Expression for the Number of Spanning Trees of Line Graphs of Arbitrary Connected Graphs*

Fengming Dong†
Mathematics and Mathematics Education
National Institute of Education
Nanyang Technological University, Singapore 637616

Weigen Yan
School of Sciences, Jimei University, Xiamen 361021, China

Abstract

For any graph G, let $t(G)$ be the number of spanning trees of G, $L(G)$ be the line graph of G and for any non-negative integer r, $S_r(G)$ be the graph obtained from G by replacing each edge e by a path of length $r+1$ connecting the two ends of e. In this paper we obtain an expression for $t(L(S_r(G)))$ in terms of spanning trees of G by a combinatorial approach. This result generalizes some known results on the relation between $t(L(S_r(G)))$ and $t(G)$ and gives an explicit expression $t(L(S_r(G))) = k^{m+s-n-1}(rk+2)^{m-n+1}t(G)$ if G is of order $n+s$ and size $m+s$ in which s vertices are of degree 1 and the others are of degree k. Thus we prove a conjecture on $t(L(S_1(G)))$ for such a graph G.

Keywords: Graph; Spanning tree; Line graph; Cayley's Foumula; Subdivision.

1 Introduction

The graphs considered in this article have no loops but may have parallel edges. For any graph G, let $V(G)$ and $E(G)$ be the vertex set and edge set of G respectively, let $S(G)$ be the graph obtained from G by inserting a new vertex to each edge in G, $L(G)$ be the line graph of G, $\mathcal{T}(G)$ be the set of spanning trees of G and $t(G) = |\mathcal{T}(G)|$. Note that for any parallel edges e and e' in G, e and e' are two vertices in $L(G)$ joined by two parallel edges. For any disjoint subsets V_1, V_2 of $V(G)$, let $E_G(V_1, V_2)$ (or simply $E(V_1, V_2)$) denote the set of those edges in $E(G)$ which have ends in V_1 and V_2 respectively, and let $E_G(V_1, V(G) - V_1)$ be simply denoted by $E_G(V_1)$. For any $u \in V(G)$, let $E_G(u)$ (or simply $E(u)$) denote the set $E_G(\{u\})$. So the degree of u in G, denoted by $d_G(u)$ (or simply $d(u)$), is equal to $|E(u)|$. For any subset U of $V(G)$, let $G[U]$ denote the subgraph of G induced by U and let $G-U$ denote

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†Corresponding author. Email: fengming.dong@nie.edu.sg

the subgraph of G induced by $V(G) - U$. For any $E' \subseteq E(G)$, let $G[E']$ be the spanning subgraph of G with edge set E', $G - E'$ be the graph $G[E(G) - E']$ and G/E' be the graph obtained from G by contracting all edges of E'.

Our paper concerns the relation between $t(G)$ and $t(L(G))$ or $t(L(S(G)))$. Such a relation was first found by Vahovskii [19], then by Kelmans [8] and was rediscovered by Cvetković, Doob and Sachs [7] for regular graphs. They showed that if G is a k-regular graph of order n and size m, then

$$t(L(G)) = k^{m-n-1}2^{m-n+1}t(G). \tag{1.1}$$

The first result on the relation between $t(G)$ and $t(L(S(G)))$ was found by Zhang, Chen and Chen [21]. They proved that if G is k-regular, then

$$t(L(S(G))) = k^{m-n-1}(k+2)^{m-n+1}t(G). \tag{1.2}$$

Yan [20] recently generalized the result of (1.1). He proved that if G is a graph of order $n+s$ and size $m+s$ in which s vertices are of degree 1 and all others are of degree k, where $k \geq 2$, then

$$t(L(G)) = k^{m+s-n-1}2^{m-n+1}t(G). \tag{1.3}$$

Yan [20] also proposed a conjecture to generalize the result of (1.2).

Conjecture 1.1 ([20]) *Let G be a connected graph of order $n+s$ and size $m+s$ in which s vertices are of degree 1 and all others are of degree k. Then*

$$t(L(S(G))) = k^{m+s-n-1}(k+2)^{m-n+1}t(G).$$

If G is a digraph, the relation between $t(G)$ and $t(L(G))$ was first obtained by Knuth [9] by an application of the Matrix-Tree Theorem and a bijective proof of the result was found by Bidkhori and Kishore [3]. Note that expressions (1.1), (1.2) and (1.3) were also obtained by the respective authors mentioned above by an application of the Matrix-Tree Theorem. To our knowledge, these results still do not have any combinatorial proofs. Some related results can be seen in [2, 6, 10, 16, 22].

For an arbitrary connected graph G and any non-negative integer r, let $S_r(G)$ denote the graph obtained from G by replacing each edge e of G by a path of length $r+1$ connecting the two ends of e. Thus $S_0(G)$ is G itself and $S_1(G)$ is the graph $S(G)$. Our main purpose in this paper is to use a combinatorial method to find an expression for $t(L(S_r(G)))$ given in Theorem 1.1.

Theorem 1.1 *For any connected graph G and any integer $r \geq 0$,*

$$t(L(S_r(G))) = \prod_{v\in V(G)} d(v)^{d(v)-2} \sum_{E'\subseteq E(G)} t(G[E'])r^{|E'|-|V(G)|+1} \prod_{e\in E(G)-E'} (d(u_e)^{-1} + d(v_e)^{-1}), \tag{1.4}$$

where $d(v) = d_G(v)$ and u_e and v_e are the two ends of e.

As $S_0(G)$ is G itself, the following expression for $t(L(G))$ is a special case of Theorem 1.1:

$$t(L(G)) = \prod_{v\in V(G)} d(v)^{d(v)-2} \sum_{T\subseteq \mathcal{T}(G)} \prod_{e\in E(G)-E(T)} (d(u_e)^{-1} + d(v_e)^{-1}). \tag{1.5}$$

The proof of Theorem 1.1 will be completed in Sections 3 and 4. In Section 3, we will show that the case $r = 0$ of Theorem 1.1 (i.e., the result (1.5)) is a special case of another result (i.e., Theorem 3.1), and in Section 4, we will prove the case $r \geq 1$ of Theorem 1.1 by applying this theorem for the case $r = 0$ (i.e., (1.5)). To establish Theorem 3.1, we need to apply a result in Section 2 (i.e., Proposition 2.3), which determines the number of spanning trees in a graph G with a clique V_0 such that $F = G - E(G[V_0])$ is a forest and every vertex in V_0 is incident with at most one edge in F. Finally, in Section 5, we will apply Theorem 1.1 to show that for any graph G mentioned in Conjecture 1.1 and any integer $r \geq 0$, we have

$$t(L(S_r(G))) = k^{m+s-n-1}(rk+2)^{m-n+1}t(G). \tag{1.6}$$

Thus (1.3) follows and Conjecture 1.1 is proved.

Note that in the proof of Theorem 1.1, we will express $t(L(S_r(G)))$ in another form (i.e., (1.8)), which is actually equivalent to (1.4).

For any graph G and any $E' \subseteq E(G)$, let $\Gamma(E')$ be the set of those mappings $g : E' \to V(G)$ such that for each $e \in E'$, $g(e) \in \{u_e, v_e\}$, where u_e and v_e are the two ends of e. Observe that

$$\sum_{g\in\Gamma(E')} \prod_{v\in V(G)} d(v)^{-|g^{-1}(v)|} = \prod_{e\in E'} (d(u_e)^{-1} + d(v_e)^{-1}). \tag{1.7}$$

Thus (1.4) and (1.5) can be replaced by the following expressions:

$$t(L(S_r(G))) = \sum_{E'\subseteq E(G)} t(G[E'])r^{|E'|-|V(G)|+1} \sum_{g\in\Gamma(E(G)-E')} \prod_{v\in V(G)} d(v)^{d(v)-2-|g^{-1}(v)|} \tag{1.8}$$

and

$$t(L(G)) = \sum_{T\in\mathcal{T}(G)} \sum_{g\in\Gamma(E(G)-E(T))} \prod_{v\in V(G)} d(v)^{d(v)-2-|g^{-1}(v)|}. \tag{1.9}$$

2 Preliminary Results

In this section, we shall establish some results which will be used in the next section to prove Theorem 1.1 for the case $r = 0$.

For any connected graph H and any forest F of H, let $\mathcal{ST}_H(F)$ be the set of those spanning trees of H containing all edges of F, and $\mathcal{SF}_H(F)$ be the set of those spanning forests of H containing all edges of F.

