Extremal C_4 -free/ C_5 -free planar graphs

Chris Dowden

London School of Economics C.Dowden@lse.ac.uk

August 7, 2018

Abstract

We study the topic of 'extremal' planar graphs, defining $\exp_{\mathcal{P}}(n, H)$ to be the maximum number of edges possible in a planar graph on n vertices that does not contain a given graph H as a subgraph. In particular, we examine the case when H is a small cycle, obtaining $\exp_{\mathcal{P}}(n, C_4) \leq \frac{15}{7}(n-2)$ for all $n \geq 4$ and $\exp_{\mathcal{P}}(n, C_5) \leq \frac{12n-33}{5}$ for all $n \geq 11$, and showing that both of these bounds are tight.

2010 Mathematical Subject Classification codes: 05C10, 05C35

1 Introduction

One of the best-known results in extremal graph theory is Turan's Theorem [5], which gives the maximum number of edges that a graph on n vertices can have without containing a K_r subgraph. The Erdos-Stone Theorem [2] then extends this to the case when K_r is replaced by an arbitrary graph H, showing that the maximum number of edges possible is $(1 + o(1)) \left(\frac{\chi(H)-2}{\chi(H)-1}\right) {n \choose 2}$, where $\chi(H)$ denotes the chromatic number of H. This latter result has been called the 'fundamental theorem of extremal graph theory' [1].

Over the last decade, a large quantity of work has been carried out in the area of 'random' planar graphs (see, for example, [3] and [4]). However, there seem to be no known results on questions analogous to the Erdos-Stone Theorem, i.e. how many edges can a planar graph on n vertices have without containing a given smaller graph? It is consequently the aim of this paper to now make a start on this topic of 'extremal' planar graphs.

Unfortunately, the case when the forbidden subgraph is a complete graph (i.e. the analogue to Turan) is fairly trivial. Since K_5 is not planar, the only meaningful cases to look at are K_3 and K_4 , and these are both straightforward: for the former, it can be observed that $K_{2,n-2}$ must be extremal (since all faces have size four when drawn in the plane), and so the extremal number of edges is

2n-4; for the latter, it suffices to note that there exist planar triangulations not containing K_4 (e.g. take a cycle of length n-2 and then add two new vertices that are both adjacent to all those in the cycle), and so the extremal number is 3n-6.

The next most natural type of graph to investigate is perhaps a cycle. Hence, in this paper we focus on the cases of C_4 and C_5 , which turn out to be much more interesting.

We begin, in Section 2, with the case when the forbidden subgraph is C_4 , obtaining a tight bound for the extremal number. In Section 3, we then produce a reasonably simple inequality for C_5 , before presenting the full C_5 result in Section 4, and demonstrating in Section 5 that the latter is tight. In Section 6, we then finish with some concluding remarks.

$2 C_4$

Let us start by introducing a convenient piece of notation:

Definition 1 Let us say that a graph is H-free if it does not contain H as a subgraph (whether induced or not), and let $ex_{\mathcal{P}}(n, H)$ denote the maximum number of edges possible in a planar H-free graph on n vertices.

In this section, we shall look at the case when $H = C_4$. In Theorem 2, we shall give a simple proof to show that $\exp(n, C_4) \leq \frac{15}{7}(n-2)$ for all $n \geq 4$, and then in Theorem 3 we shall demonstrate that this bound is actually tight, in the sense that there are infinitely many values of n for which it is attained exactly. In the following section, we shall then move on to looking at C_5 .

We begin with our aforementioned upper bound (note that in the proof, as throughout this paper, we use the term 'plane graph' to mean a particular embedding of a planar graph in the plane):

Theorem 2 $\exp_{\mathcal{P}}(n, C_4) \leq \frac{15}{7}(n-2)$ for all $n \geq 4$.

Proof Let G be an arbitrary C_4 -free connected plane graph on $n \ge 4$ vertices (clearly it suffices to consider only connected graphs). Let f_i denote the number of faces of size i in G, and let f denote $\sum_i f_i$.

of faces of size i in G, and let f denote $\sum_i f_i$. Note that $\sum_{i \leq 2} f_i = f_4 = 0$ (since G is connected, $n \geq 4$ and $G \not\supseteq C_4$), and so

$$2e(G) = 3f_3 + \sum_{i \ge 5} if_i$$

$$\geq 3f_3 + 5\sum_{i \ge 5} f_i$$

$$= 3f_3 + 5(f - f_3)$$

$$= 5f - 2f_3.$$

Hence

$$f \le \frac{2(e(G) + f_3)}{5}.$$
 (1)

Observe also that no edge of G can be in two faces of size 3 without creating a C_4 (unless $G = C_3$, which is not possible if $n \ge 4$), and so it follows that $e(G) \ge 3f_3$. Putting $f_3 \le \frac{e(G)}{3}$ into (1), we thus obtain $f \le \frac{8}{15}e(G)$. By Euler's formula, we then have $n-2 = e(G) - f \ge \frac{7}{15}e(G)$, and so

 $e(G) \le \frac{15}{7}(n-2).$

Given the rather uncomplicated nature of the proof of Theorem 2 (it only used the facts that G contained no faces of size 4 and had no edges in two faces of size 3), it is perhaps surprising that equality should be achievable. However, we shall now see that this is indeed the case:

Theorem 3 $ex_{\mathcal{P}}(n, C_4) = \frac{15}{7}(n-2)$ for $n \equiv 30 \pmod{70}$.

