gamma(w)|$ divides $\frac{p-1}{2}$. Moreover, $A_v^{\Gamma(v)} \cong Z_p : Z_{\frac{p-1}{2}}$ is a Frobenius group, and $A_{v,u}^{\Gamma(v)} \cong Z_{\frac{p-1}{2}}$ is transitive on $\Gamma(v) \cap \Gamma(u)$.*

Proof. Since Γ is 2-geodesic transitive but not 2-arc transitive, it follows that Γ is not a cycle. In particular, p is an odd prime. Let $|\Gamma(v) \cap \Gamma(u)| = x$ and $|\Gamma_2(v) \cap \Gamma(u)| = y$. Then $x + y = |\Gamma(u) \setminus \{v\}| = p - 1$. Since girth$(\Gamma) = 3$, $x \geq 1$. Since p is odd and the induced subgraph $[\Gamma(v)]$ is an undirected regular graph with $\frac{px}{2}$ edges, it follows that x is even. This together with $x + y = p - 1$ and the fact that $p - 1$ is even, implies that y is also even.

Since Γ is arc transitive, $A_v^{\Gamma(v)}$ is transitive on $\Gamma(v)$. Since p is a prime, $A_v^{\Gamma(v)}$ acts primitively on $\Gamma(v)$. By Lemma 2.1, either $A_v^{\Gamma(v)}$ is 2-transitive, or $A_v^{\Gamma(v)}$ is solvable and $A_v^{\Gamma(v)} \leq AGL(1, p)$. Since Γ is not complete, it follows that $[\Gamma(v)]$ is not a complete graph. Also since girth$(\Gamma) = 3$, $[\Gamma(v)]$ is not an empty graph and so $A_v^{\Gamma(v)}$ is not 2-transitive. Hence $A_v^{\Gamma(v)} < AGL(1, p)$. Thus $A_v^{\Gamma(v)} \cong Z_p : Z_m$ is a Frobenius group, where $m|(p - 1)$ and $m < p - 1$. Hence $m \leq \frac{p-1}{2}$.

Since Γ is vertex transitive, it follows that $A_u^{\Gamma(u)} \cong Z_p : Z_m$, and hence $A_{u,v}^{\Gamma(u)} \cong Z_m$ is semiregular on $\Gamma(u) \setminus \{v\}$ with orbits of size m. Since Γ is 2-geodesic transitive, $A_{u,v}^{\Gamma(u)}$ is transitive on $\Gamma_2(v) \cap \Gamma(u)$, and hence $y = |\Gamma_2(v) \cap \Gamma(u)| = m$, so $x = p - 1 - m = m(\frac{p-1}{m} - 1) \geq m$, and x is divisible by m.

Now again by arc transitivity, $|\Gamma(u) \cap \Gamma(w)| = |\Gamma(u) \cap \Gamma(v)| = x$. Since $|\Gamma_2(v) \cap \Gamma(u)| = m$, it follows that $|\Gamma_2(v) \cap \Gamma(u) \cap \Gamma(w)| \leq m - 1$. Since $\Gamma(w) \cap \Gamma(u) = (\Gamma(w) \cap \Gamma(u) \cap \Gamma(v)) \cup (\Gamma(w) \cap \Gamma(u) \cap \Gamma_2(v))$, it follows that