In this section, we always assume that G is a connected graph with a clique V_0 such that $F = G - E(G[V_0])$ is a forest and every vertex of V_0 is incident with at most one edge of F, as shown in Figure 1.

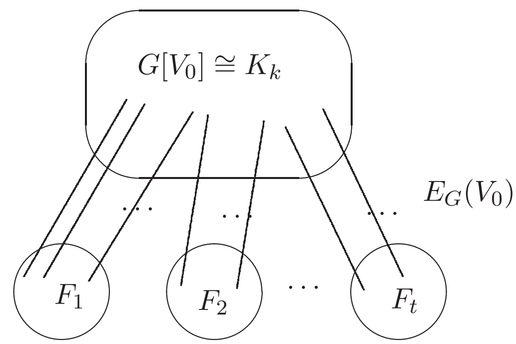


Figure 1: V_0 is a clique of G such that and $G - E(G[V_0])$ is a forest

Let $k = |V_0|$, $d = |E_G(V_0)|$, $t = c(G - V_0)$ and $F_1, F_2, \cdots, F_t$ be components of $G - V_0$. Observe that $k \geq d \geq t$, as $|E_G(v, V - V_0)| \leq 1$ holds for each $v \in V_0$ and $|E_G(V_0, V(F_i))| \geq 1$ holds for each F_i.

The main purpose in this section is to show that if $k > d$, then the set $\mathcal{ST}_G(F)$ can be equally partitioned into $\prod_{1 \leq j \leq t} |E_G(V_0, V(F_j))|$ subsets, each of which has its size $k^{k-2+t-d}$.

In the following, we divide this section into two parts.

2.1 A preliminary result on trees

In this subsection, we shall establish some results on trees which are needed for the next subsection and following sections.

Let T be any tree and V_0 be any proper subset of $V(T)$. Observe that identifying all vertices in V_0 changes T to a connected graph which is a tree if and only if $|E_T(V_0)| = c(T - V_0)$. So the following observation is obvious.

Lemma 2.1 *Let $t = c(t - V_0)$ and S be any proper subset of $E_T(V_0)$. Then the two statements below are equivalent:*

(i) $|S \cap E_T(V_0, V(F_i))| = 1$ *holds for all components $F_1, F_2, \cdots, F_t$ of $T - V_0$;*

(ii) *the graph obtained from T by removing all edges in the set $E_T(V_0) - S$ and identifying all vertices of V_0 is a tree.*

With T, V_0 given above together with a special vertex $v \in V_0$ such that $N(v) \subseteq V_0$, a subset S of $E_T(V_0)$ with the properties in Lemma 2.1 will be determined by a procedure below (i.e., Algorithm A). As S is uniquely determined by T, V_0 and v, we can denote it by $\Phi(T, V_0, v)$. Thus $|\Phi(T, V_0, v)| = t = c(T - V_0)$.

Roughly, if $t = 1$, the only edge of $\Phi(T, V_0, v)$ will be selected from $E_T(V_0)$ according to the condition that it has one end in the same component of $T[V_0]$ as v; if $t \geq 2$, the t edges of $\Phi(T, V_0, v)$ will be determined by the $t-1$ paths $P_2, P_3, \cdots, P_t$ in T, where P_j is the shortest path connecting vertices of F_1 and vertices of F_j for $j = 2, 3, \cdots, t$ and $F_1, F_2, \cdots, F_t$ are the components of $T - V_0$.

Assume that in Algorithm A, $E(T) = \{e_i : i \in I\}$ for some finite I of positive integers.

Algorithm A with input (T, V_0, v):

Step A1. Let $t = c(T - V_0)$.

Step A2. If $t = 1$, let $\Phi = \{e_j\}$, where e_j is the unique edge in the set $E_T(V_0)$ which has one end in the component of $T[V_0]$ containing v. Go to Step A5.

Step A3. (Now we have $t \geq 2$.)

A3-1. The components of $T - V_0$ are labeled as $F_1, F_2, \cdots, F_t$ such that

$$\min\{s : e_s \in E_T(V_0, F_i)\} < \min\{s' : e_{s'} \in E_T(V_0, F_{i+1})\} \tag{2.1}$$

for all $i = 1, 2, \cdots, t-1$. (In other words, these components are sorted by the minimum edge labels. For example, for the tree T in Figure 2(a), the four components F_1, F_2, F_3, F_4 of $T - V_0$ are labeled according to this rule.)

A3-2. For $j = 2, 3, \cdots, t$, determine the unique path P_j in T which is the shortest one among all those paths in T connecting vertices of F_1 to vertices of F_j.

Step A4. Let $\Phi = (E(P_2) \cap E_T(V_0, V(F_1))) \cup \bigcup_{j=2}^{t} (E(P_j) \cap E_T(V_0, V(F_j)))$.

Step A5. Output Φ.

Remarks:

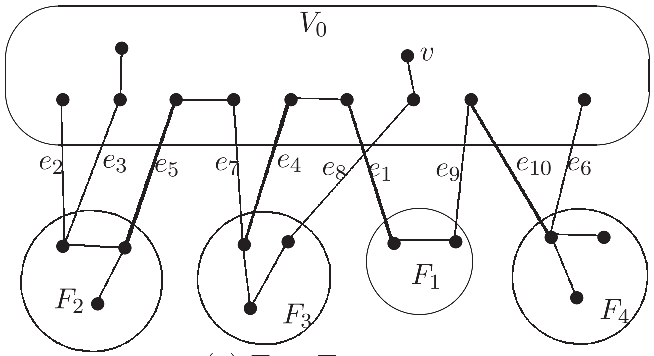
(i) Vertex v is needed only for the case that $t = 1$;

(ii) If $t = 1$, the only edge of Φ is uniquely determined as T is a tree and $T - V_0$ is connected;

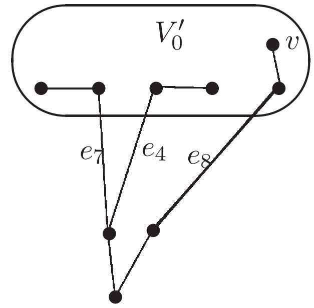
(iii) As T is a tree and F_1 and F_j are connected, P_j is actually the only path of T with its ends in F_1 and F_j respectively and every internal vertex of P_j does not belong to $V(F_1) \cup V(F_j)$. Thus for P_j chosen in Step A3,

$$|E(P_j) \cap E_G(V_0, V(F_1))| = |E(P_j) \cap E_G(V_0, V(F_j))| = 1,$$

implying that by Step A4, $|\Phi \cap E_G(V_0, V(F_j))| = 1$ for all $j = 1, 2, \cdots, t$.



(a) Tree T



(b) Tree T'

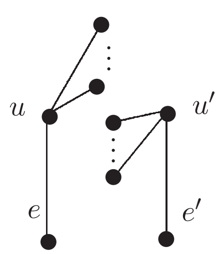
Figure 2: $\Phi(T, V_0, v) = \{e_1, e_4, e_5, e_{10}\}$ and $\Phi(T', V_0', v) = \{e_8\}$

For example, for the tree T with V_0 and v shown in Figure 2 (a), $T - V_0$ has four components F_1, F_2, F_3, F_4 labeled according to the minimum edge labels, and running Algorithm A with input (T, V_0, v) gives $\Phi(T, V_0, v) = \{e_1, e_4, e_5, e_{10}\}$, as the three paths P_2, P_3 and P_4 obtained by the algorithm have properties that $\{e_1, e_5\} \subseteq E(P_2)$, $\{e_1, e_4\} \subseteq E(P_3)$ and $\{e_9, e_{10}\} \subseteq E(P_4)$. For the tree T' in Figure 2 (b), $T' - V_0'$ has one component only and $\Phi(T', V_0', v) = \{e_8\}$. Note that vertex v is used for finding $\Phi(T', V_0', v)$ but not for finding $\Phi(T, V_0, v)$.

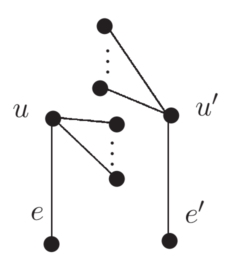
Our second purpose in this subsection is to show that if $|E_T(V_0, V(F_j))| > 1$ for some component F_j of $T - V_0$, we can find another tree T' with $V(T') = V(T)$ and $T' - E(T'[V_0]) = T - E(T[V_0])$ such that $\Phi(T', V_0, v)$ and $\Phi(T, V_0, v)$ are different only at choosing the edge joining a vertex of V_0 to a vertex in F_j.