Proof An examination of the proof of Theorem 2 shows that equality is achieved for n if and only if there exists a connected C_4 -free plane graph on n vertices for which every edge lies in one face of size 3 and one face of size 5. The icosidodecahedron (see Figure 1) is an example of such a graph with 30 vertices.



Figure 1: The icosidodecahedron.

Let us now use the notation G_0 to denote the icosidodecahedron, and, for $k \geq 1$, let us then proceed to define further plane graphs G_k of order 30 + 70kinductively via the illustration given in Figure 2. Here, the entire graph G_{k-1} is placed into the central pentagon of Figure 2, and the entire graph G_0 is then placed between the two bold pentagons of Figure 2 (in such a way that these are identified with the bold pentagons of Figure 1).

It can be checked that, for all k, G_k satisfies the specified conditions, and so we are done.



Figure 2: The graph G_k .

As an aside, we note that slightly more complicated examples can also be produced whenever $n \equiv 2 \pmod{14}$ (apart from $n \in \{2, 16\}$, for which equality is not possible).

3 C_5 — a simple inequality

In Theorem 2 of the previous section, we gave a tight bound for $\exp_{\mathcal{P}}(n, C_4)$. In this section, we shall now move on to looking at $\exp_{\mathcal{P}}(n, C_5)$.

By an exactly analogous method to the proof of Theorem 2 (with the addition of taking into account the number of shared edges possible between different C_3 faces and between C_3 faces and C_4 faces), we shall first see that it is reasonably straightforward to obtain $\exp_{\mathcal{P}}(n, C_5) \leq \frac{12}{5}(n-2)$ for all $n \geq 5$ (Lemma 4). Unfortunately, we shall then also observe that this time equality is not achievable for large n (Observation 7). In the following section, we shall consequently present a much more complicated proof that does provide a tight bound.

Readers may be tempted to skip this section and proceed straight to the full result in Section 4. However, it should be noted that the simple inequality given in Lemma 4 is actually used during the proof of the full result (in order to deal with blocks of order 5 to 10), and also that the proof of Observation 7 is intended to aid understanding of the proof of the later result, which is rather detailed.

We start with the aforementioned 'simple' inequality:

Lemma 4 $\exp_{\mathcal{P}}(n, C_5) \leq \frac{12}{5}(n-2)$ for all $n \geq 5$.

Proof The proof is by induction. It can be checked that the statement is true for n = 5, so let us assume that it is true for all $n \in \{5, 6, \ldots, k-1\}$, and let us now consider an arbitrary C_5 -free plane graph G on $k \ge 6$ vertices. Let f_i denote the number of faces of size i in G, and let f denote $\sum_i f_i$. Note that we may assume that $\delta(G) \ge 3$, since otherwise we could simply delete a vertex of minimal degree and obtain the result by applying the induction hypothesis to the resulting graph.

Let us use e_3 to denote the number of edges in G that are in at least one face of size 3.

Claim 5 $f_3 \leq \frac{e_3}{2}$.

Proof Note that each face of size 3 must be a C_3 . For each one of these faces, consider the number of edges that are also in another C_3 face. Note that it is possible for there to be two such edges (if they both belong to the same K_4 subgraph), but that it is impossible to have all three (unless $G = C_3$ or $G = K_4$, which may be ignored since $k \ge 6$). Hence, the total number of edges in two faces of size 3 is at most $\frac{1}{2}2f_3$ (dividing by two to avoid double-counting), and so we have

$$f_3 = \frac{1}{3} (\# \text{ edges in at least one face of size } 3 + \# \text{ edges in two faces of size } 3)$$
$$\leq \frac{1}{3} (e_3 + f_3),$$

from which the claim follows. End of Proof of Claim 5

Claim 6 $f_4 \leq \frac{e(G) - e_3}{2}$.

Proof Note that each face of size 4 must be a C_4 (using the fact that k > 3), and recall that each face of size 3 must be a C_3 . Observe that it is impossible for a face of size 4 and a face of size 3 to have exactly one edge in common without creating a C_5 , and that it is impossible for them to have two edges in common without leaving a vertex in the middle of the facial boundary with degree 2 (which would contradict the fact that $\delta(G) \geq 3$). Hence, the faces of size 4 must be edge-disjoint from the faces of size 3. Each edge can be in at most two faces of size 4, and each face of size 4 contains exactly four edges, so we then obtain $f_4 \leq \frac{2}{4}(e(G) - e_3) = \frac{e(G) - e_3}{2}$. End of Proof of Claim 6

Note that a face of size 5 in a plane graph must be either a C_5 or a C_3 with a pendant edge (as shown in Figure 3). Since G contains no C_5 's and has minimum degree at least 3, we thus have $f_5 = 0$. Hence (since also $\sum_{i<3} f_i = 0$), we obtain

$$2e(G) = 3f_3 + 4f_4 + \sum_{i \ge 6} if_i$$

$$\ge 3f_3 + 4f_4 + 6(f - f_3 - f_4),$$



Figure 3: A C_3 with a pendant edge.

and so

$$\begin{array}{rcl}
6f &\leq& 2e(G) + 3f_3 + 2f_4 \\
&\leq& 2e(G) + \frac{3}{2}e_3 + e(G) - e_3 \text{ by Claims 5 and 6} \\
&=& 3e(G) + \frac{1}{2}e_3 \\
&\leq& \frac{7}{2}e(G),
\end{array}$$

which gives $f \leq \frac{7}{12}e(G)$. Using Euler's formula, we then have $k-2 \geq e(G) - f \geq \frac{5}{12}e(G)$, from which the result follows.

