HADWIGER'S CONJECTURE FOR ℓ-LINK GRAPHS

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Abstract. In this paper we define and study a new family of graphs that generalises the notions of line graphs and path graphs. Let G be a graph with no loops but possibly with parallel edges. An *ℓ-link* of G is a walk of G of length $\ell \geqslant 0$ in which consecutive edges are different. We identify an ℓ-link with its reverse sequence. The *ℓ-link graph* $\mathbb{L}_\ell(G)$ of G is the graph with vertices the ℓ-links of G, such that two vertices are joined by $\mu \geqslant 0$ edges in $\mathbb{L}_\ell(G)$ if they correspond to two subsequences of each of μ $(\ell+1)$-links of G.

By revealing a recursive structure, we bound from above the chromatic number of ℓ-link graphs. As a corollary, for a given graph G and large enough ℓ, $\mathbb{L}_\ell(G)$ is 3-colourable. By investigating the shunting of ℓ-links in G, we show that the Hadwiger number of a nonempty $\mathbb{L}_\ell(G)$ is greater or equal to that of G. Hadwiger's conjecture states that the Hadwiger number of a graph is at least the chromatic number of that graph. The conjecture has been proved by Reed and Seymour (2004) for line graphs, and hence 1-link graphs. We prove the conjecture for a wide class of ℓ-link graphs.

Keywords. ℓ-link graph; path graph; chromatic number; graph minor; Hadwiger's conjecture.

1. Introduction and main results

We introduce a new family of graphs, called *ℓ-link graphs*, which generalises the notions of line graphs and path graphs. Such a graph is constructed from a certain kind of walk of length $\ell \geqslant 0$ in a given graph G. To ensure that the constructed graph is undirected, G is undirected, and we identify a walk with its reverse sequence. To avoid loops, G is loopless, and the consecutive edges in each walk are different. Such a walk is called an *ℓ-link*. For example, a 0-link is a vertex, a 1-link is an edge, and a 2-link consists of two edges with an end vertex in common. An *ℓ-path* is an ℓ-link without repeated vertices. We use $\mathscr{L}_\ell(G)$ and $\mathscr{P}_\ell(G)$ to denote the sets of ℓ-links and ℓ-paths of G respectively. There have been a number of families of graphs constructed from ℓ-links. As one of the most commonly studied graphs, the *line graph* $\mathbb{L}(G)$, introduced by Whitney [22], is the simple graph with vertex set $E(G)$, in which two vertices are adjacent if their corresponding edges are incident to a common vertex. More generally,

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the *ℓ-path graph* $\mathbb{P}_\ell(G)$ is the simple graph with vertex set $\mathscr{P}_\ell(G)$, where two vertices are adjacent if the union of their corresponding ℓ-paths forms a path or a cycle of length $\ell + 1$. Note that $\mathbb{P}_\ell(G)$ is the $\mathbb{P}_{\ell+1}$-graph of G introduced by Broersma and Hoede [4]. Inspired by these graphs, we define the *ℓ-link graph* $\mathbb{L}_\ell(G)$ of G to be the graph with vertex set $\mathscr{L}_\ell(G)$, in which two vertices are joined by $\mu \geqslant 0$ edges in $\mathbb{L}_\ell(G)$ if they correspond to two subsequences of each of μ $(\ell+1)$-links of G. More strict definitions can be found in Section 2, together with some other related graphs.

This paper studies the structure, colouring and minors of ℓ-link graphs including a proof of Hadwiger's conjecture for a wide class of ℓ-link graphs. By default $\ell \geqslant 0$ is an integer. And all graphs are finite, undirected and loopless. Parallel edges are admitted unless we specify the graph to be *simple*.

1.1. **Graph colouring.** Let $t \geqslant 0$ be an integer. A *t-colouring* of G is a map $\lambda : V(G) \to [t] := \{1, 2, \dots, t\}$ such that $\lambda(u) \neq \lambda(v)$ whenever $u, v \in V(G)$ are adjacent in G. A graph with a t-colouring is *t-colourable*. The *chromatic number* $\chi(G)$ is the minimum t such that G is t-colourable. Similarly, an *t-edge-colouring* of G is a map $\lambda : E(G) \to [t]$ such that $\lambda(e) \neq \lambda(f)$ whenever $e, f \in E(G)$ are incident to a common vertex in G. The *edge-chromatic number* $\chi'(G)$ of G is the minimum t such that G admits a t-edge-colouring. Let $\chi_\ell(G) := \chi(\mathbb{L}_\ell(G))$, and $\Delta(G)$ be the maximum degree of G. By [6, Proposition 5.2.2], $\chi_0(G) = \chi(G) \leqslant \Delta(G) + 1$. Shannon [17] proved that $\chi_1(G) = \chi'(G) \leqslant \frac{3}{2}\Delta(G)$. We prove a recursive structure for ℓ-link graphs which leads to the following upper bounds for $\chi_\ell(G)$:

Theorem 1.1. *Let G be a graph, $\chi := \chi(G)$, $\chi' := \chi'(G)$, and $\Delta := \Delta(G)$.*
- **(1)** *If $\ell \geqslant 0$ is even, then $\chi_\ell(G) \leqslant \min\{\chi, \lfloor (\frac{2}{3})^{\ell/2}(\chi - 3)\rfloor + 3\}$.*
- **(2)** *If $\ell \geqslant 1$ is odd, then $\chi_\ell(G) \leqslant \min\{\chi', \lfloor (\frac{2}{3})^{\frac{\ell-1}{2}}(\chi' - 3)\rfloor + 3\}$.*
- **(3)** *If $\ell \neq 1$, then $\chi_\ell(G) \leqslant \Delta + 1$.*
- **(4)** *If $\ell \geqslant 2$, then $\chi_\ell(G) \leqslant \chi_{\ell-2}(G)$.*

Theorem 1.1 implies that $\mathbb{L}_\ell(G)$ is 3-colourable for large enough ℓ.

Corollary 1.2. *For each graph G, $\mathbb{L}_\ell(G)$ is 3-colourable in the following cases:*
- **(1)** *$\ell \geqslant 0$ is even, and either $\chi(G) \leqslant 3$ or $\ell > 2\log_{1.5}(\chi(G) - 3)$.*
- **(2)** *$\ell \geqslant 1$ is odd, and either $\chi'(G) \leqslant 3$ or $\ell > 2\log_{1.5}(\chi'(G) - 3) + 1$.*

As explained in Section 2, this corollary is related to and implies a result by Kawai and Shibata [14].

1.2. **Graph minors.** By *contracting* an edge we mean identifying its end vertices and deleting possible resulting loops. A graph H is a *minor* of G if H can be obtained from a subgraph of G by contracting edges. An *H-minor* is a minor of G that is isomorphic to H. The *Hadwiger number* $\eta(G)$ of G is the maximum integer t such that G contains a K_t-minor. Denote by $\delta(G)$ the minimum degree of G. The *degeneracy* $\mathrm{d}(G)$ of G is the maximum $\delta(H)$ over the subgraphs H of G. We prove the following:

Theorem 1.3. *Let $\ell \geqslant 1$, and G be a graph such that $\mathbb{L}_\ell(G)$ contains at least one edge. Then $\eta(\mathbb{L}_\ell(G)) \geqslant \max\{\eta(G), \mathrm{d}(G)\}$.*

By definition $\mathbb{L}(G)$ is the underlying simple graph of $\mathbb{L}_1(G)$. And $\mathbb{L}_\ell(G) = \mathbb{P}_\ell(G)$ if $girth(G) > \{\ell, 2\}$. Thus Theorem 1.3 can be applied to path graphs.

Corollary 1.4. *Let $\ell \geqslant 1$, and G be a graph of girth at least $\ell + 1$ such that $\mathbb{P}_\ell(G)$ contains at least one edge. Then $\eta(\mathbb{P}_\ell(G)) \geqslant \max\{\eta(G), \mathrm{d}(G)\}$.*

As a far-reaching generalisation of the four-colour theorem, in 1943, Hugo Hadwiger [9] conjectured the following:

Hadwiger's conjecture: $\eta(G) \geqslant \chi(G)$ for every graph G.

Hadwiger's conjecture was proved by Robertson, Seymour and Thomas [16] for $\chi(G) \leqslant 6$. The conjecture for line graphs, or equivalently for 1-link graphs, was proved by Reed and Seymour [15]. We prove the following:

Theorem 1.5. *Hadwiger's conjecture is true for $\mathbb{L}_\ell(G)$ in the following cases:*

(1) $\ell \geqslant 1$ *and G is biconnected.*
(2) $\ell \geqslant 2$ *is an even integer.*
(3) $\mathrm{d}(G) \geqslant 3$ *and* $\ell > 2\log_{1.5} \frac{\Delta(G)-2}{\mathrm{d}(G)-2} + 3$.
(4) $\Delta(G) \geqslant 3$ *and* $\ell > 2\log_{1.5}(\Delta(G) - 2) - 3.83$.
(5) $\Delta(G) \leqslant 5$.

The corresponding results for path graphs are listed below:

Corollary 1.6. *Let G be a graph of girth at least $\ell + 1$. Then Hadwiger's conjecture holds for $\mathbb{P}_\ell(G)$ in the cases of Theorem 1.5 (1) – (5).*

2. Definitions and terminology

We now give some formal definitions. A graph G is *null* if $V(G) = \emptyset$, and *nonnull* otherwise. A nonnull graph G is *empty* if $E(G) = \emptyset$, and nonempty otherwise. A *unit* is a vertex or an edge. The subgraph of G induced by $V \subseteq V(G)$ is the maximal subgraph of G with vertex set V. And in this case, the subgraph is called an *induced subgraph* of G. For $\emptyset \neq E \subseteq E(G)$, the subgraph of G induced by $E \cup V$ is the minimal subgraph of G with edge set E, and vertex set including V.