For two distinct edges e, e' of $E_T(V_0)$ incident with u and u' respectively, where $u, u' \in V_0$, let $T(e \leftrightarrow e')$ be the graph, as shown in Figure 3, obtained from T by changing every edge (u, w) of $T[V_0]$, where $w \neq u'$, to (u', w) and every edge (u', w') of $T[V_0]$, where $w' \neq u$, to (u, w').

Roughly, $T(e \leftrightarrow e')$ is actually obtained from T by exchanging $(N_T(u) \cap V_0) - \{u'\}$ with $(N_T(u') \cap V_0) - \{u\}$. Note that u and u' are adjacent in T if and only if they are adjacent in $T(e \leftrightarrow e')$.



Part of T



Part of $T(e \leftrightarrow e')$

Figure 3: T and $T(e \leftrightarrow e')$

Let T' denote $T(e \leftrightarrow e')$ in the remainder of this subsection. There is a bijection $\tau : E(T) \to$

$E(T')$ defined below: $\tau(e) = e'$, $\tau(e') = e$, $\tau((u, w)) = (u', w)$ whenever $(u, w) \in E(T)$ for $w \in V_0 - \{u'\}$, $\tau((u', w')) = (u, w')$ whenever $(u', w') \in E(T)$ for $w' \in V_0 - \{u\}$, and $\tau(e'') = e''$ for all other edges e'' in T.

Note that T' may be not a tree, although $T' - V_0$ and $T - V_0$ are the same graph and $F_1, F_2, \cdots, F_t$ are also the components of $T' - V_0$. But T' is indeed a tree if both e and e' have ends in the same component of $T - V_0$.

Lemma 2.2 *Let e, e' be distinct edges of $E_T(V_0, V(F_i))$ for some i with $1 \leq i \leq t$.*

(i) *Then T' is a tree;*

(ii) *If $e \in \Phi(T, V_0, v)$ and either $t \geq 2$ or $N_T(v) \subseteq V_0$, then $\Phi(T', V_0, v) = (\Phi(T, V_0, v) - \{e\}) \cup \{e'\}$.*

Proof. Note that for any edge $e'' \in E(T - V_0)$, T/e'' is also a tree, T' is a tree if and T'/e'' is a tree, and more importantly, $\Phi(T, V_0, v) = \Phi(T/e'', V_0, v)$. Thus it suffices to prove this lemma only for the case that $|V(F_i)| = 1$ for all $i = 1, 2, \cdots, t$.

(i) It can be proved easily by induction on the number of edges in T.

(ii) Assume that $t = 1$. Then $N_T(v) \subseteq V_0$ and so v is not any end of e. As $e \in \Phi(T, V_0, v)$, $\Phi(T, V_0, v) = \{e\}$. By Algorithm A, e has one end (i.e., u) in the component of $T[V_0]$ containing v (i.e., the subgraph $T[V_0]$ has a path P connecting v to u). By the definition of T' (i.e., $T(e \leftrightarrow e')$), P is now changed to a path P' in $T'[V_0]$ by the mapping τ connecting v to one end of e' (i.e., u'). Thus $\Phi(T', V_0, v) = \{e'\}$ by Algorithm A. The result holds for this case.

Now assume that $t \geq 2$. For $j = 2, 3, \cdots, t$, let P_j be the only path in T with its two ends in F_1 and F_j respectively and every interval vertex of P_j does not below to $V(F_1) \cup V(F_j)$.

With the bijection $\tau : E(T) \to E(T')$ defined above, for $j = 2, 3, \cdots, t$, $\tau(E(P_j))$ is a subset of $E(T')$ and forms a path in T', denoted by P'_j. Note that the two ends of P'_j are in F_1 and F_j respectively and every interval vertex of P'_j does not below to $V(F_1) \cup V(F_j)$. Also observe that for $j = 2, 3, \cdots, t$, if $i \in \{1, j\}$, then

$$E(P'_j) \cap E_{T'}(V_0, V(F_i)) = \{e'\},$$

and if $s \in \{1, j\} - \{i\}$, then

$$E(P'_j) \cap E_{T'}(V_0, V(F_s)) = E(P_j) \cap E_T(V_0, V(F_s)).$$

Hence (ii) holds. □

2.2 Partitions of $\mathcal{ST}_G(F)$

Recall that G is a connected graph with a clique V_0 of order k such that $F = G - E(G[V_0])$ is a forest and every vertex of V_0 is incident with at most one edge of F (i.e., $d_F(v) \le 1$ for each $v \in V_0$), as shown in Figure 1. In this subsection, our main purpose is to partition $\mathcal{ST}_G(F)$ equally into $\prod_{j=1}^{t} |E_G(V_0, V(F_j))|$ subsets, where $F_1, F_2, \cdots, F_t$ are the components of $G - V_0$.

We start with the following beautiful formula for the number of spanning trees of a complete graph K_k of order k containing a given spanning forest. This result was originally due to Lovász (Problem 4 in page 29 of [11]).

Theorem 2.1 ([11]) *For any spanning forest F of K_k, if c is the number of components of F and $k_1, k_2, \cdots, k_c$ are the orders of its components, then*

$$|\mathcal{ST}_{K_k}(F)| = k^{c-2} \prod_{i=1}^{c} k_i.$$

This result naturally generalizes the well-known formula that $t(K_k) = k^{k-2}$ for any $k \ge 1$, which was first obtained by Cayley [1]. Now we apply this result to establish some results on the set $\mathcal{ST}_G(F)$ and finally partition $\mathcal{ST}_G(F)$ equally into $\prod_{j=1}^{t} |E_G(V_0, V(F_j))|$ subsets.

Recall that $d = |E_G(V_0)|$ and $k \ge d \ge t$.

Proposition 2.1 *With G, F and V_0 defined above, we have*

$$|\mathcal{ST}_G(F)| = k^{k-2+t-d} \prod_{j=1}^{t} |E_G(V_0, V(F_j))|.$$

Proof. We only need to consider the case that $E_G(V_0, V(F_j)) \ne \emptyset$ for every component F_j of $G - V_0$; otherwise, the result is trivial as $|\mathcal{ST}_G(F)| = 0$ when G is disconnected.

Observe that for any edge e of $E(G - V_0)$, we have $|\mathcal{ST}_{G/e}(F/e)| = |\mathcal{ST}_G(F)|$. Thus we may assume that every component of $G - V_0$ is a single vertex, implying that $G - V_0$ is the empty graph of t vertices, namely $x_1, x_2, \cdots, x_t$. So $E(F) = E_G(V_0)$.

For each $j = 1, 2, \cdots, t$, let $V_j = \{x \in V_0 : x \text{ is incident with } x_j\}$ and e_j be any edge joining x_j to some vertex in V_j. Let $G_0 = G[V_0]$. Note that $F/\{e_1, e_2, \cdots, e_t\}$ can be considered as a spanning forest of G_0 and

$$\mathcal{ST}_G(F) = \mathcal{ST}_{G_0}(F/\{e_1, e_2, \cdots, e_t\}).$$

As G_0 is a complete graph of order k, by Theorem 2.1,

$$|\mathcal{ST}_{G_0}(F/\{e_1, e_2, \cdots, e_t\})| = k^{c-2} \prod_{j=1}^{c} |V'_j|,$$

where c is the number of components of $F/\{e_1, e_2, \cdots, e_t\}$ and $V'_1, V'_2, \cdots, V'_c$ are vertex sets of components of $F/\{e_1, e_2, \cdots, e_t\}$. Note that

$$|V_0 - \bigcup_{j=1}^{t} V_j| = |V_0| - \sum_{k=1}^{t} |V_j| = k - |E_G(V_0)| = k - d,$$

implying that $c = k - d + t$ and the sizes of $V'_1, V'_2, \cdots, V'_c$ are equal to

$$|V_1|, \cdots, |V_t|, \underbrace{1, \cdots, 1}_{k-d}.$$

As $|V_j| = |E_G(V_0, \{x_j\})| = |E_G(V_0, V(F_j))|$, the result follows from Theorem 2.1. $\square$

Now assume that v is a vertex of V_0 with $N_G(v) \subseteq V_0$, i.e., $d_F(v) = 0$. Note that this condition is only needed for the case that $G - V_0$ is connected. Under this condition, it is obvious that $k > d$.

