# Planar digraphs without large acyclic sets 

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#### Abstract

Given a directed graph, an acyclic set is a set of vertices inducing a directed subgraph with no directed cycle. In this note we show that for all integers $n \geq g \geq 3$, there exist oriented planar graphs of order $n$ and digirth $g$ for which the size of the maximum acyclic set is at most $\left\lceil\frac{n(g-2)+1}{g-1}\right\rceil$. When $g=3$ this result disproves a conjecture of Harutyunyan and shows that a question of Albertson is best possible.


## 1 Introduction

An oriented graph is a digraph $D$ without loops and multiple arcs. An acyclic set in $D$ is a set of vertices which induces a directed subgraph without directed cycles. The complement of an acyclic set of $D$ is a feedback vertex set of $D$. A question of Albertson, which was the problem of the month on Mohar's web page [6] and was listed as a "Research Experience for Graduate Students" by West [11], asks whether every oriented planar graph on $n$ vertices has an acyclic set of size at least $\frac{n}{2}$. There are three independent strengthenings of this question in the literature. In the following, we discuss them briefly.

Conjecture 1 (Harutyunyan [3] [4]) Every oriented planar graph of order $n$ has an acyclic set of size at least $\frac{3 n}{5}$.

The digirth of a directed graph is the length of a smallest directed cycle. Golowich and Rolnick [5] showed that a oriented planar graph of digirth $g$ has an acyclic set of size at least $\max \left(\frac{n(g-3)+6}{g}, \frac{n(2 g-3)+6}{3 g}\right)$, in particular proving Conjecture 1 for oriented planar graphs of digirth 8 .

A lower bound of $\frac{n}{2}$ for the size of an acyclic set in an oriented planar graph would immediately follow from any of the following two conjectures.

Conjecture 2 (Neumann-Lara [7]) Every oriented planar graph can be vertex-partitioned into two acyclic sets.

Harutyunyan and Mohar [4] proved Conjecture 2 for oriented planar graphs of digirth 5. The undirected analogue of Conjecture 2 is false. Indeed, it is equivalent to a conjecture of Tait [9], saying that every 3 -connected planar cubic graph has a Hamiltonian cycle, which was disproved by Tutte [10]. However, the following question remains open:

Conjecture 3 (Albertson and Berman [1]) Every simple undirected planar graph of order $n$ has an induced forest of order at least $\frac{n}{2}$.

There are many graphs showing that Conjecture 3, if true, is best-possible, e.g., $K_{4}$ and the octahedron. The best-known lower bound for the order of a largest induced forest in a planar graph is $\frac{2 n}{5}$ and follows from Borodin's result on acyclic vertex-coloring of undirected planar graphs [2].

We summarize the discussion in Table 1

[^0]| digirth $g$ | 3 | 4 | 5 | $\geq 6$ |
| :--- | :---: | :---: | :---: | :---: |
| acyclic set | $\frac{2 n}{5}[2]$ | $\frac{5 n+6}{12}[5]$ | $\frac{n}{2}[4]$ | $\frac{n(g-3)+6}{g}[5]$ |

Table 1: Lower bounds for acyclic sets in oriented planar graphs.

## 2 The construction

In this section, we construct oriented planar graphs of a given digirth with no large acyclic sets. The most important case of this result is the one when the digirth is 3 . Here our result implies that, if true, for odd $n$ the lower bound of $\frac{n}{2}$ in Albertson's question is best possible, while it might be improved by at most 1 in the even case, see Problem 1. In particular, this disproves Conjecture 1 .

Theorem 1 For all integers $n \geq g \geq 3$, there exists an n-vertex oriented planar graph with digirth $g$ in which the maximum acyclic set has size $n-\left\lfloor\frac{n-1}{g-1}\right\rfloor=\left\lceil\frac{n(g-2)+1}{g-1}\right\rceil$.

## Proof

Let $g \geq 3$ be fixed. We inductively show that for any $f \geq 1$ there exists a oriented planar graph $D_{f}$ such that: $D_{f}$ has digirth $g$, order $f(g-1)+1$, and a minimum vertex feedback set of size $f$. Moreover, $D_{f}$ has two vertices $x$ and $y$ on a common face $F$, which do not simultaneously appear in any minimum feedback vertex set. For $D_{1}$ we take a directed cycle of order $g$. This clearly satisfies all conditions. If $f>1$, take the plane digraph $D_{f-1}$, add a directed path $s_{1}, \ldots, s_{g-1}$ into its face $F$ and add $\operatorname{arcs} x s_{1}, y s_{1}, s_{g-1} x$, and $s_{g-1} y$. See Figure 1 .


Figure 1: The construction in Theorem 1
Since in $D_{f-1}$ no minimum feedback vertex set uses both $x$ and $y$, in order to hit the two newly created directed cycles, an additional vertex $z$ is needed. Thus, $D_{f}$ has a minimum feedback vertex set of size at least $f$. Now, choosing $z \in\left\{s_{1}, \ldots, s_{g-1}\right\}$, indeed gives a feedback vertex set of size $f$. Note that in $D_{f}$ there is no minimum feedback vertex containing two vertices from $\left\{s_{1}, \ldots, s_{g-1}\right\}$ and each such pair of vertices lies on a common face. It only remains to check the order of $D_{f}$. Since we added a total of $g-1$ new vertices, $D_{f}$ has $f(g-1)+1=: n$ vertices, i.e., $f=\frac{n-1}{g-1}$. Thus, the largest acyclic set in $D_{f}$ is of size $n-\frac{n-1}{g-1}$. Therefore, this construction proves the theorem for the case when $g-1$ divides $n-1$.

If $n-1$ is not divisible by $g-1$ then we do our construction for the largest integer $n^{\prime} \leq n$ such that $n^{\prime}-1$ is divisible by $g-1$, that is $n^{\prime}=(g-1)\left\lfloor\frac{n-1}{g-1}\right\rfloor+1$. We add $n-n^{\prime}$ independent vertices to this graph. Now, the largest acyclic set of the obtained graph is of size $n^{\prime}-\frac{n^{\prime}-1}{g-1}+\left(n-n^{\prime}\right)=n-\left\lfloor\frac{n-1}{g-1}\right\rfloor$.

By Theorem 1 for even $n$, there exist $n$-vertex oriented planar graphs in which every acyclic set has size at most $\frac{n}{2}+1$. A computer check, using tools from Sage [8], shows that there are ten planar triangulations with $n$ vertices ( $n$ is even and $n \leq 10$ ) that are tight examples for Conjecture 3 . Furthermore, for all orientations of these examples, the largest directed acyclic set is of size at least $\frac{n}{2}+1$. We wonder if the following is true:

Problem 1 If a largest induced forest in a simple undirected planar graph $G$ on $n$ vertices is of size $a \leq \frac{n}{2}$, then for every orientation $D$ of $G$ there is an acyclic set of size at least $a+1$.

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