Figure 4 gives an example of a graph for which equality in Lemma 4 is attained with n = 7. However, we shall now see (in Observation 7) that equality



Figure 4: A graph achieving equality in Lemma 4 for n = 7.

is not possible for larger values of n.

Readers may proceed straight to Section 4 if they so wish. However, it is hoped that the ideas presented in the proof of Observation 7, and in particular the introduction of a graph G', will prove helpful in understanding the later detailed arguments.

Observation 7 ex_{\mathcal{P}} $(n, C_5) < \frac{12}{5}(n-2)$ for all $n \ge 8$.

Proof A careful examination of the proof of Lemma 4 shows that equality is possible only if the plane graph G is connected and consists entirely of K_4 's and faces of size 6, combined in such a way that no two K_4 's share an edge and no two faces of size 6 share an edge (note that the outside face in Figure 4 has size 6). It will consequently be convenient for us to consider the plane graph G' formed from G by deleting the central vertex from each K_4 , along with all



Figure 5: The graph G' formed from the graph G in Figure 4.

incident edges, to leave a C_3 (see Figure 5 for an example). Hence, G' will then consist entirely of faces of size 3 and 6, with every edge of G' lying on the boundary between a face of size 3 and a face of size 6.

Let us use f' to denote the total number of faces in G', f'_3 to denote the number of faces in G' that have size 3, and f'_6 to denote the number of faces that have size 6. Then we consequently obtain $3f'_3 = e(G') = 6f'_6$, and $f' = f'_3 + f'_6 = \frac{e(G')}{3} + \frac{e(G')}{6} = \frac{e(G')}{2}$. Thus, Euler's formula gives $|G'| - 2 = e(G') - f' = \frac{e(G')}{2}$, and hence e(G') = 2|G'| - 4.

Crucially, this implies that the average degree in G' is less than 4. However, vertices of degree 0 or 1 are clearly impossible, vertices of degree 3 are also not possible (every vertex must have even degree, since the adjacent faces are *alternately* size 3 and size 6), and it can be checked that a vertex of degree 2 is only possible for the special case shown in Figure 5 (using the fact that if the faces adjacent to such a vertex v are cycles u_1vu_2 and $u_1vu_2u_3u_4u_5$, then $u_1u_2u_3u_4u_5$ would be a C_5). Hence, we obtain a contradiction.

4 C_5 — the full result

In this section, we shall improve on the results of Lemma 4 and Observation 7 by showing that $\exp(n, C_5) \leq \frac{12n-33}{5}$ for all $n \geq 11$ (see Theorem 8). As the details are very lengthy, a sketch of the proof is also provided. In the following section, we shall then demonstrate that this bound is tight, in the sense that there are infinitely many values of n for which it is attained exactly.

Theorem 8 $ex_{\mathcal{P}}(n, C_5) \leq \frac{12n-33}{5}$ for all $n \geq 11$.

Sketch of Proof Part I: We take a C_5 -free plane graph G, and use induction to deal with the cases when $\delta(G) \leq 2$ or $\kappa(G) \leq 1$, where $\kappa(G)$ denotes the vertex-connectivity of G (Lemma 4 is utilised for this latter case). This allows us to simplify some of the details in later arguments.

Part II: Similarly to with the proof of Observation 7, we now form a new plane graph G' by deleting any edges that lie between two C_3 faces, and we again show that $e(G') \leq 2|G'| - 4$ (this time, the proof is complicated by the fact that we do not assume that G achieves equality in Lemma 4, and so we need to allow for the presence of faces of size 4), and hence that G' contains vertices of degree less than 4.

Part III: We examine the case when e(G') = 2|G'| - 4, and deduce that equality implies that either (i) G' consists solely of C_4 faces or (ii) every edge in G' lies on the boundary between one C_3 face and one C_6 face. We then show that the former would imply that G = G', and so e(G) would also only be 2|G| - 4, while the latter is impossible (as in the proof of Observation 7).

Part IV: We then examine the remaining case when $e(G') \leq 2|G'|-5$, noting that this implies a more considerable lower bound on the number of vertices of degree 2 or 3 in G'. We find that the presence of each such vertex forces fewer faces of small size (to avoid a C_5), and we consequently obtain detailed inequalities which essentially express the exact amount of negative impact that each of these vertices has on e(G). Crucially, when combining this with our lower bound for the number of such vertices, we find that the best we can do is to obtain $e(G) = \frac{12|G|-33}{5}$.

