# On lower bounds for the matching number of subcubic graphs 

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#### Abstract

We give a complete description of the set of triples $(\alpha, \beta, \gamma)$ of real numbers with the following property. There exists a constant $K$ such that $\alpha n_{3}+\beta n_{2}+\gamma n_{1}-K$ is a lower bound for the matching number $\nu(G)$ of every connected subcubic graph $G$, where $n_{i}$ denotes the number of vertices of degree $i$ for each $i$.


Keywords: matching, subcubic graph, polyhedron

## 1 Introduction

A graph is said to be subcubic if its maximum degree is at most three. In this paper we consider lower bounds for the maximum size $\nu(G)$ of a matching in subcubic graphs $G$.

Various lower bounds on $\nu(G)$ for subcubic graphs $G$ appear in the literature. For example, the following theorem is due to Biedl, Demaine, Duncan, Fleischer and Kobourov [1]. Here $n_{i}$ denotes the number of vertices of degree $i$ in $G$, and $\ell_{2}$ denotes the number of end-blocks in the block-cutvertex tree of $G$.

[^0]Theorem 1. Let $G$ be a connected graph with $n$ vertices.

1. If $G$ is cubic then $\nu(G) \geq 4(n-1) / 9$.
2. If $G$ is subcubic then $\nu(G) \geq n_{3} / 2+n_{2} / 3+n_{1} / 2-\ell_{2} / 3$, and $\nu(G) \geq$ $(n-1) / 3$.

They also asked whether $\nu(G) \geq\left(3 n+n_{2}\right) / 9$ for every subcubic graph. It will turn out below that this is not the case.

Generalisations of [1] to regular graphs of higher degree were given by Henning and Yeo in [5] (see also O and West [7]). Lower bounds in terms of other parameters of $G$ have been given, for example, in [7] and 4].

Our aim in this paper is to give a complete description of the set $L$ of 3-tuples of real coefficients $(\alpha, \beta, \gamma)$ for which there exists a constant $K$ such that $\nu(G) \geq \alpha n_{3}+\beta n_{2}+\gamma n_{1}-K$ for every connected subcubic graph $G$. (Note that this is equivalent to saying $\nu(G) \geq \alpha n_{3}+\beta n_{2}+\gamma n_{1}-K c(G)$ for every subcubic graph $G$, where $c(G)$ denotes the number of components of $G$.) Our work here is similar in spirit to a result of Chvátal and McDiarmid [2], who addressed a similar question for cover numbers of hypergraphs in terms of their number of vertices and number of edges. We will find, as in [2], that $L$ is a convex set, but in contrast to [2] where the number of extreme points is infinite, in our case $L$ is a certain 3 -dimensional polyhedron with a relatively simple description.

We define the polyhedron $P \subset \mathbb{R}^{3}$ to be the intersection of the six halfspaces

$$
\begin{aligned}
x_{3} & \leq 4 / 9 \\
x_{2} & \leq 1 / 2, \\
x_{3}+x_{1} & \leq 2 / 3, \\
x_{3}+3 x_{2} / 2 & \leq 1, \\
x_{3}+x_{2}+x_{1} & \leq 1 \\
x_{3}+x_{2} / 6 & \leq 1 / 2 .
\end{aligned}
$$

We let $P_{+}$be the intersection of $P$ with the nonnegative orthant $[0, \infty)^{3}$ in $\mathbb{R}^{3}$. It is easily seen that $P$ is unbounded. However, it follows from the first three inequalities above that $P_{+}$is a bounded subset of the nonnegative orthant.

The main aim of this paper is to prove the following theorem.

Theorem 2. $P=L$.
We will prove that $P \subseteq L$ in Section 2, and $L \subseteq P$ in Section 4.
Our proof that $P \subseteq L$ will need the fact that five specific points belong to $L$. This is a consequence of the following stronger result, which we prove in Section 3.

Theorem 3. Let $G$ be a subcubic graph with $c=c(G)$ components. Then

$$
\begin{align*}
& \nu(G) \geq n_{2} / 2+n_{1} / 2-c / 2  \tag{1}\\
& \nu(G) \geq n_{2} / 3+2 n_{1} / 3-c,  \tag{2}\\
& \nu(G) \geq n_{3} / 4+n_{2} / 2+n_{1} / 4-c / 2  \tag{3}\\
& \nu(G) \geq 7 n_{3} / 16+3 n_{2} / 8+3 n_{1} / 16-c / 8  \tag{4}\\
& \nu(G) \geq 4 n_{3} / 9+n_{2} / 3+2 n_{1} / 9-c / 9 \tag{5}
\end{align*}
$$

All five of these bounds are sharp: (4) is attained by the triangle, (1) and (3) by any odd cycle, and (1), (2) and (5) by the claw $K_{1,3}$. Furthermore, for a subcubic graph $G$, each of the bounds is sharp for $G$ if and only if it is sharp for every component of $G$. We will give further connected, sharp examples for (11), (21), (3), (5) in Section 4. The proof of Theorem 3 is given in Section 3, where we will also note the following corollary concerning the constant $K$ from the definition of $L$.

Corollary 4. Let $(\alpha, \beta, \gamma)$ be an element of $P$.

1. If $\alpha \geq 0$ then $\nu(G) \geq \alpha n_{3}+\beta n_{2}+\gamma n_{1}-1$ for every connected subcubic graph $G$.
2. If $\alpha<0$ then $\nu(G) \geq \alpha n_{3}+\beta n_{2}+\gamma n_{1}-(2|\alpha|+1)$ for every connected subcubic graph $G$.