Recall that for any tree T of $\mathcal{ST}_G(F)$, $\Phi(T, V_0, v)$ is a subset of $E_G(V_0)$ with the property that $|\Phi(T, V_0, v) \cap E_G(V_0, V(F_j))| = 1$ for each $j = 1, 2, \cdots, t$. For each subset S of $E_G(V_0)$ with the property that $|S \cap E_G(V_0, V(F_j))| = 1$ for each $j = 1, 2, \cdots, t$, let $\mathcal{ST}_G(F, S, v)$ denote the set of those spanning trees $T \in \mathcal{ST}_G(F)$ with $\Phi(T, V_0, v) = S$. Thus $\mathcal{ST}_G(F)$ is partitioned into $\prod_{j=1}^{t} |E_G(V_0, V(F_j))|$ subsets $\mathcal{ST}_G(F, S, v)$'s. The following result shows that all these sets $\mathcal{ST}_G(F, S, v)$'s have the same size.

The following result shows that $|\mathcal{ST}_G(F, S, v)|$ is independent of S.

Proposition 2.2 *Assume that $k > d$ and $N(v) \subseteq V_0$. For any subset S of $E_G(V_0)$ with $|S \cap E_G(V_0, V(F_j))| = 1$ for each component F_j of $G - V_0$, we have*

$$|\mathcal{ST}_G(F, S, v)| = k^{k-2+t-d}.$$

Proof. There are exactly $\prod_{j=1}^{t} |E_G(V_0, V(F_j))|$ subsets S of $E_G(V_0)$ with the property that $|S \cap E_G(V_0, V(F_j))| = 1$ for each component F_j of $G - V_0$. By Proposition 2.1, we only need to show that $|\mathcal{ST}_G(F, S, v)| = |\mathcal{ST}_G(F, S', v)|$ holds for any two such sets S and S'. Thus it suffices to show that $|\mathcal{ST}_G(F, S, v)| = |\mathcal{ST}_G(F, S', v)|$ holds for any two such sets S and S' with $|S - S'| = 1$, i.e., S and S' have exactly $t - 1$ same edges.

Let S be such a subset of $E_G(V_0)$ mentioned above. Assume that e, e' are distinct edges in $E_G(V_0, V(F_j))$ for some j with $e \in S$ and $e' \notin S$. Let $S' = (S - \{e\}) \cup \{e'\}$. It remains to show that $|\mathcal{ST}_G(F, S, v)| = |\mathcal{ST}_G(F, S', v)|$.

For any $T \in \mathcal{ST}_G(F, S, v)$, let T' be the tree $T(e \leftrightarrow e')$. By Lemma 2.2, we have $\Phi(T', V_0, v) = (\Phi(T, V_0, v) - \{e\}) \cup \{e'\}$, implying that $T' \in \mathcal{ST}_G(F, S', v)$.

Let ϕ be the mapping from $\mathcal{ST}_G(F, S, v)$ to $\mathcal{ST}_G(F, S', v)$ defined by $\phi(T) = T(e \leftrightarrow e')$. It is obvious that ϕ is an onto mapping, and $\phi' : T' \to T'(e' \leftrightarrow e)$ is also an onto mapping from $\mathcal{ST}_G(F, S', v)$ to $\mathcal{ST}_G(F, S, v)$.

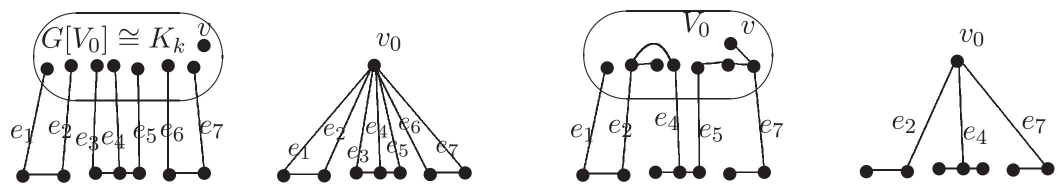
Thus $|\mathcal{ST}_G(F,S,v)| = |\mathcal{ST}_G(F,S',v)|$ and the result follows. $\square$

We end this section with an application of Proposition 2.2 to deduce another result.

Let G' be the graph obtained from G by contracting all edges in $G[V_0]$. Then V_0 becomes a vertex in G', denoted by v_0. Thus $V(G') = (V(G)-V_0)\cup\{v_0\}$, and $E(G')$ and $E(G)-E(G[V_0])$ are the same although for each edge $e \in E_G(V_0)$, its end in V_0 is changed to v_0 when e becomes an edge in G'. An example for G and G' is shown in Figure 4 (a) and (b).

Let T' be any spanning tree of G' with $E(G'-v_0) \subseteq E(T')$, i.e., $T' \in \mathcal{ST}_{G'}(F')$ for $F' = G'-v_0$. Thus $|E_{T'}(v_0)| = t$, i.e., $E_{T'}(v_0)$ has exactly t edges, corresponding to t edges in G, one from $E_G(V_0, V(F_j))$ for each component F_j of $G-V_0$. An example for T' is shown in Figure 4 (d).

Let D be any subset of $E_G(V_0)-E_{T'}(v_0)$ and let $\mathcal{ST}_G(V_0,T',D,v)$ be the set of those spanning trees T of G such that (i) $T-V_0$ and $T'-v_0$ are the same graph; (ii) $E_T(V_0)$ is the disjoint union of D and $E_{T'}(v_0)$ and (iii) $\Phi(T,V_0,v) = E_{T'}(v_0)$. Thus $\mathcal{ST}_G(V_0,T',D,v) \subseteq \mathcal{ST}_G(F)$ if and only if $D = E_G(V_0) - E_{T'}(v_0)$. For example, the tree T in Figure 4 (c) belongs to $\mathcal{ST}_G(V_0,T',D,v)$ with $D=\{e_1,e_5\}$, but $T \notin \mathcal{ST}_G(F)$, as $E(F) \not\subseteq E(T)$.



(a) G with $k \geq 8$ (b) G' (c) T (d) T'

Figure 4: A tree T in $\mathcal{ST}_G(V_0,T',D,v)$ with $D=\{e_1,e_5\}$

Proposition 2.3 *With T' and D given above, we have*

$$|\mathcal{ST}_G(V_0,T',D,v)| = k^{k-2-|D|}.$$

Proof. Let G^* denote the graph $G-D'$, where $D' = E_G(V_0)-(D\cup E_{T'}(v_0))$. Observe that

$$\mathcal{ST}_G(V_0,T',D,v) = \mathcal{ST}_{G^*}(F^*, E_{T'}(v_0), v),$$

where $F^* = G^* - E(G^*[V_0])$, i.e., $F^* = F - D'$. Also note that $c(G^*-V_0) = c(G-V_0) = t$ and

$$|E_{G^*}(V_0)| = |E_{T'}(v_0)\cup D| = t + |D|.$$

By Proposition 2.2, we have

$$|\mathcal{ST}_{G^*}(F^*, E_{T'}(v_0), v)| = k^{k-2+t-(t+|D|)} = k^{k-2-|D|}.$$

Thus the result holds. $\square$

3 Proving Theorem 1.1 for $r = 0$

In this section, we shall prove Theorem 1.1 for the case $r = 0$ (i.e., the result of (1.9) or equivalently (1.5)) is a special case of another result (i.e., Theorem 3.1).

Let u be any vertex in a simple graph G. Assume that $E_G(u) = \{(u, u_i) : 1 \le i \le s\}$, where $s = d_G(u)$. If G' is the graph obtained from $G - u$ by adding a complete graph K_s with vertices $w_1, w_2, \cdots, w_s$ and adding s new edges (w_i, u_i) for $i = 1, 2, \cdots, s$, then G' is said to be obtained from G by a *clique-insertion at u*. The clique-insertion is a graph operation playing an important role in the study of vertex-transitive graphs (see [12, 14]). The *clique-inserted graph* of G, denoted by $C(G)$, is obtained from G by operating clique-insertion at every vertex of G. Note that the clique-inserted graph of G is also called *the para-line graph* of G (see [18]). An example for $C(G)$ is shown in Figure 5.

Let M be the set of those edges in $E(C(G))$ which are not in the inserted cliques. So M consists of all edges in $E(G)$ and thus can be considered as the same as $E(G)$. Observe that $C(G)$ has the following properties:

(i) M is a matching of $C(G)$;

(ii) $L(G)$ is the graph $C(G)/M$ and thus $t(L(G)) = |\mathcal{ST}_{C(G)}(M)|$;

(iii) each component of $C(G) - M$ is a complete graph.