Full Proof

Part I Our proof is by induction. It can be checked systematically that the result $\exp_{\mathcal{P}}(n, C_5) \leq \frac{12n-33}{5}$ holds for $n \in \{11, 12, 13\}$ (this turns out to be not as onerous as it might at first appear!), so let $k \geq 14$ and let us assume that the result holds for all $n \in \{11, 12, \ldots, k-3, k-2, k-1\}$.

Let G be a C_5 -free plane graph on k vertices, and let us deal in this part of the proof with the cases when $\delta(G) \leq 2$ or $\kappa(G) \leq 1$. For the former, we may simply delete a vertex of minimum degree and then use the induction hypothesis to obtain the result. For the latter, by considering an arbitrary block in G (which will have k - i vertices for some $i \geq 1$) and the union of all other blocks, we may obtain

$$e(G) \leq \max_{i \geq 1} \{ \exp(i+1, C_5) + \exp(k-i, C_5) \}$$

$$\leq \max_{i \geq 4} \left\{ 1 + \frac{12(k-1) - 33}{5}, 3 + \frac{12(k-2) - 33}{5}, 6 + \frac{12(k-3) - 33}{5}, -\frac{12}{5}(i-1) + \frac{12}{5}(k-i-2) \right\}$$

 $= \frac{12k - 36}{5}.$

Hence, for the remainder of the proof, we may assume that we have $\kappa(G) \geq 2$ and $\delta(G) \geq 3$. Note that the condition that there are no cut-vertices implies that all faces of G must be cycles. We shall often use this fact implicitly in the rest of the proof.

Part II Let G' denote the plane graph formed from G by (a) deleting any edges that lie on two C_3 faces in G and (b) also deleting any isolated vertices that are thus produced (see Figure 6 for an example of this). In this part of the proof, we shall work towards showing that $e(G') \leq 2|G'| - 4$ by paying particular attention to the number of faces in G' that are C_3 's or C_4 's.



Figure 6: Constructing the graph G' from the graph G.

Observe that if two C_3 faces in G share an edge, then (since G is C_5 -free) these two faces must either be edge-disjoint to all other C_3 faces in G or must form part of a K_4 consisting of three adjacent C_3 faces in G. In the former case, the two C_3 faces in G will be replaced by a C_4 face in G'; in the latter case, the K_4 in G will be replaced by a C_3 face in G'.

Thus, all faces of G' will be cycles, as with G. For the rest of the proof, it will be extremely important to note further that all faces of size greater than 4 in G' were also faces in G; that all C_4 faces in G' were either also faces in G or are the remains of two adjacent C_3 faces of G; and that all C_3 faces in G' were either faces in G or formed the outside of a K_4 in G. It follows from the latter observation that there will now be no edges that lie on two C_3 faces (we may discount the possibility that $G' = C_3$, since this would imply that $G = K_4$ and hence that k = 4).

Let f'_i denote the number of C_i faces in G', and let e'_i denote the number of edges of G' that are in at least one C_i face. Then, since there is no double-counting, we have

$$f_3' = \frac{1}{3}e_3'.$$
 (2)

Claim 9 A C_4 face in G' cannot share an edge with a C_3 face.

Proof Clearly, a C_4 face cannot share exactly one edge with a C_3 face without creating a C_5 . If a C_4 face shares two edges with a C_3 face, then it must look as shown in Figure 7, where a C_4 face $F_1 = v_1 v_2 v_3 v_4$ and a C_3 face $F_2 = v_2 v_3 v_4$



Figure 7: A C_4 face and a C_3 face that share two edges.

share the edges v_2v_3 and v_3v_4 . However, note that if such a situation were to occur in G', then (since $\deg_G(v_3) \ge 3$) it must be that either (a) F_2 formed the outside of a K_4 in G, in which case we could then find a $C_5 \subset G$ (see Figure 8), or (b) F_1 is the remains of two C_3 faces $v_1v_2v_3$ and $v_1v_3v_4$, in which case the



Figure 8: A $C_5 \subset G$ in case (a).

vertex v_3 and all edges incident to it should have been deleted when G' was constructed. Thus, it must be that such a situation does not occur. End of **Proof of Claim 9**

Note that $f'_4 \leq \frac{e'_4}{2}$, with equality if and only if every edge in a C_4 face is actually in two such faces. From Claim 9, we know that all C_4 faces in G' must be edge-disjoint from all C_3 faces, and so it follows that

$$f_4' \le \frac{e(G') - e_3'}{2},\tag{3}$$

with equality if and only if every edge not in a C_3 face is in two C_4 faces.

Let f' denote $\sum_i f'_i$. Then we have

$$2e(G') \geq 3f'_3 + 4f'_4 + 6(f' - f'_3 - f'_4)$$

and so

$$\begin{array}{rcl} 6f' & \leq & 2e(G') + 3f'_3 + 2f'_4 \\ & \leq & 2e(G') + e'_3 + e(G') - e'_3 \text{ by (2) and (3)} \\ & = & 3e(G'). \end{array}$$

Thus, $f' \leq \frac{1}{2}e(G')$.

Hence, since Euler's formula then gives $e(G') = |G'| - 2 + f' \le |G'| - 2 + \frac{1}{2}e(G')$, we obtain $\frac{1}{2}e(G') \le |G'| - 2$, i.e. $e(G') \le 2|G'| - 4$.

