CONNECTEDNESS AND ISOMORPHISM PROPERTIES OF THE ZIG-ZAG PRODUCT OF GRAPHS

DANIELE D'ANGELI, ALFREDO DONNO, AND ECATERINA SAVA-HUSS

ABSTRACT. In this paper we investigate the connectedness and the isomorphism problems for zig-zag products of two graphs. A sufficient condition for the zig-zag product of two graphs to be connected is provided, reducing to the study of the connectedness property of a new graph which depends only on the second factor of the graph product. We show that, when the second factor is a cycle graph, the study of the isomorphism problem for the zig-zag product is equivalent to the study of the same problem for the associated pseudo-replacement graph. The latter is defined in a natural way, by a construction generalizing the classical replacement product, and its degree is smaller than the degree of the zig-zag product graph.

Two particular classes of products are studied in detail: the zig-zag product of a complete graph with a cycle graph, and the zig-zag product of a 4-regular graph with the cycle graph of length 4. Furthermore, an example coming from the theory of Schreier graphs associated with the action of self-similar groups is also considered: the graph products are completely determined and their spectral analysis is developed.

Mathematics Subject Classification (2010): 05C60, 05C76, 05C78.

1. Introduction

The fruitful idea of constructing new graphs starting from smaller factor graphs is very popular in Mathematics and it has been largely studied and developed in the literature for its theoretical interest, as well as for its numerous applications in several branches like Combinatorics, Probability, Theoretical Computer Science, Statistical Mechanics.

This paper is devoted to the study of the connectedness and isomorphism properties of zig-zag products of graphs. This combinatorial construction, which applies to regular graphs, was introduced in [25] by O. Reingold, S. Vadhan and A. Wigderson, in order to provide new sequences of constant degree expanders of arbitrary size. Informally, a graph is expander if it is simultaneously sparse, i.e., it has relatively few edges, and highly connected. What is mostly fascinating about expander graphs, is the fact that the expansion property can be described from several points of view - combinatorial, algebraic and probabilistic. Expander graphs have many interesting applications in different areas of Computer Science, such as design and analysis of communication networks and error

Date: September 24, 2018, preprint.

Key words and phrases. Zig-zag product, replacement product, parity block, parity block decomposition, connected component, pseudo-replacement, double cycle graph.

correcting codes, as well as in many computational problems, by playing a crucial role also in Statistical Physics, Computational Group Theory, and Optimization [20, 22].

The zig-zag product is strictly related to a simpler construction, called replacement product of graphs. The replacement and the zig-zag product play an important role in Geometric Group Theory, since it turns out that, when applied to Cayley graphs of two finite groups, they provide the Cayley graph of the semidirect product of these groups [2]. Further results about the relationship between graph products and group operations are given in [14].

The structure of the paper is as follows. In Section 2 we recall the definition and the basic properties of the replacement and zig-zag product of graphs. In Section 3, we attack the connectedness problem for zig-zag products of regular graphs. In Section 4, we focus our attention on the classification of the isomorphism classes of zig-zag products in the case where the second factor graph is a cycle graph of even length. In this context, we prove that the connected components of the zig-zag product are in one-to-one correspondence with the so-called parity blocks, introduced in Subsection 4.1. These are subgraphs of the first factor of the zig-zag product, considered together with the bi-labelling of its edges. The isomorphism problem is treated by associating with any parity block (and so with any connected component of the zig-zag product) a new simpler graph, that we call pseudo-replacement graph. The pseudo-replacement graph contains, in general, less vertices and edges, and has a smaller degree than the corresponding connected component of the zig-zag product. Nevertheless, it completely encodes the isomorphism properties of each connected component (see Subsection 4.2). In the case where the cycle graph has length 4, we show that the structure of the zig-zag product is very regular: it consists of highly symmetric graphs that we call double cycle graphs (see Subsection 5.2). In particular, this implies that the zig-zag product of graphs is not injective, as two non isomorphic graphs can produce isomorphic zig-zag products. In this setting, we are also able to perform a complete spectral analysis, by using the fact that the adjacency matrices of the double cycle graphs are circulant.

Interesting sequences of increasing regular graphs can be obtained by considering the Schreier graphs of the action of groups generated by finite automata. In Section 6, we describe an application of the zig-zag product to this setting. The class of automata groups became very popular after the introduction of the Grigorchuk group, that was the first example of a group with intermediate growth (see [16] for the definition and further references). Surprising deep connections between groups generated by automata, complex dynamics, fractal geometry have been discovered, and they constitute a very exciting topic of investigation in modern mathematics [23, 24]. In particular, sequences of finite Schreier graphs represent a discrete approximation of fractal limit objects associated with such groups. This point of view can also be exploited in the study of models coming from Statistical Mechanics [8, 9, 4, 15].

The main results achieved in the current paper can be summarized as follows.

- A sufficient condition for the connectedness of the zig-zag product $G_1 \boxtimes G_2$ is given in terms of the connectedness of a new graph \mathcal{N} , called the neighborhood graph of G_2 . The construction of \mathcal{N} depends only on the structure of G_2 and the number of vertices equals the number of vertices of G_2 (Theorem 3.1).
- There exists a one-to-one correspondence between the parity blocks in the parity block decomposition of G_1 , and the connected components of the graph $G_1 \boxtimes G_2$ (Theorem 4.1).
- There exists a one-to-one correspondence between the isomorphism classes of the connected components of the zig-zag product and the isomorphism classes of the corresponding pseudo-replacement graphs (Theorem 4.2).
- In the case $G_2 \simeq C_4$, the connected components of the zig-zag product are isomorphic to double cycle graphs DC_n , for some n (Proposition 5.3).
- If $\{\Gamma_n\}_{n\geq 1}$ is the sequence of Schreier graphs associated with the action of the Basilica group, then, for each $n\geq 1$, the graph $\Gamma_n(\mathbb{Z})C_4$ is connected and isomorphic to the double cycle graph $DC_{2^{n+1}}$ (Proposition 6.1); the spectral analysis of the graphs $\Gamma_n(\mathbb{Z})C_4$ is explicitly performed (Theorem 6.1).

2. Preliminaries

In this section we introduce the replacement and the zig-zag product of two regular graphs. For this, we recall first some basic definitions and properties of regular graphs, and we fix the notation for the rest of the paper.

Let G = (V, E) be a finite undirected graph, where V and E denote the vertex set and the edge set of G, respectively. In other words, the elements of the edge set E are unordered pairs of type $e = \{u, v\}$, with $u, v \in V$. If $e = \{u, v\} \in E$, we say that the vertices u and v are adjacent in G, and we use the notation $u \sim v$. We will also say that the edge e joins u and v. Loops and multi-edges are also allowed. A path in G is a sequence $\{u_0, u_1, \ldots, u_t\}$ of vertices of V such that $u_i \sim u_{i+1}$. The graph G is connected if, for every $u, v \in V$, there exists a path u_0, u_1, \ldots, u_t in G such that $u_0 = u$ and $u_t = v$. The degree of a vertex $v \in V$ of G is defined as $deg(v) = |\{e \in E : v \in e\}|$. We assume that a loop at the vertex v counts twice in the degree of v. We say that G is a regular graph of degree d, or a d-regular graph, if deg(v) = d for every $v \in V$.

Let |V| = n and denote by $A_G = (a_{u,v})_{u,v \in V}$ the adjacency matrix of G, that is, the square matrix of size n indexed by V, whose entry $a_{u,v}$ equals the number of edges joining u and v. Note that $a_{u,u} = 2k$ if there are k loops at the vertex u. As the graph G is undirected, A_G is a symmetric matrix, so that it admits n real eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. One has $deg(u) = \sum_{v \in V} a_{u,v}$: in particular, the d-regularity condition can be rewritten as $\sum_{v \in V} a_{u,v} = d$, for each $u \in V$. For a d-regular graph G, the normalized adjacency matrix is defined as $A'_G = \frac{1}{d}A_G$. It is known [5, 11] that, if G = (V, E) is a d-regular graph, with |V| = n, and A_G is its adjacency matrix, then d is an eigenvalue of A_G . Its multiplicity as an eigenvalue of A_G equals the number of connected components of G, and any other eigenvalue λ_i satisfies the condition $|\lambda_i| \leq d$, for each $i = 1, \ldots, n$.

2.1. Replacement product of graphs. The replacement product of two graphs is a simple and intuitive construction, which is well known in the literature, where it was often used in order to reduce the vertex degree without losing the connectivity property. It has been widely used in many areas including Combinatorics, Probability, Group theory, in the study of expander graphs and graph-based coding schemes [20, 21]. It is worth mentioning that Gromov studied the second eigenvalue of an iterated replacement product of a d-dimensional cube with a lower dimensional cube [18].

Let us introduce some notation. Let G = (V, E) be a finite connected d-regular graph (loops and multi-edges are allowed). Suppose that we have a set of d colors (labels), that we identify with the set of natural numbers $[d] := \{1, 2, \ldots, d\}$. We assume that, for each vertex $v \in V$, the edges incident to v are labelled by a color $h \in [d]$ near v, and that any two distinct edges issuing from v have a different color near v. A rotation map $\text{Rot}_G: V \times [d] \longrightarrow V \times [d]$ is defined by

$$Rot_G(v, h) = (w, k), \quad \forall v \in V, h \in [d],$$

if there exists an edge joining v and w in G, which is colored by the color h near v and by the color k near w. We may have $h \neq k$. Moreover, it follows from the definition that the composition $\text{Rot}_G \circ \text{Rot}_G$ is the identity map. Since an edge of G joining the vertices u and v is colored by some color h near u and by some color k near v, we will say that the graph G is bi-labelled.

Definition 2.1. Let $G_1 = (V_1, E_1)$ be a connected d_1 -regular graph, and let $G_2 = (V_2, E_2)$ be a connected d_2 -regular graph, satisfying the condition $|V_2| = d_1$. The replacement product $G_1(\widehat{\mathfrak{D}}G_2)$ is the regular graph of degree $d_2 + 1$ with vertex set $V_1 \times V_2$, that we can identify with the set $V_1 \times [d_1]$, and whose edges are described by the following rotation map:

$$Rot_{G_1(\widehat{\Gamma})G_2}((v,k),i) = \begin{cases} ((v,m),j) & \text{if } i \in [d_2] \text{ and } Rot_{G_2}(k,i) = (m,j) \\ (Rot_{G_1}(v,k),i) & \text{if } i = d_2 + 1, \end{cases}$$

for all $v \in V_1, k \in [d_1], i \in [d_2 + 1]$.

One can imagine that the vertex set of $G_1(\widehat{\Gamma})G_2$ is partitioned into clouds, which are indexed by the vertices of G_1 , where by definition the v-cloud, for $v \in V_1$, consists of vertices $(v,1),(v,2),\ldots,(v,d_1)$. Within this construction, the idea is to put a copy of G_2 around each vertex v of G_1 , while keeping edges of both G_1 and G_2 . Every vertex of $G_1(\widehat{\Gamma})G_2$ will be connected to its original neighbors within its cloud (by edges coming from G_2), but also to one vertex of a different cloud, according to the rotation map of G_1 . Note that the degree of $G_1(\widehat{\Gamma})G_2$ depends only on the degree of the second factor graph G_2 .

Remark 2.1. Notice that the definition of $G_1 \oplus G_2$ depends on the bi-labelling of G_1 . In general, there may exist two different bi-labellings of G_1 , such that the associated replacement products are non isomorphic graphs [1, Example 2.3].

2.2. **Zig-zag product of graphs.** The zig-zag product of two graphs was introduced in [25] as a construction which produces, starting from a large graph G_1 and a small graph G_2 , a new graph $G_1(\mathbb{Z})G_2$. This new graph inherits the size from the large graph G_1 , the degree from the small graph G_2 , and the expansion property from both graphs. The most important feature of the zig-zag product is that $G_1(\mathbb{Z})G_2$ is a good expander if both G_1 and G_2 are; see Reingold, Vadhan, Wigderson [25, Theorem 3.2]. There it is explicitly described how iteration of the zig-zag construction, together with the standard squaring, provides an infinite family of constant-degree expander graphs, starting from a particular graph representing the building block of this construction.

