

Regular graphs of odd degree are antimagic

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Abstract

An antimagic labeling of a graph G with m edges is a bijection from $E(G)$ to $\{1, 2, \dots, m\}$ such that for all vertices u and v , the sum of labels on edges incident to u differs from that for edges incident to v . Hartsfield and Ringel conjectured that every connected graph other than the single edge K_2 has an antimagic labeling. We prove this conjecture for regular graphs of odd degree.

1 Introduction

A *magic square* of order n is a $n \times n$ arrangement of the integers $\{1, 2, \dots, n^2\}$ so that the sums of the entries in each row, each column, and along the two main diagonals are equal. These squares were known to the Chinese as early as the fourth century B.C. and have been widely studied in recreational mathematics [4].

A *labeling* of a graph G with m edges is a bijection from $E(G)$ to $\{1, 2, \dots, m\}$. Given a labeling of a graph, the *vertex sum* at a vertex v is the sum of the labels on edges incident to v . A labeling is *magic* if all vertex sums are equal. Magic labelings take their name from their connection with magic squares, since a magic square of order n naturally gives rise to a magic labeling of the complete bipartite graph $K_{n,n}$ (vertices in one part correspond to rows of the square, and vertices in the other correspond to columns). Finally, a labeling of a graph is *antimagic* if all its vertex sums are different. We call a graph antimagic (magic) if it has an antimagic (magic) labeling.

It is easy to find many graphs that are not magic (for example, forests). However, graphs that are not antimagic are rare. In fact, Hartsfield and Ringel conjectured the following.

Conjecture 1 ([5]). *Every connected graph other than K_2 is antimagic.*

Hartsfield and Ringel also explicitly conjectured that all trees other than K_2 are antimagic. Both conjectures remain wide open; however, much progress has been made. The first major result on antimagic labelings was due to Alon, Kaplan, Lev, Roditty, and

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Yuster [1]. They showed that there exists a constant c such that if G is an n -vertex graph with minimum degree $\delta \geq c \log n$, then G is antimagic. This proof relies on a combination of combinatorial ideas, probabilistic tools, and methods from analytic number theory. They also proved that graphs with maximum degree $\Delta \geq n - 2$ are antimagic. Yilma [9] later extended this result to show that graphs with $\Delta \geq n - 3$ are antimagic. His proof finds a breadth-first spanning tree T rooted at a vertex of maximum degree; he labels all edges outside of T first, then uses the largest $n - 1$ labels on T to guarantee an antimagic labeling.

Hefetz [6] used algebraic tools to show that a graph is antimagic if it has 3^k vertices and a C_3 -factor. Hefetz, Saluz, and Tran [7] generalized this approach to show that a graph is antimagic if it has p^k vertices and a C_p -factor (where p is an odd prime). Cranston [2] used Hall's marriage theorem to show that regular bipartite graphs are antimagic. Liang and Zhu [8] labeled edges in order of decreasing distance from a central vertex (breaking ties carefully) to show that 3-regular graphs are antimagic.

Perhaps the most interesting result is that of Eccles [3], who recently improved on the work of Alon et.al. He showed that if a graph has no isolated edges or vertices and has average degree at least 4468, then it is antimagic. He conjectures that, under the same first condition, average degree at least $\sqrt{2}$ implies that a graph is antimagic. This much stronger conjecture immediately implies Conjecture 1, since a connected n -vertex graph has at least $n - 1$ edges, and so for $n \geq 4$ has average degree at least $2(n - 1)/n = 2 - 2/n > \sqrt{2}$.

In this note, we prove that every k -regular graph with k odd and $k \geq 3$ is antimagic.

2 Main Result

A *trail* is a walk in G that may reuse vertices but may not reuse edges; a trail is *open* if it starts and ends at distinct vertices, and is *even (odd)* if its length is even (odd). For a set of vertices U and a function σ we write $\sigma(U)$ to denote $\{\sigma(u) : u \in U\}$. For a subgraph or trail H , we write $d_H(v)$ for the degree of v in H . We begin with an easy decomposition result for bipartite graphs.

Helpful Lemma. *Let G be a bipartite graph with parts U and W . There exists a function $\sigma : U \rightarrow E(G)$ and a set $\mathcal{T} = \{T_1, T_2, \dots\}$ such that $\sigma(u)$ is incident to u for all $u \in U$ and \mathcal{T} is a collection of edge-disjoint open trails with at most one trail ending at each vertex and with $(\bigcup_{T \in \mathcal{T}} E(T)) \cap \sigma(U) = \emptyset$ and $\bigcup_{T \in \mathcal{T}} E(T) \cup \sigma(U) = E(G)$. In other words, we can partition $E(G)$ into \mathcal{T} and $\sigma(U)$.*

Proof. We first choose $\sigma(U)$ arbitrarily, and let $\widehat{E} = E(G) \setminus \sigma(U)$. We form a greedy trail decomposition of \widehat{E} as follows. Start at an arbitrary vertex and keep walking (using unused edges of \widehat{E}) as long as possible. When you reach a vertex with no unused edges, start a trail at another vertex. Repeat this process until all edges are used up. This gives a decomposition \mathcal{T} of \widehat{E} , but it might contain a closed trail.