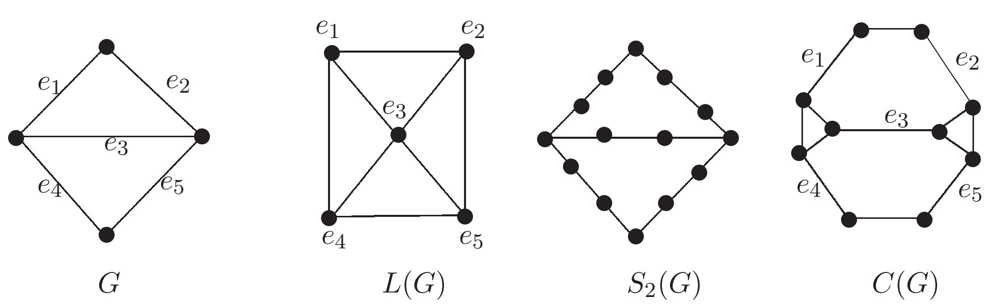


Figure 5: Line graph $L(G)$ and clique-inserted graph $C(G)$

From observation (iii) above, $C(G)$ is in a type of connected graphs with a matching whose removal yields components which are all complete graphs. As $t(L(G)) = |\mathcal{ST}_{C(G)}(M)|$ holds for any connected graph G with M defined above, we now extend our problem to finding an expression for $|\mathcal{ST}_Q(M)|$, where Q is an arbitrary connected graph and M is any matching of Q such that all components of $Q - M$ are complete graphs.

Throughout this section, we assume

(i) Q is a simple and connected graph with a matching M such that all components $Q_1, Q_2, \cdots, Q_n$ of $Q - M$ are complete graphs;

(ii) for $i = 1, 2, \cdots, n$, $V_i = V(Q_i) = \{v_{i,j} : j = 1, 2, \cdots, k_i\}$, where $k_i = |V_i|$;

(iii) $M = \{e_1, e_2, \cdots, e_m\}$ and M_i is the set of those edges of M which have one end in V_i and $m_i = |M_i|$ for $i = 1, 2, \cdots, n$;

(iv) $v_{i,j}$ is incident with an edge of M_i if and only if $1 \le j \le m_i$;

(v) Q^* is the graph obtained from Q by contracting all edges of Q_i for all $i = 1, 2, \cdots, n$. Thus each Q_i is converted to a vertex in Q^* denoted by v_i.

With the above assumptions, we observe that $V(Q^*) = \{v_1, v_2, \cdots, v_n\}$ and $E(Q^*) = M$. As M is a matching of Q and Q is connected, we have $1 \le m_i \le k_i$. If $k_i > m_i$, then vertex $v_{i,j}$ is not incident with any edge of M for all $j : m_i < j \le k_i$. If $k_i = m_i$ for all $i = 1, 2, \cdots, n$, then $|\mathcal{ST}_Q(M)| = t(L(Q^*))$. Thus result (1.9) is a special case of Theorem 3.1 which is the main result to be established in this section.

Theorem 3.1 *For Q, Q^* and M defined above, we have*

$$|\mathcal{ST}_Q(M)| = \sum_{T \in \mathcal{T}(Q^*)} \sum_{f \in \Gamma(E(Q^*) - E(T))} \prod_{i=1}^{n} k_i^{k_i - 2 - |f^{-1}(v_i)|}. \tag{3.1}$$

To prove Theorem 3.1, by the following result, we only need to consider the case that $k_i > m_i$ for all $i = 1, 2, \cdots, n$.

Proposition 3.1 *Theorem 3.1 holds if it holds whenever $k_i > m_i$ for all $i = 1, 2, \cdots, n$.*

Proof. Assume that M is fixed and so all m_i's are fixed. Without loss of generality, we only need to show that with k_i, where $k_i \ge m_i$, to be fixed for all $i = 2, \cdots, n$, if (3.1) holds for every integer k_1 with $k_1 \ge m_1 + 1$, then it also holds for the case $k_1 = m_1$.

For any integer $k_1 \ge m_1$, let

$$\gamma(k_1) = |\mathcal{ST}_Q(M)|.$$

By the assumption, for any $k_1 \ge m_1 + 1$, (3.1) holds and thus

$$\gamma(k_1) = \sum_{T \in \mathcal{T}(Q^*)} \sum_{f \in \Gamma(E(Q^*) - E(T))} k_1^{k_1 - 2 - |f^{-1}(v_1)|} \prod_{i=2}^{n} k_i^{k_i - 2 - |f^{-1}(v_i)|} = \sum_{s=0}^{m_1 - 1} a_s k_1^{k_1 - 2 - s}, \tag{3.2}$$

where

$$a_s = \sum_{T \in \mathcal{T}(Q^*)} \sum_{\substack{f \in \Gamma(E(Q^*) - E(T)) \\ |f^{-1}(v_1)| = s}} \prod_{i=2}^{n} k_i^{k_i - 2 - |f^{-1}(v_i)|}. \tag{3.3}$$

It is clear that a_s is independent of the value of k_1.

Now let Q' be the graph $Q - E(Q_1) - \{v_{1,j} : m_1 < j \le k_1\}$. So Q' is independent of k_1. Note that for every $T \in \mathcal{ST}_Q(M)$, $F = T - E(T[V_1]) - \{v_{1,j} : m_1 < j \le k_1\}$ is a member of $\mathcal{SF}_{Q'}(M)$, i.e., a spanning forest of Q' containing all edges of M, since $v_{1,j}$ is not incident with any edge of M for all $j : m_1 < j \le k_1$. Thus $\mathcal{ST}_Q(M)$ can be partitioned into

$$\mathcal{ST}_Q(M) = \bigcup_{F \in \mathcal{SF}_{Q'}(M)} \mathcal{ST}_{Q''}(F),$$

where $Q'' = Q[E(F) \cup E(Q_1)]$. It is possible that $\mathcal{ST}_{Q''}(F) = \emptyset$ for some $F \in \mathcal{SF}_{Q'}(M)$. But $\mathcal{ST}_{Q''}(F') \cap \mathcal{ST}_{Q''}(F'') = \emptyset$ for distinct $F', F'' \in \mathcal{SF}_{Q'}(M)$, implying that for any $k_1 = |V_1| \ge m_1$,

$$\gamma(k_1) = \sum_{F \in \mathcal{SF}_{Q'}(M)} |\mathcal{ST}_{Q''}(F)|.$$

By Proposition 2.1, for any $F \in \mathcal{SF}_{Q'}(M)$, if $F/\{v_{1,j} : 1 \le j \le m_1\}$ is connected, then

$$|\mathcal{ST}_{Q''}(F)| = k_1^{k_1-2+c(F-V_1)-m_1} \prod_{j=1}^{c(F-V_1)} |E_F(V_1, V(F_j))|,$$

where $F_1, F_2, \cdots, F_{c(F-V_1)}$ are the components of $F - V_1$. Let $\mathcal{SF}^c_{Q'}(M)$ denote the set of those $F \in \mathcal{SF}_{Q'}(M)$ such that $F/\{v_{1,j} : 1 \le j \le m_1\}$ is connected. Thus, for any $k_1 \ge m_1$, we have

$$\begin{aligned}
\gamma(k_1) &= \sum_{F \in \mathcal{SF}^c_{Q'}(M)} k_1^{k_1-2+c(F-V_1)-m_1} \prod_{j=1}^{c(F-V_1)} |E_F(V_1, V(F_j))| \\
&= \sum_{s=0}^{m_1-1} b_s k_1^{k_1-2-s}, \qquad (3.4)
\end{aligned}$$

where

$$b_s = \sum_{\substack{F \in \mathcal{SF}^c_{Q'}(M) \\ c(F-V_1)=m_1-s}} \prod_{j=1}^{c(F-V_1)} |E_F(V_1, V(F_j))|. \qquad (3.5)$$

As Q' is independent of k_1, for any $F \in \mathcal{SF}^c_{Q'}(M)$, the expression $\prod_{j=1}^{c(F-V_1)} |E_F(V_1, V(F_j))|$ is independent of $k_1 = |V_1|$ and hence b_s is independent of k_1.