Definition 2.2. Let $G_1 = (V_1, E_1)$ be a connected d_1 -regular graph, and let $G_2 = (V_2, E_2)$ be a connected d_2 -regular graph such that $|V_2| = d_1$ (as usual, graphs are allowed to have loops or multi-edges). Let Rot_{G_1} (resp. Rot_{G_2}) be the rotation map of G_1 (resp. G_2). The zig-zag product $G_1 \odot G_2$ is a regular graph of degree d_2^2 with vertex set $V_1 \times V_2$, that we identify with the set $V_1 \times [d_1]$, and whose edges are described by the rotation map

$$Rot_{G_1(\mathbb{Z})G_2}((v,k),(i,j)) = ((w,l),(j',i')),$$

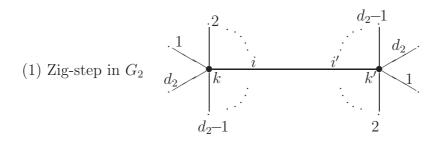
for all $v \in V_1, k \in [d_1], i, j \in [d_2], if:$

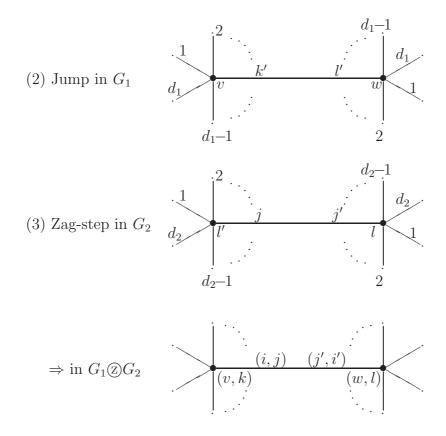
- (1) $Rot_{G_2}(k,i) = (k',i'),$
- (2) $Rot_{G_1}(v, k') = (w, l'),$
- (3) $Rot_{G_2}(l',j) = (l,j'),$

where $w \in V_1, l, k', l' \in [d_1]$ and $i', j' \in [d_2]$.

Observe that labels in $G_1 \boxtimes G_2$ are elements of $[d_2]^2$. As in the case of the replacement product, the vertex set of $G_1 \boxtimes G_2$ is partitioned into clouds, indexed by the vertices of G_1 . By definition the v-cloud consists of vertices $(v, 1), (v, 2), \ldots, (v, d_1)$, for every $v \in V_1$. Two vertices (v, k) and (w, l) of $G_1 \boxtimes G_2$ are adjacent in $G_1 \boxtimes G_2$ if it is possible to go from (v, k) to (w, l) by a sequence of three steps of the following form:

- (1) a first step "zig" within the initial cloud, from the vertex (v, k) to the vertex (v, k'), described by $Rot_{G_2}(k, i) = (k', i')$;
- (2) a second step jumping from the v-cloud to the w-cloud, from the vertex (v, k') to the vertex (w, l'), described by $Rot_{G_1}(v, k') = (w, l')$;
- (3) a third step "zag" within the new cloud, from the vertex (w, l') to the vertex (w, l), described by $Rot_{G_2}(l', j) = (l, j')$.





From the definition of the replacement and the zig-zag product it follows that the edges of $G_1 \boxtimes G_2$ arise from paths of length 3 in $G_1 \oplus G_2$ of type:

- (1) a first step within one cloud (the zig-step);
- (2) a second step which is a jump to a new cloud;
- (3) a third step within the new cloud (the zag-step).

In other words, $G_1 \otimes G_2$ is a regular subgraph of the graph obtained by taking the third power of $G_1(\widehat{r})G_2$. This fact can be explicitly expressed in terms of normalized adjacency matrices. More precisely, let A'_1 (resp. A'_2) be the normalized adjacency matrix of the graph G_1 (resp. G_2), and suppose that $|V_1| = n_1$. Then the normalized adjacency matrix of $G_1 \otimes G_2$ is $M \otimes = \widetilde{A}_2 \widetilde{A}_1 \widetilde{A}_2$ (see [25]), with $\widetilde{A}_2 = I_{n_1} \otimes A'_2$, where the symbol \otimes denotes the tensor product, or Kronecker product, and \widetilde{A}_1 is the permutation matrix on $V_1 \times [d_1]$ associated with the map Rot_{G_1} , i.e,

$$\widetilde{A}_{1(v,k),(w,l)} = \begin{cases} 1 & \text{if } \operatorname{Rot}_{G_1}(v,k) = (w,l) \\ 0 & \text{otherwise.} \end{cases}$$

The matrix \widetilde{A}_1 has exactly one entry 1 in each row and each column and 0's elsewhere. Note that \widetilde{A}_1 and \widetilde{A}_2 are both symmetric matrices, due to the undirectedness of G_1 and G_2 . On the other hand, it is easy to check that the normalized adjacency matrix of $G_1(\widehat{\Gamma})G_2$ is $M_{\widehat{\Gamma}} = \frac{\widetilde{A}_1 + d_2\widetilde{A}_2}{d_2 + 1}$, and that the following decomposition holds:

$$M_{(\underline{\Gamma})}^3 = \frac{d_2^2}{(d_2+1)^3} M_{(\underline{Z})} + \left(1 - \frac{d_2^2}{(d_2+1)^3}\right) C,$$

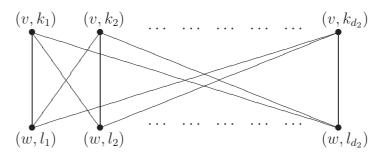
where C is the normalized adjacency matrix of a regular graph.

Remark 2.2. Note that the replacement product $G_1 \ \Box G_2$ and the zig-zag product $G_1 \ \Box G_2$ are defined for finite connected regular graphs G_1 and G_2 . It follows from the definition of replacement product that the graph $G_1 \ \Box G_2$ is connected; on the other hand, the connectedness of G_1 and G_2 does not ensure the connectedness of the graph $G_1 \ \Box G_2$. One of the goals of the current work is to investigate this property for the zig-zag construction.

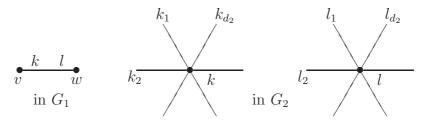
Recall that a complete bipartite graph G = (V, E) is a graph whose vertex set can be partitioned into two subsets U_1 and U_2 such that, for every two vertices $u_1 \in U_1$, $u_2 \in U_2$, one has $\{u_1, u_2\} \in E$, but there is no edge joining two vertices belonging to the same subset U_i . A complete bipartite graph is usually denoted by $K_{m,n}$, if $|U_1| = m$ and $|U_2| = n$.

The following basic result will be very useful for the rest of the paper. It shows that the graph $G_1 \boxtimes G_2$ consists of unions of special "elementary blocks", each isomorphic to a complete bipartite graph.

Lemma 2.1. Let G_1 be a d_1 -regular graph, and let G_2 be a d_2 -regular graph on d_1 vertices. Suppose that the vertices v and w are adjacent in G_1 , with $Rot_{G_1}(v,k) = (w,l)$, and $k,l \in [d_1]$. Let $\{k_1,\ldots,k_{d_2}\}$ be the set of vertices adjacent to k in G_2 ; similarly, let $\{l_1,\ldots,l_{d_2}\}$ be the set of vertices adjacent to l in G_2 . Then the edge connecting v and w in G_1 produces in $G_1 \odot G_2$ the following subgraph isomorphic to K_{d_2,d_2} :

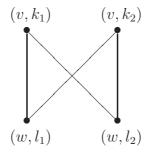


Proof. The hypothesis ensures that the graphs G_1 and G_2 contain the subgraphs depicted below.



Now it suffices to apply the definition of the zig-zag product in order to get the assertion.

Remark 2.3. In the case where G_2 is a 2-regular graph, for instance, if G_2 is a cycle graph, we call the graph $K_{2,2}$ the *papillon graph*. We will say that the vertices $(v, k_1), (v, k_2), (w, l_1), (w, l_2)$ form the papillon graph depicted below.



Example 2.1. Let $G_1 = (V_1, E_1)$ be the 3-dimensional Hamming cube, so that $V_1 = \{0, 1\}^3$ is the set of binary words of length 3, and two words $u = x_0x_1x_2$ and $v = y_0y_1y_2$ are adjacent if and only if $x_i = y_i$ for all but one index $i \in \{0, 1, 2\}$. Observe that G_1 can be interpreted as the Cayley graph of the group \mathbb{Z}_2^3 with respect to the generating set $\mathbf{E}_3 = \{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2\}$, where \mathbf{e}_i denotes the triple with 1 at the *i*-th coordinate and 0 elsewhere. Now let $G_2 = (V_2, E_2)$ be the cycle graph of length 3, which can be interpreted as the Cayley graph of the cyclic group $\mathbb{Z}_3 = \{0, 1, 2\}$, with respect to the generating set $\{\pm 1\}$ (see Figure 1).

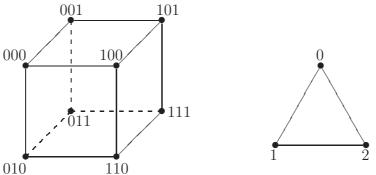


FIGURE 1. The graphs $Cay(\mathbb{Z}_2^3, \mathbb{E}_3)$ and $Cay(\mathbb{Z}_3, \{\pm 1\})$.

Having these interpretations in our mind, we label the edges of G_1 as follows: the edge connecting two vertices u and v is labelled by i both near u and v, if the corresponding words $u = x_0x_1x_2$ and $v = y_0y_1y_2$ differ in the i-th letter (this corresponds to moving by using the generator \mathbf{e}_i in the Cayley graph of \mathbb{Z}_2^3). Similarly, we label the edges of G_2 in such a way that $\text{Rot}_{G_2}(u, \pm 1) = (u \pm 1, \mp 1)$, where the integers $u, u \pm 1, \pm 1, \mp 1$ are taken modulo 3 (this corresponds to moving by using the generators ± 1 in the Cayley graph of \mathbb{Z}_3). In the replacement product, every vertex of \mathbb{Z}_2^3 is replaced by a cloud of 3 vertices representing a copy of \mathbb{Z}_3 . Moreover, each vertex of any cloud is connected to exactly one vertex of a neighboring cloud according with the following rule: the vertex (v,i) is connected to the vertex $(v+\mathbf{e}_i,i)$. The replacement product is depicted in Figure 2. It is worth mentioning that the replacement product $Cay(\mathbb{Z}_2^3, \mathbf{E}_3)$ $(\mathbf{C}ay(\mathbb{Z}_3, \{\pm 1\})$ is

isomorphic to the Cayley graph of the semidirect product $\mathbb{Z}_2^3 \times \mathbb{Z}_3$, with respect to the generating set $\{(000, \pm 1), (\mathbf{e}_0, 0)\}$ (see [13]).

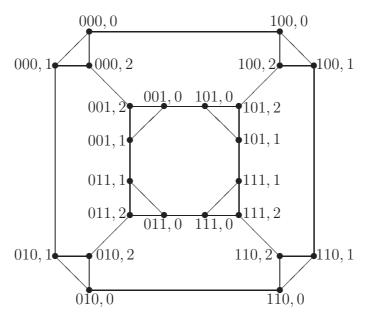


FIGURE 2. The graph $Cay(\mathbb{Z}_2^3, \mathbb{E}_3)$ (r) $Cay(\mathbb{Z}_3, \{\pm 1\})$.