Suppose that \mathcal{T} contains a closed trail T_1 . If any vertex v of T_1 has an open trail T_2 that ends at v , then we splice T_1 and T_2 together, by starting at v , following all the edges of T_1 , then following the edges of T_2 . If no vertex of T_1 is the endpoint of an open trail in \mathcal{T} , then

choose $u \in U \cap V(T_1)$ arbitrarily. Let w be a successor of u on T_1 and let v be such that $\sigma(u) = uv$. We redefine $\sigma(u) := uw$, and redefine $T_1 := T_1 - uw + uv$. Now T_1 is an open trail, since $d_{T_1}(w)$ is odd.

By repeating this process for each closed trail in \mathcal{T} , we reach a collection of open trails. If any vertex v is the endpoint of at least two open trails, then we merge them together, by walking along one to end at v , then walking along another starting from v . Merging two trails reduces the number of trails, and “opening up” a closed trail (as described above), does not increase this number. So iterating these merging and opening up steps gives the desired partition of $E(G)$ into \mathcal{T} and $\sigma(U)$. \square

Now we prove our main result. Our proof builds heavily on that of Liang and Zhu [8], who showed that 3-regular graphs are antimagic.

Main Theorem. *Every k -regular graph with k odd and $k \geq 3$ is antimagic.*

Proof. Suppose that G and H are both antimagic k -regular graphs and that $|E(G)| = m$. Given antimagic labelings for G and H , we get an antimagic labeling of $G \cup H$ by increasing the label on each edge of H by m . Thus, we need only consider connected graphs.

Choose an arbitrary vertex v^* and let V_i denote the set of vertices at distance exactly i from v^* ; let p be the furthest distance of a vertex from v^* . Let $G[V_i]$ denote the subgraph induced by V_i and $G[V_i, V_{i-1}]$ denote the induced bipartite subgraph with parts V_i and V_{i-1} .

For each i , we apply the Helpful Lemma to $G[V_i, V_{i-1}]$ with $U = V_i$ and $W = V_{i-1}$ to get a partition of $E(G[V_i, V_{i-1}])$ into an edge set $\sigma(V_i)$ and a collection of edge-disjoint open trails. Let $G_\sigma[V_i, V_{i-1}] = G[V_i, V_{i-1}] \setminus \sigma(V_i)$. Let $E_i = E(G[V_i])$, let $E'_i = E(G_\sigma[V_i, V_{i-1}])$, and let $E''_i = \sigma(V_i)$; note that E'_i and E''_i partition $E(G[V_i, V_{i-1}])$. Given a labeling f of the edges, we denote the total sum of labels on edges incident to vertex v by $t(v) = \sum_{e \in E(v)} f(e)$, where $E(v)$ denotes the set of edges incident to v . Similarly, we denote the partial sum at v (omitting the label on $\sigma(v)$) by $p(v) = \sum_{e \in E(v) \setminus \{\sigma(v)\}} f(e) = t(v) - f(\sigma(v))$.

We now outline the proof. We will label the edges in the order $E_p, E'_p, E''_p, \dots, E_1, E'_1, E''_1$, using the smallest unused labels on each edge set when we come to it. In other words, we use the $|E_p|$ smallest labels on E_p , the $|E'_p|$ next smallest labels on E'_p , the $|E''_p|$ next smallest labels after that on E''_p , etc. (Note that the labels assigned to each of these edge sets span an interval.) This label assignment immediately gives that if $i \geq j + 2$ and $u \in V_i$ and $w \in V_j$, then $t(u) < t(w)$ since G is regular and the edges incident to u have smaller labels than the edges incident to w . Thus, we need only ensure that $t(u) \neq t(w)$ when either (i) $u, w \in V_i$ or (ii) $u \in V_i$ and $w \in V_{i-1}$. We handle these two cases by specifying more precisely how to assign the label to each edge of these $3p$ edge sets.

We label the edges of each E_i arbitrarily from its assigned labels. We now specify how to label each E''_i ; in the process, we handle Case (i). Suppose that for some i , we have already labeled the edges of $E_p, E'_p, E''_p, \dots, E_i, E'_i$. As a result, $p(u)$ is already determined for each $u \in V_i$. We may name the vertices of V_i as u_1, u_2, u_3, \dots so that $p(u_1) \leq p(u_2) \leq p(u_3) \leq \dots$. Now we use the smallest label for E''_i on $\sigma(u_1)$, the next smallest on $\sigma(u_2)$, etc. This ensures that $t(u_j) < t(u_{j+1})$ for all $u_j \in V_i$.