Part III In this part of the proof, we shall deal with the case when e(G') = 2|G'| - 4. As mentioned in the sketch of the proof, we shall find that our argument splits into two subcases depending on whether G' consists solely of C_4 faces or is instead comprised of a mixture of C_3 's and C_6 's.

Let us start by noting (from an examination of Part II) that equality in $e(G') \leq 2|G'| - 4$ implies that $f' = f'_3 + f'_4 + f'_6$ and that every edge of G' not in a C_3 face is in two C_4 faces.

Let $E'_{i,j}$ denote the set of edges of G' between a C_i face and a C_j face, and let $E'_{i,i}$ denote the set of edges of G' between two C_i faces. Recall that $E'_{3,3} = \emptyset$, and also that $E'_{3,4} = \emptyset$. Hence, $E'_{3,6} \cup E'_{4,4} = E(G')$.

and also that $E'_{3,4} = \emptyset$. Hence, $E'_{3,6} \cup E'_{4,4} = E(G')$. If $E'_{3,6}$ and $E'_{4,4}$ are both non-empty, then there exists a face with edges in both $E'_{3,6}$ and $E'_{4,4}$, which is clearly absurd. Thus, either $E'_{3,6} = E(G')$ or $E'_{4,4} = E(G')$.

Let us first consider the case $E'_{4,4} = E(G')$, when every edge in G' is shared by two C_4 faces.

Claim 10 A C_4 face in G' can only share an edge with another C_4 face if neither is the remains of two adjacent C_3 faces of G.

Proof First, note that we may discount the possibility that $G' = C_4$, since this would imply that G also had only 4 vertices. Hence, two C_4 faces in G' can only share at most two edges. In the case when two C_4 faces share exactly one edge, it is clear that neither can be the remains of two adjacent C_3 faces without creating a C_5 in G. If two C_4 faces share two edges, then they must look as shown in Figure 9, where the C_4 faces $v_1v_2v_3v_4$ and $v_2v_3v_4v_5$ share the edges



Figure 9: Two C_4 faces that share two edges.

 v_2v_3 and v_3v_4 . However, note that the condition $\delta(G) \geq 3$ implies that we must then have either $v_1v_3 \in E(G)$ or $v_3v_5 \in E(G)$, and both of these would produce a C_5 . End of Proof of Claim 10

From Claim 10, it follows that if $E'_{4,4} = E(G')$ then there can be no edges in $E(G') \setminus E(G)$, and hence $E(G) = 2k - 4 < \frac{12k - 33}{5}$, and so we would be done.

Let us now consider the case when $E'_{3,6} = E(G')$, i.e. when all edges in G'lie on the boundary between a C_3 face and a C_6 face. In particular, let us consider the vertices with $\deg_{G'} \leq 3$ (note that such vertices must exist, since E(G') < 2|G'|), and let us copy the proof of Observation 7. Observe firstly that there can't be any vertices with degree less than 2, since all faces in G' are cycles. Secondly, note that if a vertex v of degree 2 exists, then it must lie on the boundary of a C_3 face v, v_1, v_2 and a C_6 face $v, v_1, v_3, v_4, v_5, v_2$, as shown in Figure 10, in which case v_1, v_3, v_4, v_5, v_2 would be a cycle of size 5, which is a contradiction. Finally, note that if $\deg(v) = 3$ and $\Gamma(v) = \{v_1, v_2, v_3\}$, then it is impossible for all the edges vv_1, vv_2 and vv_3 to be in a C_3 face without having two adjacent such faces, which would also be a contradiction.



Figure 10: A vertex v of degree 2 in the case when $E'_{3,6} = E(G')$.

Part IV In this final part of the proof, we shall deal with the remaining case when $e(G') \leq 2|G'| - 5$. The proof will follow from a very careful examination of the impact of vertices of degree 2 and 3 in G'. We shall look first at vertices of degree 3, obtaining equation (5). We shall then consider vertices of degree 2, producing (via equations (6) and (7) and Claim 12) Claim 11 and equation (9). Finally, by plugging these newly derived inequalities into our 'standard' framework (i.e. the fact that 2e(G) equals the sum of all face sizes, plus Euler's formula), we obtain the result.

Let d'_i denote the number of vertices of degree i in G', and recall that there can't be any vertices with degree less than 2, since all faces in G' are cycles. Note that we must then have

$$2d'_{2} + d'_{3} \geq \begin{cases} 10 & \text{if } e(G') = 2|G'| - 5\\ 12 & \text{if } e(G') \le 2|G'| - 6 \end{cases}$$
(4)

since $2d'_2 + d'_3 \ge \sum_{i>2} (4-i)d'_i = 4|G'| - 2e(G').$

Consider a vertex v of degree 3 in G', and, in particular, consider the three faces that it lies on. Note that if any of these is a C_3 or is a C_4 that was formed from two adjacent C_3 's in G, then neither of the others can have size less than 6 (this follows from Claims 9 and 10 and the recollection that G' contains no adjacent C_3 faces). Hence, it follows that either (i) none of the three edges incident to v were in any C_3 faces in G or (ii) there is at least one edge incident to v that was in two faces of size at least 6 in G.