For more accurate analysis, we need to define *ℓ-arcs*. An *ℓ-arc* (or *$*$-arc* if we ignore the length) of G is an alternating sequence $\vec{L} := (v_0, e_1, \dots, e_\ell, v_\ell)$ of units of G such that the end vertices of $e_i \in E(G)$ are v_{i-1} and v_i for $i \in [\ell]$, and that $e_i \neq e_{i+1}$ for $i \in [\ell - 1]$. The *direction* of $\vec{L}$ is its vertex sequence $(v_0, v_1, \dots, v_\ell)$.

In algebraic graph theory, ℓ-arcs in simple graphs have been widely studied [18, 19, 21, 3]. Note that $\vec{L}$ and its *reverse* $-\vec{L} := (v_\ell, e_\ell, \dots, e_1, v_0)$ are different unless $\ell = 0$. The ℓ-link (or $*$-*link* if the length is ignored) $L := [v_0, e_1, \dots, e_\ell, v_\ell]$ is obtained by taking $\vec{L}$ and $-\vec{L}$ as a single object. For $0 \leqslant i \leqslant j \leqslant \ell$, the $(j-i)$-arc $\vec{L}(i,j) := (v_i, e_{i+1}, \dots, e_j, v_j)$ and the $(j-i)$-link $\vec{L}[i,j] := [v_i, e_{i+1}, \dots, e_j, v_j]$ are called *segments* of $\vec{L}$ and L respectively. We may write $\vec{L}(j,i) := -\vec{L}(i,j)$, and $\vec{L}[j,i] := \vec{L}[i,j]$. These segments are called *middle segments* if $i + j = \ell$. L is called an ℓ-*cycle* if $\ell \geqslant 2$, $v_0 = v_\ell$ and $\vec{L}[0, \ell-1]$ is an $(\ell-1)$-path. Denote by $\vec{\mathscr{L}_\ell}(G)$ and $\mathscr{C}_\ell(G)$ the sets of ℓ-arcs and ℓ-cycles of G respectively. Usually, $\vec{e_i} := (v_{i-1}, e_i, v_i)$ is called an *arc* for short. In particular, v_0, v_ℓ, e_1, e_ℓ, $\vec{e}_1$ and $\vec{e}_\ell$ are called the *tail vertex*, *head vertex*, *tail edge*, *head edge*, *tail arc*, and *head arc* of $\vec{L}$ respectively.

Godsil and Royle [8] defined the ℓ-*arc graph* $\mathbb{A}_\ell(G)$ to be the digraph with vertex set $\vec{\mathscr{L}_\ell}(G)$, such that there is an arc, labeled by $\vec{Q}$, from $\vec{Q}(0,\ell)$ to $\vec{Q}(1, \ell+1)$ in $\mathbb{A}_\ell(G)$ for every $\vec{Q} \in \vec{L}_{\ell+1}(G)$. The t-*dipole graph* D_t is the graph consists of two vertices and $t \geqslant 1$ edges between them. (See Figure 1(a) for D_3, and Figure 1(b) the 1-arc graph of D_3.) The ℓ^{th} *iterated line digraph* $\mathbb{A}^\ell(G)$ is $\mathbb{A}_1(G)$ if $\ell = 1$, and $\mathbb{A}_1(\mathbb{A}^{\ell-1}(G))$ if $\ell \geqslant 2$ (see [2]). Examples of undirected graphs constructed from ℓ-arcs can be found in [12, 11].

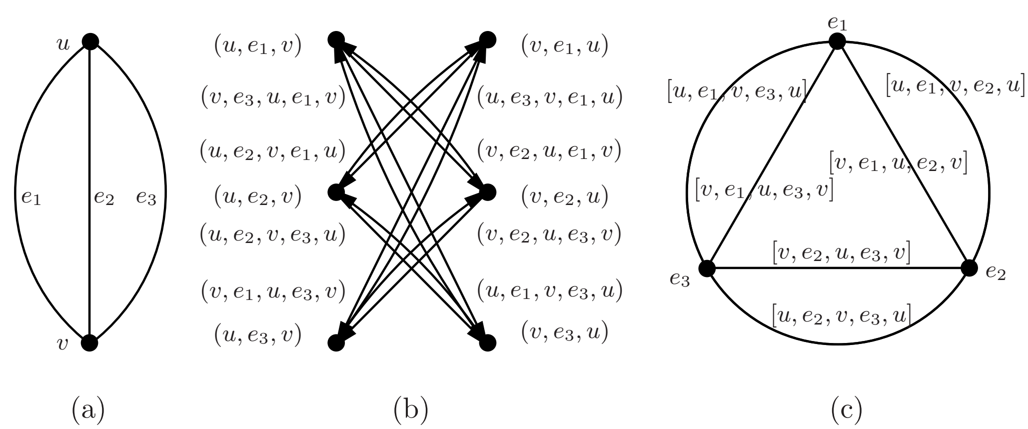

FIGURE 1. (a) D_3 (b) $\mathbb{A}_1(D_3)$ (c) $\mathbb{L}_1(D_3)$

Shunting of ℓ-arcs was introduced by Tutte [20]. We extend this motion to ℓ-links. For $\ell, s \geqslant 0$, and $\vec{Q} \in \vec{\mathscr{L}}_{\ell+s}(G)$, let $\vec{L}_i := \vec{Q}(i, \ell+i)$ for $i \in [0,s]$, and $\vec{Q}_i := \vec{L}(i-1, \ell+i)$ for $i \in [s]$. Let $Q^{[\ell]} := [L_0, Q_1, L_1, \dots, L_{s-1}, Q_s, L_s]$. We say L_0 can be *shunted* to L_s through $\vec{Q}$ or Q. $Q^{\{\ell\}} := \{L_0, L_1, \dots, L_s\}$ is the set of *images* during this shunting. For $L, R \in \mathscr{L}_\ell(G)$, we say L can be *shunted* to R if there are ℓ-links $L = L_0, L_1, \dots, L_s = R$ such that L_{i-1} can be shunted to L_i

through some $*$-arc $\vec{Q}_i$ for $i \in [s]$. In Figure 2, $[u_0, f_0, v_0, e_0, v_1]$ can be shunted to $[v_1, e_0, v_0, e_1, v_1]$ through $(u_0, f_0, v_0, e_0, v_1, f_1, u_1)$ and $(u_1, f_1, v_1, e_0, v_0, e_1, v_1)$.

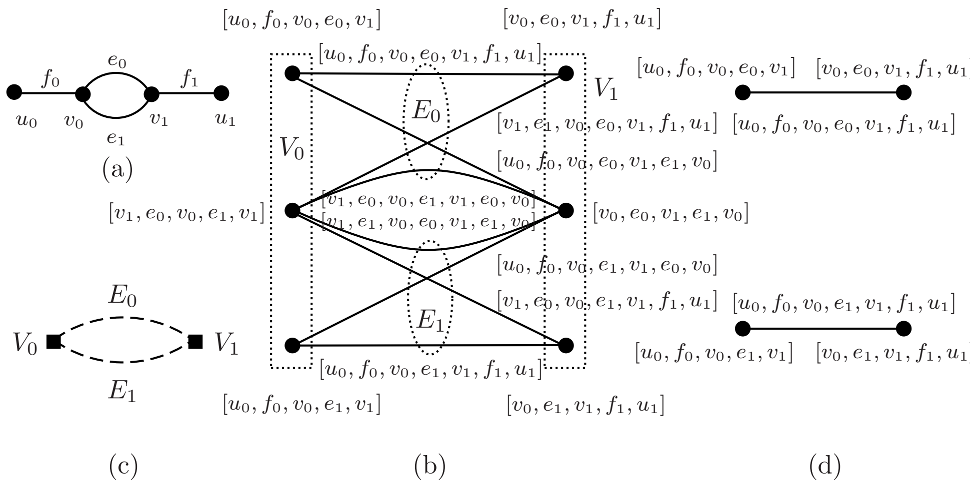

FIGURE 2. (a) G (b) $H := \mathbb{L}_2(G)$ (c) $H_{(\mathcal{V},\mathcal{E})}$ (d) $\mathbb{P}_2(G)$

For $L, R \in \mathscr{L}_\ell(G)$ and $\mathscr{Q} \subseteq \mathscr{L}_{\ell+1}(G)$, denote by $\mathscr{Q}(L, R)$ the set of $Q \in \mathscr{Q}$ such that L can be shunted to R through Q. We show in Section 3 that $|\mathscr{Q}(L, R)|$ is 0 or 1 if G is simple, and can be up to 2 if $\ell \geqslant 1$ and G contains parallel edges. A more formal definition of ℓ-link graphs is given below:

Definition 2.1. Let $\mathscr{L} \subseteq \mathscr{L}_\ell(G)$, and $\mathscr{Q} \subseteq \mathscr{L}_{\ell+1}(G)$. The *partial ℓ-link graph* $\mathbb{L}(G, \mathscr{L}, \mathscr{Q})$ of G, with respect to $\mathscr{L}$ and $\mathscr{Q}$, is the graph with vertex set $\mathscr{L}$, such that $L, R \in \mathscr{L}$ are joined by exactly $|\mathscr{Q}(L, R)|$ edges. In particular, $\mathbb{L}_\ell(G) = \mathbb{L}(G, \mathscr{L}_\ell(G), \mathscr{L}_{\ell+1}(G))$ is the ℓ-link graph of G.