The zig-zag product $G_1 \otimes G_2$ is represented in Figure 3. It is known that it can be regarded as the Cayley graph of the group $\mathbb{Z}_2^3 \rtimes \mathbb{Z}_3$, with respect to the generating set $\{(\mathbf{e}_1, 2), (\mathbf{e}_1, 0), (\mathbf{e}_2, 0), (\mathbf{e}_2, 1)\}$ (see [13]). Observe that one edge in this graph is obtained as a sequence of three steps in the graph $G_1(\widehat{\mathbf{r}})G_2$ in Figure 2: the first step within some initial cloud, the second step jumping to a new cloud and finally a third step within the new cloud. For instance, the three steps $(111, 1) \to (111, 2) \to (110, 2) \to (110, 0)$ produce the edge connecting the vertices (111, 1) and (110, 0) in $G_1(\widehat{\mathbf{z}})G_2$. The papillon subgraphs structure of $G_1(\widehat{\mathbf{z}})G_2$ is well-rendered in Figure 3.

3. Connectedness

In this section we discuss the connectedness problem of the zig-zag product of graphs. We will see (Example 5.3), that in general such graph product is not connected. The upcoming result relates the investigation of the connectedness properties of the graph $G_1 \boxtimes G_2$ to the study of a new graph that one constructs starting from G_2 , independently of G_1 .

Like before, let $G_1 = (V_1, E_1)$ be a d-regular connected graph and $G_2 = (V_2, E_2)$, with $d = |V_2|$. For any $h \in V_2$, we put $N_h = \{h' \in V_2 : h \sim h'\}$. Now we associate with G_2 a new graph $\mathcal{N} = (\mathcal{V}, \mathcal{E})$, called the *neighborhood graph* of G_2 .

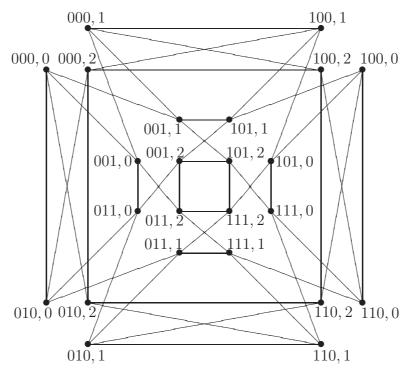


FIGURE 3. The graph $Cay(\mathbb{Z}_2^3, \mathbb{E}_3) \otimes Cay(\mathbb{Z}_3, \{\pm 1\})$.

The neighborhood graph $\mathcal{N} = (\mathcal{V}, \mathcal{E})$ of G_2 is defined by:

• $\mathcal{V} = \{N_h, h \in V_2\};$ • $\mathcal{E} = \{\{N_h, N_k\} : N_h \cap N_k \neq \emptyset, h \neq k\}.$

In other words, two vertices N_h , N_k of \mathcal{N} are adjacent if h and k have at least a common neighbor in G_2 . We do not put any label on the edges of \mathcal{N} .

Theorem 3.1. Let $G_1 = (V_1, E_1)$ be a d-regular graph and let $G_2 = (V_2, E_2)$, with $d = |V_2|$. If the neighborhood graph \mathcal{N} of G_2 is connected, then $G_1 \boxtimes G_2$ is connected as well.

Proof. It is enough to show that for all $h, h' \in V_2$ and $v \in V_1$, the vertices (v, h) and (v, h') are in the same connected component of $G_1 \boxtimes G_2$. In fact, if $v \neq v'$ are adjacent vertices in G_1 , there exist $k, k' \in V_2$ such that $\operatorname{Rot}_{G_1}(v, k) = (v', k')$. This implies that the vertices (v, k) and (v', k') are connected in $G_1 \boxtimes G_2$, for all $k \in V_2$ and $k' \in V_3$. Hence, if two vertices v and v are connected by a path in v, there exists a path in v connecting v, v and v are in the same connected component of v and v are in the same connected component of v and v are very $v \in V_1, h, h' \in V_2$, we get the assertion.

So, let us show now that (v, h) and (v, h') belong to the same connected component. Since \mathcal{N} is connected, there exists a sequence N_{h_1}, \ldots, N_{h_i} such that $h \in N_{h_1}, h' \in N_{h_i}$ and $N_{h_j} \cap N_{h_{j+1}} \neq \emptyset$, for $j = 1, \ldots, i-1$. Let $k_j \in N_{h_j} \cap N_{h_{j+1}}$. Notice that (v, h) and (v, k_1) are connected in $G_1 \boxtimes G_2$, since they have the common neighbor (w, s), where $\text{Rot}_{G_1}(v, h_1) = (w, s')$ and $s \in N_{s'}$. The same can be said for (v, k_1) and (v, k_2) , as they share a neighbor of the form (u, t) where $\text{Rot}_{G_1}(v, h_2) = (u, t')$ and $t \in N_{t'}$. By using the same argument, we can say that (v, k_{i-1}) and (v, h') are in the same connected component of $G_1 \boxtimes G_2$. This ensures that $(v, h), (v, k_1), \ldots, (v, k_{i-1}), (v, h')$ (and in particular (v, h) and (v, h')) are in the same connected component of $G_1 \boxtimes G_2$ and this concludes the proof.

From now on, the complete graph on n vertices will be denoted by K_n , and the cycle graph on n vertices will be denoted by C_n .

Corollary 3.1. Let G = (V, E) be a d-regular graph. Then for every $d \geq 3$, the graph $G \boxtimes K_d$ is connected. If $d \geq 3$ is odd, then the graph $G \boxtimes C_d$ is connected.

Proof. It suffices to observe that the neighborhood graph associated with K_d is isomorphic to K_d itself; similarly, it is straightforward to check that the neighborhood graph associated with C_d is isomorphic to C_d .

Remark 3.1. Notice that the condition of the previous theorem is not a necessary condition. In the case of the Schreier graphs $\{\Gamma_n\}_{n\geq 1}$ of the Basilica group discussed in Section 6, the zig-zag product $\Gamma_n \boxtimes C_4$ is connected, for every $n\geq 1$, even if the neighborhood graph \mathcal{N} associated with the cycle graph C_4 consists of two connected components.

4. Isomorphism properties

In what follows we focus our attention on the case when the factor graph G_2 in the zig-zag product $G_1 \boxtimes G_2$ is a cycle graph. This assumption allows us to give precise results about the structure of the connected components of $G_1 \boxtimes G_2$.

We have seen in Corollary 3.1 that if $d \geq 3$ is an odd integer, then the zig-zag product $G(\mathbb{Z})C_d$ is always connected, independently of the bi-labelling of the edges of G. For this reason, our analysis will be restricted to the case $G(\mathbb{Z})C_d$, with an even d.

- 4.1. **Parity blocks.** Let G = (V, E) be a d-regular bi-labelled graph, where d is an even natural number; recall that an edge joining two vertices u and w in G is colored by some color i near u and by some color j near w. As usual, we identify the set of colors with the set [d]. Let $[d_e]$ (resp. $[d_o]$) be the subset of [d] consisting of the even (resp. odd) numbers from 1 to d, so that $[d] = [d_e] \sqcup [d_o]$. Given $v \in V$, and chosen one of the sets $[d_i]$, $i \in \{e, o\}$, the parity block P(v, i) = (V(v, i), E(v, i)) is the subgraph of G defined as follows:
 - V(v,i) is the set of all vertices $w \in V$ with the property that there exists a path $\mathcal{P} = \{v = v_0, v_1, \dots, v_{n-1}, w = v_n\}$ in G such that the following parity properties are satisfied:

- (1) $\operatorname{Rot}_G(v_k, i_k) = (v_{k+1}, j_k)$, for $k = 0, \dots, n-1$;
- (2) $i_0 \in [d_i];$
- (3) $i_{k+1} \equiv j_k \pmod{2}$;
- E(v, i) consists of the edges joining two consecutive vertices v_k and v_{k+1} in \mathcal{P} , and bi-labelled according with the bi-labelling of G, described by the rotation map $\text{Rot}_G(v_k, i_k) = (v_{k+1}, j_k)$.

A vertex $w \in P(v, i)$ is said to be *even* or with parity e (resp. odd or with parity o) in P(v, i) if the path $\{v = v_0, v_1, \dots, v_{n-1}, w = v_n\}$ is such that $j_{n-1} \in [d_e]$ (resp. $j_{n-1} \in [d_o]$). If a vertex w is both even and odd, we will say that w is odden or with parity e - o. In other words, the vertex w is odden if and only if P(w, e) and P(w, o) coincide.

Since G is finite, G decomposes into a finite number of P(v, i)'s, in the sense that every edge in E belongs to some graph P(v, i) for some $v \in V$ and $i \in \{e, o\}$. We write $G = \bigcup_j P(v_j, i_j)$ where j runs over an opportune finite index set. Notice that a vertex which is either even or odd in a parity block has degree d/2 in that parity block, whereas an odden vertex has degree d in the parity block.

Lemma 4.1. Let P(v, i) be a parity block of G = (V, E). If w is a vertex in G with parity i_w in P(v, i), then $P(v, i) = P(w, i_w)$.

Proof. By definition w has parity i_w in P(v, i) if there exists a path $\{v = v_0, v_1, \ldots, v_{n-1}, w = v_n\}$ in P(v, i), with $\text{Rot}_G(v_k, i_k) = (v_{k+1}, j_k)$, such that $i_0 \in [d_i]$ and $j_{n-1} \in [d_{i_w}]$. In order to prove our statement, it is enough to show that $v \in P(w, i_w)$. Notice that the inverse path $\{w = w_0, w_1, \ldots, w_{n-1}, v = w_n\}$, with $w_i = v_{n-i}$, satisfies the parity conditions, with the property that $v \in P(w, i_w)$ with parity $i_v = i$, as $\text{Rot}_G(w_{n-1}, j_0) = (v, i_0)$.

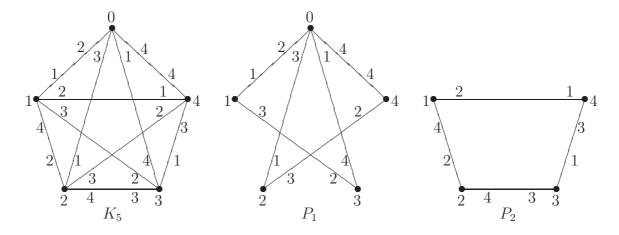
This result ensures that, given a bi-labelled graph G, a decomposition of G into parity blocks is uniquely determined, and we are allowed to use the notation $G = \cup_j P_j$, without explicitly expressing the dependence of the parity blocks on the particular vertices. Given a bi-labelled graph G, we will refer to its decomposition into parity blocks as its parity block decomposition. We will often write $|P_j|$ to denote the number of vertices belonging to the parity block P_j .

Example 4.1. Consider the graph K_5 endowed with the bi-labelling in the figure below. Its parity block decomposition consists of two parity blocks P_1 and P_2 , with

$$P_1 = P(0, e) = P(0, o) = P(1, o) = P(2, o) = P(3, e) = P(4, e)$$

 $P_2 = P(1, e) = P(2, e) = P(3, o) = P(4, o),$

and so $K_5 = P_1 \cup P_2$. Observe that the vertex 0 is odden, the vertices 1, 2 are odd, and the vertices 3, 4 are even in P_1 ; the vertices 1 and 2 are even and the vertices 3 and 4 are odd in P_2 .



In what follows, we will use the convention that the sum of two elements $i, j \in [d]$ is given by

(1)
$$i + j = \begin{cases} i + j, & \text{if } i + j \le d; \\ i + j - d, & \text{if } i + j > d. \end{cases}$$

The interest in the parity block decomposition is justified by the following result.

Theorem 4.1. Let G = (V, E) be a regular graph of even degree $d \ge 3$, with parity block decomposition $G = \bigcup_{j \in J} P_j$. Let C_d be the cycle graph of length d, with vertex set [d]. Then there is a one-to-one correspondence between the set of parity blocks $\{P_j\}_{j \in J}$ and the set of connected components $\{S_i\}_{i \in J}$ of the zig-zag product $G (C_d)$.