$$x \leq |\Gamma(w) \cap \Gamma(u) \cap \Gamma(v)| + (m - 1). \qquad (*)$$

Let $z = |\Gamma(v) \cap \Gamma(w)|$ and $n = |\Gamma_2(v)|$. Since Γ is 2-geodesic transitive, z, n are independent of v, w and, counting edges between $\Gamma(v)$ and $\Gamma_2(v)$ we have $pm = nz$. Now $z \leq |\Gamma(v)| = p$. Suppose first that $z = p$. Then $m = n$ and $\Gamma(v) = \Gamma(w)$, and so for distinct $w_1, w_2 \in \Gamma_2(v)$, $d_\Gamma(w_1, w_2) = 2$. Since Γ is 2-geodesic transitive, it follows that $\Gamma(v) = \Gamma(v')$ whenever $d_\Gamma(v, v') = 2$. Thus diam$(\Gamma) = 2$, $V\Gamma = \{v\} \cup \Gamma(v) \cup \Gamma_2(v)$ and $|V\Gamma| = 1+p+m$. Let $\Delta = \{v\} \cup \Gamma_2(v)$. Then for distinct $v_1, v_1' \in \Delta$, $d_\Gamma(v_1, v_1') = 2$; for any $v_1'' \in V\Gamma \setminus \Delta$, v_1, v_1'' are adjacent. Thus, for any $v_1 \in \Delta$, $\Delta = \{v_1\} \cup \Gamma_2(v_1)$. It follows that Δ is a block of imprimitivity for A of size $m+1$. Hence $(m+1)|(p+m+1)$, so $(m+1)|p$. Since $m|(p-1)$, it follows that $m+1 = p$ which contradicts the inequality $m \leq \frac{p-1}{2}$.

Thus $z < p$, and so z divides m, as $pm = nz$. Since $|\Gamma(w) \cap \Gamma(u) \cap \Gamma(v)| \leq z$, it follows from $(*)$ that $x \leq z + (m - 1) \leq 2m - 1 < 2m$. Since x is divisible by m and $x \geq m$ we have $x = m$. Thus $2m = x + y = p - 1$, so $x = y = m = \frac{p-1}{2}$, and since x is even, $p \equiv 1 \pmod 4$. Also $x = m$ implies that $A_{v,u}^{\Gamma(v)}$ is transitive on $\Gamma(v) \cap \Gamma(u)$. Finally, since $nz = pm = p(\frac{p-1}{2})$ and $z < p$, it follows that z divides $\frac{p-1}{2}$. $\Box$

Lemma 4.2 *For $v \in V\Gamma$, the stabiliser $A_v \cong Z_p : Z_{\frac{p-1}{2}}$ is a Frobenius group.*

Proof. Suppose that (v, u) is an arc of Γ. Then by Lemma 4.1, $A_v^{\Gamma(v)} \cong Z_p : Z_{\frac{p-1}{2}}$ is a Frobenius group, and $A_{v,u}^{\Gamma(v)} \cong Z_{\frac{p-1}{2}}$ is regular on $\Gamma(v) \cap \Gamma(u)$. Let K be the kernel of the action of A_v on $\Gamma(v)$. Let $u' \in \Gamma(v) \cap \Gamma(u)$ and $x \in K$. Then $x \in A_{v,u,u'}$. Since $A_{u,v}^{\Gamma(u)} \cong Z_{\frac{p-1}{2}}$ is semiregular on $\Gamma(u) \setminus \{v\}$, it follows that x fixes all vertices of $\Gamma(u)$. Since x also fixes all vertices of $\Gamma(v)$, this argument for each $u \in \Gamma(v)$ shows that x fixes all vertices of $\Gamma_2(v)$. Since Γ is connected, x fixes all vertices of Γ, and hence $x = 1$. Thus $K = 1$, so $A_v \cong Z_p : Z_{\frac{p-1}{2}}$ is a Frobenius group. $\Box$

Lemma 4.3 *Let (v, u, w) be a 2-geodesic of Γ. Then $|\Gamma(v) \cap \Gamma(w)| = \frac{p-1}{2}$, $|\Gamma_2(v) \cap \Gamma(w) \cap \Gamma(u)| = \frac{p-1}{4}$, $|\Gamma_2(v)| = p$, and $|\Gamma_2(v) \cap \Gamma(w)| = \frac{p-1}{2}$.*