Remark. We assign exclusively to each edge of $\mathbb{L}_\ell(G)$ between $L, R \in \mathscr{L}_\ell(G)$ a $Q \in \mathscr{L}_{\ell+1}(G)$ such that L can be shunted to R through Q, and refer to this edge simply as Q. In this sense, $Q^{[\ell]} := [L, Q, R]$ is a 1-link of $\mathbb{L}_\ell(G)$.

For example, the 1-link graph of D_3 can be seen in Figure 1(c). A 2-link graph is given in Figure 2(b), and a 2-path graph is depicted in Figure 2(d).

Reed and Seymour [15] pointed out that proving Hadwiger's conjecture for line graphs of multigraphs is more difficult than for that of simple graphs. This motivates us to work on the ℓ-link graphs of multigraphs. Diestel [6, page 28] explained that, in some situations, it is more natural to develop graph theory for multigraphs. The observation below follows from the definitions:

Observation 2.2. *$\mathbb{L}_0(G) = G$, $\mathbb{P}_1(G) = \mathbb{L}(G)$, and $\mathbb{P}_\ell(G)$ is the underlying simple graph of $\mathbb{L}_\ell(G)$ for $\ell \in \{0, 1\}$. For $\ell \geqslant 2$, $\mathbb{P}_\ell(G) = \mathbb{L}(G, \mathscr{P}_\ell(G), \mathscr{P}_{\ell+1}(G) \cup \mathscr{C}_{\ell+1}(G))$ is an induced subgraph of $\mathbb{L}_\ell(G)$. If G is simple, then $\mathbb{P}_\ell(G) = \mathbb{L}_\ell(G)$ for $\ell \in \{0, 1, 2\}$. Further, $\mathbb{P}_\ell(G) = \mathbb{L}_\ell(G)$ if girth$(G) > \max\{\ell, 2\}$.*

Let $\vec{Q} \in \vec{\mathscr{L}}_{\ell+s}(G)$, and $[L_0, Q_1, L_1, \dots, L_{s-1}, Q_s, L_s] := Q^{[\ell]}$. From Definition 2.1, for $i \in [s]$, Q_i is an edge of $H := \mathbb{L}_\ell(G)$ between $L_{i-1}, L_i \in V(H)$. So $Q^{[\ell]}$ is an s-link of H. In Figure 2(b), $[u_0, f_0, v_0, e_0, v_1, e_1, v_0, e_0, v_1]^{[2]} = [[u_0, f_0, v_0, e_0, v_1], [u_0, f_0, v_0, e_0, v_1, e_1, v_0], [v_0, e_0, v_1, e_1, v_0], [v_0, e_0, v_1, e_1, v_0, e_0, v_1], [v_1, e_1, v_0, e_0, v_1]]$ is a 2-path of H.

We say H is *homomorphic* to G, written $H \to G$, if there is an injection $\alpha : V(H) \cup E(H) \to V(G) \cup E(G)$ such that for $w \in V(H)$, $f \in E(H)$ and $[u, e, v] \in \mathscr{L}_1(H)$, their images $w^\alpha \in V(G)$, $f^\alpha \in E(G)$ and $[u^\alpha, e^\alpha, v^\alpha] \in \mathscr{L}_1(G)$. In this case, α is called a *homomorphism* from H to G. The definition here is a generalisation of the one for simple graphs by Godsil and Royle [8, Page 6]. A bijective homomorphism is an *isomorphism*. By Hell and Nešetřil [10], $\chi(H) \leqslant \chi(G)$ if $H \to G$. For instance, $\vec{L} \mapsto L$ for $\vec{L} \in \vec{\mathscr{L}}_\ell(G) \cup \vec{\mathscr{L}}_{\ell+1}(G)$ can be seen as a homomorphism from $\mathbb{A}_\ell(G)$ to $\mathbb{L}_\ell(G)$. By Bang-Jensen and Gutin [1], $\mathbb{A}_\ell(G) \cong \mathbb{A}^\ell(G)$. So $\chi(\mathbb{A}^\ell(G)) = \chi(\mathbb{A}_\ell(G)) \leqslant \chi(\mathbb{L}_\ell(G)) \leqslant \chi_\ell(G)$. We emphasize that $\chi(\mathbb{A}^\ell(G))$ might be much less than $\chi_\ell(G)$. For example, as depicted in Figure 1, when $t \geqslant 3$, $\chi(\mathbb{A}^\ell(D_t)) = 2 < t = \chi_\ell(D_t)$. Kawai and Shibata proved that $\mathbb{A}^\ell(G)$ is 3-colourable for large enough ℓ. By the analysis above, Corollary 1.2 implies this result.

A graph homomorphism from H is usually represented by a vertex partition $\mathcal{V}$ and an edge partition $\mathcal{E}$ of H such that: **(a)** each part of $\mathcal{V}$ is an independent set of H, and **(b)** each part of $\mathcal{E}$ is incident to exactly two parts of $\mathcal{V}$. In this situation, for different $U, V \in \mathcal{V}$, define $\mu(U, V)$ to be the number of parts of $\mathcal{E}$ incident to both U and V. The *quotient graph* $H_{(\mathcal{V},\mathcal{E})}$ of H is defined to be the graph with vertex set $\mathcal{V}$, and for every pair of different $U, V \in \mathcal{V}$, there are exactly $\mu(U, V)$ edges between them. To avoid ambiguity, for $V \in \mathcal{V}$ and $E \in \mathcal{E}$, we use $V_\mathcal{V}$ and $E_\mathcal{E}$ to denote the corresponding vertex and edge of $H_{(\mathcal{V},\mathcal{E})}$, which defines a graph homomorphism from H to $H_{(\mathcal{V},\mathcal{E})}$. Sometimes, we only need the underlying simple graph $H_\mathcal{V}$ of $H_{(\mathcal{V},\mathcal{E})}$.

For $\ell \geqslant 2$, there is a natural partition in an ℓ-link graph. For each $R \in \mathscr{L}_{\ell-2}(G)$, let $\mathscr{L}_\ell(R)$ be the set of ℓ-links of G with middle segment R. Clearly, $\mathcal{V}_\ell(G) := \{\mathscr{L}_\ell(R) \neq \emptyset | R \in \mathscr{L}_{\ell-2}(G)\}$ is a vertex partition of $\mathbb{L}_\ell(G)$. And $\mathcal{E}_\ell(G) := \{\mathscr{L}_{\ell+1}(P) \neq \emptyset | P \in \mathscr{L}_{\ell-1}(G)\}$ is an edge partition of $\mathbb{L}_\ell(G)$. Consider the 2-link graph H in Figure 2(b). The vertex and edge partitions of H are indicated by the dotted rectangles and ellipses respectively. The corresponding quotient graph is given in Figure 2(c).

Special partitions are required to describe the structure of ℓ-link graphs. Let H be a graph admitting partitions $\mathcal{V}$ of $V(H)$ and $\mathcal{E}$ of $E(H)$ that satisfy **(a)** and **(b)** above. $(\mathcal{V}, \mathcal{E})$ is called an *almost standard partition* of H if further:

(c) each part of $\mathcal{E}$ induces a complete bipartite subgraph of H,

(d) each vertex of H is incident to at most two parts of $\mathcal{E}$,

(e) for each $V \in \mathcal{V}$, and different $E, F \in \mathcal{E}$, V contains at most one vertex incident to both E and F.

If $\ell \geqslant 2$ is an even integer, and G is a simple graph, then $\mathbb{L}_\ell(G)$ is isomorphic to the $(2, \ell/2)$-*double star graph* of G introduced by Jia [11]. While this paper focuses on the combinatorial properties including connectedness, colouring and minors of $\mathbb{L}_\ell(G)$, a series of companion papers have been composed to contribute to the recognition and determination problems and algorithms. For example, a joint work by Ellingham and Jia [7] shows that, for a given graph H, there is at most one pair (G, ℓ), where $\ell \geqslant 2$, and G is a simple graph of minimum degree at least 3, such that $\mathbb{L}_\ell(G)$ is isomorphic to H. Moreover, such a pair can be determined from H in linear time.

3. General structure of ℓ-link graphs

We begin by determining some basic properties of ℓ-link graphs, including their multiplicity and connectedness. The work in this section forms the basis for our main results on colouring and minors of ℓ-link graphs.

Let us first fix some concepts by two observations.

Observation 3.1. *The number of edges of $\mathbb{L}_\ell(G)$ is equal to the number of vertices of $\mathbb{L}_{\ell+1}(G)$. In particular, if G is r-regular for some $r \geqslant 2$, then this number is $|E(G)|(r-1)^\ell$. If further $\ell \geqslant 1$, then $\mathbb{L}_\ell(G)$ is $2(r-1)$-regular.*

Proof. Let G be r-regular, $n := |V(G)|$ and $m := |E(G)|$. We prove that $|\mathscr{L}_{\ell+1}(G)| = m(r-1)^\ell$ by induction on ℓ. It is trivial for $\ell = 0$. For $\ell = 1$, $|\mathscr{L}_2(v)| = \binom{r}{2}$, and hence $|\mathscr{L}_2(G)| = \binom{r}{2}n = m(r-1)$. Inductively assume $|\mathscr{L}_{\ell-1}(G)| = m(r-1)^{\ell-2}$ for some $\ell \geqslant 2$. For each $R \in \mathscr{L}_{\ell-1}(G)$, we have $|\mathscr{L}_{\ell+1}(R)| = (r-1)^2$ since $r \geqslant 2$. Thus $|\mathscr{L}_{\ell+1}(G)| = |\mathscr{L}_{\ell-1}(G)|(r-1)^2 = m(r-1)^\ell$ as desired. The other assertions follow from the definitions. ∎