Thus, we have

$$2\left(\frac{1}{3}(\# \text{ edges not in a } C_3 \text{ face in } G) + \# \text{ edges in two faces of size at least 6 in } G\right) \geq d'_3$$

(where the multiplication of the left-hand side by 2 is to allow for possible double-counting of an edge). If we let e_i denote the number of edges of G that are in at least one C_i face, then this corresponds to

$$2\left(\frac{1}{3}(e(G) - e_3) + (e(G) - e_3 - e_4)\right) \ge d'_3.$$
 (5)

Now consider a vertex u of degree 2 in G', and let $\Gamma_{G'}(u) = \{u_1, u_2\}$. It must be that either (i) u lies on a C_3 face uu_1u_2 (formed from the outside of

a K_4 in G), as shown in Figure 11, or (ii) u lies on a C_4 face $uu_1u_3u_2$ formed



Figure 11: The vertex u in case (i).

from two adjacent C_3 faces $T_1 = uu_1u_3$ and $T_2 = uu_2u_3$, as shown in Figure 12.



Figure 12: The vertex u in case (ii).

In case (i), it will be important to note that the other face containing u must have size at least 7. In case (ii), it will be important to note that the other faces bordering T_1 and T_2 in G cannot have been C_3 's themselves (since only the edge uu_3 was deleted when G' was constructed), and so T_1 and T_2 are two C_3 faces in G that both have exactly one edge (uu_3) that is also in another C_3 face in G.

Before proceeding, let us also take care to allow for any possible multiplecounting. For case (i), note that it is possible for a face of size $i \ge 7$ to be adjacent in G' to i-6 faces of size 3; for case (ii), note that both vertices u and u_3 in Figure 12 can have degree 2 in G'.

Thus, allowing for this multiple-counting, we obtain

$$2\left(\frac{1}{2}(\# C_3 \text{ faces in } G \text{ that have exactly one}\right)$$

edge that is also in another C_3 face in G)
 $+\# C_7$ faces in $G + 2(\# C_8 \text{ faces in } G)$
 $+3(\# C_9 \text{ faces in } G) + \dots \geq d'_2.$ (6)

Let f_i denote the number of C_i faces in G, and let $f_{3,1}$ denote the number of C_3 faces in G that have exactly one edge that is also in another C_3 face in G. Then (6) translates to

$$f_{3,1} + \sum_{i \ge 7} (i-6)f_i \ge d'_2.$$
(7)

Claim 11 If e(G') = 2|G'| - 5, then $\sum_{i \ge 7} (i-6)f_i \ge d'_2 - 2$.

Proof Note that, since $G \not\supseteq C_5$, $f_{3,1}$ must be equal to 2l, where l is precisely the number of C_4 faces of G' that were formed from the remains of two adjacent C_3 faces of G.

Let $e'_{4,1}$ denote the number of edges of G' that are in exactly one C_4 face in G'. Then, by Claim 10, we have $e'_{4,1} \ge 4l$. Note that $e'_4 - e'_{4,1}$ will be precisely the number of edges of G' that are in exactly two C_4 faces in G', and thus

$$f'_{4} = \frac{e'_{4,1} + 2(e'_{4} - e'_{4,1})}{4}$$

$$= \frac{2e'_{4} - e'_{4,1}}{4}$$

$$\leq \frac{2e'_{4} - 4l}{4}$$

$$= \frac{e'_{4}}{2} - l.$$
(8)

We know that $2e(G') \ge 3f'_3 + 4f'_4 + 6(f' - f'_3 - f'_4)$, and so

$$\begin{array}{rcl} 6f' &\leq & 2e(G') + 3f'_3 + 2f'_4 \\ &\leq & 2e(G') + e'_3 + e'_4 - 2l \text{ using } (2) \text{ and } (8) \\ &\leq & 2e(G') + e'_3 + e(G') - e'_3 - 2l \text{ by Claim 9} \\ &= & 3e(G') - 2l. \end{array}$$

Hence, $f' \leq \frac{1}{2}e(G') - \frac{l}{3}$.

Euler's formula e(G') = |G'| - 2 + f' then gives $e(G') \le 2(|G'| - (2 + \frac{l}{3}))$. If e(G') = 2|G'| - 5, this implies that $l \le 1$ (since l is an integer), and hence that $f_{3,1} \le 2$. The result then follows from (7). End of Proof of Claim 11

Claim 12 $f_{3,1} \leq 2e_3 - 4f_3$.

Proof Note that it is impossible for all three edges in a C_3 face in G to also be in another C_3 face in G without creating a C_5 unless $G = C_3$ or $G = K_4$, both of which may be ignored since we know that k > 4. Thus,

$$f_{3} = \frac{1}{3} (\# \text{ edges in at least one } C_{3} \text{ face in } G + \# \text{ edges in two } C_{3} \text{ faces in } G)$$

$$\leq \frac{1}{3} \left(e_{3} + \frac{1}{2} (f_{3,1} + 2(f_{3} - f_{3,1})) \right)$$

$$= \frac{1}{3} e_{3} + \frac{1}{3} f_{3} - \frac{1}{6} f_{3,1},$$

from which the claim follows. End of Proof of Claim 12

By (7) and Claim 12, we have

$$2e_3 - 4f_3 + \sum_{i \ge 7} (i-6)f_i \ge d'_2.$$
(9)