Note in particular that if $G$ is a connected subcubic graph then $\nu(G) \geq$ $\alpha n_{3}+\beta n_{2}+\gamma n_{1}-1$ for every $(\alpha, \beta, \gamma) \in P_{+}$. Note also that if we consider $G=K_{1,3}$ and $(\alpha, \beta, \gamma)=(-\lambda, 0, \lambda+2 / 3)$ (which is in $P$ for all $\left.\lambda \geq 0\right)$, then the first bound in Lemma 4 is sharp for $\lambda=0$, and the second is sharp for all $\lambda>0$.

In the other direction, the fact that $L \subseteq P$ is a consequence of the following result, which we will prove in Section 4.

Theorem 5. If $(\alpha, \beta, \gamma) \notin P$ then for every constant $K$ there exists a connected subcubic graph $G$ such that $\nu(G)<\alpha n_{3}+\beta n_{2}+\gamma n_{1}-K$.

Our results generalize previous work. For example, the first bound in Theorem 1 is a special case of (5); the bound $\nu \geq(n-1) / 3$ follows from a convex combination of (2) and (5). On the other hand, the answer to the question of Biedl, Demaine, Duncan, Fleischer and Kobourov [1] as to whether $\nu(G) \geq\left(3 n+n_{2}\right) / 9$ for every subcubic graph is negative by Theorem 2. the vector $(1 / 3,4 / 9,1 / 3)$ is not in $P$ as it violates the inequality $x_{1}+x_{2}+$ $x_{3} \leq 1$, and Example 3 in Section 4 is a counterexample.

## $2 \quad P \subseteq L$

In this section we prove one direction of Theorem 2, namely that $P \subseteq L$ (leaving aside the proof of Theorem 3, which we defer to the next section). We will prove that $P \subseteq L$ in two steps. We first show that it is enough to consider just $P_{+}$, and then prove that $P_{+} \subseteq L$.

We begin with the following simple but useful observation.
Lemma 6. In any connected subcubic graph $G$ we have $n_{3} \geq n_{1}-2$.
Proof. Let $T$ be a spanning tree of $G$, and let $t_{i}$ denote the number of vertices of degree $i$ in $T$. Then $t_{1} \geq n_{1}, t_{3} \leq n_{3}$, and $t_{1}=t_{3}+2$. Thus $n_{3} \geq n_{1}-2$.

Next we note some closure properties of $L$.
Lemma 7. 1. L is convex.
2. $L$ is downward closed: if $\left(a_{3}, a_{2}, a_{1}\right) \in L$ and $b_{i} \leq a_{i}$ for all $i$ then $\left(b_{3}, b_{2}, b_{1}\right) \in L$.
3. If $\left(x_{3}, x_{2}, x_{1}\right) \in L$ then $\left(x_{3}-\lambda, x_{2}, x_{1}+\lambda\right) \in L$ for all $\lambda \geq 0$.

Proof. Suppose that $\mathbf{a}=\left(a_{3}, a_{2}, a_{1}\right), \mathbf{b}=\left(b_{3}, b_{2}, b_{1}\right)$ lie in $L$, with associated constants $K_{a}, K_{b}$. Thus for every subcubic graph $G$, say with parameters $\mathbf{n}=\left(n_{3}, n_{2}, n_{1}\right)$ and matching number $\nu$, we have $\mathbf{a} \cdot \mathbf{n} \leq \nu+K_{a}$ and $\mathbf{b} \cdot \mathbf{n} \leq$ $\nu+K_{b}$. Suppose that $\lambda \in[0,1]$ and $\mathbf{c}=\lambda \mathbf{a}+(1-\lambda) \mathbf{b}$. Then

$$
\begin{aligned}
\mathbf{c} \cdot \mathbf{n} & =\lambda \mathbf{a} \cdot \mathbf{n}+(1-\lambda) \mathbf{b} \cdot \mathbf{n} \\
& \leq \lambda\left(\nu+K_{a}\right)+(1-\lambda)\left(\nu+K_{b}\right) \\
& =\nu+\lambda K_{a}+(1-\lambda) K_{b} .
\end{aligned}
$$

It follows that $\mathbf{c} \in L$, with associated constant $\lambda K_{a}+(1-\lambda) K_{b}$. Thus $L$ is convex.

For the second claim, simply note that if $\mathbf{a} \in P$ with associated constant $K$, then for every subcubic graph $G$, say with parameters $\mathbf{n}=\left(n_{3}, n_{2}, n_{1}\right)$ and matching number $\nu$, we have $\mathbf{b} \cdot \mathbf{n} \leq \mathbf{a} \cdot \mathbf{n} \leq \nu+K$, so $\mathbf{b} \in L$ with associated constant $K$.

Now for the final part. Let $K$ be such that $\nu(G) \geq x_{3} n_{3}+x_{2} n_{2}+x_{1} n_{1}-K$ for every connected subcubic graph $G$. By Lemma6 we have $n_{3} \geq n_{1}-2$, and so $\left(x_{3}-\lambda\right) n_{3}+x_{2} n_{2}+\left(x_{1}+\lambda\right) n_{1}-(K+2 \lambda) \leq x_{3} n_{3}+x_{2} n_{2}+x_{1} n_{1}-K \leq \nu(G)$, which shows that $\left(x_{3}-\lambda, x_{2}, x_{1}+\lambda\right) \in L$.