Observation 3.2. *Let $n, m \geqslant 2$. If $\ell \geqslant 1$ is odd, then $\mathbb{L}_\ell(K_{n,m})$ is $(n+m-2)$-regular with order $nm[(n-1)(m-1)]^{\frac{\ell-1}{2}}$. If $\ell \geqslant 2$ is even, then $\mathbb{L}_\ell(K_{n,m})$ has average degree $\frac{4(n-1)(m-1)}{n+m-2}$, and order $\frac{1}{2}nm(n+m-2)[(n-1)(m-1)]^{\frac{\ell}{2}-1}$.*

Proof. Let $\ell \geqslant 1$ be odd, and L be an ℓ-link of $K_{n,m}$ with middle edge incident to a vertex u of degree n in $K_{n,m}$. It is not difficult to see that L can be shunted in one step to $n-1$ ℓ-links whose middle edge is incident to u. By symmetry, each vertex of $\mathbb{L}_\ell(K_{n,m})$ is incident to $(n-1)+(m-1) = n+m-2$ edges. Now we prove $|\mathscr{L}_\ell(K_{n,m})| = nm[(n-1)(m-1)]^{\frac{\ell-1}{2}}$ by induction on ℓ. Clearly, $|\mathscr{L}_1(K_{n,m})| = |E(K_{n,m})| = nm$. Inductively assume $|\mathscr{L}_{\ell-2}(K_{n,m})| = nm[(n-1)(m-1)]^{\frac{\ell-3}{2}}$ for some $\ell \geqslant 3$. For each $R \in \mathscr{L}_{\ell-2}(K_{n,m})$, we have $|\mathscr{L}_\ell(R)| = (n-1)(m-1)$. So $|\mathscr{L}_\ell(K_{n,m})| = |\mathscr{L}_{\ell-2}(K_{n,m})|(n-1)(m-1) = nm[(n-1)(m-1)]^{\frac{\ell-1}{2}}$ as desired. The even ℓ case is similar. ∎

3.1. **Loops and multiplicity.** Our next observation is a prerequisite for the study of the chromatic number since it indicates that ℓ-link graphs are loopless.

Observation 3.3. *For each* $(\ell+1)$*-arc* $\vec{Q}$*, we have* $\vec{Q}[0,\ell] \neq \vec{Q}[1,\ell+1]$.

Proof. Let G be a graph, and $\vec{Q} := (v_0, e_1, \dots, e_{\ell+1}, v_{\ell+1}) \in \vec{\mathscr{L}}_{\ell+1}(G)$. Since G is loopless, $v_0 \neq v_1$ and hence $\vec{Q}(0,\ell) \neq \vec{Q}(1,\ell+1)$. So the statement holds for $\ell = 0$. Now let $\ell \geqslant 1$. Suppose for a contradiction that $\vec{Q}(0,\ell) = -\vec{Q}(1,\ell+1)$. Then $v_i = v_{\ell+1-i}$ and $e_{i+1} = e_{\ell+1-i}$ for $i \in \{0,1,\dots,\ell\}$. If $\ell = 2s$ for some integer $s \geqslant 1$, then $v_s = v_{s+1}$, contradicting that G is loopless. If $\ell = 2s+1$ for some $s \geqslant 0$, then $e_{s+1} = e_{s+2}$, contradicting the definition of a $*$-arc. ∎

The following statement indicates that, for each $\ell \geqslant 1$, $\mathbb{L}_\ell(G)$ is simple if G is simple, and has multiplicity exactly 2 otherwise.

Observation 3.4. *Let G be a graph, $\ell \geqslant 1$, and $L_0, L_1 \in \mathscr{L}_\ell(G)$. Then L_0 can be shunted to L_1 through two $(\ell+1)$-links of G if and only if G contains a 2-cycle $O := [v_0, e_0, v_1, e_1, v_0]$, such that one of the following cases holds:*

- **(1)** *$\ell \geqslant 1$ is odd, and $L_i = [v_i, e_i, v_{1-i}, e_{1-i}, \dots, v_i, e_i, v_{1-i}] \in \mathscr{L}_\ell(O)$ for $i \in \{0,1\}$. In this case, $[v_i, e_i, v_{1-i}, e_{1-i}, \dots, v_{1-i}, e_{1-i}, v_i] \in \mathscr{L}_{\ell+1}(O)$, for $i \in \{0,1\}$, are the only two $(\ell+1)$-links available for the shunting.*
- **(2)** *$\ell \geqslant 2$ is even, and $L_i = [v_i, e_i, v_{1-i}, e_{1-i}, \dots, v_{1-i}, e_{1-i}, v_i] \in \mathscr{L}_\ell(O)$ for $i \in \{0,1\}$. In this case, $[v_i, e_i, v_{1-i}, e_{1-i}, \dots, v_i, e_i, v_{1-i}] \in \mathscr{L}_{\ell+1}(O)$, for $i \in \{0,1\}$, are the only two $(\ell+1)$-links available for the shunting.*

Proof. $(\Leftarrow)$ is trivial. For $(\Rightarrow)$, since L_0 can be shunted to L_1, there exists $\vec{L} := (v_0, e_0, v_1, \dots, v_\ell, e_\ell, v_{\ell+1}) \in \vec{\mathscr{L}}_{\ell+1}(G)$ such that $L_i = \vec{L}[i, \ell+i]$ for $i \in \{0,1\}$. Let $\vec{R} \in \vec{\mathscr{L}}_{\ell+1}(G) \backslash \{\vec{L}\}$ such that $L_i = \vec{R}[i, \ell+i]$. Then $\vec{L}(i,\ell+i)$ equals $\vec{R}(i,\ell+i)$ or $\vec{R}(\ell+i, i)$. Suppose for a contradiction that $\vec{L}(0,\ell) = \vec{R}(0,\ell)$. Then $\vec{L}(1,\ell) = \vec{R}(1,\ell)$. Since $\vec{L} \neq \vec{R}$, we have $\vec{L}(1,\ell+1) \neq \vec{R}(1,\ell+1)$. Thus $\vec{L}(1,\ell+1) = \vec{R}(\ell+1,1)$, and hence $\vec{L}(2,\ell+1) = \vec{R}(\ell,1) = \vec{L}(\ell,1)$, contradicting Observation 3.3. So $\vec{L}(0,\ell) = \vec{R}(\ell,0)$. Similarly, $\vec{L}(1,\ell) = \vec{R}(\ell+1,1)$. Consequently, $\vec{L}(0,\ell-1) = \vec{R}(\ell,1) = \vec{L}(2,\ell+1)$; that is, $v_j = v_0$ and $e_j = e_0$ if $j \in [0,\ell]$ is even, while $v_j = v_1$ and $e_j = e_1$ if $j \in [0,\ell+1]$ is odd. ∎

3.2. **Connectedness.** This subsection characterises when $\mathbb{L}_\ell(G)$ is connected. A middle segment of $L \in \mathscr{L}_\ell(G)$ is a *middle unit*, written c_L, if it is a unit of G. Note that c_L is a vertex if ℓ is even, and is an edge otherwise. Denote by $G(\ell)$ the subgraph of G induced by the middle units of ℓ-links of G.

The lemma below is important in dealing with the connectedness of ℓ-link graphs. Before stating it, we define a *conjunction* operation, which is an extension of an operation by Biggs [3, Chapter 17]. Let $\vec{L} := (v_0, e_1, v_1, \dots, e_\ell, v_\ell) \in \vec{\mathscr{L}}_\ell(G)$ and $\vec{R} := (u_0, f_1, u_1, \dots, f_s, u_s) \in \vec{\mathscr{L}}_s(G)$ such that $v_\ell = u_0$ and $e_\ell \neq f_1$.

The *conjunction* of $\vec{L}$ and $\vec{R}$ is $(\vec{L}.\vec{R}) := (v_0, e_1, \ldots, e_\ell, v_\ell = u_0, f_1, \ldots, f_s, u_s) \in \vec{\mathscr{L}}_{\ell+s}(G)$ or $[\vec{L}.\vec{R}] := [v_0, e_1, \ldots, e_\ell, v_\ell = u_0, f_1, \ldots, f_s, u_s] \in \mathscr{L}_{\ell+s}(G)$.

Lemma 3.5. *Let $\ell, s \geqslant 0$, and G be a connected graph. Then $G(\ell)$ is connected. And each s-link of $G(\ell)$ is a middle segment of a $(2\lfloor \frac{\ell}{2} \rfloor + s)$-link of G. Moreover, for ℓ-links L and R of G, there is an ℓ-link L' with middle unit c_L, and an ℓ-link R' with middle unit c_R, such that L' can be shunted to R'.*

Proof. For $\ell \in \{0, 1\}$, since G is connected, $G(\ell) = G$ and the lemma holds. Let $\ell := 2m \geqslant 2$ be even. $u, v \in V(G(\ell))$ if and only if they are middle vertices of some $\vec{L}, \vec{R} \in \vec{\mathscr{L}}_\ell(G)$ respectively. Since G is connected, there exists some $\vec{P} \in \vec{\mathscr{L}}_s(G)$ from (u, e, u_1) to (v_{s-1}, f, v). By Observation 3.3, $\vec{L}[m-1, m] \neq \vec{L}[m, m+1]$. For such an s-arc $\vec{P}$, without loss of generality, $e \neq \vec{L}[m-1, m]$, and similarly, $f \neq \vec{R}[m, m+1]$. Then $\vec{P}$ is a middle segment of $\vec{Q} := (\vec{L}(0, m).\vec{P}.\vec{R}(m, 2m)) \in \vec{\mathscr{L}}_{\ell+s}(G)$. So $\vec{P} \in \vec{\mathscr{L}}_s(G(\ell))$. And $L' := \vec{Q}[0, \ell]$ can be shunted to $R' := \vec{Q}[s, \ell + s]$ through $\vec{Q}$. The odd ℓ case is similar. ∎

Sufficient conditions for $\mathbb{A}_\ell(G)$ to be strongly connected can be found in [8, Page 76]. The following corollary of Lemma 3.5 reveals a strong relationship between the shunting of ℓ-links and the connectedness of ℓ-link graphs.