By (3.2) and (3.4), for every integer k_1 with $k_1 \ge m_1 + 1$, we have

$$\sum_{s=0}^{m_1-1} a_s k_1^{k_1-2-s} = \sum_{s=0}^{m_1-1} b_s k_1^{k_1-2-s}, \qquad (3.6)$$

where a_s and b_s are independent of k_1 for all $s = 0, 1, 2, \cdots, m_1 - 1$. Considering sufficiently large values of k_1 in (3.6), we come to the conclusion that $a_s = b_s$ for all $s = 0, 1, \cdots, m_1$, implying that

$$\begin{aligned}
\gamma(m_1) &= \sum_{s=0}^{m_1-1} b_s m_1^{m_1-2-s} = \sum_{s=0}^{m_1-1} a_s m_1^{m_1-2-s} \\
&= \sum_{T \in \mathcal{T}(Q^*)} \sum_{f \in \Gamma(E(Q^*)-E(T))} m_1^{m_1-2-|f^{-1}(v_1)|} \prod_{i=2}^{n} k_i^{k_i-2-|f^{-1}(v_i)|},
\end{aligned}$$

implying that (3.1) holds for $k_1 = m_1$. Hence the result holds. $\square$

In the remainder of this section, we assume that $k_i \geq m_i + 1$ for all i with $1 \leq i \leq n$. Thus vertex v_{i,k_i} is not incident with any edge of M for each i. We will complete the proof of Theorem 3.1 by the approach explained in the two steps below:

(a) $\mathcal{ST}_Q(M)$ will be partitioned into $t(Q^*)2^{m-n+1}$ subsets denoted by $\Delta(T_0, f)$'s, corresponding to $t(Q^*)2^{m-n+1}$ ordered pairs (T_0, f), where $T_0 \in \mathcal{T}(Q^*)$ and $f \in \Gamma(E(Q^*) - E(T_0))$;

(b) then we show that for any given $T_0 \in \mathcal{T}(Q^*)$ and $f \in \Gamma(E(Q^*) - E(T_0))$,

$$|\Delta(T_0, f)| = \prod_{i=1}^{n} k_i^{k_i - 2 - |f^{-1}(v_i)|}.$$

Step (a) above will be done by Algorithm B below which determines a spanning tree T_0 of Q^* and a mapping $f \in \Gamma(E(Q^*) - E(T_0))$ for any given $T \in \mathcal{ST}_Q(M)$.

Algorithm B ($T \in \mathcal{ST}_Q(M)$):

Step B1. Let T_n be T;

Step B2. for $i = n, n-1, \cdots, 1$, let $D_i = E_{T_i}(V_i) - \Phi(T_i, V_i, v_{i,k_i})$ and T_{i-1} be the graph obtained from T_i by deleting all edges in $D_i \cup E(T_i[V_i])$ and identifying all vertices of V_i as one, denoted by v_i, which is a vertex of Q^*;

Step B3. output T_0 and f, where f is a mapping from $D_1 \cup D_2 \cup \cdots \cup D_n$ to $V(Q^*)$ defined by $f(e) = v_i$ whenever $e \in D_i$.

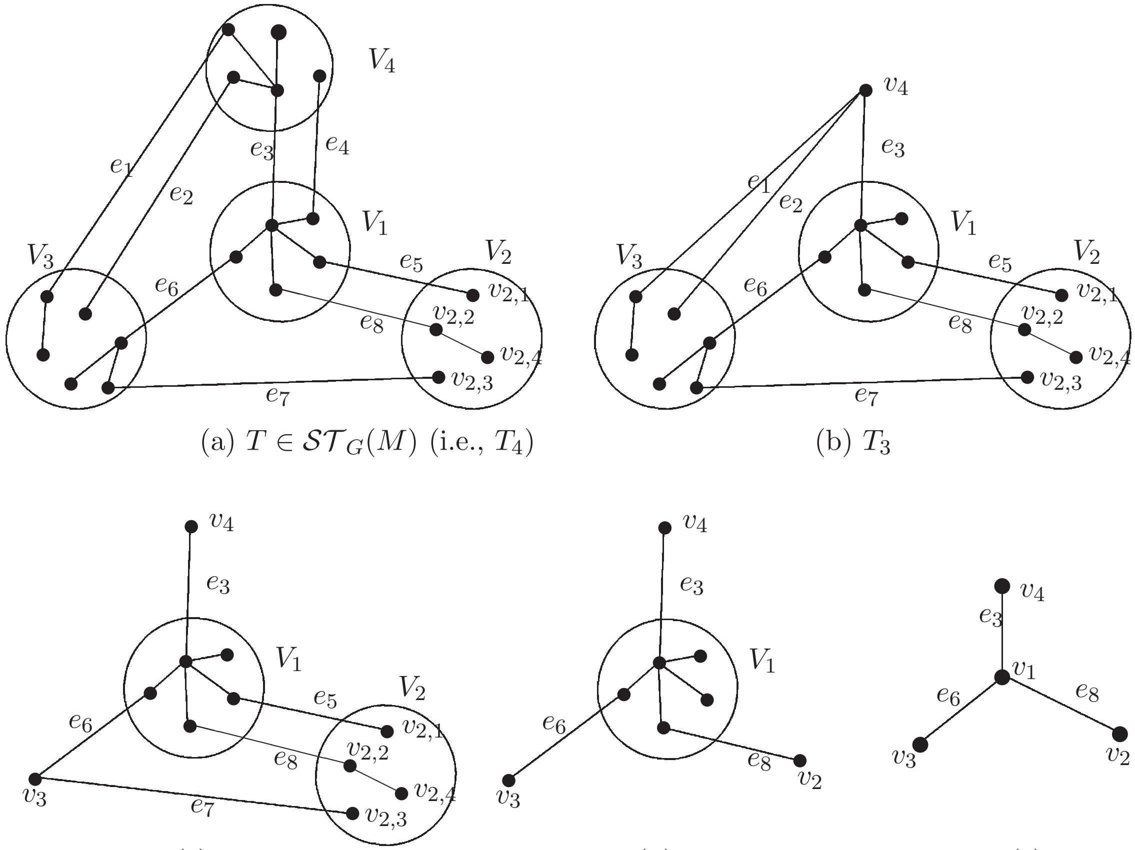
By Lemma 2.1, each graph T_i produced in the process of running Algorithm B is indeed a tree and thus T_0 is a tree in $\mathcal{T}(Q^*)$. It is also clear that $D_1 \cup D_2 \cup \cdots \cup D_n = E(Q^*) - E(T_0)$ and so the mapping f output by Algorithm B belongs to $\Gamma(E(Q^*) - E(T_0))$.

An example is presented below. Let T be a tree in $\mathcal{ST}_Q(M)$ as shown in Figure 6(a), where Q is a connected graph with a matching $M = \{e_1, e_2, \cdots, e_8\}$ such that $Q - M$ has four components Q_1, Q_2, Q_3 and Q_4 isomorphic to complete graphs of orders $5, 4, 6, 5$ respectively. If we run Algorithm B with this tree T as its input, then we have T_3, T_2, T_1 and T_0 as shown in Figure 6 and thus

$$D_4 = \{e_4\}, D_3 = \{e_1, e_2\}, D_2 = \{e_5, e_7\}, D_1 = \emptyset,$$

implying that the mapping $f \in \Gamma(E(Q^*) - E(T_0))$ output by Algorithm B, where $E(Q^*) - E(T_0) = \{e_1, e_2, e_4, e_5, e_7\}$, is the one given below:

$$f(e_1) = f(e_2) = v_3, f(e_4) = v_4, f(e_5) = f(e_7) = v_2.$$



(a) $T \in \mathcal{ST}_G(M)$ (i.e., T_4) (b) T_3

(c) T_2 (d) T_1 (e) T_0

Figure 6: $T \in \mathcal{ST}_Q(M)$ (i.e., T_4) and T_3, T_2, T_1, T_0

Let ψ be a mapping from $\mathcal{ST}_Q(M)$ to the following set of ordered pair (T_0, f)'s:

$$\{(T_0, f) : T_0 \in \mathcal{T}(Q^*), f \in \Gamma(E(Q) - E(T_0))\},$$

defined by $\psi(T) = (T_0, f)$ if T_0 and f are output by running Algorithm B with input T. For any $T_0 \in \mathcal{T}(Q^*)$ and $f \in \Gamma(E(Q) - E(T_0))$, let $\Delta(T_0, f) = \psi^{-1}(T_0, f)$. Thus $\mathcal{ST}_Q(M)$ is partitioned into $t(Q^*)2^{m-n+1}$ subsets $\Delta(T_0, f)$'s, where $T_0 \in \mathcal{T}(Q^*)$ and $f \in \Gamma(E(Q) - E(T_0))$.