The next lemma will allow us to restrict our attention to $P_{+}$.
Lemma 8. If $P_{+} \subseteq L$ then $P \subseteq L$.
Proof. Consider $x=\left(x_{3}, x_{2}, x_{1}\right) \in P \backslash L$. Our aim is to find a point in $P_{+} \backslash L$. If each $x_{i}$ is non-negative then $x$ is such a point, so we assume the contrary.

First suppose $x_{2}<0$. We claim that $x^{\prime}=\left(x_{3}, 0, x_{1}\right) \in P$. Since $x \in P$, the first and third inequalities defining $P$ are immediate for $x^{\prime}$, and the second is trivial. The fourth and sixth inequalities follow from the first, and the fifth follows from the third. Therefore $x^{\prime} \in P$. Now if $x^{\prime} \in L$ then $x \in L$ because $L$ is downward closed, contradicting our choice of $x$. Thus $x^{\prime} \in P \backslash L$.

Therefore we may assume that $x_{2} \geq 0$. Next we consider the case in which $x_{3}<0$. Set $\lambda=-x_{3}$ and let $x^{\prime}=\left(x_{3}+\lambda, x_{2}, x_{1}-\lambda\right)=\left(0, x_{2}, x_{1}+x_{3}\right)$. We claim that $x^{\prime} \in P$. The first inequality for $P$ is trivial, and the second, third and fifth are true because $x \in P$. The fourth and sixth inequalities are implied by the second. Thus $x^{\prime} \in P$. If $x^{\prime} \in L$ then by Lemma 7 the point $\left(x_{3}+\lambda-\lambda, x_{2}, x_{1}-\lambda+\lambda\right)=x \in L$, contradicting our choice of $x$. Therefore $x^{\prime} \in P \backslash L$ and we may assume $x_{3} \geq 0$.

Finally suppose $x_{1}<0$. Then we claim $x^{\prime}=\left(x_{3}, x_{2}, 0\right) \in P \backslash L$. To check $x^{\prime} \in P$ observe that the first, second, fourth and sixth inequalities are true because $x \in P$. The third follows from the first and the fifth follows from the first and second. Again we may conclude $x^{\prime} \notin L$ because $L$ is downward closed. Hence $x^{\prime} \in P \backslash L$ as required, completing the proof that $P_{+} \subseteq L$ implies $P \subseteq L$.

It is therefore enough to prove that $P_{+} \subseteq L$. Since $L$ is a convex set, it is enough to show that the extreme points of $P_{+}$all belong to $L$. The extreme
points of $P_{+}\left(\right.$written as $\left.\left(x_{3}, x_{2}, x_{1}\right)\right)$ are

$$
\begin{array}{r}
\{(0,1 / 2,1 / 2),(0,1 / 3,2 / 3),(1 / 4,1 / 2,1 / 4),(7 / 16,3 / 8,3 / 16), \\
(4 / 9,1 / 3,2 / 9),(1 / 4,1 / 2,0),(7 / 16,3 / 8,0),(0,1 / 2,0),(4 / 9,0,0), \\
(0,0,0),(4 / 9,1 / 3,0),(0,0,2 / 3),(4 / 9,0,2 / 9)\} .
\end{array}
$$

This can be verified by hand, or (as we did) by using a computational package such as polymake [3].

Our aim is then to show that all thirteen extreme points of $P_{+}$belong to $L$. Since $L$ is downward closed, it is enough to consider the points that do not lie below any others: for instance, $(7 / 16,3 / 8,0)$ lies below $(7 / 16,3 / 8,3 / 16)$, so $(7 / 16,3 / 8,3 / 16) \in L$ implies that $(7 / 16,3 / 8,0) \in L$. This leaves us with the following five points:
$\{(0,1 / 2,1 / 2),(0,1 / 3,2 / 3),(1 / 4,1 / 2,1 / 4),(7 / 16,3 / 8,3 / 16),(4 / 9,1 / 3,2 / 9)\}$.
The fact that these points all belong to $L$ follows from Theorem 3, which we prove in the next section. We conclude that $P \subseteq L$.

## 3 Proofs of Theorem 3 and Corollary 4

First we show how Corollary 4 follows from Theorem 3.
Proof. Let $G$ be a connected subcubic graph. Observe that by Theorem 3 and monotonicity, we have $\nu(G) \geq \alpha n_{3}+\beta n_{2}+\gamma n_{1}-1$ for each extreme point $(\alpha, \beta, \gamma)$ of $P_{+}$. By convexity, the same inequality holds for every point $(\alpha, \beta, \gamma) \in P_{+}$.

Now suppose $(\alpha, \beta, \gamma) \in P$ and $\alpha \geq 0$. Then (arguing as in the proof of Lemma (8) we know that $\left(\alpha, \beta^{\prime}, \gamma^{\prime}\right) \in P_{+}$where $\beta^{\prime}=\max \{\beta, 0\}$ and $\gamma^{\prime}=$ $\max \{\gamma, 0\}$. Hence

$$
\nu(G) \geq \alpha n_{3}+\beta^{\prime} n_{2}+\gamma^{\prime} n_{1}-1 \geq \alpha n_{3}+\beta n_{2}+\gamma n_{1}-1
$$

If $\alpha<0$ then set $\lambda=|\alpha|$. Then as in the proof of Lemma 8 we find that $(\alpha+\lambda, \beta, \gamma-\lambda)=(0, \beta, \gamma-\lambda) \in P$. Hence by the previous paragraph $\nu(G) \geq \beta n_{2}+(\gamma-\lambda) n_{1}-1$. By Lemma 6 we have $2 \lambda \geq \lambda n_{1}-\lambda n_{3}$. Summing these two inequalities and rearranging gives $\nu(G) \geq \alpha n_{3}+\beta n_{2}+\gamma n_{1}-(2 \lambda+1)$ as required.