Proof. Let P be a parity block in the parity block decomposition of G and let $v \in P$ with parity i, where $i \in \{e, o\}$. Notice that P contains all edges issuing from v and labelled by $j \in [d_i]$ near v. Given $j \in [d_i]$, let $w \in P$ with $Rot_G(v, j) = (w, h)$. Lemma 2.1 implies that the vertices (v, j - 1), (v, j + 1) and (w, h - 1), (w, h + 1) form a papillon subgraph in $G \otimes C_d$, so that these vertices belong to the same connected component of $G \otimes C_d$.

Moreover, one has that all vertices of the type (v, k), with $k \in [d] \setminus [d_i]$, belong to the same connected component in $G \boxtimes C_d$. To see that, it is enough to observe that (v, k) belongs exactly to two papillon subgraphs, which connect (v, k) to (v, k - 2) and to (v, k + 2): by iteration, this implies that there is a path connecting all (v, k)'s with $k \in [d] \setminus [d_i]$. Hence, we have shown that, if there exists in P a path from v (with parity i) to w (with parity j not necessarily different from i), then the vertices (v, i'), for every $i' \in [d] \setminus [d_i]$, and (w, j'), for every $j' \in [d] \setminus [d_j]$, belong to the same connected component of $G \boxtimes C_d$. In order to complete the proof, we show that, if v has parity i in P_i and w has parity j in P_j , and $P_i \neq P_j$, then the vertices of type (v, i'), $i' \in [d] \setminus [d_i]$ and (w, j'), $j' \in [d] \setminus [d_j]$ belong to distinct connected components S_i and S_j of $G \boxtimes C_d$. If this is not the case, so that (v, i') and (w, j') are in the same connected component, then there is a path $P = \{(v, i') = (v_0, i'_0), (v_1, i'_1), \dots, (w, j') = (v_n, i'_n)\}$ in $G \boxtimes C_d$. The vertices (v_k, i'_k) and (v_{k+1}, i'_{k+1}) are connected if and only if they belong to a papillon graph, corresponding to the edge $\{v_k, v_{k+1}\}$ in G, such that $Rot_G(v_k, i_k) = (v_{k+1}, j_k)$ with $i_k \in \{i'_k \pm 1\}$ and

 $j_k \in \{i'_{k+1} \pm 1\}$. We want to show that the path \mathcal{P} in $G \supseteq C_d$ can be projected onto G, giving rise to a path connecting v and w satisfying the parity properties: this will give a contradiction. In other words, we claim that for each $k = 0, 1, \ldots, n$, the vertex v_k has parity t_k , with $i'_k \in [d] \setminus [d_{t_k}]$, in the parity block P_i . This follows by observing that any $(v_k, i'_k) \in \mathcal{P}$ exactly belongs to two papillon subgraphs, containing the vertices $(v_k, i'_k), (v_k, i'_{k-2})$ and $(v_k, i'_k), (v_k, i'_{k-2})$, respectively. The corresponding edges $\{v_{k-1}, v_k\}$ and $\{v_k, v_{k+1}\}$ in G satisfy $\text{Rot}_G(v_{k-1}, i_{k-1}) = (v_k, j_{k-1})$ and $\text{Rot}_G(v_k, i_k) = (v_{k+1}, j_k)$. Since $i_k, j_{k-1} \in \{i'_k \pm 1\}$, we have $j_{k-1} \equiv i_k \pmod{2}$ and the proof is completed. \square

Remark 4.1. We have shown in the proof of Theorem 4.1 that, if a vertex v in a parity block P is even (resp. odd), so that the elements in $[d_e]$ (resp. $[d_o]$) label the edges issuing from that vertex in P, then in the connected component of $G \boxtimes C_d$ associated with that parity block we find all vertices (v, j), with $j \in [d_o]$ (resp. $[d_e]$). Analogously, if a vertex v in a parity block P is odden, then in the connected component of $G \boxtimes C_d$ associated with that parity block we find the vertices (v, j), with j belonging to both the sets $[d_o]$ and $[d_e]$.

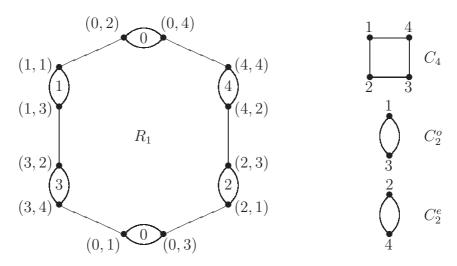
4.2. Isomorphism classification via pseudo-replacement graphs. Once we have described a method for distinguishing the connected components of the zig-zag product, it is straightforward to investigate the isomorphism classes problem. As before, G is a regular graph of even degree: in order to simplify the exposition, we put the degree of G equal to 2d, where d is a positive integer.

We introduce now new graphs, called the *pseudo-replacement* graphs, which are in a one-to-one correspondence with the parity blocks of a parity block decomposition of G (and so with the connected components of $G \otimes C_{2d}$ by virtue of Theorem 4.1). We will see that the isomorphism problem for the connected components of $G \otimes C_{2d}$ is equivalent to the analogous problem for such graphs.

Let P be a parity block in the block decomposition of G. The *pseudo-replacement graph* R associated with P in $G \odot C_{2d}$ is the graph obtained as described below. We distinguish two cases.

- (1) Suppose that each vertex $v_i \in P$ is either even or odd, so that the degree of v_i in P is d. Let C_d^e (resp. C_d^o) be the cycle graph of length d with vertex set $[(2d)_e]$ (resp. $[(2d)_o]$). Then the pseudo-replacement graph associated with the parity block P is the graph R in which every even (resp. odd) vertex v_i is replaced by a copy of the graph C_d^e (resp. C_d^o): this implies that R has d|P| vertices. The vertex j of C_d^e (resp. C_d^o) belonging to the copy associated with v_i will be denoted by (v_i, j) ; therefore, the vertex (v_i, j) is joint to (v_k, h) in R if either $\text{Rot}_G(v_i, j) = (v_k, h)$ or i = k and $h = j \pm 2$.
- (2) If v is odden, one associates with v two vertices v_e , v_o and two disjoint copies C_d^e and C_d^o , and for each of them we proceed as in the case (1).

Example 4.2. The picture below represents the pseudo-replacement graph R_1 corresponding to the parity block P_1 of Example 4.1. Recall that the vertex 0 is odden; the vertices 1, 2 are odd; the vertices 3, 4 are even.



Remark 4.2. The name pseudo-replacement is justified by the fact that, if P contains no odden vertex (so that P is a d-regular graph) and C' is a graph isomorphic to C_d^e (or C_d^o), then R is isomorphic to $P(\widehat{\mathbf{r}})C'$. Notice that the number of vertices of R is

 $2d \cdot |\{\text{odden vertices in } P\}| + d \cdot |\{\text{non odden vertices in } P\}|.$

The next result shows that it is enough to consider the graphs R in order to study the isomorphism classes of the connected components S of the zig-zag product.

Theorem 4.2. Let G = (V, E) be a 2d-regular graph and C_{2d} be the cycle graph of length 2d. Let $G = \bigcup_{k \in J} P_k$ be the parity block decomposition of G. For each $k \in J$, let S_k be the connected component of $G \boxtimes C_{2d}$ associated with P_k , and let R_k be the pseudo-replacement graph associated with P_k . Then $S_k \simeq S_{k'}$ if and only if $R_k \simeq R_{k'}$.

Proof. Let $\phi: S_k \to S_{k'}$ be an isomorphism. By Lemma 2.1, S_k is the union of a finite number of papillon graphs. In particular, let (v,i), (v,i+2) and (w,j), (w,j+2) be joint in a papillon graph, corresponding to an edge connecting v and w in P_k , such that $Rot_G(v, i+1) = (w, j+1)$. The image of such a papillon graph under ϕ will consist of the four vertices (v',i'),(v',i'+2) and (w',j'),(w',j'+2). Let $\psi:R_k\to R_{k'}$ such that $\psi(v,i+1)=(v',i'+1)$. Let us show that the map ψ is indeed a bijection. The injectivity follows from the fact that a pair of vertices (v,i), (v,i+2) in a papillon graph uniquely determines the vertex (v, i+1) in R_k . The map ψ is surjective by the hypothesis on ϕ . Hence, it is enough to prove that ψ preserves adjacency. We have that (v, i+1)and (w, j + 1) are joint by an edge in R_k if and only if the vertices v, w in P_k satisfy $Rot_G(v, i+1) = (w, j+1)$. By Lemma 2.1, this is equivalent to saying that there is a papillon graph in S_k containing (v,i), (v,i+2) and (w,j), (w,j+2). Since ϕ is an isomorphism, this is true if and only if there is a papillon graph consisting of vertices (v',i'),(v',i'+2) and (w',j'),(w',j'+2) in $S_{k'}$. This fact is equivalent to saying that in $R_{k'}$ there is an edge connecting (v', i'+1) and (w', j'+1). We have proven in this way that ψ preserves the adjacency.

Conversely, suppose that there exists an isomorphism $\psi: R_k \to R_{k'}$ and let $v \in P_k$, $i \in [2d]$ such that $\psi(v, i + 1) = (v', i' + 1)$, for some $v' \in P_{k'}$, $i' \in [2d]$. Notice that the

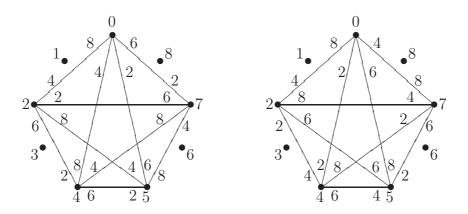
vertices (v, i+1) and (v', i'+1) univocally correspond to the sets of vertices (v, i), (v, i+2) in S_k and (v', i'), (v', i'+2) in $S_{k'}$, and each of these pairs of vertices univocally determine two papillon graphs in S_k and $S_{k'}$, respectively. Since ψ is an isomorphism, one has $\psi(v, i+3) \sim \psi(v, i+1)$ and $\psi(v, i-1) \sim \psi(v, i+1)$, as $(v, i+3) \sim (v, i+1)$ and $(v, i-1) \sim (v, i+1)$ in R_k . We define a map $\phi: S_k \to S_{k'}$ such that

$$\phi(v,i) = \begin{cases} (v',i') & \text{if } \psi(v,i+3) = (v',i'+3) \\ (v',i'+2) & \text{if } \psi(v,i+3) = (v',i'-1). \end{cases}$$

As before, the bijectivity of ψ ensures the bijectivity of ϕ . Let us show now that ϕ preserves the adjacency relation. We have that (v,i) and (w,j) are adjacent in S_k if and only if there exists in S_k a papillon graph containing (v,i), (v,i*2) and (w,j), (w,j*2) for some $*, * \in \{+, -\}$. Suppose, without loss of generality, that this papillon graph consists of vertices (v,i), (v,i+2) and (w,j), (w,j+2). This is equivalent to stating that there is an edge joining (v,i+1) and (w,j+1) in R_k ; since ψ is an isomorphism, this implies that there is an edge joining (v',i'+1) and (w',j'+1) in $R_{k'}$. Therefore, in the corresponding papillon graph of the connected component $S_{k'}$ associated with $R_{k'}$, there exists an edge joining (v',i') and (w',j'). This gives the assertion.

We have proven in Theorem 4.2 that the isomorphism classes of the connected components $\{S_i\}$ of $G \otimes C_{2d}$ are characterized by the isomorphism classes of the replacement graphs $\{R_i\}$. On the other hand, it is not true, in general, that the isomorphism classes of the connected components $\{S_i\}$ of the zig-zag product $G \otimes C_{2d}$ are characterized by the isomorphism classes of the parity blocks $\{P_i\}$, regarded as non bi-labelled graphs, in the parity block decomposition of G.