Corollary 3.6. *For a connected graph G, $\mathbb{L}_\ell(G)$ is connected if and only if any two ℓ-links of G with the same middle unit can be shunted to each other.*

We now present our main result of this section, which plays a key role in dealing with the graph minors of ℓ-link graphs in Section 5.

Lemma 3.7. *Let G be a graph, and X be a connected subgraph of $G(\ell)$. Then for every pair of ℓ-links L and R of X, L can be shunted to R under the restriction that in each step, the middle unit of the image of L belongs to X.*

Proof. First we consider the case that c_L is in R. Then there is a common segment Q of L and R of maximum length containing c_L. Without loss of generality, assign directions to L and R such that $\vec{L} = (\vec{L}_0.\vec{Q}.\vec{L}_1)$ and $\vec{R} = (\vec{R}_1.\vec{Q}.\vec{R}_0)$, where $\vec{L}_i \in \vec{\mathscr{L}}_{\ell_i}(X)$ and $\vec{R}_i \in \vec{\mathscr{L}}_{s_i}(X)$ for $i \in \{0, 1\}$ such that $s_1 \geqslant s_0$. Then $\ell \geqslant \ell_0 + \ell_1 = s_0 + s_1 \geqslant s_1$. Let x be the head vertex and e be the head edge of $\vec{L}$. Since c_L is in Q, $\ell_0 \leqslant \ell/2$. Since X is a subgraph of $G(\ell)$, by Lemma 3.5, there exists $\vec{L}_2 \in \vec{\mathscr{L}}_{\ell_0}(G)$ with tail vertex x and tail edge different from e. Let y be the tail vertex and f be the tail edge of $\vec{R}$. Then there exits $\vec{R}_2 \in \vec{\mathscr{L}}_{s_0}(G)$ with head vertex y and head edge different from f. We can shunt L to R first through $(\vec{L}.\vec{L}_2) \in \vec{\mathscr{L}}_{\ell+\ell_0}(G)$, then $-(\vec{R}_2.\vec{R}_1.\vec{Q}.\vec{L}_1.\vec{L}_2) \in \vec{\mathscr{L}}_{\ell+\ell_0+\ell_1}(G)$, and finally $(\vec{R}_2.\vec{R}) \in \vec{\mathscr{L}}_{\ell+s_0}(G)$. Since $\ell_0 \leqslant \ell/2$ and $s_0 \leqslant s_1 \leqslant \ell/2$, the middle unit of each image is inside L or R.

Secondly, we consider the case that c_L is not in R. Then there exists a segment Q of L of maximum length that contains c_L, and is edge-disjoint with R. Since X is connected, there exists a shortest $*$-arc $\vec{P}$ from a vertex v of R to a vertex u of L. Then P is edge-disjoint with Q because of its minimality. Without loss of generality, assign directions to L and R such that u separates $\vec{L}$ into $(\vec{L}_0.\vec{L}_1)$ with c_L on L_1, and v separates $\vec{R}$ into $(\vec{R}_1.\vec{R}_0)$, where L_i is of length ℓ_i while R_i is of length s_i for $i \in \{0, 1\}$, such that $s_1 \geqslant s_0$. Then $\ell_0, s_0 \leqslant \ell/2$. Let x be the head vertex and e be the head edge of $\vec{L}$. Since $\ell_0 \leqslant \ell/2$ and X is a subgraph of $G(\ell)$, by Lemma 3.5, there exists an ℓ_0-arc $\vec{L}_2$ of G with tail vertex x and tail edge different from e. Let y be the tail vertex and f be the tail edge of $\vec{R}$. Then there exits an s_0-arc $\vec{R}_2$ of G with head vertex y and head edge different from f. Now we can shunt L to R through $(\vec{L}.\vec{L}_2)$, $-(\vec{R}_2.\vec{R}_1.\vec{P}.\vec{L}_1.\vec{L}_2)$ and $(\vec{R}_2.\vec{R})$ consecutively. One can check that in this process the middle unit of each image belongs to L, P or R. ∎

From Lemma 3.7, the set of ℓ-links of a connected $G(\ell)$ serves as a 'hub' in the shunting of ℓ-links of G. More explicitly, for $L, R \in \mathscr{L}_\ell(G)$, if we can shunt L to $L' \in \mathscr{L}_\ell(G(\ell))$, and R to $R' \in \mathscr{L}_\ell(G(\ell))$, then L can be shunted to R since L' can be shunted to R'. Thus we have the following corollary which provides a more efficient way to test the connectedness of ℓ-link graphs.

Corollary 3.8. *Let G be a graph. Then $\mathbb{L}_\ell(G)$ is connected if and only if $G(\ell)$ is connected, and each ℓ-link of G can be shunted to an ℓ-link of $G(\ell)$.*

4. Chromatic number of ℓ-link graphs

In this section, we reveal a recursive structure of ℓ-link graphs, which leads to an upper bound for the chromatic number of ℓ-link graphs.

Lemma 4.1. *Let G be a graph and $\ell \geqslant 2$ be an integer. Then $(\mathcal{V}, \mathcal{E}) := (\mathcal{V}_\ell(G), \mathcal{E}_\ell(G))$ is an almost standard partition of $H := \mathbb{L}_\ell(G)$. Further, $H_{(\mathcal{V},\mathcal{E})}$ is isomorphic to an induced subgraph of $\mathbb{L}_{\ell-2}(G)$.*

Proof. First we verify that $(\mathcal{V}, \mathcal{E})$ is an almost standard partition of H.

(a) We prove that, for each $R \in \mathscr{L}_{\ell-2}(G)$, $V := \mathscr{L}_\ell(R) \in \mathcal{V}$ is an independent set of H. Suppose not. Then there are $\vec{L}, \vec{L}' \in \vec{\mathscr{L}}_\ell(G)$ such that $L, L' \in V$, and L can be shunted to L' in one step. Then $R = \vec{L}[1, \ell-1]$ can be shunted to $R = \vec{L}'[1, \ell-1]$ in one step, contradicting Observation 3.3.

(b) Here we show that each $E \in \mathcal{E}$ is incident to exactly two parts of $\mathcal{V}$. By definition there exists $P \in \mathscr{L}_{\ell-1}(G)$ with $\mathscr{L}_{\ell+1}(P) = E$. Let $\{L, R\} := P^{\{\ell-2\}}$. Then $\mathscr{L}_\ell(L)$ and $\mathscr{L}_\ell(R)$ are the only two parts of $\mathcal{V}$ incident to E.

(c) We explain that each $E \in \mathcal{E}$ is the edge set of a complete bipartite subgraph of H. By definition there exists $\vec{P} \in \vec{\mathscr{L}}_{\ell-1}(G)$ with $\mathscr{L}_{\ell+1}(P) = E$.

Let $A := \{[\vec{e}.\vec{P}] \in \mathscr{L}_\ell(G)\}$ and $B := \{[\vec{P}.\vec{f}] \in \mathscr{L}_\ell(G)\}$. One can check that E induces a complete bipartite subgraph of H with bipartition $A \cup B$.

(d) We prove that each $v \in V(H)$ is incident to at most two parts of $\mathcal{E}$. By definition there exists $Q \in \mathscr{L}_\ell(G)$ with $Q = v$. Then the set of edge parts of $\mathcal{E}$ incident to v is $\{\mathscr{L}_{\ell+1}(L) \neq \emptyset | L \in Q^{\{\ell-1\}}\}$ with cardinality at most 2.

(e) Let v be a vertex of $V \in \mathcal{V}$ incident to different $E, F \in \mathcal{E}$. We explain that v is uniquely determined by V, E and F. By definition there exists $\vec{P} \in \vec{\mathscr{L}}_{\ell-2}(G)$ such that $V = \mathscr{L}_\ell(P)$. There also exists $Q := [\vec{e}_1.\vec{P}.\vec{e}_\ell] \in \mathscr{L}_\ell(P)$ such that $v = Q$. Besides, there are $L, R \in \mathscr{L}_{\ell-1}(G)$ such that $E = \mathscr{L}_{\ell+1}(L)$ and $F = \mathscr{L}_{\ell+1}(R)$. Then $\{L, R\} = Q^{\{\ell-1\}}$ since $L \neq R$. Note that Q is uniquely determined by $Q^{\{\ell-1\}}$ and $c_Q = c_P$. Thus it is uniquely determined by $E = \mathscr{L}_{\ell+1}(L), F = \mathscr{L}_{\ell+1}(R)$ and $V = \mathscr{L}_\ell(P)$.

Now we show that $H_{(\mathcal{V},\mathcal{E})}$ is isomorphic to an induced subgraph of $\mathbb{L}_{\ell-2}(G)$. Let X be the subgraph of $\mathbb{L}_{\ell-2}(G)$ of vertices $L \in \mathscr{L}_{\ell-2}(G)$ such that $\mathscr{L}_\ell(L) \neq \emptyset$, and edges $Q \in \mathscr{L}_{\ell-1}(G)$ such that $\mathscr{L}_{\ell+1}(Q) \neq \emptyset$. One can check that X is an induced subgraph of $\mathbb{L}_{\ell-2}(G)$. An isomorphism from $H_{(\mathcal{V},\mathcal{E})}$ to X can be defined as the injection sending $\mathscr{L}_\ell(L) \neq \emptyset$ to L, and $\mathscr{L}_{\ell+1}(Q) \neq \emptyset$ to Q. ∎

Below we give an interesting algorithm for colouring a class of graphs.