We shall now use (5), (9) and Claim 11 to obtain our result. Letting f denote $\sum_i f_i$, we know that $2e(G) = 3f_3 + 4f_4 + 6(f - f_3 - f_4) + \sum_{i \ge 7} (i - 6)f_i$, so we have

$$\begin{array}{lll} 6f &=& 2e(G)+3f_3+2f_4-\sum_{i\geq 7}(i-6)f_i\\ &\leq& 2e(G)+3f_3+e_4-\sum_{i\geq 7}(i-6)f_i\\ &=& 2e(G)+\frac{3}{2}e_3+e_4-\sum_{i\geq 7}(i-6)f_i-\frac{3}{4}(2e_3-4f_3)\\ &=& 2e(G)+\frac{3}{2}e_3+e(G)-e_3-\sum_{i\geq 7}(i-6)f_i-\frac{3}{4}(2e_3-4f_3)-(e(G)-e_3-e_4)\\ &=& 3e(G)+\frac{1}{2}e_3-\sum_{i\geq 7}(i-6)f_i-\frac{3}{4}(2e_3-4f_3)-(e(G)-e_3-e_4)\\ &=& \frac{7}{2}e(G)-\sum_{i\geq 7}(i-6)f_i-\frac{3}{4}(2e_3-4f_3)-(e(G)-e_3-e_4)-\frac{1}{2}(e(G)-e_3)\\ &=& \frac{7}{2}e(G)-\frac{3}{4}\left(2e_3-4f_3+\sum_{i\geq 7}(i-6)f_i\right)-\left(\frac{1}{3}(e(G)-e_3)+(e(G)-e_3-e_4)\right)\\ &-\frac{1}{4}\sum_{i\geq 7}(i-6)f_i-\frac{1}{6}(e(G)-e_3)\\ &\leq& \frac{7}{2}e(G)-\frac{3}{4}d_2'-\frac{1}{2}d_3'-\frac{1}{4}\sum_{i\geq 7}(i-6)f_i \text{ by (5) and (9).} \end{array}$$

Thus, $f \leq \frac{7}{12}e(G) - \frac{1}{8}d'_2 - \frac{1}{12}d'_3 - \frac{1}{24}\sum_{i\geq 7}(i-6)f_i$, and so Euler's formula e(G) = k-2+f then implies that

$$e(G) \leq \frac{12}{5} \left(k - \left(2 + \frac{1}{8}d'_2 + \frac{1}{12}d'_3 + \frac{1}{24}\sum_{i\geq 7}(i-6)f_i\right)\right)$$

=
$$\frac{12k - \left(24 + \frac{3}{2}d'_2 + d'_3 + \frac{1}{2}\sum_{i\geq 7}(i-6)f_i\right)}{5}.$$

It now only remains to show that $\frac{3}{2}d'_2 + d'_3 + \frac{1}{2}\sum_{i\geq 7}(i-6)f_i \geq 9$. If $e(G') \leq 2|G'|-6$, then $2d'_2+d'_3 \geq 12$ (from (4)), and so certainly $\frac{3}{2}d'_2+d'_3 \geq 9$. If e(G') = 2|G'|-5, then recall (from Claim 11) that we have $\sum_{i\geq 7}(i-6)f_i \geq d'_2-2$, and hence $\frac{3}{2}d'_2 + d'_3 + \frac{1}{2}\sum_{i\geq 7}(i-6)f_i \geq 2d'_2 + d'_3 - 1 \geq 9$ (using (4)).

$5 \quad C_5 - extremal graphs$

In the previous section, we showed that $\exp_{\mathcal{P}}(n, C_5) \leq \frac{12n-33}{5}$ for all $n \geq 11$. In this section we shall now complete matters (in Theorem 14) by demonstrating

that this bound is tight.

We start with a lemma that will prove useful:

Lemma 13 For infinitely many values of k, there exists a plane triangulation T_k with vertex set $\{v_1, v_2, \ldots, v_k\}$ satisfying (i) $\deg(v_i) = 4$ for $i \le 6$, (ii) $\deg(v_i) = 6$ for i > 6, and (iii) $E(T_k) \supset \{v_1v_2, v_3v_4, v_5v_6\}$.

Proof Note first that the triangulation T_6 shown in Figure 13 certainly satisfies the conditions for k = 6.



Figure 13: The triangulation T_6 .

We now proceed inductively. Given a triangulation satisfying the conditions, let us construct a larger triangulation by subdividing all the edges and inserting triangles between the new vertices, as shown in Figure 14.



Figure 14: Constructing a larger triangulation.

It can be observed that conditions (i) and (ii) will also be satisfied by the new triangulation, but (due to the subdividing of edges) not condition (iii). However, we may then simply modify the new triangulation into one that does satisfy all three conditions by applying the local transformation shown in Figure 15 (which includes some relabelling of the vertices) at the relevant three places. \Box

We may now proceed with our construction of extremal graphs:

Theorem 14 There exist infinitely many values of n for which $ex_{\mathcal{P}}(n, C_5) = \frac{12n-33}{5}$.

Proof We shall use the triangulations guaranteed by Lemma 13 to construct specific plane graphs with 15k + 9 vertices and $\frac{12(15k+9)-33}{5}$ edges, and we shall then show that these graphs are C_5 -free.