The remainder of this section is devoted to the proof of Theorem 3.
Lemma 9. Let $G$ be a connected subcubic graph with $n$ vertices. Suppose $\nu(G) \geq(n-1) / 2$. Then $G$ satisfies Theorem 3.

Proof. Bounds (11) and (3) are immediate. Bound (4) holds unless $7 n / 16-$ $1 / 8>n / 2-1 / 2$, which implies $n \leq 5$. If (5) fails to hold then $4 n / 9-1 / 9>$ $n / 2-1 / 2$, which means $n \leq 6$. These cases are easily checked. For (2), using Lemma 6 we find $n_{1} \leq n_{3}+2 \leq n-n_{1}+2$, and hence $n_{1} \leq 1+n / 2$. Thus $n_{2} / 3+2 n_{1} / 3-1 \leq n / 3+n_{1} / 3-1 \leq n / 2+1 / 3-1$.

In particular, if $G$ has a perfect matching or if $G$ is hypomatchable (meaning $G-v$ has a perfect matching for every $v \in V(G))$ then Theorem 3 holds.

In our proof we will make use of the Gallai-Edmonds structure theorem (see, for instance, [6]). In the statement below, the sets $A, B$ and $C$ are defined as follows (here $\Gamma(A)$ denotes the neighbourhood of $A$ ).

- $A=\{v \in V(G): \nu(G-v)=\nu(G)\}$,
- $B=\Gamma(A) \backslash A$,
- $C=V(G) \backslash(A \cup B)$.

Theorem 10. (Gallai-Edmonds) Let $G$ be a graph. Then

1. every component of $G[A]$ is hypomatchable,
2. every component of $G[C]$ has a perfect matching,
3. every $X \subseteq B$ has neighbours in at least $|X|+1$ components of $G[A]$.

One consequence of Theorem 10 is that we may assume $B \neq \emptyset$, otherwise each component of $G$ has a perfect matching or is hypomatchable, in which case we are done by Lemma 9. Note also that Part (3) implies that each vertex of $B$ has degree at least two.

It is easy to check that all the bounds in Theorem 3 hold for graphs with at most three vertices, so we assume $G$ has $n \geq 4$ vertices and that the theorem is true for graphs with fewer than $n$ vertices. Since we may consider each component separately, we may assume $G$ is connected. Choose a vertex $v \in B$, and consider $G-v$. Since $v \notin A$ we know $\nu(G-v)=\nu(G)-1$. Let $t_{i}$ denote the number of neighbours of $v$ of degree $i$ for $i=1,2,3$. Let $U$ denote
the set of neighbours of $v$ of degree 1 , so $|U|=t_{1}$. Then $G^{\prime}=G-v-U$ satisfies $\nu\left(G^{\prime}\right)=\nu(G)-1$.

Let $n_{i}^{\prime}$ denote the number of vertices of degree $i$ in $G^{\prime}$. Since each degree-3 neighbour of $v$ becomes a degree- 2 vertex, the number of degree- 3 vertices drops by $t_{3}$, plus one more if $v$ itself has degree 3 . Thus $n_{3}^{\prime}=n_{3}-t_{3}-(d(v)-$ $2)=n_{3}-t_{3}-\left(t_{1}+t_{2}+t_{3}-2\right)=n_{3}-2 t_{3}-t_{2}-t_{1}+2$. Each degree- 2 neighbour of $v$ becomes a degree- 1 vertex, and if $v$ has degree 2 then the number of degree- 2 vertices drops by one more. Hence $n_{2}^{\prime}=n_{2}+t_{3}-t_{2}-(3-d(v))=$ $\left.n_{2}+t_{3}-t_{2}-\left(3-t_{1}-t_{2}-t_{3}\right)\right)=n_{2}+2 t_{3}+t_{1}-3$. Finally $n_{1}^{\prime}=n_{1}-t_{1}+t_{2}$, and $c^{\prime} \leq t_{3}+t_{2}$. Then by the induction hypothesis,

1. $\nu\left(G^{\prime}\right) \geq n_{2}^{\prime} / 2+n_{1}^{\prime} / 2-c^{\prime} / 2$

$$
\begin{aligned}
& \geq n_{2} / 2+\left(2 t_{3}+t_{1}-3\right) / 2+n_{1} / 2+\left(t_{2}-t_{1}\right) / 2-\left(t_{3}+t_{2}\right) / 2 \\
& =n_{2} / 2+n_{1} / 2-1 / 2+\left(t_{3}-2\right) / 2
\end{aligned}
$$

2. $\nu\left(G^{\prime}\right) \geq n_{2}^{\prime} / 3+2 n_{1}^{\prime} / 3-c^{\prime}$

$$
\begin{aligned}
& \geq n_{2} / 3+\left(2 t_{3}+t_{1}-3\right) / 3+2 n_{1} / 3+2\left(t_{2}-t_{1}\right) / 3-\left(t_{3}+t_{2}\right) \\
& =n_{2} / 3+2 n_{1} / 3-1-\left(t_{3}+t_{2}+t_{1}\right) / 3,
\end{aligned}
$$