Lemma 4.2. *Let H be a graph with a t-colouring such that each vertex of H is adjacent to at most $r \geqslant 0$ differently coloured vertices. Then $\chi(H) \leqslant \lfloor \frac{tr}{r+1} \rfloor + 1$.*

Proof. The result is trivial for $t = 0$ since, in this case, $\chi(H) = 0$. If $r + 1 \geqslant t \geqslant 1$, then $\lfloor \frac{tr}{r+1} \rfloor + 1 = t$, and the lemma holds since $t \geqslant \chi(H)$.

Now assume $t \geqslant r + 2 \geqslant 2$. Let $U_1, U_2, \dots, U_t$ be the colour classes of the given colouring. For $i \in [t]$, denote by i the colour assigned to vertices in U_i. Run the following algorithm: For $j = 1, \dots, t$, and for each $u \in U_{t-j+1}$, let $s \in [t]$ be the minimum integer that is not the colour of a neighbour of u in H; if $s < t - j + 1$, then recolour u by s.

In the algorithm above, denote by C_i the set of colours used by the vertices in U_i for $i \in [t]$. Let $k := \lfloor \frac{t-1}{r+1} \rfloor$. Then $t - 1 \geqslant k(r + 1) \geqslant k \geqslant 1$. We claim that after $j \in [0, k]$ steps, $C_{t-i+1} \subseteq [ir + 1]$ for $i \in [j]$, and $C_i = \{i\}$ for $i \in [t - j]$. This is trivial for $j = 0$. Inductively assume it holds for some $j \in [0, k - 1]$. In the $(j + 1)^{th}$ step, we change the colour of each $u \in U_{t-j}$ from $t - j$ to the minimum $s \in [t]$ that is not used by the neighbourhood of u. It is enough to show that $s \leqslant (j + 1)r + 1$.

First suppose that all neighbours of u are in $\bigcup_{i\in[t-j-1]} U_i$. By the analysis above, $t - j - 1 \geqslant t - k \geqslant kr + 1 \geqslant r + 1$. So at least one part of $\mathcal{S} := \{U_i | i \in [t - j - 1]\}$ contains no neighbour of u. From the induction hypothesis, $C_i = \{i\}$ for $i \in [t - j - 1]$. Hence at least one colour in $[r + 1]$ is not used by the neighbourhood of u; that is, $s \leqslant r + 1 \leqslant (j + 1)r + 1$.

Now suppose that u has at least one neighbour in $\bigcup_{i\in[t-j+1,t]} U_i$. By the induction hypothesis, $\bigcup_{i\in[t-j+1,t]} C_i \subseteq [jr+1]$. At the same time, u has neighbours in at most $r-1$ parts of $\mathcal{S}$. So the colours possessed by the neighbourhood of u are contained in $[jr+1+r-1] = [(j+1)r]$. Thus $s \leqslant (j+1)r+1$. This proves our claim.

The claim above indicates that, after the k^{th} step, $C_{t-i+1} \subseteq [ir+1]$ for $i \in [k]$, and $C_i = \{i\}$ for $i \in [t-k]$. Hence we have a $(t-k)$-colouring of H since $t-k \geqslant kr+1$. Therefore, $\chi(H) \leqslant t-k = \lceil \frac{tr+1}{r+1} \rceil = \lfloor \frac{tr}{r+1} \rfloor + 1$. ∎

Lemma 4.1 indicates that $\mathbb{L}_\ell(G)$ is homomorphic to $\mathbb{L}_{\ell-2}(G)$ for $\ell \geqslant 2$. So by [5, Proposition 1.1], $\chi_\ell(G) \leqslant \chi_{\ell-2}(G)$. By Lemma 4.1, every vertex of $\mathbb{L}_\ell(G)$ has neighbours in at most two parts of $\mathcal{V}_\ell(G)$, which enables us to improve the upper bound on $\chi_\ell(G)$.

Lemma 4.3. *Let G be a graph, and $\ell \geqslant 2$. Then $\chi_\ell(G) \leqslant \lfloor \frac{2}{3}\chi_{\ell-2}(G) \rfloor + 1$.*

Proof. By Lemma 4.1, $(\mathcal{V}, \mathcal{E}) := (\mathcal{V}_\ell(G), \mathcal{E}_\ell(G))$ is an almost standard partition of $H := \mathbb{L}_\ell(G)$. So each vertex of H has neighbours in at most two parts of $\mathcal{V}$. Further, $H_\mathcal{V}$ is a subgraph of $\mathbb{L}_{\ell-2}(G)$. So $\chi_\ell(G) \leqslant \chi := \chi(H_\mathcal{V}) \leqslant \chi_{\ell-2}(G)$.

We now construct a χ-colouring of H such that each vertex of H is adjacent to at most two differently coloured vertices. By definition $H_\mathcal{V}$ admits a χ-colouring with colour classes $K_1, \dots, K_\chi$. For $i \in [\chi]$, assign the colour i to each vertex of H in $U_i := \bigcup_{V_\mathcal{V} \in K_i} V$. One can check that this is a desired colouring. In Lemma 4.3, letting $t = \chi$ and $r = 2$ yields that $\chi_\ell(G) \leqslant \lfloor \frac{2}{3}\chi \rfloor + 1$. Recall that $\chi \leqslant \chi_{\ell-2}(G)$. Thus the lemma follows. ∎

As shown below, Lemma 4.3 can be applied recursively to produce an upper bound for $\chi_\ell(G)$ in terms of $\chi(G)$ or $\chi'(G)$.

Proof of Theorem 1.1. When $\ell \in \{0,1\}$, it is trivial for **(1)(2)** and **(4)**. By [6, Proposition 5.2.2], $\chi_0 = \chi \leqslant \Delta+1$. So **(3)** holds. Now let $\ell \geqslant 2$. By Lemma 4.1, $H := \mathbb{L}_\ell(G)$ admits an almost standard partition $(\mathcal{V}, \mathcal{E}) := (\mathcal{V}_\ell(G), \mathcal{E}_\ell(G))$, such that $H_{(\mathcal{V},\mathcal{E})}$ is an induced subgraph of $\mathbb{L}_{\ell-2}(G)$. By definition each part of $\mathcal{V}$ is an independent set of H. So $H \to \mathbb{L}_{\ell-2}(G)$, and $\chi_\ell \leqslant \chi_{\ell-2}$. This proves **(4)**. Moreover, each vertex of H has neighbours in at most two parts of $\mathcal{V}$. By Lemma 4.3, $\chi_\ell := \chi_\ell(G) \leqslant \frac{2\chi_{\ell-2}}{3} + 1$. Continue the analysis, we have $\chi_\ell \leqslant \chi_{\ell-2i}$, and $\chi_\ell - 3 \leqslant (\frac{2}{3})^i(\chi_{\ell-2i} - 3)$ for $1 \leqslant i \leqslant \lfloor \ell/2 \rfloor$. Therefore, if ℓ is even, then $\chi_\ell \leqslant \chi_0 = \chi \leqslant \Delta + 1$, and $\chi_\ell - 3 \leqslant (\frac{2}{3})^{\ell/2}(\chi - 3)$. Thus **(1)** holds. Now let $\ell \geqslant 3$ be odd. Then $\chi_\ell \leqslant \chi_1 = \chi'$, and $\chi_\ell - 3 \leqslant (\frac{2}{3})^{\frac{\ell-1}{2}}(\chi' - 3)$. This verifies **(2)**. As a consequence, $\chi_\ell \leqslant \chi_3 \leqslant \frac{2}{3}(\chi' - 3) + 3 = \frac{2}{3}\chi' + 1$. By Shannon [17], $\chi' \leqslant \frac{3}{2}\Delta$. So $\chi_\ell \leqslant \Delta + 1$, and hence **(3)** holds. ∎

The following corollary of Theorem 1.1 implies that Hadwiger's conjecture is true for $\mathbb{L}_\ell(G)$ if G is regular and $\ell \geqslant 4$.

Corollary 4.4. *Let G be a graph with $\Delta := \Delta(G) \geqslant 3$. Then $\chi_\ell(G) \leqslant 3$ for all $\ell > 2\log_{1.5}(\Delta - 2) + 3$. Further, Hadwiger's conjecture holds for $\mathbb{L}_\ell(G)$ if $\ell > 2\log_{1.5}(\Delta - 2) - 3.83$, or $\mathrm{d} := \mathrm{d}(G) \geqslant 3$ and $\ell > 2\log_{1.5}\frac{\Delta-2}{\mathrm{d}-2} + 3$.*

Proof. By Theorem 1.1, for each $t \geqslant 3$, $\chi_\ell := \chi_\ell(G) \leqslant t$ if $(\frac{2}{3})^{\ell/2}(\Delta-2) < t-2$ and $(\frac{2}{3})^{\frac{\ell-1}{2}}(\frac{3}{2}\Delta-3) < t-2$. Solving these inequalities gives $\ell > 2\log_{1.5}(\Delta-2) - 2\log_{1.5}(t-2)+3$. Thus $\chi_\ell \leqslant 3$ if $\ell > 2\log_{1.5}(\Delta-2)+3$. So the first statement holds. By Robertson et al. [16] and Theorem 1.3, Hadwiger's conjecture holds for $\mathbb{L}_\ell(G)$ if $\ell \geqslant 1$ and $\chi_\ell \leqslant \max\{6, \mathrm{d}\}$. Letting $t = 6$ gives that $\ell > 2\log_{1.5}(\Delta - 2) - 4\log_{1.5} 2 + 3$. Letting $t = \mathrm{d} \geqslant 3$ gives that $\ell > 2\log_{1.5}\frac{\Delta-2}{\mathrm{d}-2} + 3$. So the corollary holds since $4\log_{1.5} 2 - 3 > 3.83$. ∎

Proof of Theorem 1.5(3)(4)(5). **(3)** and **(4)** follow from Corollary 4.4. Now consider **(5)**. By Reed and Seymour [15], Hadwiger's conjecture holds for $\mathbb{L}_1(G)$. If $\ell \geqslant 2$ and $\Delta \leqslant 5$, by Theorem 1.1(3), $\chi_\ell(G) \leqslant 6$. In this case, Hadwiger's conjecture holds for $\mathbb{L}_\ell(G)$ by Robertson et al. [16]. ∎

5. Complete minors of ℓ-link graphs

It has been proved in the last section that Hadwiger's conjecture is true for $\mathbb{L}_\ell(G)$ if ℓ is large enough. In this section, we further investigate the minors, especially the complete minors, of ℓ-link graphs. To see the intuition of our method, let v be a vertex of degree t in G. Then $\mathbb{L}_1(G)$ contains a K_t-subgraph whose vertices correspond to the edges of G incident to v. For $\ell \geqslant 2$, roughly speaking, we extend v to a subgraph X of diameter less than ℓ, and extend each edge incident to v to an ℓ-link of G starting from a vertex of X. By studying the shunting of these ℓ-links, we find a K_t-minor in $\mathbb{L}_\ell(G)$.