Example 4.3. The following parity blocks in K_9 are isomorphic as non-labelled graphs; however, one can show that the associated pseudo-replacement graphs are not isomorphic.



On the other hand, in the particular case d=4, one has that two pseudo-replacement graphs are isomorphic if and only if they are associated with two parity blocks of the same size, as the following corollary shows.

Corollary 4.1. Let G be a 4-regular graph, with parity block decomposition $G = \bigcup_{j \in J} P_j$, and let $\{S_j\}_{j \in J}$ be the set of connected components of $G(\mathbb{Z})C_4$. If $|P_i| = |P_j|$ then $S_i \simeq S_j$.

Proof. Consider the vertex v in P(v,i), with $i \in \{e,o\}$. Since the degree of G is 4, the set $[4_i]$ contains only two indices, let they be j_1, j_2 (actually, one has $[4_e] = \{2, 4\}$ and $[4_o] = \{1,3\}$) satisfying $Rot_G(v,j_1) = (w,h)$ and $Rot_G(v,j_2) = (u,k)$, for some $w, u \in P(v, i)$ and $h, k \in [4]$. Therefore, the path $\{u, v, w\}$ is contained in the parity block P(v,i). The same argument applies to any other vertex in the same block, and we can construct in this way an ordered sequence of vertices v_1, v_2, \ldots of P(v, i). Since G is finite, there exists n such that $v_{1+i} = v_{n+i}$ for all $i \geq 0$. In fact, if this is not the case, there exists j > 1 such that $v_{j+i} = v_{n+i}$ for each $i \geq 0$, which is absurd since v_j would have degree 3, but the vertices of P(v,i) must have degree 2 or 4. Observe that, if v_k is an odden vertex, then it occurs twice in the sequence. On the other hand, if v_k is either odd or even, then it appears only once. Hence, any parity block P is determined by a sequence of vertices $\{v=v_1,\ldots,v_n=v\}$ in G. In particular, R is constituted by an alternate sequence of simple edges and cycles of length 2 (as in the figure of Example 4.2), whose number is equal to twice the number of odden vertices plus the number of non odden vertices. As a consequence, the structure of R depends only on the size of P, and this concludes the proof.

Note that the sequence of vertices $\{v_1 = v, v_2, \ldots, v_n = v\}$ constitutes an Eulerian circuit in the parity block P(v, i), satisfying the further property that $\text{Rot}_G(v_i, h_i) = (v_{i+1}, k_i)$, with $k_i \equiv h_{i+1} \pmod{2}$, for every $i = 1, \ldots, n-2$, and with $k_{n-1} \equiv h_1 \pmod{2}$. Moreover, a vertex which is either odd or even is visited once by the cycle, whereas an odden vertex is visited twice. We will call such a circuit a spanning path of the parity block P(v, i).

Remark 4.3. It is known that every connected 2d-regular graph G = (V, E) admits an Eulerian circuit; this ensures that, given a connected 2d-regular graph G, there exists at least a bi-labelling of the edges of G such that the graph $G \otimes C_{2d}$ is connected. We will see an explicit application of this property in Proposition 5.2, where G is the complete graph.

4.3. Explicit description of isomorphisms of pseudo-replacement graphs. In this section we describe the isomorphisms between pseudo-replacement graphs associated with two isomorphic (as non-labelled graphs) parity blocks. We have seen in Example 4.3 that two parity blocks which are isomorphic as non-labelled graphs can be associated with two non isomorphic pseudo-replacement graphs (and so with non isomorphic connected components). Therefore, in what follows, given a 2d-regular graph G, and considering the zig-zag product $G \supseteq C_{2d}$, we focus on a parity block P, and the natural question arising in this context is the following. Given two distinct bi-labellings $\mathcal{L}, \mathcal{L}'$ of P (and the relative pseudo-replacement graphs R, R' associated with P), under which conditions on \mathcal{L} and \mathcal{L}' are the corresponding connected components S and S' of $G \supseteq C_{2d}$ isomorphic?

In order to answer this question we need to fix some notations. In what follows, $D_d = \langle a, b | a^d = b^2 = 1 \rangle$ denotes the dihedral group with 2d elements. Notice that D_d can

be identified with the automorphism group of the cycle graphs C_d^e and C_d^o , with vertices given by $[(2d)_e]$ or $[(2d)_o]$, respectively. Observe that there exists a natural bijection $\theta: [(2d)_o] \to [(2d)_e]$ defined by $\theta(i) = i + 1$, for each $i \in [(2d)_o]$. Given an automorphism $\phi \in D_d \simeq Aut(C_d^e) \simeq Aut(C_d^o)$, we put

$$\overline{\phi}(i) = \begin{cases} \phi(\theta(i)), & \text{if } i \in C_d^o \\ \phi(\theta^{-1}(i)), & \text{if } i \in C_d^e. \end{cases}$$

We recall that if w is an odden vertex in a parity block P, so that its degree is 2d in P, then in the corresponding pseudo-replacement R, w is replaced by two subgraphs isomorphic to C_d^o and C_d^e , respectively. The vertices of C_d^o and C_d^e are denoted by (w, i) and (w, j), with $i \in [(2d)_o]$ and $j \in [(2d)_e]$, respectively.

Let V and W be the subsets of vertices of P having degree d and 2d, respectively. Any automorphism f of P bijectively maps V into V, and W into W. For each vertex $w \in W$, let us introduce a permutation $\varepsilon_w \in Sym(\{e,o\})$ that will enable us to take into account the possibility of switching the subgraphs C_d^e and C_d^o associated with an odden vertex. Now for any $v \in V$ with parity i, let $g_v \in Aut(C_d^i)$, and for any $w \in W$ let $g_w^e \in Aut(C_d^e)$ and $g_w^o \in Aut(C_d^o)$. Finally, for each $w \in W$, we put:

$$\widetilde{g}_{w}^{\varepsilon_{w}(i)}(h) = \begin{cases} g_{w}^{i}(h), & \text{if } h \text{ has parity } i \text{ and } \varepsilon_{w} = id, \\ \\ \overline{g_{w}^{\varepsilon_{w}(i)}}(h), & \text{if } h \text{ has parity } i \text{ and } \varepsilon_{w} \neq id. \end{cases}$$

Theorem 4.3. Let G be a 2d-regular graph. With the above notations, let us define the map $F: R \to R'$ as

$$F(u,k) = \begin{cases} (f(u), g_u(k)), & \text{if } u \in V, u \text{ and } f(u) \text{ have parity } i; \\ (f(u), \overline{g_u}(k)), & \text{if } u \in V, u \text{ and } f(u) \text{ have different parities} \end{cases}$$

and

$$F(u,k) = (f(u), \widetilde{g}_u^{\varepsilon_u(i)}(k)), \quad if \ u \in W \ and \ k \in [(2d)_i].$$

If the conditions

- (1) $\operatorname{Rot}_G(u,h) = (v,k) \implies \operatorname{Rot}_G(f(u),g_u(h)) = (f(v),g_v(k)), \quad \text{if } u,v \in V;$
- (2) $\operatorname{Rot}_{G}(u,h) = (v,k) \Longrightarrow \operatorname{Rot}_{G}(f(u), \widetilde{g}_{u}^{\varepsilon_{u}(i)}(h)) = (f(v), g_{v}(k))$ $if \ u \in W, v \in V, h \in [(2d)_{i}];$
- (3) $\operatorname{Rot}_{G}(u,h) = (v,k) \Longrightarrow \operatorname{Rot}_{G}(f(u),\widetilde{g}_{u}^{\varepsilon_{u}(i)}(h)) = (f(v),\widetilde{g}_{v}^{\varepsilon_{v}(j)}(k))$ $if \ u,v \in W, h \in [(2d)_{i}], k \in [(2d)_{j}]$

hold, then F is an isomorphism between R and R'.

Proof. It is clear that, once fixed $f \in Aut(P)$, together with the automorphisms g_u^i for $u \in W$, and the automorphisms g_u for $u \in V$, the map F is a bijection between the set of vertices of R and R'.

Let us prove that F preserves the adjacency relations. Let $u \in V$ and suppose, without loss of generality, that u is even. In particular, since f is an automorphism of P, one has $f(u) \in V$. We can suppose that also f(u) is even. Take a pair of adjacent vertices in R. We have two possibilities. The first one is that the vertices have the form (u, h) and $(u, h \pm 2)$: therefore $(f(u), g_u(h))$ and $(f(u), g_u(h \pm 2)) = (f(u), g_u(h) \pm 2)$ are adjacent vertices in R'. The second possibility is that the adjacent vertices in R are (v, h) and (\widetilde{v}, h) , with $v \neq \widetilde{v}$, and $v, \widetilde{v} \in V$, so that the vertices f(v) and $f(\widetilde{v})$ are adjacent in P, with $f(v) \neq f(\widetilde{v})$. Moreover, by the definition of R, it must be that $Rot_G(v, h) = (\widetilde{v}, h)$. Condition (1) implies that $Rot_G(f(v), g_v(h)) = (f(\widetilde{v}), g_{\widetilde{v}}(h))$, and so F(v, h) and $F(\widetilde{v}, h)$ are adjacent in R'.

Consider now the case when the adjacent vertices in R have the form (w, h) and (v, k), with $w \in W$ and $v \in V$. By the construction of R, it must be $Rot_G(w, h) = (v, k)$, with $h \in [(2d)_i]$, for some $i \in \{e, o\}$. Moreover the definition of F gives $F(w, h) = (f(w), \tilde{g}_w^{\varepsilon_w(i)}(h))$ and $F(v, k) = (f(v), g_v(k))$. By condition (2), the vertices F(w, h) and F(v, k) are adjacent in R', and so F preserves the adjacency relation also in this case.

The case of two adjacent vertices (w, h) and (w', k) in R, with $w, w' \in W$, can be similarly discussed by using condition (3), and this completes the proof.

5. Special cases

For a better understanding of the results on zig-zag product $G_1(\mathbb{Z})G_2$ obtained in the previous sections, we consider here two particular cases.

- (1) $G_1 = K_{2d+1}$ and $G_2 = C_{2d}$.
- (2) $G_2 = C_4$.

5.1. The case of the complete graph. Theorem 4.1 shows that, fixing a bi-labelling of K_{2d+1} , the connected components of $K_{2d+1} extbf{\omega} C_{2d}$ are in bijection with the set of parity blocks $\{P_j\}$ in the parity block decomposition of K_{2d+1} . Here, one has a constraint on the size of the parity blocks. First of all, recall that if P is a parity block of the graph K_{2d+1} , then all its vertices have degree either d or 2d. The following proposition holds.

Proposition 5.1. Consider the complete graph K_{2d+1} on 2d+1 vertices, endowed with a bi-labelling of its edges, and let P be a parity block in the parity block decomposition of K_{2d+1} . Let p be the number of vertices in P, and let i be the number of vertices of degree 2d in P. Then:

• if i > 0, then p = 2d + 1 and i satisfies

$$(i-1)(d-1) \equiv 0 \pmod{2};$$

• if i = 0, then p satisfies

$$p \ge d+1$$
 and $pd \equiv 0 \pmod{2}$.

Viceversa, given a positive integer i satisfying $(1-i)(d-1) \equiv 0 \pmod{2}$, there exists a bi-labelling of the edges of K_{2d+1} such that the associated parity block decomposition contains a parity block of 2d+1 vertices, with i vertices of degree 2d; similarly, given an integer $p \geq d+1$ satisfying $pd \equiv 0 \pmod{2}$, there exists a bi-labelling of the edges of K_{2d+1} such that the associated parity block decomposition contains a parity block of p vertices, all having degree d.