3. $\nu\left(G^{\prime}\right) \geq n_{3}^{\prime} / 4+n_{2}^{\prime} / 2+n_{1}^{\prime} / 4-c^{\prime} / 2$

$$
\begin{aligned}
& \geq n_{3} / 4+\left(2-2 t_{3}-t_{2}-t_{1}\right) / 4+n_{2} / 2+\left(2 t_{3}+t_{1}-3\right) / 2+n_{1} / 4 \\
& \quad \quad \quad\left(t_{2}-t_{1}\right) / 4-\left(t_{3}+t_{2}\right) / 2 \\
& =n_{3} / 4+n_{2} / 2+n_{1} / 4-1 / 2-\left(t_{2}+1\right) / 2
\end{aligned}
$$

4. $\nu\left(G^{\prime}\right) \geq 7 n_{3}^{\prime} / 16+3 n_{2}^{\prime} / 8+3 n_{1}^{\prime} / 16-c^{\prime} / 8$

$$
\begin{aligned}
& \geq 7 n_{3} / 16+7\left(2-2 t_{3}-t_{2}-t_{1}\right) / 16+3 n_{2} / 8+3\left(2 t_{3}+t_{1}-3\right) / 8 \\
& \quad \quad+3 n_{1} / 16+3\left(t_{2}-t_{1}\right) / 16-\left(t_{3}+t_{2}\right) / 8 \\
& =7 n_{3} / 16+3 n_{2} / 8+3 n_{1} / 16-1 / 4-t_{3} / 4-3 t_{2} / 8-t_{1} / 4 \\
& =\left[7 n_{3} / 16+3 n_{2} / 8+3 n_{1} / 16-1 / 8\right]-\left(4 t_{3}+6 t_{2}+4 t_{1}+2\right) / 16,
\end{aligned}
$$

5. $\nu\left(G^{\prime}\right) \geq 4 n_{3}^{\prime} / 9+n_{2}^{\prime} / 3+2 n_{1}^{\prime} / 9-c^{\prime} / 9$

$$
\begin{aligned}
& \geq 4 n_{3} / 9+4\left(2-2 t_{3}-t_{2}-t_{1}\right) / 9+n_{2} / 3+\left(2 t_{3}+t_{1}-3\right) / 3 \\
& \quad+2 n_{1} / 9+2\left(t_{2}-t_{1}\right) / 9-\left(t_{3}+t_{2}\right) / 9 \\
& =4 n_{3} / 9+n_{2} / 3+2 n_{1} / 9-1 / 9-\left(t_{3}+t_{2}+t_{1}\right) / 3 .
\end{aligned}
$$

Since $\nu(G)=\nu\left(G^{\prime}\right)+1$ and $t_{3}+t_{2}+t_{1} \leq 3$ it follows from the calculations above that bounds (1), (2) and (5) hold for $G$. (In fact (2) alternatively follows from (5) together with Lemma 7(3)).

We now focus on bounds (3) and (4). Note that in these cases, our inductive statement gives

$$
\nu\left(G^{\prime}\right) \geq n_{3} / 4+n_{2} / 2+n_{1} / 4-1 / 2-\left(t_{2}+1\right) / 2
$$

and

$$
\nu\left(G^{\prime}\right) \geq\left[7 n_{3} / 16+3 n_{2} / 8+3 n_{1} / 16-1 / 8\right]-\left(4 t_{3}+6 t_{2}+4 t_{1}+2\right) / 16
$$

First we note some consequences of Theorem 10 and the above calculations.

Lemma 11. 1. Every $v \in B$ has at least two neighbours in $A$.
2. If $x \in A$ has exactly two neighbours $u$ and $w$, and if $u \in B$, then $w \in B$ as well.
3. If (4) fails for $G$ then every $v \in B$ has degree 3.
4. If one of (3) and (4) fails for $G$ then every $v \in B$ has at least two degree-2 neighbours.

Proof. We have already noted that the first statement is immediate from Theorem 10(3). To verify the second claim, observe that if $w \in A$ then $u$ and $w$ are both in a component $H$ of $G[A]$, which is hypomatchable by Theorem 10. But $x$ has degree 1 in $H$, which is not possible in a hypomatchable component. Thus $w \in B$.

If (3) fails then $t_{2} \geq 2$; if (4) fails then $4 t_{3}+6 t_{2}+4 t_{1} \geq 15$ and so (as $d(v) \leq 3)$ we have $t_{2} \geq 2$ and $t_{1}+t_{2}+t_{3}=3$. The last two assertions follow immediately, as the same calculation holds for any vertex of $B$.

Next we derive some elementary facts about the neighbours of degree-2 vertices.

Lemma 12. Suppose $G$ fails to satisfy one of (3) and (4). Then no two degree-2 vertices of $G$ are adjacent. Furthermore every vertex of $B$ has degree 3.

Proof. Recall our assumption that $G$ has at least four vertices. If $G$ is a 4cycle then (3) and (4) are satisfied (by Lemma 9), so let us assume otherwise. Suppose $u$ and $w$ are adjacent degree- 2 vertices.

If $u$ and $w$ are not in a triangle or 4-cycle then suppressing $u$ and $w$ (i.e. if $u^{\prime}$ and $v^{\prime}$ are the other neighbours of $u, v$ then we replace the path $u^{\prime} u v v^{\prime}$ by the edge $u^{\prime} v^{\prime}$ ) gives a connected graph $G^{\prime}$ with $\nu\left(G^{\prime}\right)=\nu(G)-1$, $n_{3}^{\prime}=n_{3}, n_{2}^{\prime}=n_{2}-2$, and $n_{1}^{\prime}=n_{1}$. Then by the induction hypothesis for (3)), $\nu\left(G^{\prime}\right) \geq n_{3}^{\prime} / 4+n_{2}^{\prime} / 2+n_{1}^{\prime} / 4-1 / 2=n_{3} / 4+n_{2} / 2+n_{1} / 4-1 / 2-1$, showing $G$ satisfies (3). For (4) we have by induction $\nu\left(G^{\prime}\right) \geq 7 n_{3}^{\prime} / 16+3 n_{2}^{\prime} / 8+3 n_{1}^{\prime} / 16-$ $1 / 8=7 n_{3} / 16+3 n_{2} / 8+3 n_{1} / 16-1 / 8-6 / 8$, which also suffices.