For subgraphs X, Y of G, let $\vec{E}(X,Y)$ be the set of arcs of G from $V(X)$ to $V(Y)$, and $E(X,Y)$ be the set of edges of G between $V(X)$ and $V(Y)$.

Lemma 5.1. *Let $\ell \geqslant 1$ be an integer, G be a graph, and X be a subgraph of G with $\mathrm{diam}(X) < \ell$ such that $Y := G - V(X)$ is connected. If $t := |E(X,Y)| \geqslant 2$, then $\mathbb{L}_\ell(G)$ contains a K_t-minor.*

Proof. Let $\vec{e}_1, \dots, \vec{e}_t$ be distinct arcs in $\vec{E}(Y,X)$. Say $\vec{e}_i = (y_i, e_i, x_i)$ for $i \in [t]$. Since $\mathrm{diam}(X) < \ell$, there is a dipath $\vec{P}_{ij}$ of X from x_i to x_j of length $\ell_{ij} \leqslant \ell - 1$ such that $P_{ij} = P_{ji}$. Since Y is connected, it contains a dipath $\vec{Q}_{ij}$ from y_i to y_j. Since $t \geqslant 2$, $O_i := [\vec{P}_{i\,i'}.-\vec{e}_{i'}.\vec{Q}_{i'\,i}.\vec{e}_i]$ is a cycle of G, where $i' := (i \bmod t) + 1$. Thus $H := \mathbb{L}_\ell(G)$ contains a cycle $\mathbb{L}_\ell(O_1)$, and hence a K_2-minor. Now let $t \geqslant 3$, and $\vec{L}_i \in \vec{\mathscr{L}}_\ell(O_i)$ with head arc $\vec{e}_i$. Then $[\vec{L}_i.\vec{P}_{ij}]^{[\ell]} \in \mathscr{L}_{\ell_{ij}}(H)$. And the union of the units of $[\vec{L}_i.\vec{P}_{ij}]^{[\ell]}$ over $j \in [t]$ is a connected subgraph X_i of H. In the remainder of the proof, for distinct $i, j \in [t]$, we show that X_i and X_j are disjoint. Further, we construct a path in H between X_i and X_j that is internally disjoint with its counterparts, and has no inner vertex in any of $V(X_1), \ldots, V(X_t)$. Then by contracting each X_i into a vertex, and each path into an edge, we obtain a K_t-minor of H.

First of all, assume for a contradiction that there are different $i, j \in [t]$ such that X_i and X_j share a common vertex that corresponds to an ℓ-link R of G. Then by definition, there exists some $p \in [t]$ such that R can be obtained by shunting L_i along $(\vec{L}_i.\vec{P}_{ip})$ by some $s_i \leqslant \ell_{ip}$ steps. So $R = [\vec{L}_i(s_i, \ell).\vec{P}_{ip}(0, s_i)]$. Similarly, there are $q \in [t]$ and $s_j \leqslant \ell_{jq}$ such that $R = [\vec{L}_j(s_j, \ell).\vec{P}_{jq}(0, s_j)]$. Recall that $E(X) \cap E(X, Y) = E(Y) \cap E(X, Y) = \emptyset$. So $e_i = \vec{L}_i[\ell - 1, \ell]$ and $e_j = \vec{L}_j[\ell - 1, \ell]$ belong to both L_i and L_j. By the definition of O_i, this happens if and only if $i = j'$ and $j = i'$, which is impossible since $t \geqslant 3$.

Secondly, for different $i, j \in [t]$, we define a path of H between X_i and X_j. Clearly, L_i can be shunted to L_j through $\vec{R}'_{ij} := (\vec{L}_i.\vec{P}_{ij}. - \vec{L}_j)$ in G. In this shunting, $L'_i := [\vec{L}_i(\ell_{ij}, \ell).\vec{P}_{ij}]$ is the last image corresponding to a vertex of X_i, while $L'_j := [\vec{P}_{ij}.\vec{L}_j(\ell, \ell_{ij})]$ is the first image corresponding to a vertex of X_j. Further, L'_i can be shunted to L'_j through $\vec{R}_{ij} := (\vec{L}_i(\ell_{ij}, \ell).\vec{P}_{ij}.\vec{L}_j(\ell, \ell_{ij})) \in \vec{\mathscr{L}}_{2\ell - \ell_{ij}}(G)$, which is a subsequence of $\vec{R}'_{ij}$. Then $R^{[\ell]}_{ij}$ is an $(\ell - \ell_{ij})$-path of H between X_i and X_j. We show that for each $p \in [t]$, X_p contains no inner vertex of $R^{[\ell]}_{ij}$. When $\ell - \ell_{ij} = 1$, $R^{[\ell]}_{ij}$ contains no inner vertex. Now assume $\ell - \ell_{ij} \geqslant 2$. Each inner vertex of $R^{[\ell]}_{ij}$ corresponds to some $Q_{ij} := [\vec{L}_i(s_i, \ell).\vec{P}_{ij}.\vec{L}_j(\ell, \ell + \ell_{ij} - s_i)] \in \mathscr{L}_\ell(G)$, where $\ell_{ij} + 1 \leqslant s_i \leqslant \ell - 1$. Assume for a contradiction that for some $p \in [t]$, X_p contains a vertex corresponding to Q_{ij}. By definition there exists $q \in [t]$ such that $Q_{ij} = [\vec{L}_p(s_p, \ell).\vec{P}_{pq}(0, s_p)]$, where $0 \leqslant s_p \leqslant \ell_{pq}$. Without loss of generality, $(\vec{L}_i(s_i, \ell).\vec{P}_{ij}.\vec{L}_j(\ell, \ell + \ell_{ij} - s_i)) = (\vec{L}_p(s_p, \ell).\vec{P}_{pq}(0, s_p))$. Since e_j and e_p are not in P_{pq}, hence $\vec{e}_j$ belongs to $-\vec{L}_p$ and $\vec{e}_p$ belongs to $-\vec{L}_j$. By the definition of $\vec{L}_i$, this happens only when $j = p'$ and $p = j'$, contradicting $t \geqslant 3$.

We now show that $R^{[\ell]}_{ij}$ and $R^{[\ell]}_{pq}$ are internally disjoint, where $i \neq j$, $p \neq q$ and $\{i, j\} \neq \{p, q\}$. Suppose not. Then by the analysis above, there are s_i and s_p with $\ell_{ij}+1 \leqslant s_i \leqslant \ell-1$ and $\ell_{pq}+1 \leqslant s_p \leqslant \ell-1$ such that $Q_{ij} = Q_{pq}$. Without loss of generality, $(\vec{L}_i(s_i, \ell).\vec{P}_{ij}.\vec{L}_j(\ell, \ell + \ell_{ij} - s_i)) = (\vec{L}_p(s_p, \ell).\vec{P}_{pq}.\vec{L}_q(\ell, \ell + \ell_{pq} - s_p))$. If $s_i = s_p$, then $\vec{e}_i = \vec{e}_p$ and $\vec{e}_j = \vec{e}_q$ since $E(X) \cap E(X, Y) = \emptyset$; that is, $i = p$ and $j = q$, contradicting $\{i, j\} \neq \{p, q\}$. Otherwise, with no loss of generality, $s_i > s_p$. Then $\vec{e}_q$ and $\vec{e}_i$ belong to $\vec{L}_j$ and $\vec{L}_p$ respectively; that is, $i = p$ and $j = q$, again contradicting $\{i, j\} \neq \{p, q\}$.

In summary, $X_1, \ldots, X_t$ are vertex-disjoint connected subgraphs, which are pairwise connected by internally disjoint $*$-links $R^{[\ell]}_{ij}$ of H, such that no inner vertex of $R_{ij}^{[\ell]}$ is in $V(X_1) \cup \ldots \cup V(X_t)$. So by contracting each X_i to a vertex, and $R_{ij}^{[\ell]}$ to an edge, we obtain a K_t-minor of H. ∎

Lemma 5.2. *Let $\ell \geqslant 1$, G be a graph, and X be a subgraph of G with* $\mathrm{diam}(X) < \ell$ *such that $Y := G - V(X)$ is connected and contains a cycle. Let $t := |E(X,Y)|$. Then $\mathbb{L}_\ell(G)$ contains a K_{t+1}-minor.*

Proof. Let O be a cycle of Y. Then $H := \mathbb{L}_\ell(G)$ contains a cycle $\mathbb{L}_\ell(O)$ and hence a K_2-minor. Now assume $t \geqslant 2$. Let $\vec{e}_1, \ldots, \vec{e}_t$ be distinct arcs in $\vec{E}(Y,X)$. Say $\vec{e}_i = (y_i, e_i, x_i)$ for $i \in [t]$. Since Y is connected, there is a dipath $\vec{P}_i$ of Y of minimum length $s_i \geqslant 0$ from some vertex z_i of O to y_i. Let $\vec{Q}_i$ be an ℓ-arc of O with head vertex z_i. Then $\vec{L}_i := (\vec{Q}_i.\vec{P}_i.\vec{e}_i)(s_i+1, \ell+s_i+1) \in \vec{\mathscr{L}}_\ell(G)$. Since $\mathrm{diam}(X) \leqslant \ell - 1$, there is a dipath $\vec{P}_{ij}$ of X of length $\ell_{ij} \leqslant \ell - 1$ from x_i to x_j such that $P_{ij} = P_{ji}$.