The proof of Theorem 3.1 now remains to determine the size of $\Delta(T_0, f)$ below.

Proposition 3.2 *For any $T_0 \in \mathcal{T}(Q^*)$ and $f \in \Gamma(E(Q^*) - E(T_0))$, we have*

$$|\Delta(T_0, f)| = \prod_{i=1}^{n} k_i^{k_i - 2 - |f^{-1}(v_i)|}.$$

Proof. Let $D_i = f^{-1}(v_i) = \{e \in M - E(T_0) : f(e) = v_i\}$ for $i = 1, 2, \cdots, n$. So $D_i \subseteq M_i$. By Algorithm B, T is a member of $\Delta(T_0, f)$ if and only if there exist trees $T_1, T_2, \cdots, T_{n-1}$ such that for $i = n, n-1, \cdots, 1$, the following properties hold, where T_n is the tree T:

(P1) $V(T_i) = (V(T_{i-1}) - \{v_i\}) \cup V_i$;

(P2) $T_i - V_i$ and $T_{i-1} - v_i$ are the same graph; and

(P3) $E_{T_{i-1}}(v_i) = \Phi(T_i, V_i, v_{i,k_i}) = E_{T_i}(V_i, V(T_i) - V_i) - D_i$ and $D_i \subseteq E_{T_i}(V_i, V(T_i) - V_i)$.

Let $U_i = \bigcup\limits_{1\le j\le i} V_j \cup \{v_{i+1}, \cdots, v_n\}$. Observe that if properties (P1), (P2) and (P3) hold for all i with $1 \le i \le n$, then $V(T_i) = U_i$ for all $i = 0, 1, \cdots, n$.

Now let $\Delta_0 = \{T_0\}$. Define sets $\Delta_1, \Delta_2, \cdots, \Delta_n$ as follows. For $i = 1, 2, \cdots, n$, let

$$\Delta_i = \bigcup_{T_{i-1}\in\Delta_{i-1}} \Psi(T_{i-1}),$$

where $\Psi(T_{i-1})$ is the set of all those spanning trees T_i of H_i such that properties (P1), (P2) and (P3) hold for T_i and T_{i-1} and H_i is the graph with $V(H_i) = U_i$ such that V_i is a clique of H_i, $H_i - V_i$ is the same as $T_{i-1} - v_i$ and $E_{H_i}(V_i) = E_{T_{i-1}}(v_i) \cup D_i$. Note that for each edge $e \in E_{H_i}(V_i)$, e is actually also an edge in Q and we assume that e joins the same pair of vertices as it does in Q unless e as an edge of Q has one end in some V_j with $j > i$, while in this case this end of e in H_i is v_j.

By (P1), (P2) and (P3), T_{i-1} is uniquely determined by any $T_i \in \Psi(T_{i-1})$. Thus $\Psi(T'_{i-1}) \cap \Psi(T''_{i-1}) = \emptyset$ for any distinct members T'_{i-1} and T''_{i-1} of Δ_{i-1}. For any $T_{i-1} \in \Delta_{i-1}$, observe that $\Psi(T_{i-1})$ is actually the set $\mathcal{ST}_{H_i}(V_i, T_{i-1}, D_i, v_{i,k_i})$, and thus by Proposition 2.3, we have

$$|\Psi(T_{i-1})| = k_i^{k_i-2-|D_i|}.$$

Hence $|\Delta_i| = k_i^{k_i-2-|D_i|}|\Delta_{i-1}|$ for all $i = 1, 2, \cdots, n$. As $\Delta(T_0, f) = \Delta_n$, the result holds. □

We end this section with a proof of Theorem 3.1.

Proof of Theorem 3.1: By Proposition 3.1, we may assume that $k_i > m_i$ for all $i = 1, 2, \cdots, n$. By the definition of ψ and $\Delta(T_0, f) = \psi^{-1}(T_0, f)$, we have

$$\mathcal{ST}_Q(M) = \bigcup_{\substack{T_0\in\mathcal{T}(H)\\ f\in\Delta(E(H)-E(T_0))}} \Delta(T_0, f),$$

where the union gives a partition of $\mathcal{ST}_Q(M)$. Thus

$$|\mathcal{ST}_Q(M)| = \sum_{\substack{T_0\in\mathcal{T}(H)\\ f\in\Gamma(E(H)-E(T_0))}} |\Delta(T_0, f)| = \sum_{\substack{T_0\in\mathcal{T}(H)\\ f\in\Gamma(E(H)-E(T_0))}} \prod_{i=1}^{n} k_i^{k_i-2-|f^{-1}(v_i)|},$$

where the last step follows from Proposition 3.2. Hence Theorem 3.1 holds. □

4 Proving Theorem 1.1 for $r \geq 1$

In this section, we shall prove Theorem 1.1 for the case $r \geq 1$.

For any graph G and edge e in G, let $G-e$ and G/e be the graphs obtained from G by deleting e and contracting e respectively. The following result is obvious.

Lemma 4.1 ([4, 5]) *For any graph G and edge e in G, we have*

$$t(G) = t(G-e) + t(G/e).$$

In particular, if e is a bridge of G, then $t(G) = t(G/e)$.

For any edge e in G, let $G_{\bullet e}$ be the graph obtained from G by inserting a vertex on e and G_{-e} be the graph obtained from $G-e$ by attaching a pendent edge to each end of e, as shown in Figure 7. Similarly, for any $E' \subseteq E(G)$, let $G_{\bullet E'}$ be the graph obtained from G by inserting a vertex on each edge of E' and $G_{-E'}$ be the graph obtained from $G-E'$ by attaching a pendent edge to each end of e for all $e \in E'$. Clearly $G_{\bullet E'}$ is the graph $S(G)$ when $E' = E(G)$.

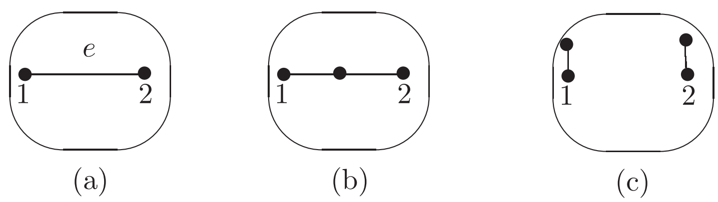


Figure 7: (a) G with edge e (b) The graph $G_{\bullet e}$ (c) The graph G_{-e}

By the definition of the line graph, the following lemma follows from Lemma 4.1.

Lemma 4.2 *Let G be any graph and e be an edge in G. Then*

$$t(L(G_{\bullet e})) = t(L(G)) + t(L(G_{-e})).$$

In particular, if e is a bridge of G, then $t(L(G_{\bullet e})) = t(L(G))$.

For any edge e in G and any non-negative integer r, let $G_{r\bullet e}$ be the graph obtained from G by inserting r new vertices on e, i.e., replacing e by a path of length $r+1$ connecting the two ends of e. For any subset F of $E(G)$, let $G_{r\bullet F}$ be the graph obtained from G by replacing each edge e of F by a path of length $r+1$ connecting the two ends of e.

Lemma 4.3 *Let G be any graph and F be any subset of $E(G)$. Then, for any $r \geq 0$,*

$$t(L(G_{r\bullet F})) = \sum_{E' \subseteq F} r^{|E'|} t(L(G_{-E'})). \tag{4.1}$$

Proof. Note that for any two vertices u, v in a graph H, if $N_H(u) = \{v\}$ and $d_H(v) = 2$, then $t(L(H)) = t(L(H - u))$. Thus, for any edge e of G and any positive integer r, by Lemma 4.2, we have

$$t(L(G_{r\bullet e})) = t(L(G_{(r-1)\bullet e})) + t(L(G_{-e})), \tag{4.2}$$

where $G_{0\bullet e}$ is G. Applying (4.2) repeatedly deduces that

$$t(L(G_{r\bullet e})) = t(L(G)) + rt(L(G_{-e})). \tag{4.3}$$

Note that (4.1) is obvious for $F = \emptyset$ or $r = 0$. Now assume that $e \in F$ and $r \geq 1$. By induction, we have

$$t(L(G_{r\bullet F-\{e\}})) = \sum_{E'\subseteq F-\{e\}} r^{|E'|} t(L(G_{-E'})). \tag{4.4}$$

By (4.3), we have

$$t(L(G_{r\bullet F})) = t(L(G_{r\bullet F-\{e\}})) + rt(L((G_{r\bullet F-\{e\}})_{-e})). \tag{4.5}$$

Thus (4.1) follows immediately from (4.4). □

We are now ready to prove Theorem 1.1 for the case $r \geq 1$.