Proof. Let $z = |\Gamma(v) \cap \Gamma(w)|$ and $n = |\Gamma_2(v)|$. By Lemma 4.1, $|\Gamma(u) \cap \Gamma_2(v)| = \frac{p-1}{2}$ and $z | \frac{p-1}{2}$. Counting the edges between $\Gamma(v)$ and $\Gamma_2(v)$ gives $\frac{p-1}{2} p = nz$. By Lemma 4.2, $A_{v,u} = Z_{\frac{p-1}{2}}$, and by Lemma 4.1, $A_{v,u}$ is transitive on $\Gamma(v) \cap \Gamma(u)$, so $[\Gamma(u)]$ is A_u-arc transitive. Since p is a prime, it follows by Lemma 2.2 that $[\Gamma(u)]$ is a Paley graph $P(p)$. Since $v, w \in \Gamma(u)$ are not adjacent, by Lemma 2.3, $|\Gamma(v) \cap \Gamma(u) \cap \Gamma(w)| = \frac{p-1}{4}$, hence $z \geq \frac{p-1}{4} + 1$. Since $z | \frac{p-1}{2}$, it follows that $z = \frac{p-1}{2}$. Hence $n = p$. Thus, $|\Gamma(v) \cap \Gamma(w)| = \frac{p-1}{2}$ and $|\Gamma_2(v)| = p$.

By Lemma 4.1, we have $|\Gamma(v) \cap \Gamma(u)| = \frac{p-1}{2}$. Since Γ is arc transitive, it follows that $|\Gamma(v_1) \cap \Gamma(v_2)| = \frac{p-1}{2}$ for every arc (v_1, v_2). Thus, $|\Gamma(u) \cap \Gamma(w)| = \frac{p-1}{2}$. Since $\Gamma(u) \cap \Gamma(w) = (\Gamma(v) \cap \Gamma(u) \cap \Gamma(w)) \cup (\Gamma_2(v) \cap \Gamma(u) \cap \Gamma(w))$ where $\Gamma(v) \cap \Gamma(u) \cap \Gamma(w)$ and $\Gamma_2(v) \cap \Gamma(u) \cap \Gamma(w)$ are disjoint, and since $|\Gamma(v) \cap \Gamma(u) \cap \Gamma(w)| = \frac{p-1}{4}$, it follows that $|\Gamma_2(v) \cap \Gamma(u) \cap \Gamma(w)| = \frac{p-1}{2} - \frac{p-1}{4} = \frac{p-1}{4}$. Since $A_v = Z_p : Z_{\frac{p-1}{2}}$, it follows that $A_{v,w} = Z_{\frac{p-1}{2}}$ and $A_{v,w}$ is semiregular on $\Gamma_2(v) \setminus \{w\}$ with orbits of size $\frac{p-1}{2}$. Since $\Gamma_2(v) \cap \Gamma(w) \subseteq \Gamma(w) \setminus \Gamma(v)$ (of size $\frac{p-1}{2}$) and since $|\Gamma_2(v) \cap \Gamma(w) \cap \Gamma(u)| = \frac{p-1}{4} > 0$, it follows that $|\Gamma_2(v) \cap \Gamma(w)| = \frac{p-1}{2}$. $\square$

Lemma 4.4 *Let v be a vertex of Γ. Then $|\Gamma_3(v)| = 1$ and $\mathrm{diam}(\Gamma) = 3$, so Γ is antipodal with fibres of size 2. Further, Γ is geodesic transitive.*

Proof. Suppose that (v, u, w) is a 2-geodesic of Γ. Then by Lemma 4.3, $|\Gamma(v) \cap \Gamma(w)| = \frac{p-1}{2}$ and $|\Gamma_2(v) \cap \Gamma(w)| = \frac{p-1}{2}$. Hence $|\Gamma_3(v) \cap \Gamma(w)| = p - |\Gamma(v) \cap \Gamma(w)| - |\Gamma_2(v) \cap \Gamma(w)| = 1$. Since Γ is 2-geodesic transitive, it follows that $|\Gamma_3(v) \cap \Gamma(w_1)| = 1$ for all $w_1 \in \Gamma_2(v)$. Thus Γ is 3-geodesic transitive.