Proof. Let i > 0 be the number of vertices of degree 2d in P. We must have p = 2d + 1, so that there are p - i = 2d + 1 - i vertices of degree d in P. We want to establish for which value of i such a configuration is allowed. First of all, it is easy to check that it must be either $i \le d$ or i = p = 2d + 1. In fact, if i > d, then necessarily it must be i = 2d + 1 (since the remaining 2d + 1 - i vertices of P would have degree at least d + 1). The configuration i = 2d + 1 is allowed, as shown in Example 5.1 in the case of K_5 .

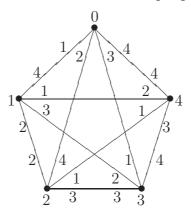
Let us restrict now to the case $0 < i \le d$. The problem of establishing for which values of i this situation can occur, is equivalent to the problem of the existence of a (d-i)-regular graph on 2d+1-i vertices. On the other hand, it is known that there exists a k-regular graph on m vertices if and only if $m \ge k+1$ and mk is an even integer [27, Exercise 8.8]. In our case, these conditions become $2d-i+1 \ge d-i+1$ (always verified) and $(2d+1-i)(d-i) \equiv 0 \pmod{2}$. We can summarize what we said above, by stating that a parity block of the complete graph K_{2d+1} has i vertices of degree 2d, and 2d+1-i vertices of degree d, with $0 < i \le d$, if and only if the integer i satisfies the condition

$$(1-i)(d-i) \equiv 0 \pmod{2}.$$

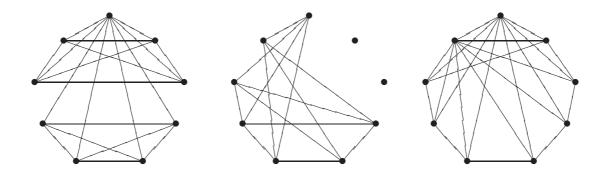
Now consider the case i = 0 and let p be the number of vertices of P. Observe that in this case, the problem of establishing which values of p are allowed is equivalent to the problem of the existence of a d-regular graph on p vertices. The argument above says that such a configuration is possible if and only if

$$p \ge d+1$$
 and $pd \equiv 0 \pmod{2}$.

Example 5.1. In the picture below the graph K_5 has a bi-labelling such that its parity decomposition consists of only one parity block: therefore, we have in this case d = 2 and p = i = 5. In other words, all the vertices of the unique parity block are odden.



The following pictures represent: the case d = 4, i = 1, p = 9; the case d = 4, i = 0, p = 7; the case d = 4, i = 2, p = 9.



We recall now the classical definition and the basic properties of circulant and block circulant matrices, which will play a special role in the description of the adjacency matrix of the graphs in Proposition 5.2 and in the investigation of the spectral properties of a particular sequence of zig-zag product graphs studied in Section 6.

Definition 5.1. A circulant matrix C of size n is a square matrix of the form

(2)
$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ c_2 & c_3 & \cdots & c_{n-1} & c_0 & c_1 \\ c_1 & c_2 & c_3 & \cdots & c_{n-1} & c_0 \end{pmatrix},$$

with $c_i \in \mathbb{C}$, for every $i = 0, 1, \ldots, n-1$.

The spectral analysis of circulant matrices is well known [12]. More precisely, it is known that the eigenvectors of the matrix C are the vectors $\mathbf{v}_j = (1, w^j, w^{2j}, \dots, w^{(n-1)j})$, for $j = 0, 1, \dots, n-1$, where

$$w = \exp\left(\frac{2\pi i}{n}\right), \qquad i^2 = -1.$$

The associated eigenvalues are the complex numbers

$$\lambda_j = \sum_{k=0}^{n-1} c_k w^{jk}, \qquad j = 0, 1, \dots, n-1.$$

More generally, a block circulant matrix of type (n, m) is a matrix of the form (2), where c_i is a square matrix of size m with entries in \mathbb{C} , for every $i = 0, 1, \ldots, n-1$. For the spectral analysis of a block circulant matrix one can refer, for instance, to the paper [26].

In the following proposition, we explicitly describe a particular bi-labelling of the graph K_{2d+1} such that $K_{2d+1} \otimes C_{2d}$ is connected for each $d \geq 1$. We will use the notation $V(K_{2d+1}) = \{0, 1, \ldots, 2d\}$ and $V(C_{2d}) = \{1, \ldots, 2d\}$. The vertices of $K_{2d+1} \otimes C_{2d}$ will be pairs of type (i, j), with $i \in V(K_{2d+1})$ and $j \in V(C_{2d})$, with the convention that the sum in the first coordinate is mod 2d + 1, and in the second component the sum is given by equation (1), where d is now replaced by 2d.

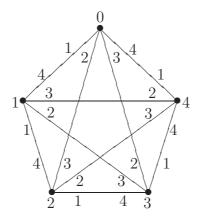
Proposition 5.2. Let \mathcal{L} be the bi-labelling of K_{2d+1} such that, with the above notation:

$$Rot_{K_{2d+1}}(i,j) = (i+j,2d-j+1).$$

Then $K_{2d+1} \boxtimes C_{2d}$ is connected and there exists an ordering of the vertices of $K_{2d+1} \boxtimes C_{2d}$ such that the associated adjacency matrix is a block circulant matrix.

Proof. By using the symmetry of our construction and the fact that the graph K_{2d+1} is complete, we have just to prove, by Theorem 4.1, that the parity block decomposition of K_{2d+1} consists of only one parity block P such that deg(i) = 2d in P, for every $i \in V(K_{2d+1})$. Without loss of generality, we can suppose that the vertex i is odd in P. Furthermore, by the construction of \mathcal{L} , we have that the condition $Rot_{K_{2d+1}}(i,j)=(i',j')$ implies j + j' = 2d + 1, and so j, j' have different parities. Note that in K_{2d+1} the path $\{v = i, v_1 = i + 3, v_2 = i + 2, i = v\}$ described by $Rot_{K_{2d+1}}(i,3) = (i + 3, 2d - 2),$ $Rot_{K_{2d+1}}(i+3,2d) = (i+2,1), Rot_{K_{2d+1}}(i+2,2d-1) = (i,2)$ satisfies the parity conditions. This implies that i is also even, so that i is odden. We can repeat the same construction for every other vertex, possibly reversing the path for vertices with even parity, so that every vertex of P has degree 2d. The second statement follows by observing that $Rot_{K_{2d+1}}(i,j) =$ (i+j,2d-j+1) implies $\operatorname{Rot}_{K_{2d+1}}(i+1,j)=(i+j+1,2d-j+1),$ and so the lexicographic order of the vertices $\{(i,j): i \in V(K_{2d+1}), j \in V(C_{2d})\}$ of $K_{2d+1} \otimes C_{2d}$ produces a block circulant adjacency matrix. More precisely, there are 2d+1 blocks, indexed by the vertices of K_{2d+1} , each of size 2d.

Example 5.2. In the following picture, the bi-labelling \mathcal{L} is represented in the case of K_5 . Observe that the labels around each vertex of the graph, regarded as integer numbers in $\{1, 2, 3, 4\}$, increase in anticlockwise sense.



The adjacency matrix of the graph $K_5 \boxtimes C_4$, with the lexicographic order of its vertices, is given by

$$\begin{pmatrix} C_0 & C_1 & C_2 & C_3 & C_4 \\ C_4 & C_0 & C_1 & C_2 & C_3 \\ C_3 & C_4 & C_0 & C_1 & C_2 \\ C_2 & C_3 & C_4 & C_0 & C_1 \\ C_1 & C_2 & C_3 & C_4 & C_0 \end{pmatrix},$$

where

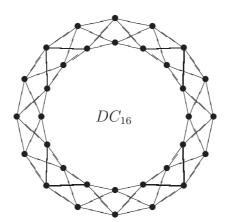
$$C_1 = C_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \qquad C_2 = C_4 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and C_0 is the zero matrix of size 4.

5.2. The case of a 4-regular graph. In this subsection we are interested in the special case of the zig-zag product $G(\mathbb{Z})C_4$, where G is a 4-regular graph. This particular choice forces the structure of the zig-zag product $G(\mathbb{Z})C_4$ to be highly regular. Actually, one can order the vertices of $G(\mathbb{Z})C_4$ in such a way that the graph is a disjoint union of connected components, whose adjacency matrices are all circulant; therefore, a complete spectral analysis can be performed in this case. We will see an application to the case of Schreier graphs associated with group actions in Section 6.

Notice that Corollary 4.1 implies that the isomorphism classes of the connected components of $G \odot C_4$ are determined by the size of the corresponding parity blocks of G. We recall that the pseudo-replacement graphs are given by an alternate sequence of simple edges and cycles of length 2.

In what follows, a double cycle graph DC_n of length n is a 4-regular graph with 2n vertices in which any vertex belongs exactly to two papillon graphs. The picture below represents the graph DC_{16} .

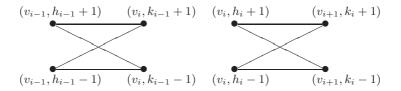


The following proposition shows that the connected components of the zig-zag product $G(\mathbb{Z})C_4$ are isomorphic to double cycle graphs.

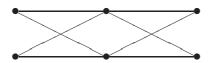
Proposition 5.3. Let G be a 4-regular graph and let $\{v = v_0, v_1, \ldots, v_n = v\}$ be a path spanning a parity block P of the parity block decomposition of G. Then the corresponding connected component S in $G(\mathbb{Z})C_4$ is isomorphic to the double cycle graph DC_n . More precisely, if $Rot_G(v_i, h_i) = (v_{i+1}, k_i)$, then $(v_i, h_i \pm 1)$ and $(v_{i\pm 1}, h_{i\pm 1} \pm 1)$ form two adjacent papillon graphs in DC_n .

Proof. Two consecutive edges $\{v_{i-1}, v_i\}$ and $\{v_i, v_{i+1}\}$ in the path spanning P

produce the following papillon subgraphs, respectively:

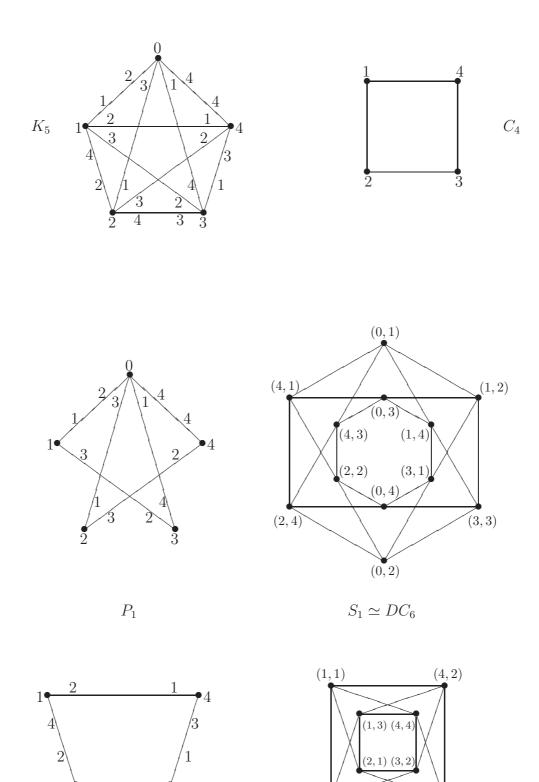


Notice that the sets $\{h_i \pm 1\}$ and $\{k_{i-1} \pm 1\}$ coincide, since the spanning path must satisfy the parity properties. Therefore, we can identify the pair of vertices $(v_i, k_{i-1} \pm 1)$ with the pair of vertices $(v_i, h_i \pm 1)$, getting in $G \supseteq C_4$ the following subgraph obtained by gluing together two single papillon subgraphs.



Since the path spanning P contains n vertices, the assertion follows from Corollary 4.1. \square

Example 5.3. In this example, we consider the graph $G = K_5$ endowed with a bi-labelling \mathcal{L} such that $K_5 = P_1 \cup P_2$. The corresponding connected components S_1 and S_2 of $K_5 \otimes C_4$ are given as well.