Finally, we specify how to label each E'_i ; in the process, we handle Case (ii). That is, we ensure that if $u \in V_i$ and $w \in V_{i-1}$, then $t(u) \neq t(w)$. Let $\{s, s+1, \dots, \ell-1, \ell\}$ be the set of labels to be used on E'_i . Recall that G is k -regular for odd $k \geq 3$, and let $t = (k-1)/2$. We will ensure that $p(u) \leq t(s+\ell)$ and that $p(w) \geq t(s+\ell)$. Now since $f(\sigma(u)) < f(\sigma(w))$, we get that $t(u) < t(w)$. The details follow.

Let \mathcal{T} be the set of open trails partitioning E'_i (from the Helpful Lemma). Again, let $\{s, s+1, \dots, \ell-1, \ell\}$ be the labels assigned to E'_i . We label each trail so that every pair of successive labels (on a trail) incident to a vertex $u \in V_i$ has sum at most $s+\ell$ and each pair of successive labels incident to a vertex $w \in V_{i-1}$ has sum at least $s+\ell$. This ensures that $p(u) \leq t(s+\ell)$ and $p(w) \geq t(s+\ell)$.

We first label each even trail, then label the odd trails, taken together in pairs (possibly with a single odd trail last). Suppose that we have already labeled some even number $2r$ of edges in the set E'_i and the remaining labels available for this edge set are $\{s+r, s+r+1, \dots, \ell-r-1, \ell-r\}$. We have three possibilities. (1) Suppose first that $T \in \mathcal{T}$ is an even trail with both endpoints in V_{i-1} . We assign the labels: $s+r, \ell-r, s+r+1, \ell-r-1, \dots$ successively along the trail. Now every two successive edges incident to $u \in V_i$ have sum $s+\ell$ and every two successive edges incident to $w \in V_{i-1}$ have sum $s+\ell+1$. (2) Suppose instead that $T \in \mathcal{T}$ is an even trail with both endpoints in V_i . Now we assign the labels: $\ell-r, s+r, \ell-r-1, s+r+1, \dots$ successively along the trail. Now every two successive edges incident to $u \in V_i$ have sum $s+\ell-1$ and every two successive edges incident to $w \in V_{i-1}$ have sum $s+\ell$. (3) Finally, suppose that $T_1, T_2 \in \mathcal{T}$ are odd trails with lengths $2a+1$ and $2b+1$. Beginning at a vertex in V_i , we label the edges of T_1 with $\ell-r, s+r, \ell-r-1, \dots, s+r+a-1, \ell-r-a$. Here the successive pairs of labels incident to $u \in V_i$ sum to $s+\ell-1$ and the pairs incident to $w \in V_{i-1}$ sum to $s+\ell$. Finally, beginning at a vertex in V_{i-1} , we label the edges of T_2 with $s+r+a, \ell-r-a-1, s+r+a+1, \dots, s+r+a+b$. Again the successive pairs incident to $u \in V_i$ sum to $s+\ell-1$ and the successive pairs incident to $w \in V_{i-1}$ sum to $s+\ell$. If we have a single odd trail left at the end, we treat it like a trail of length $2a+1$ above.

All that remains is to verify that for $u \in V_i$ and $w \in V_{i-1}$ we have $p(u) \leq t(s+\ell)$ and $p(w) \geq t(s+\ell)$. We consider $p(u)$, and the analysis for $p(w)$ is nearly identical. Recall that $d_G(u) = k = 2t+1$. If $d_{E'_i}(u) = 2t$, then the desired inequality holds, since each of the t pairs of successive edges on trails through u have label sum at most $s+\ell$. If u is the end of some trail T in \mathcal{T} , then let e be the final edge of T incident to u ; note that $f(e) \leq \ell$. But now, we have $d_{E'_i}(u)$ is odd, so u has some incident edge (in fact, an odd number of them) in $E'_{i+1} \cup E''_{i+1} \cup E_i$; this edge has label less than s . Thus, the sum of this label and $f(e)$ is less than $s+\ell$. If u has additional incident edges in $E'_{i+1} \cup E''_{i+1} \cup E_i$, then each edge has label less than s ; thus, each pair of these edges has label sum less than $s+\ell$. So $p(u) \leq t(s+\ell)$, as desired. For each $w \in V_{i-1}$, the analysis to show that $p(w) \geq t(s+\ell)$ is nearly identical to that above; the only difference is that all edges incident to w that are not in E'_i are in $E''_i \cup E_{i-1} \cup E'_{i-1}$, so each such edge has label larger than ℓ . This completes the proof. \square

We remark in closing that the proof easily translates to an efficient (polynomial time) algorithm to find an antimagic labeling. We thank Mike Barrus for his careful reading of this manuscript and detailed feedback.

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