Clearly, $[\vec{L}_i.\vec{P}_{ij}]^{[\ell]}$ is an ℓ_{ij}-link of H. And the union of the units of $[\vec{L}_i.\vec{P}_{ij}]^{[\ell]}$ over $j \in [t]$ induces a connected subgraph X_i of H. For different $i, j \in [t]$, let $R_{ij} := [\vec{L}_i(\ell_{ij}, \ell).\vec{P}_{ij}.\vec{L}_j(\ell, \ell_{ij})] = R_{ji} \in \mathscr{L}_{2\ell-\ell_{ij}}(G)$. Then $R_{ij}^{[\ell]}$ is an $(\ell - \ell_{ij})$-path of H between X_i and X_j. As in the proof of Lemma 5.1, it is easy to check that $X_1, \ldots, X_t$ are vertex-disjoint connected subgraphs of H, which are pairwise connected by internally disjoint paths $R_{ij}^{[\ell]}$. Further, no inner vertex of $R_{ij}^{[\ell]}$ is in $V(X_1) \cup \ldots \cup V(X_t)$. So a K_t-minor of H is obtained accordingly.

Finally, let Z be the connected subgraph of H induced by the units of $\mathbb{L}_\ell(O)$ and $[\vec{Q}_i.\vec{P}_i]^{[\ell]}$ over $i \in [t]$. Then Z is vertex-disjoint with X_i and with the paths $R_{ij}^{[\ell]}$. Moreover, Z sends an edge $(\vec{Q}_i.\vec{P}_i.\vec{e}_i)(s_i, \ell+s_i+1)^{[\ell]}$ to each X_i. Thus H contains a K_{t+1}-minor. ∎

In the following, we use the 'hub' (described after Lemma 3.7) to construct certain minors in ℓ-link graphs.

Corollary 5.3. *Let $\ell \geqslant 0$, G be a graph, M be a minor of $G(\ell)$ such that each branch set contains an ℓ-link. Then $\mathbb{L}_\ell(G)$ contains an M-minor.*

Proof. Let $X_1, \ldots, X_t$ be the branch sets of an M-minor of $G(\ell)$ such that X_i contains an ℓ-link for each $i \in [t]$. For any connected subgraph Y of $G(\ell)$ contains at least one ℓ-link, let $\mathbb{L}_\ell(G,Y)$ be the subgraph of $H := \mathbb{L}_\ell(G)$ induced by the ℓ-links of G of which the middle units are in Y. Let $H(Y)$ be the union of the components of $\mathbb{L}_\ell(G,Y)$ which contains at least one vertex corresponding to an ℓ-link of Y. By Lemma 3.7, $H(Y)$ is connected.

By definition each edge of M corresponds to an edge e of $G(\ell)$ between two different branch sets, say X_i and X_j. Let Y be the graph consisting of X_i, X_j and e. Then $H(X_i)$ and $H(X_j)$ are vertex-disjoint since X_i and X_j are vertex-disjoint. By the analysis above, $H(X_i)$ and $H(X_j)$ are connected subgraphs of the connected graph $H(Y)$. Thus there is a path Q of $H(Y)$ joining $H(X_i)$ and

$H(X_j)$ only at end vertices. Further, if ℓ is even, then Q is an edge; otherwise, Q is a 2-path whose middle vertex corresponds to an ℓ-link L of Y such that $c_L = e$. This implies that Q is internally disjoint with its counterparts and has no inner vertex in any branch set. Then, by contracting each $H(X_i)$ to a vertex, and Q to an edge, we obtain an M-minor of H. ∎

Now we are ready to give a lower bound for the Hadwiger number of $\mathbb{L}_\ell(G)$.

Proof of Theorem 1.3. Since $H := \mathbb{L}_\ell(G)$ contains an edge, $t := \eta(H) \geqslant 2$. We first show that $t \geqslant \mathrm{d} := \mathrm{d}(G)$. By definition there exists a subgraph X of G of $\delta(X) = \mathrm{d}$. We may assume that $\mathrm{d} \geqslant 3$. Then X contains an $(\ell-1)$-link P such that $\mathscr{L}(P) \neq \emptyset$. By Lemma 4.1, $\mathscr{L}^{[\ell]}(P)$ is the edge set of a complete bipartite subgraph of H with a $K_{\mathrm{d}-1,\mathrm{d}-1}$-subgraph. By Zelinka [24], $K_{\mathrm{d}-1,\mathrm{d}-1}$ contains a K_{d}-minor. Thus $t \geqslant \mathrm{d}$ as desired.

We now show that $t \geqslant \eta := \eta(G)$. If $\eta = 3$, then G contains a cycle O of length at least 3, and H contains a K_3-minor contracted from $\mathbb{L}_\ell(O)$. Now assume that G is connected with $\eta \geqslant 4$. Repeatedly delete vertices of degree 1 in G until $\delta(G) \geqslant 2$. Then $G = G(\ell)$. Clearly, this process does not reduce the Hadwiger number of G. So G contains branch sets of a K_η-minor covering $V(G)$ (see [23]). If every branch set contains an ℓ-link, then the statement follows from Corollary 5.3. Otherwise, there exists some branch set X with $\mathrm{diam}(X) < \ell$. Since $\eta \geqslant 4$, $Y := G - V(X)$ is connected and contains a cycle. Thus by Lemma 5.2, H contains a K_η-minor since $|E(X,Y)| \geqslant \eta - 1$. ∎

Here we prove Hadwiger's conjecture for $\mathbb{L}_\ell(G)$ for even $\ell \geqslant 2$.

Proof of Theorem 1.5(2). Let $\mathrm{d} := \mathrm{d}(G)$, $\ell \geqslant 2$ be an even integer, and $H := \mathbb{L}_\ell(G)$. By [6, Proposition 5.2.2], $\chi := \chi(G) \leqslant \mathrm{d}+1$. So by Theorem 1.1, $\chi(H) \leqslant \min\{\mathrm{d}+1, \frac{2}{3}\mathrm{d}+\frac{5}{3}\}$. If $\mathrm{d} \leqslant 4$, then $\chi(H) \leqslant 5$. By Robertson et al. [16], Hadwiger's conjecture holds for H in this case. Otherwise, $\mathrm{d} \geqslant 5$. By Theorem 1.3, $\eta(H) \geqslant \mathrm{d} \geqslant \frac{2}{3}\mathrm{d}+\frac{5}{3} \geqslant \chi(H)$ and the statement follows. ∎

We end this paper by proving Hadwiger's conjecture for ℓ-link graphs of biconnected graphs for $\ell \geqslant 1$.

Proof of Theorem 1.5(1). By Reed and Seymour [15], Hadwiger's conjecture holds for $H := \mathbb{L}_\ell(G)$ for $\ell = 1$. By Theorem 1.5(2), the conjecture is true if $\ell \geqslant 2$ is even. So we only need to consider the situation that $\ell \geqslant 3$ is odd. If G is a cycle, then H is a cycle and the conjecture holds [9]. Now let v be a vertex of G with degree $\Delta := \Delta(G) \geqslant 3$. By Theorem 1.1, $\chi(H) \leqslant \Delta + 1$. Since G is biconnected, $Y := G - v$ is connected. By Lemma 5.2, if Y contains a cycle, then $\eta(H) \geqslant \Delta + 1 \geqslant \chi(H)$. Now assume that Y is a tree, which implies that G is K_4-minor free. By Lemma 5.1, $\eta(H) \geqslant \Delta$. By Theorem 1.1, $\chi(H) \leqslant \chi' := \chi'(G)$. So it is enough to show that $\chi' = \Delta$.

Let $U := \{u \in V(Y) | \deg_Y(u) \leqslant 1\}$. Then $|U| \geqslant \Delta(Y)$. Let $\hat{G}$ be the underlying simple graph of G, $t := \deg_{\hat{G}}(v) \geqslant 1$ and $\hat{\Delta} := \Delta(\hat{G}) \geqslant t$. Since G is biconnected, $U \subseteq N_G(v)$. So $t \geqslant |U| \geqslant \Delta(Y)$. Let $u \in U$. When $|U| = 1$, $t = \deg_{\hat{G}}(u) = 1$. When $|U| \geqslant 2$, $\deg_{\hat{G}}(u) = 2 \leqslant |U| \leqslant t$. Thus $t = \hat{\Delta}$. Juvan et al. [13] proved that the edge-chromatic number of a K_4-minor free simple graph equals the maximum degree of this graph. So $\hat{\chi}' := \chi'(\hat{G}) = \hat{\Delta}$ since $\hat{G}$ is simple and K_4-minor free. Note that all parallel edges of G are incident to v. So $\chi' = \hat{\chi}' + \deg_G(v) - t = \hat{\Delta} + \Delta - \hat{\Delta} = \Delta$ as desired. ∎

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