(2,3)

 P_2

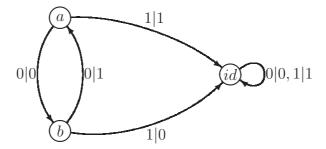
 $S_2 \simeq DC_4$

(3,4)

6. Schreier Graphs: An application

Let $X = \{0, 1\}$ be a binary alphabet. Denote by X^0 the set consisting of the empty word, and by $X^n = \{w = x_1 \dots x_n : x_i \in X\}$ the set of words of length n over the alphabet X, for each $n \geq 1$. Put $X^* = \bigcup_{n \geq 0} X^n$ and let $X^{\infty} = \{w = x_1 x_2 \dots\}$ be the set of infinite words over X.

Consider the so-called Basilica group B acting on the set $X^* \cup X^{\infty}$, which is the group generated by the following three-state automaton:



The states a and b of the automaton are the generators of the group, whereas id represents the identity action: therefore, it can be read from the automaton that the action of a and b is given by

$$a(0w) = 0b(w)$$

$$a(1w) = 1w$$

$$b(0w) = 1a(w)$$

$$b(1w) = 0w,$$

for every $w \in X^* \cup X^{\infty}$. In particular, B maps X^n into X^n , for each $n \ge 1$, and X^{∞} into X^{∞} . Furthermore, it is easy to check that the action of B on X^n is transitive for every n.

The Basilica group belongs to the important class of self-similar groups and was introduced by R. Grigorchuk and A. Żuk [17]. It is a remarkable fact due to Nekrashevych [23] that it can be described as the iterated monodromy group $IMG(z^2-1)$ of the complex polynomial z^2-1 . Moreover, B is the first example of an amenable group (a highly non-trivial and deep result of Bartholdi and Virág [3]) not belonging to the class SG of subexponentially amenable groups, which is the smallest class containing all groups of subexponential growth and closed after taking subgroups, quotients, extensions and direct unions. In [6], the action of the Basilica group on the set $X^* \cup X^{\infty}$ is studied from the point of view of Gelfand pairs theory.

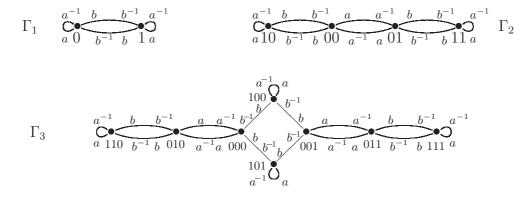
For each $n \geq 1$, let Γ_n be the (orbital) Schreier graph associated with the action of B on X^n . By definition, the vertices of Γ_n are the elements of X^n , and two vertices u, v are connected by an edge labelled by s close to u and by s^{-1} close to v if s(u) = v (so that $s^{-1}(v) = u$). Here, we are assuming $s \in \{a, b\}$. Observe that Γ_n is a connected graph on 2^n vertices, since B acts transitively on each level; moreover, there are 4 edges issuing from every vertex, so that Γ_n is a 4-regular graph, and the labels near v are given by $a^{\pm 1}$, $b^{\pm 1}$, for every $v \in X^n$.

Similarly, one can consider the action of B on X^{∞} and define the (orbital) infinite Schreier graph Γ_{ξ} , describing the orbit $B(\xi)$ of the element $\xi \in X^{\infty}$ under the action of the generators of B. Note that, even though the group acts transitively on X^n , for each $n \geq 1$, there exist uncountably many orbits in X^{∞} under the action of B. The infinite Schreier graph Γ_{ξ} can be approximated (as a rooted graph) by finite Schreier graphs Γ_n , as $n \to \infty$, in the compact space of rooted graphs of uniformly bounded degree endowed with pointed Gromov-Hausdorff convergence [19, Chapter 3]: if $\xi = x_1 x_2 \ldots \in X^{\infty}$ and $\xi_n = x_1 \ldots x_n$ is its prefix of length n, then one has

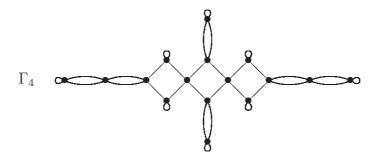
$$\lim_{n\to\infty}(\Gamma_n,\xi_n)=(\Gamma_\xi,\xi),$$

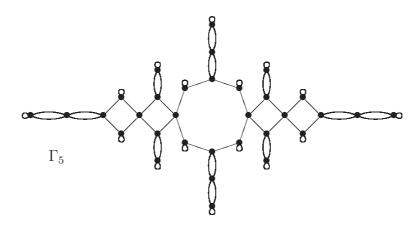
where we denote by (Γ_n, ξ_n) the graph Γ_n regarded as a graph rooted at the vertex ξ_n , and by (Γ_{ξ}, ξ) the graph Γ_{ξ} regarded as a graph rooted at the vertex ξ .

In [7], finite and infinite Schreier graphs of the Basilica group are investigated. Precise substitutional rules allowing to construct recursively the sequence of finite Schreier graphs are provided, and a topological (up to isomorphism of rooted graphs) classification of the infinite Schreier graphs is given there. In the following pictures, the graphs $\Gamma_1, \Gamma_2, \Gamma_3$ are depicted.

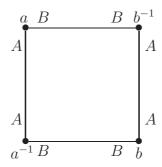


For instance, the fact that the edge connecting 000 and 101 in Γ_3 is labelled by b near 000 and by b^{-1} near 101 means that b(000) = 101, so that $b^{-1}(101) = 000$. The graphs Γ_4 and Γ_5 are represented below (the bi-labelling of the edges and the words corresponding to each vertex are omitted in order to simplify the picture).





As the graph Γ_n is a 4-regular graph, for each $n \geq 1$, it is natural to construct the sequence $\{\Gamma_n \boxtimes C_4\}_{n\geq 1}$. We label the graph C_4 as follows:



We identify in a natural manner the ordered set $\{a, a^{-1}, b, b^{-1}\}$ with the ordered set $[4] = \{1, 2, 3, 4\}$. Then the following result holds.

Proposition 6.1. Let Γ_n be the Schreier graph of the action of the Basilica group on $\{0,1\}^n$. Then the parity block decomposition of Γ_n consists of only one parity block, so that the graph $\Gamma_n(\mathbb{Z})C_4$ is a connected graph, isomorphic to the double cycle graph $DC_{2^{n+1}}$.

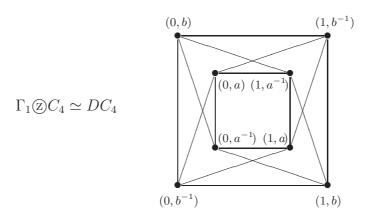
Proof. Take an arbitrary vertex $v \in \Gamma_n$: by the construction of Γ_n , we have

$$\operatorname{Rot}_{\Gamma_n}(v, b^{-1}) = (b^{-1}(v), b)$$
 and $\operatorname{Rot}_{\Gamma_n}(b^{-1}(v), a) = (ab^{-1}(v), a^{-1}).$

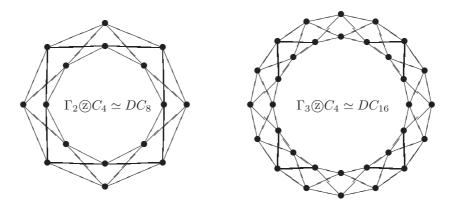
Observe that the vertices a and b in C_4 have been identified with the integers 1 and 3, respectively: this ensures that if we move in Γ_n by using the generators b^{-1} and a alternately, we do not leave our parity block, since the parity properties are satisfied. By iterating this argument, we describe a path in Γ_n , consisting of the vertices that one obtains starting from v and applying alternately b^{-1} , a, b^{-1} , a and so on. On the other hand, it is not difficult to prove, by induction on n, that the action of the product ab^{-1} on X^n has order 2^n , so that the construction produces a spanning path of Γ_n , of length 2^{n+1} , where each vertex of Γ_n occurs twice. Therefore, we get a unique parity block coinciding with Γ_n ; by Proposition 5.3, we conclude that the zig-zag product $\Gamma_n \otimes C_4$ consists of only one connected component isomorphic to $DC_{2^{n+1}}$.

Corollary 6.1. Let Γ be the Schreier graph of the action of a group G, generated by the symmetric set $S = \{a^{\pm 1}, b^{\pm 1}\}$, on a set X with |X| = n, so that $\operatorname{Rot}_{\Gamma}(x, s) = (s(x), s^{-1})$, with $s \in S$. Let C_4 be the cycle graph whose vertices are labelled as above. Then the graph $\Gamma(\mathbb{Z})C_4$ is connected if and only if the action of ab^{-1} is transitive on X. If this is the case, one has $\Gamma(\mathbb{Z})C_4 \simeq DC_{2n}$.

Example 6.1. The zig-zag product $\Gamma_1(z)C_4$ gives rise to the following graph:



For n = 2, 3, we get the double cycle graphs DC_8 and DC_{16} , respectively (labels are omitted in the pictures below).



Remark 6.1. Observe that we get a connected zig-zag product $\Gamma_n \boxtimes C_4$ for every n, even though the neighborhood graph \mathcal{N} associated with C_4 is disconnected, according with the fact that the condition in Theorem 3.1 is only sufficient but not necessary in order to have the connectedness property.

Theorem 6.1. For every $n \geq 1$, the spectrum of the graph $\Gamma_n(\mathbb{Z})C_4 \simeq DC_{2^{n+1}}$ is given by

$$\Sigma_n = \left\{ \underbrace{0, \dots, 0}_{2^{n+1} \text{ times}}, 4\cos\left(\frac{\pi j}{2^n}\right), j = 0, 1, \dots, 2^{n+1} - 1 \right\}.$$

Let $\mathbf{u}_0 = (1, -1)$, $\mathbf{u}_1 = (1, 1)$ and $\mathbf{v}_j = (1, w^j, w^{2j}, \dots, w^{(2^{n+1}-1)j})$. Then for each $j = 0, 1, \dots, 2^{n+1} - 1$, $\mathbf{u}_0 \otimes \mathbf{v}_j$ is an eigenvector associated with the eigenvalue 0, and $\mathbf{u}_1 \otimes \mathbf{v}_j$ is an eigenvector associated with the eigenvalue $4 \cos \left(\frac{\pi j}{2^n}\right)$.