Proof of Theorem 1.1 for $r \geq 1$: Assume that $r \geq 1$. By Lemma 4.3, we have

$$t(L(S_r(G))) = \sum_{E'\subseteq E(G)} r^{|E'|} t(L(G_{-E'})). \tag{4.6}$$

The above summation needs only to take those subsets E' of $E(G)$ with $t(L(G_{-E'})) > 0$ (i.e. $G - E'$ is connected). Now let E' be any fixed subset of $E(G)$ such that $G - E'$ is connected and let H denote $G_{-E'}$. By Theorem 1.1 for $r = 0$ (i.e., (1.9)),

$$t(L(G_{-E'})) = \sum_{T'\in\mathcal{T}(H)} \sum_{g\in\Gamma(E(H)-E(T'))} \prod_{v\in V(H)} d_H(v)^{d_H(v)-2-|g^{-1}(v)|}. \tag{4.7}$$

Observe that $V(G) \subseteq V(H)$. For any $v \in V(H)$, if $v \in V(G)$, then $d_H(v) = d_G(v)$; otherwise, $d_H(v) = 1$. Thus

$$\prod_{v\in V(H)} d_H(v)^{d_H(v)-2-|g^{-1}(v)|} = \prod_{v\in V(G)} d_G(v)^{d_G(v)-2-|g^{-1}(v)|}. \tag{4.8}$$

For each $T' \in \mathcal{T}(H)$, T' contains all pendent edges in H and so T' corresponds to T, where $T = T'[V(G)]$, which is a spanning tree of $G - E'$. Thus $E(H) - E(T') = E(G - E') - E(T)$ and

$$t(L(G_{-E'})) = \sum_{T\in\mathcal{T}(G-E')} \sum_{g\in\Gamma(E(G-E')-E(T))} \prod_{v\in V(G)} d_G(v)^{d_G(v)-2-|g^{-1}(v)|}. \tag{4.9}$$

By (4.6) and (4.9),

$$t(L(S_r(G))) = \sum_{E'\in E(G)} r^{|E'|} \sum_{T\in\mathcal{T}(G-E')} \sum_{g\in\Gamma(E(G-E')-E(T))} \prod_{v\in V(G)} d_G(v)^{d_G(v)-2-|g^{-1}(v)|}. \tag{4.10}$$

By replacing $E(G) - E' - E(T)$ by E'', (4.10) implies that

$$\begin{aligned} t(L(S_r(G))) &= \sum_{E''\subseteq E(G)} \sum_{T'\in \mathcal{T}(G-E'')} \sum_{g\in\Gamma(E'')} r^{|E(G)|-|E''|-|E(T')|} \prod_{v\in V(G)} d_G(v)^{d_G(v)-2-|g^{-1}(v)|} \\ &= \sum_{E''\subseteq E(G)} r^{|E(G)|-|E''|-|V(G)|+1} t(G-E'') \sum_{g\in\Gamma(E'')} \prod_{v\in V(G)} d_G(v)^{d_G(v)-2-|g^{-1}(v)|} \\ &= \sum_{E'''\subseteq E(G)} r^{|E'''|-|V(G)|+1} t(G[E''']) \sum_{g\in\Gamma(E-E''')} \prod_{v\in V(G)} d_G(v)^{d_G(v)-2-|g^{-1}(v)|}. \end{aligned}$$

Hence the case $r \geq 1$ of Theorem 1.1 holds. □

5 Proof of Conjecture 1.1

Now we turn back to those connected graphs G mentioned in Conjecture 1.1 and apply the following result and Theorem 1.1 to deduce a relation between $t(L(S_r(G)))$ and $t(G)$. The case $r = 1$ of this relation is exactly the conclusion of Conjecture 1.1.

Lemma 5.1 *Let H be any connected graph of order n and size m. For any integer i with $0 \leq i \leq m - n + 1$, we have*

$$\binom{m-n+1}{i} t(H) = \sum_{\substack{E'\subseteq E(H)\\ |E'|=i}} t(H - E').$$

Proof. We prove this result by providing two different methods to determining the size of the following set:

$$\Theta = \{(T, E') : T \text{ is a spanning tree of } H \text{ and } E' \subseteq E(H) - E(T) \text{ with } |E'| = i\}.$$

Note that for each spanning tree T of H, as $|E(H)| = m$ and $|E(T)| = n - 1$, the number of subsets E' of $E(H) - E(T)$ with $|E'| = i$ is $\binom{m-n+1}{i}$. On the other hand, for each $E' \subseteq E(H)$ with $|E'| = i$, there are exactly $t(H - E')$ spanning trees T of G such that $E' \subseteq E(H) - E(T)$. Thus the result holds. □

We now deduce the following consequence of Theorem 1.1 for those connected graphs G mentioned in Conjecture 1.1.

Corollary 5.1 *Let G be a connected graph of order $n + s$ and size $m + s$ in which s vertices are of degree 1 and all others are of degree k, where $k \geq 2$. Then, for any $r \geq 0$,*

$$t(L(S_r(G))) = k^{m+s-n-1}(rk + 2)^{m-n+1} t(G).$$

Proof. For any $E' \subseteq E(G)$ with $t(G[E']) \neq 0$, E' contains every bridge of G, and so $d(u_e) = d(v_e) = k$ for all $e \in E(G) - E'$. By Theorem 1.1, we have

$$
\begin{aligned}
t(L(S_r(G))) &= (k^{k-2})^n \sum_{E' \subseteq E(G)} t(G[E']) r^{|E'|-(n+s)+1} (2k^{-1})^{(m+s)-|E'|} \\
&= (k^{k-2})^n r^{-(n+s)+1} (2k^{-1})^{(m+s)} \sum_{E' \subseteq E(G)} t(G[E']) r^{|E'|} (2k^{-1})^{-|E'|} \\
&= (k^{k-2})^n r^{-(n+s)+1} (2k^{-1})^{(m+s)} \sum_{E'' \subseteq E(G)} t(G - E'') r^{|E(G)|-|E''|} (2k^{-1})^{|E''|-|E(G)|} \\
&= (k^{k-2})^n r^{-(n+s)+1} (2k^{-1})^{(m+s)} \sum_{j=0}^{m-n+1} r^{m+s-j} (2k^{-1})^{j-m-s} \sum_{\substack{E'' \subseteq E(G) \\ |E''|=j}} t(G - E'') \\
&= (k^{k-2})^n \sum_{j=0}^{m-n+1} r^{m-n+1-j} (2k^{-1})^j \binom{m-n+1}{j} t(G) \qquad \text{(by Lemma 5.1)} \\
&= (k^{k-2})^n (r + 2k^{-1})^{m-n+1} t(G) \\
&= k^{n(k-2)-(m-n+1)} (kr+2)^{m-n+1} t(G) \\
&= k^{m+s-n-1} (kr+2)^{m-n+1} t(G),
\end{aligned}
$$

where the last expression follows from the equality $2(m+s) = kn + s$ by the given conditions on G. Hence the result is obtained. □

Notice that (1.3) is the special case of Corollary 5.1 for $r = 0$ while the conclusion of Conjecture 1.1 is the special case of Corollary 5.1 for $r = 1$.

We end this section with the the following result on some special bipartite graphs, which can be obtained by applying Lemma 5.1 and the case $r = 0$ of Theorem 1.1.

Corollary 5.2 *Let $G = (A, B; E)$ be a connected bipartite graph of order n and size m such that $d(x) \in \{1, a\}$ for all $x \in A$ and $d(y) \in \{1, b\}$ for all $y \in B$, where $a \geq 2$ and $b \geq 2$. Then*

$$t(L(G)) = a^{(a-2)n_1} b^{(b-2)n_2} (a^{-1} + b^{-1})^{m-n+1} t(G),$$

where n_1 is the number of vertices x in A with $d(x) = a$ and n_2 is the number of vertices y in B with $d(y) = b$.

The result of Corollary 5.2 in the case that G is an (a, b)-semiregular bipartite graph was originally due to Cvetković (see Theorem 3.9 in [13], §5.2 of [15], or [17]).

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