Figure 15: Modifying the triangulation to satisfy condition (iii).

Let T_k be a plane triangulation satisfying the conditions of Lemma 13, and let E^* denote the set of edges $\{v_1v_2, v_3v_4, v_5v_6\}$. Let us now construct a new plane graph G by (i) subdividing all edges in $E(T_k) \setminus E^*$ and (ii) replacing all edges in E^* with the 'diamond-holder' structure shown in Figure 16. An



Figure 16: Replacing an edge with a 'diamond-holder'.

example of this is shown in Figure 17.

Note that the k original vertices (shown as large vertices in Figure 17) will now all have degree 6 (including the six that only had degree 4 in T_k), and so each can be thought of as forming the centre of a 'star' of size six (i.e. $K_{1,6}$). Hence, G consists of k of these stars, together with the three 'diamonds' (i.e. K_4 minus an edge) from the centres of the diamond-holders.

Finally, let us now replace each star with the 'snowflake' structure shown in Figure 18, where the central vertex is replaced by a hexagon and the edges are replaced by K_4 's. Thus, the final graph G^* will consist of k snowflakes and three diamonds.

Observe that each diamond contains 4 vertices, 2 of which will be shared with snowflakes, and 5 edges; each snowflake contains 18 vertices, 6 of which will be shared with other snowflakes/diamonds, and 36 edges.

Hence, allowing for double-counting of the shared vertices, we find that G^* contains $3(2+\frac{2}{2}) + k(12+\frac{6}{2}) = 15k+9$ vertices and $3 \times 5 + 36k = \frac{12(15k+9)-33}{5}$ edges, thus achieving the bound of Theorem 8.

It now only remains to show that G^* is C_5 -free.

To see this, let us first examine the earlier graph G. Due to the subdividing of the edges of T_k (and the resulting doubling of all cycle lengths), note that Gcan contain no cycles of length less than six apart from those (of length three and four) contained entirely within a diamond-holder.



Figure 17: Constructing a new graph G from a triangulation T_{15} .



Figure 18: Replacing a star with a 'snowflake'.

Now observe that any cycle in G^* of length less than six must either induce a cycle of length less than six in G or must be contained entirely within a single snowflake.

Hence, any C_5 in G^* must either be contained entirely within the structure formed from a diamond-holder (see Figure 19) or must be contained within one snowflake. However, it can be seen that Figures 18 and 19 are both C_5 -free, and hence so is G^* .

As an aside, let us remark that it is also possible to produce graphs attaining equality in Theorem 8 via alternative constructions to that used in Theorem 14 (for example, it is possible to obtain such graphs when $n \in \{14, 39, 54\}$).

6 Concluding remarks

In this paper, we have made a start on the topic of 'extremal' planar graphs, defining $ex_{\mathcal{P}}(n, H)$ to be the maximum number of edges possible in a planar



Figure 19: The structure formed from a diamond-holder.

graph on *n* vertices that does not contain a given graph *H* as a subgraph. We observed that the case when *H* is a complete graph is straightforward, and consequently focused on the more exciting examples of C_4 and C_5 , obtaining the tight bounds $\exp_{\mathcal{P}}(n, C_4) \leq \frac{15}{7}(n-2)$ for all $n \geq 4$ (in Section 2) and $\exp_{\mathcal{P}}(n, C_5) \leq \frac{12n-33}{5}$ for all $n \geq 11$ (in Section 4).

The next step would seem to be to look at larger cycles. As the C_5 proof is rather intricate, it may be impractical to try to obtain exact results, but it would perhaps be reasonable to investigate the leading term in the extremal numbers and to work towards a general formula for this.

Of course, there are also many other options to choose for the forbidden subgraph — the simplest would perhaps be K_4 minus one edge. In particular, it would be interesting to discover whether or not the chromatic number ever plays a role, as in the Erdos-Stone Theorem.

Another angle to take on the problem would be to examine which graphs have extremal number 3n - 6. We have observed that K_4 certainly falls into this category, and it would be interesting to develop necessary and sufficient conditions for all such graphs.

There are also various ways to extend the topic further. For example, one natural idea would be to investigate extremal problems on other surfaces, such as the torus, while another option would be to forbid several subgraphs at once.

There seems to be a large amount of uncharted territory here, and it is hoped that this paper has made a useful start.

Acknowledgements

This work was largely produced at the Ecole Polytechnique as part of the European Research Council funded project ERC StG 208471 ExploreMaps, and I am grateful for this support. I would also like to thank the referees for some very helpful comments.

References

- [1] B. Bollobas, Modern Graph Theory, Springer, New York, 1998, page 120.
- [2] P. Erdős, A. Stone, On the structure of linear graphs, Bulletin of the American Mathematical Society 52 (1946), 1087–1091.
- [3] O. Giménez, M. Noy, Asymptotic enumeration and limit laws of planar graphs, Journal of the American Mathematical Society **22** (2009), 309–329.
- [4] C. McDiarmid, A. Steger, D. Welsh, Random planar graphs, Journal of Combinatorial Theory Series B 93 (2005), 187–205.
- [5] P. Turán, On an extremal problem in graph theory, Matematikai és Fizikai Lapok 48 (1941), 436–452.