If $u w x$ is a triangle then form $G^{\prime}$ by removing $u$ and $w$. Then $\nu\left(G^{\prime}\right)=$ $\nu(G)-1, n_{3}^{\prime}=n_{3}-1, n_{2}^{\prime}=n_{2}-2, n_{1}^{\prime}=n_{1}+1$, and $c^{\prime}=1$. For (3) we get $\nu\left(G^{\prime}\right) \geq n_{3}^{\prime} / 4+n_{2}^{\prime} / 2+n_{1}^{\prime} / 4-1 / 2=n_{3} / 4-1 / 4+n_{2} / 2-1+n_{1} / 4+1 / 4-1 / 2=$ $\left[n_{3} / 4+n_{2} / 2+n_{1} / 4-1 / 2\right]-1$, showing $G$ satisfies (3). For (4) we have by induction $\nu\left(G^{\prime}\right) \geq 7 n_{3} / 16-7 / 16+3 n_{2} / 8-6 / 8+3 n_{1} / 16+3 / 16-1 / 8=$ $\left[7 n_{3} / 16+3 n_{2} / 8+3 n_{1} / 16-1 / 8\right]-1$, as needed.

If $u$ and $w$ are in a 4 -cycle $u w x z$ then by assumption (say) $x$ has degree 3 . Form $G^{\prime}$ by removing $u$ and $w$, so that $\nu\left(G^{\prime}\right)=\nu(G)-1$. If $d(z)=3$ then $G^{\prime}$ has $n_{3}^{\prime}=n_{3}-2, n_{2}^{\prime}=n_{2}, n_{1}^{\prime}=n_{1}$, and $c^{\prime}=1$. Then using induction for (3) we find $\nu\left(G^{\prime}\right) \geq n_{3}^{\prime} / 4+n_{2}^{\prime} / 2+n_{1}^{\prime} / 4-1 / 2=\left(n_{3}-2\right) / 4+n_{2} / 2+n_{1} / 4-1 / 2=$ $n_{3} / 4+n_{2} / 2+n_{1} / 4-1 / 2-1 / 2$, which suffices. For (4) we get $\nu\left(G^{\prime}\right) \geq 7 n_{3} / 16-$ $14 / 16+3 n_{2} / 8+3 n_{1} / 16-1 / 8=\left[7 n_{3} / 16+3 n_{2} / 8+3 n_{1} / 16-1 / 8\right]-14 / 16$ as required.

If $d(z)=2$ the parameters become $n_{3}^{\prime}=n_{3}-1, n_{2}^{\prime}=n_{2}-2$, and $n_{1}^{\prime}=n_{1}+1$, giving for (3) $\nu\left(G^{\prime}\right) \geq n_{3}^{\prime} / 4+n_{2}^{\prime} / 2+n_{1}^{\prime} / 4-1 / 2=\left(n_{3}-1\right) / 4+$ $\left(n_{2}-2\right) / 2+n_{1} / 4=1 / 4-1 / 2+n_{3} / 4+n_{2} / 2+n_{1} / 4-1 / 2-1$ as needed. For (4) we get $\nu\left(G^{\prime}\right) \geq 7 n_{3} / 16-7 / 16+3 n_{2} / 8-6 / 8+3 n_{1} / 16+3 / 16-1 / 8=$ $\left[7 n_{3} / 16+3 n_{2} / 8+3 n_{1} / 16-1 / 8\right]-1$. This completes the proof of the first statement. The second statement now follows using Lemma 11(3),(4).

Lemma 13. Suppose $G$ fails to satisfy one of (3) and (4). Then each degree2 vertex $w$ has two degree-3 neighbours.

Proof. Lemma 12 tells us that $w$ has no degree-2 neighbours. Suppose for a contradiction that $w$ has a degree-1 neighbour $x$. Then (recalling $G$ has at least four vertices) $G^{\prime}-\{w, x\}$ has $\nu\left(G^{\prime}\right)=\nu(G)-1, n_{3}^{\prime}=n_{3}-1$, $n_{2}^{\prime}=n_{2}, n_{1}^{\prime}=n_{1}-1$, and $c^{\prime}=1$. Then using induction for (3) gives $\nu\left(G^{\prime}\right) \geq n_{3}^{\prime} / 4+n_{2}^{\prime} / 2+n_{1}^{\prime} / 4-c^{\prime} / 2 \geq n_{3} / 4-1 / 4+n_{2} / 2+n_{1} / 4-1 / 4-1 / 2=$ $\left[n_{3} / 4+n_{2} / 2+n_{1} / 4-1 / 2\right]-1 / 2$, which suffices. For (4) we get $\nu\left(G^{\prime}\right) \geq$
$7 n_{3}^{\prime} / 16+3 n_{2}^{\prime} / 8+3 n_{1}^{\prime} / 16-c^{\prime} / 8 \geq 7 n_{3} / 16-7 / 16+3 n_{2} / 8+3 n_{1} / 16-3 / 16-1 / 8=$ $\left[7 n_{3} / 16+3 n_{2} / 8+3 n_{1} / 16-1 / 8\right]-10 / 16$.