Proof. It can be deduced from the structure of the graph $DC_{2^{n+1}}$ that the adjacency matrix of the graph $\Gamma_n \boxtimes C_4$ is a circulant matrix of size 2^{n+2} . More precisely, we choose the following order of the vertices of $\Gamma_n \boxtimes C_4$:

$$(0^{n}, a); (b^{-1}(0^{n}), a^{-1}); (ab^{-1}(0^{n}), a); (b^{-1}ab^{-1}(0^{n}), a^{-1}); \dots; ((ab^{-1})^{2^{n}-1}(0^{n}), a);$$

$$(b^{-1}(ab^{-1})^{2^{n}-1}(0^{n}), a^{-1}); (0^{n}, b); (b^{-1}(0^{n}), b^{-1}); (ab^{-1}(0^{n}), b); (b^{-1}ab^{-1}(0^{n}), b^{-1}); \dots$$

$$\dots; ((ab^{-1})^{2^{n}-1}(0^{n}), b); (b^{-1}(ab^{-1})^{2^{n}-1}(0^{n}), b^{-1}).$$

In other words, we are applying alternately b^{-1} and a to the word 0^n (observe that the 2^n -iteration of ab^{-1} is the identity on $\{0,1\}^n$), with an alternation of a, a^{-1} in the second coordinate of the first 2^n vertices (the inner cycle) and of b, b^{-1} in the second coordinate of the second 2^n vertices (the outer cycle). It is straightforward to check that, with this ordering of the vertices, the adjacency matrix M_n of the graph $\Gamma_n \boxtimes C_4$, for each $n \ge 1$, is given by $U \otimes \widetilde{M}_n$, where $U = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and \widetilde{M}_n is the square matrix of size 2^{n+1} of type

$$\widetilde{M}_n = \begin{pmatrix} 0 & 1 & 0 & & \dots & 0 & 1 \\ 1 & 0 & 1 & & & & 0 & 0 \\ 0 & 1 & 0 & 1 & & & & \\ & & 1 & 0 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 1 & & \\ & & & 1 & 0 & 1 & 0 \\ 0 & 0 & & & & 1 & 0 & 1 \\ 1 & 0 & & \dots & & 0 & 1 & 0 \end{pmatrix}.$$

The matrix \widetilde{M}_n is a circulant matrix, according with Definition 5.1. More precisely, \widetilde{M}_n is a circulant matrix of size 2^{n+1} , satisfying

$$c_k = \begin{cases} 1, & \text{if } k = 1, 2^{n+1} - 1\\ 0, & \text{otherwise.} \end{cases}$$

We deduce that, for every $j = 0, 1, \dots, 2^{n+1} - 1$, the vector

$$\mathbf{v}_i = (1, w^j, w^{2j}, \dots, w^{(2^{n+1}-1)j}),$$

where $w = \exp\left(\frac{2\pi i}{2^{n+1}}\right)$, and $i^2 = -1$, is an eigenvector of \widetilde{M}_n with associated eigenvalue

$$\lambda_j = (-1)^j \cdot 2\cos\left(\pi j - \frac{\pi j}{2^n}\right).$$

On the other hand, we have

$$\cos\left(\pi j - \frac{\pi j}{2^n}\right) = \cos(\pi j)\cos\left(\frac{\pi j}{2^n}\right) + \sin(\pi j)\sin\left(\frac{\pi j}{2^n}\right) = (-1)^j\cos\left(\frac{\pi j}{2^n}\right),$$

so that the j-th eigenvalue of \widetilde{M}_n , for $j = 0, 1, \dots, 2^{n+1} - 1$, is

$$\lambda_j = 2\cos\left(\frac{\pi j}{2^n}\right).$$

Notice that the eigenvalues of the matrix U are $\mu_0 = 0$, with eigenvector $\mathbf{u}_0 = (1, -1)$, and $\mu_1 = 2$, with eigenvector $\mathbf{u}_1 = (1, 1)$. As $M_n = U \otimes \widetilde{M}_n$, the eigenvectors of the matrix M_n are given by $\mathbf{u}_i \otimes \mathbf{v}_j$, with associated eigenvalue $\mu_i \lambda_j$, for i = 0, 1 and $j = 0, 1, \ldots, 2^{n+1} - 1$. This gives the assertion.

Observe that in [25] the authors defined the zig-zag product $G_1 \boxtimes G_2$ of two finite connected regular graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. The definition can be naturally extended to the case where G_1 is an infinite regular graph, and the degree of G_1 is equal to $|V_2|$. The analysis of zig-zag products of infinite graphs will be considered in the upcoming paper [10]. In the following example, we will consider the infinite case, where G_1 is the infinite Schreier graph of a word in $\{0,1\}^{\infty}$ under the action of the Basilica group, and G_2 is the cycle graph of length 4.

Example 6.2. It is not difficult to see that the zig-zag product $\Gamma_{\xi} \boxtimes C_4$, where Γ_{ξ} is the infinite 4-regular graph describing the orbit of the vertex ξ , with $\xi \in \{0,1\}^{\infty}$, and $\xi \notin B(0^{\infty})$, is an infinite connected 4-regular graph isomorphic to the following graph:



Incidentally, this also shows that the graphs $\Gamma_{\xi}(\mathbb{Z})C_4$ and $\Gamma_{\eta}(\mathbb{Z})C_4$ may be isomorphic, even if the graphs Γ_{ξ} and Γ_{η} are not isomorphic. More precisely, we have that the uncountably many graphs $\Gamma_{\xi}(\mathbb{Z})C_4$ are all isomorphic, for every $\xi \notin B(0^{\infty})$. This property implies that the zig-zag construction is not injective even in the infinite context.

On the other hand, it is easy to check that the zig-zag product of the graph $\Gamma_{0\infty}$ with the cycle graph of length 4 consists of two infinite connected components, each isomorphic to the graph DC_{∞} .

We can summarize these results as follows. Choose a root in the graph C_4 , and let us denote it by v_0 . As usual, denote by (Γ_n, v) the graph Γ_n rooted at the vertex v. Let $\xi = x_1 x_2 x_3 \ldots \in \{0, 1\}^{\infty}$, and let $\xi_n = x_1 x_2 \ldots x_n$, then:

(1) if $\xi \notin B(0^{\infty})$, then

$$\lim_{n\to\infty} (\Gamma_{\xi_n}, \xi_n) \textcircled{z}(C_4, v_0) = (DC_{\infty}, (\xi, v_0));$$

(2) if $\xi \in B(0^{\infty})$, then

$$\lim_{n \to \infty} (\Gamma_{\xi_n}, \xi_n) \textcircled{2}(C_4, v_0) = (DC_{\infty}, (\xi, v_0)) \cup (DC_{\infty}, (\xi, v_0 + 1)).$$

ACKNOWLEDGMENTS

The authors want to thank Wolfgang Woess and Tullio Ceccherini-Silberstein for useful and stimulating discussions on the subjects of this paper. Daniele D'Angeli was supported by Austrian Science Fund (FWF) P24028-N18. Alfredo Donno was partially supported by the European Science Foundation (Research Project RGLIS 4915). Ecaterina Sava-Huss was supported by Austrian Science Fund (FWF): W1230.

REFERENCES

- [1] A. Abdollahi and A. Loghman, On one-factorizations of replacement products, to appear in *Filomat*, Published by Faculty of Sciences and Mathematics, University of Niš, Serbia, available at http://www.pmf.ni.ac.rs/filomat
- [2] N. Alon, A. Lubotzky and A. Wigderson, Semi-direct product in groups and zig-zag product in graphs: connections and applications (extended abstract), in: "42-nd IEEE Symposium on Foundations of Computer Science, Las Vegas, NV, 2001", 630–637. IEEE Computer Society, Los Alamitos, CA, 2001.
- [3] L. Bartholdi and B. Virág, Amenability via random walks, *Duke Math Journal* **130** (2005), no. 1, 39–56.
- [4] T. Ceccherini-Silberstein, A. Donno and D. Iacono, Tutte polynomial of the Schreier graphs of the Grigorchuk group and the Basilica group, in: *Ischia Group Theory 2010 (Proceedings of the Conference)* (M. Bianchi, P. Longobardi, M. Maj and C. M. Scoppola editors), World Scientific 2011, 45–68.
- [5] T. Ceccherini-Silberstein, F. Scarabotti and F. Tolli, Harmonic Analysis on Finite Groups: Representation theory, Gelfand pairs and Markov chains. *Cambridge Studies in Advanced Mathematics*, **108**, Cambridge University Press, 2008. xiv + 440 pp.
- [6] D. D'Angeli and A. Donno, Self-similar groups and finite Gelfand pairs, *Algebra Discrete Math.*, no. 2, (2007), 54–69.
- [7] D. D'Angeli, A. Donno, M. Matter and T. Nagnibeda, Schreier graphs of the Basilica group, J. Mod. Dyn. 4 (2010), no. 1, 167–205.
- [8] D. D'Angeli, A. Donno and T. Nagnibeda, Counting dimer coverings on self-similar Schreier graphs, European J. Combin. 33 (2012), no. 7, 1484–1513.
- [9] D. D'Angeli, A. Donno and T. Nagnibeda, Partition functions of the Ising model on some self-similar Schreier graphs, in: *Progress in Probability: Random Walks, Boundaries and Spectra* (D. Lenz, F. Sobieczky and W. Woess editors), **64** (2011), 277–304, Springer Basel.
- [10] D. D'Angeli, A. Donno and E. Sava-Huss, Zig-zag products of infinite graphs. In preparation.
- [11] G. Davidoff, P. Sarnak and A. Valette, Elementary number theory, group theory, and Ramanujan graphs. *London Mathematical Society Student Texts*, **55**. Cambridge University Press, Cambridge, 2003. x + 144 pp.
- [12] P. J. Davis, Circulant matrices. A Wiley-Interscience Publication. *Pure and Applied Mathematics*. John Wiley & Sons, New York-Chichester-Brisbane, 1979. xv + 250 pp.
- [13] A. Donno, Replacement and zig-zag products, Cayley graphs and Lamplighter random walk, *Int. J. Group Theory* 2 (2013) No. 1, 11–35.
- [14] A. Donno, Generalized wreath products of graphs and groups, to appear in *Graphs and Combinatorics*, published online at http://link.springer.com/article/10.1007/s00373-014-1414-4, DOI 10.1007/s00373-014-1414-4
- [15] A. Donno and D. Iacono, The Tutte polynomial of the Sierpiński and Hanoi graphs, Adv. Geom., Vol. 13 (2013), Issue 4, 663–694.
- [16] R. I. Grigorchuk, Solved and unsolved problems around one group, Infinite groups: geometric, combinatorial and dynamical aspects, 117–218, *Progr. Math.*, **248**, Birkhäuser, Basel, 2005.

- [17] R. I. Grigorchuk and A. Żuk, On a torsion-free weakly branch group defined by a three-state automaton. *Internat. J. Algebra Comput.*, **12**, (2002), no. 1, 223–246.
- [18] M. Gromov, Filling Riemannian manifolds, J. Differential Geom. 18 (1983), no. 1, 1–147.
- [19] M. Gromov, Structures métriques pour les variétés riemanniennes, *Textes Mathématiques*, J. Lafontaine and P. Pansu (Eds.), 1. CEDIC, Paris, 1981. iv+152 pp. ISBN: 2-7124-0714-8.
- [20] S. Hoory, N. Linial and A. Wigderson, Expander graphs and their application, *Bull. Amer. Math. Soc.* (N.S.) 43 (2006), no. 4, 439–561.
- [21] C. A. Kelley, D. Sridhara and J. Rosenthal, Zig-zag and replacement product graphs and LDPC codes, Adv. Math. Commun. 2 (2008), no. 4, 347–372.
- [22] A. Lubotzky, Expander graphs in pure and applied mathematics, Bull. Amer. Math. Soc. 49 (2012), no. 1, 113–162.
- [23] V. Nekrashevych, Self-similar Groups, Volume 117 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 2005. xii + 231 pp.
- [24] V. Nekrashevych and A. Teplyaev, Groups and analysis on fractals. In: "Analysis on Graphs and its Applications", Proceedings of Symposia in Pure Mathematics, Amer. Math. Soc., 77, 143–180. Amer. Math. Soc., Providence (2008).
- [25] O. Reingold, S. Vadhan and A. Wigderson, Entropy Waves, the Zig-Zag Graph Product, and New Constant-Degree Expanders, *Ann. of Math.* (2) **155** (2002), no. 1, 157–187.
- [26] G. J. Tee, Eigenvectors of block circulant and alternating circulant matrices, Res. Lett. Inf. Math. Sci. 8 (2005), 123–142.
- [27] I. Tomescu, Problems in Combinatorics and Graph Theory. Translated from the Romanian by R. A. Melter. Wiley Interscience Series in Discrete Mathematics. *John Wiley & Sons, Ltd. Chichester*, 1985. xvii + 335 pp.

Institut für mathematische Strukturtheorie (Math C), Technische Universität Graz Steyrergasse 30, 8010 Graz, Austria

E-mail address: dangeli@math.tugraz.at

Università degli Studi Niccolò Cusano - Via Don Carlo Gnocchi, 3 00166 Roma, Italia, Tel.: +39 06 45678356, Fax: +39 06 45678379

E-mail address: alfredo.donno@unicusano.it

Institut für mathematische Strukturtheorie (Math C), Technische Universität Graz Steyrergasse 30, 8010 Graz, Austria

E-mail address: sava-huss@tugraz.at