Let $\Gamma_3(v) \cap \Gamma(w) = \{v'\}$, $n = |\Gamma_3(v)|$ and $i = |\Gamma_2(v) \cap \Gamma(v')|$. Counting edges between $\Gamma_2(v)$ and $\Gamma_3(v)$, we have $p = ni$. Since $[\Gamma(w)]$ is a Paley graph and $u, v' \in \Gamma(w)$ are not adjacent, it follows from Lemma 2.3 that $|\Gamma(u) \cap \Gamma(w) \cap \Gamma(v')| = \frac{p-1}{4}$. Since $\Gamma(u) \cap \Gamma_2(v)$ contains these $\frac{p-1}{4}$ vertices as well as w, we have $i \geq \frac{p+3}{4} > 1$. Thus $i = p$ and $n = 1$, that is, $|\Gamma_3(v)| = 1$. Since $|\Gamma_2(v) \cap \Gamma(v')| = p$ and $|\Gamma_2(v)| = p$, it follows that $\Gamma_2(v) = \Gamma(v')$, so $\mathrm{diam}(\Gamma) = 3$ and Γ is antipodal with fibres of size 2. Therefore Γ is geodesic transitive. $\square$

We are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Let Γ be a connected non-complete graph of prime valency p. Suppose first that Γ is 2-geodesic transitive. If $\mathrm{girth}(\Gamma) \geq 4$, then every 2-arc is a 2-geodesic, so Γ is 2-arc transitive. Now assume that $\mathrm{girth}(\Gamma) = 3$. Let $v \in V\Gamma$. Then it follows from Lemmas 4.1 to 4.4 that $p \equiv 1 \pmod 4$, $|\Gamma_2(v)| = p$, $|\Gamma_3(v)| = 1$ and $\mathrm{diam}(\Gamma) = 3$. Thus, $V\Gamma = \{v\} \cup \Gamma(v) \cup \Gamma_2(v) \cup \{v'\}$, where $\Gamma_3(v) = \{v'\}$, $\Gamma(v) = \Gamma_2(v')$ and $\Gamma_2(v) = \Gamma(v')$, and also $|V\Gamma| = 2p + 2$. Further, by Lemma 4.4, Γ is antipodal and geodesic transitive.

Let $\mathcal{B} = \{\Delta_1, \Delta_2, \ldots, \Delta_{p+1}\}$ where $\Delta_i = \{u_i, u_i'\}$ such that $d_\Gamma(u_i, u_i') = 3$. Then each Δ_i is a block for $A := \mathrm{Aut}\Gamma$ of size 2 on $V\Gamma$. Further, for each $j \neq i$, u_i is adjacent to exactly one vertex of Δ_j, and u_i' is adjacent to the other. The quotient graph $\Sigma = \Gamma_\mathcal{B}$ is therefore a complete graph K_{p+1} and Γ is a cover of Σ. In particular, the map σ

such that $u_i^\sigma = u_i'$ and $u_i'^\sigma = u_i$ for all i is an automorphism of Γ of order 2, and fixes each of the Δ_i setwise.

We now determine the automorphism group A. By Lemma 4.2, $A_v \cong Z_p{:}Z_{\frac{p-1}{2}}$ is a Frobenius group, and so $|A| = |A_v|.|V\Gamma| = p(p+1)(p-1)$. Let K be the kernel of A acting on $\mathcal{B}$. Then A is an extension of K by the factor group $A^{\mathcal{B}}$. Since Γ is a cover of Σ, the kernel K is semiregular on $V\Gamma$, and hence has order at most 2. Since the involution σ defined above lies in K, it follows that $K \cong Z_2$. Thus $|A^{\mathcal{B}}| = |A/K| = \frac{p(p+1)(p-1)}{2}$.