Call a degree-3 vertex $v \in G \operatorname{good}$ if it has two degree-2 neighbours that do not have a common neighbour different from $v$. Observe that if $v$ has three degree-2 neighbours then either $v$ is good, or $G=K_{2,3}$, in which case (3) and (4) hold.

Lemma 14. Suppose $G$ fails to satisfy one of (3) and (4). Then every good vertex $v$ of $G$ has three degree-2 neighbours, all of which are in different components of $G-v$.

Proof. Let $w$ and $x$ be degree- 2 neighbours that are not adjacent and have no common neighbour other than $v$. As before, we write $t_{i}$ for the number of degree $i$ neighbours of $v$, and $U$ for the set of degree 1 neighbours of $v$. Let $G^{\prime}$ be the graph obtained by removing $\{v\} \cup U$ and identifying $w$ and $x$ into a new vertex of degree 2 . Then $\nu\left(G^{\prime}\right)=\nu(G)-1, n_{3}^{\prime}=n_{3}-t_{3}-1$, $n_{2}^{\prime}=n_{2}-t_{2}+t_{3}+1, n_{1}^{\prime}=n_{1}-t_{1}+t_{2}-2$, and $c^{\prime} \leq 2-t_{1}$.

The computation for (3) becomes $\nu\left(G^{\prime}\right) \geq n_{3}^{\prime} / 4+n_{2}^{\prime} / 2+n_{1}^{\prime} / 4-c^{\prime} / 2 \geq$ $n_{3} / 4-t_{3} / 4-1 / 4+n_{2} / 2+\left(t_{3}+1-t_{2}\right) / 2+n_{1} / 4+\left(t_{2}-t_{1}-2\right) / 4-\left(2-t_{1}\right) / 2=$ $\left[n_{3} / 4+n_{2} / 2+n_{1} / 4-1 / 2\right]+t_{3} / 4-t_{2} / 4+t_{1} / 4-3 / 4$. Then (3) holds unless $t_{2}=3$ and $c^{\prime}=2$.

For (4) we get $\nu\left(G^{\prime}\right) \geq 7 n_{3}^{\prime} / 16+3 n_{2}^{\prime} / 8+3 n_{1}^{\prime} / 16-c^{\prime} / 8 \geq 7 n_{3} / 16-7 t_{3} / 16-$ $7 / 16+3 n_{2} / 8+3\left(t_{3}+1-t_{2}\right) / 8+3 n_{1} / 16+3\left(t_{2}-t_{1}-2\right) / 16-\left(2-t_{1}\right) / 8=$ $\left[7 n_{3} / 16+3 n_{2} / 8+3 n_{1} / 16-1 / 8\right]-\left(t_{3}+3 t_{2}+t_{1}+9\right) / 16$, so (4) holds unless $t_{2}=3$ and $c^{\prime}=2$.

Hence in both cases we may assume that $t_{2}=3$ and so $c^{\prime}=2$. Let $y$ be the third neighbour of $v$. Since $c^{\prime}=2$ we know that $y$ is in a different component of $G^{\prime}$ (and hence of $G-v$ ) to $w$ and $x$. In particular, $y$ is not adjacent to $w$ or $x$ and does not share a second common neighbour with either of them. Thus we could apply the above argument with $w$ and $y$ and find that $x$ is in a different component of $G-v$ from both $w$ and $y$. This completes the proof.

We may now complete the proof for (3).
Lemma 15. G satisfies (3).
Proof. Suppose the contrary. If any degree-3 vertex has another degree3 vertex in its neighbourhood, then we may verify (3) by considering the
graph $G^{\prime}$ obtained by deleting an edge joining two degree-3 vertices. In this case $n_{3}^{\prime}=n_{3}-2, n_{2}^{\prime}=n_{2}+2, n_{1}^{\prime}=n_{1}$ and $c^{\prime} \leq 2$. Hence using induction we get $\nu(G) \geq \nu\left(G^{\prime}\right) \geq n_{3}^{\prime} / 4+n_{2}^{\prime} / 2+n_{1}^{\prime} / 4-2 / 2=n_{3} / 4+n_{2} / 2+n_{1} / 4-1 / 2$, proving (3) as required.

Thus we may assume no two degree-3 vertices are adjacent. Next we check that no degree-3 vertex has two degree-1 neighbours. If on the contrary $x$ has degree-1 neighbours $v$ and $w$, and a third neighbour $z$ (which necessarily has degree 2 , or else $G$ is $K_{1,3}$ and satisfies (3)), form $G^{\prime}$ by removing $v, w$, and $x$. Then $n_{3}^{\prime}=n_{3}-1, n_{2}^{\prime}=n_{2}-1, n_{1}^{\prime}=n_{1}-1, c^{\prime}=1$ and $\nu(G)=\nu\left(G^{\prime}\right)+1$. Therefore by induction $\nu(G) \geq n_{3}^{\prime} / 4+n_{2}^{\prime} / 2+n_{1}^{\prime} / 4-c^{\prime} / 2+1 \geq\left[n_{3} / 4+n_{2} / 2+\right.$ $\left.n_{1} / 4-1 / 2\right]-1 / 4-1 / 2-1 / 4+1$, showing (3) holds. Thus every degree-3 vertex has at least two degree-2 neighbours.