Since Γ is arc transitive, the quotient graph $\Sigma = K_{p+1}$ is $A^{\mathcal{B}}$-arc transitive. Thus, $A^{\mathcal{B}}$ is 2-transitive on the vertex set $\mathcal{B}$, and the point stabiliser $(A^{\mathcal{B}})_{\Delta_1} = KA_{u_1}/K \cong A_{u_1} \cong Z_p{:}Z_{\frac{p-1}{2}}$ is a Frobenius group, so $A^{\mathcal{B}}$ is a Zassenhaus group. Since $|A^{\mathcal{B}}| = \frac{p(p+1)(p-1)}{2}$ and $A^{\mathcal{B}}$ is not 3-transitive on $\mathcal{B}$, by [8, Theorem 11.16], $A^{\mathcal{B}} \cong PSL(2,p)$. Therefore, we have

$$A = K.A^{\mathcal{B}} = Z_2.\mathrm{PSL}(2,p).$$

Suppose that the extension of Z_2 by $PSL(2,p)$ is non-split. Then $A = SL(2,p)$ has only one involution, which lies in the center of A. However, the stabiliser $(A^{\mathcal{B}})_{\Delta_1} \cong Z_p : Z_{\frac{p-1}{2}}$ is of even order and has trivial center, which is a contradiction. So the extension $K.A^{\mathcal{B}}$ is split, and $A \cong Z_2 \times PSL(2,p)$. It now follows from Theorem 1.2 (a) that $\Gamma \in \mathcal{C}(p)$.

Conversely, if Γ is 2-arc transitive, then it is 2-geodesic transitive. If $\Gamma \in \mathcal{C}(p)$, then by Theorem 1.2 (b), Γ is 2-geodesic transitive. $\Box$

References

[1] R. W. Baddeley, Two-arc transitive graphs and twisted wreath products. *J. Algebraic Combin.* **2** (1993), 215–237.

[2] N. L. Biggs, Algebraic Graph Theory, Cambridge University Press, New York, (1974).

[3] A. E. Brouwer, A. M. Cohen and A. Neumaier, Distance-Regular Graphs, Springer Verlag, Berlin, Heidelberg, New York, (1989).

[4] A. Devillers, W. Jin, C. H. Li and C. E. Praeger, Line graphs and geodesic transitivity, *Ars Math. Contemp.* **6** (2013), 13–20.

[5] J. D. Dixon and B. Mortimer, Permutation groups, Springer, New York, (1996).

[6] C. D. Godsil, R. A. Liebler and C. E. Praeger, Antipodal distance transitive covers of complete graphs, *European J. Combin.* **19** (1998), 455–478.

[7] C. D. Godsil and G. F. Royle, Algebraic Graph Theory, Springer, New York, Berlin, Heidelberg, (2001).

[8] B. Huppert and N. Blackburn, Finite Groups III, Springer, New York, Berlin, Heidelberg, (1982).

[9] A. A. Ivanov and C. E. Praeger, On finite affine 2-arc transitive graphs. *European J. Combin.* **14** (1993), 421–444.

[10] W. Jin, W. J. Liu and S. J. Xu, Finite 2-geodesic transitive graphs of valency 6, submitted.

[11] C. H. Li, Finite CI-groups are soluble, *Bull. London Math. Soc.* **31** (1999), 419–423.

[12] P. Lorimer, Vertex transitive graphs: symmetric graphs of prime valency, *J. Graph Theory* **8** (1984), 55–68.

[13] R. E. A. C. Paley, On orthogonal matrices, *J. Math. Phys.* **12** (1933), 311–320.

[14] C. E. Praeger, An O'Nan Scott theorem for finite quasiprimitive permutation groups and an application to 2-arc transitive graphs, *J. London Math. Soc.* **(2) 47** (1993), 227–239.

[15] D. E. Taylor, Two-graphs and doubly transitive groups, *J. Combin. Theory A* **61** (1992), 113–122.

[16] W. T. Tutte, A family of cubical graphs, *Proc. Cambridge Philos. Soc.* **43** (1947), 459–474.

[17] W. T. Tutte, On the symmetry of cubic graphs, *Canad. J. Math.* **11** (1959), 621–624.

[18] R. Weiss, s-transitive graphs, *Colloquia Mathematica Societatis Janos Bolyai, Algebraic methods in graph theory, szeged (Hungary)* **25** (1978), 827–847.

[19] R. Weiss, The non-existence of 8-transitive graphs, *Combinatorica* **1** (1981), 309–311.