Suppose a degree- 2 vertex $w$ has neighbours $v$ and $z$ (which both have degree 3 by Lemma (13). If $v$ is good then $z$ is also good, since otherwise every other degree- 2 neighbour of $z$ (at least one of which exists) is also a degree-2 neighbour of $v$, and would therefore be in the same component of $G-v$ as $w$, contradicting Lemma 14. Therefore there are no good vertices at all, since otherwise (since $G$ has at least one degree- 3 vertex, in $B$ ) by Lemma 14 we would find that $G$ is a subdivision of a connected 3-regular graph, but removing any degree-3 vertex results in 3 components. This is not possible since, in particular, every connected graph has a vertex whose removal leaves a connected graph.

Since $G$ has no good vertices, in particular no degree-3 vertex can have three degree- 2 neighbours. So every degree 3 vertex has exactly two degree 2 neighbours. It follows that $G$ is a cycle (of even length) with a pendant edge attached to every second vertex (these are the graphs $G_{3}(t)$ in Example 3 in the next section). But (3) holds for this graph, completing the proof.

We are left to verify (4). We need one more technical lemma.
Lemma 16. No vertex in $B$ is good.
Proof. Suppose on the contrary that $B$ contains good vertices. Let $v \in B$ be a good vertex. Let $W$ be the union of the vertex sets of all paths of the form $v w_{1} w_{2} \ldots w_{r}$ where $r \geq 1$, each $w_{i}$ with $i$ odd is a degree- 2 vertex in $A$, and each $w_{i}$ with $i$ even is in $B$. Let $H$ be the subgraph of $G$ induced by $W$. Then $H$ is connected.

We claim that each vertex of $W \cap B$ is good. To verify this, consider a good vertex $w \in W \cap B$ (for example $w=v$ ). By Lemma 11(1) we know $w$
has at least two neighbours $u$ and $x$ in $A$, and $d(u)=d(x)=2$ by Lemma 14 . Also, Lemma 11(2) implies that the other neighbour $z$ of $u$ is in $B$ and hence is in $W \cap B$. Thus $d(z)=3$ by Lemma 11(3). If $z$ were not good then every degree- 2 neighbour of $z$ different from $u$ (at least one of which exists, by Lemma 11(4)) would be a degree- 2 neighbour of $w$, and would hence be in the same component of $G-w$ as $u$, contradicting Lemma 14. Hence $z$ is good. Applying this observation repeatedly (moving along the paths used to define $H$ ) we find that every vertex of $W \cap B$ is good.

By Lemma 11(2) we know that $A \cap W$ is independent, and each $x \in A \cap W$ has exactly two neighbours in $B \cap W$. Since each $w \in B \cap W$ is good, it has three degree-2 neighbours in $G$ by Lemma 14, at least two of which are in $A$ by Lemma 11(1). So by Lemma 11(3) we know $B \cap W$ is independent. Therefore $H$ is the subdivision of a connected subcubic graph $J$ with vertex set $B \cap W$ and minimum degree at least 2. (Note that $J$ has no multiple edges by Lemma 14 and the fact that each $w \in B \cap W$ is good.)

Since each $w \in B \cap W$ is good, the graph $J$ has the property that $J-y$ has $d(y) \geq 2$ components for every vertex $y$ of $J$. Such a graph cannot exist, so the proof is complete.

We may therefore assume that no vertex in $B$ has three degree-2 neighbours. Choose $v \in B$. By Lemma 12 we have $d(v)=3$, and by Lemma 11(4) we know that $v$ has at least two degree-2 neighbours, say $w$ and $x$. By Lemma 11(1) at least one of them, say $w$, is in $A$. Since $v$ is not good, the other neighbour $z$ of $v$ is not a degree- 2 vertex, and $w$ and $x$ have another common neighbour $y$. By Lemma 11(2) we know $y$ is in $B$. Then by Lemma 11(3) we have that $y$ has another neighbour $u$, and $d(u) \neq 2$ since $y$ is not good. Since (4) holds for $K_{4}$ with one edge deleted, we may assume $u \neq v$. If $G$ consists of a 4 -cycle plus two pendant edges attached to nonadjacent vertices then (4) holds, so we may assume without loss of generality that $z$ has degree 3 .

If $z=u$ remove $v, w, x, y$. Then $n_{3}^{\prime}=n_{3}-3, n_{2}^{\prime}=n_{2}-2, n_{1}^{\prime}=n_{1}+1$, $c^{\prime}=1$ and $\nu\left(G^{\prime}\right)=\nu(G)-2$. Then by induction $\nu(G) \geq \nu\left(G^{\prime}\right)+2 \geq$ $7 n_{3} / 16+3 n_{2} / 8+3 n_{1} / 16-1 / 8-30 / 16+2$, which implies our result.

If $u$ has degree 1 we remove $u, v, w, x, y$. Then $n_{3}^{\prime}=n_{3}-3, n_{2}^{\prime}=n_{2}-1$, $n_{1}^{\prime}=n_{1}-1, c^{\prime}=1$ and $\nu\left(G^{\prime}\right)=\nu(G)-2$. Then by induction $\nu(G) \geq$ $\nu\left(G^{\prime}\right)+2 \geq 7 n_{3} / 16+3 n_{2} / 8+3 n_{1} / 16-1 / 8-30 / 16+2$, as needed.

Otherwise $z \neq u$, and $d(z)=d(u)=3$. In this case we remove $v, w, x, y$. Then $n_{3}^{\prime}=n_{3}-4, n_{2}^{\prime}=n_{2}, n_{1}^{\prime}=n_{1}, c^{\prime} \leq 2$ and $\nu\left(G^{\prime}\right)=\nu(G)-2$. Then by
induction $\nu(G) \geq \nu\left(G^{\prime}\right)+2 \geq 7 n_{3} / 16+3 n_{2} / 8+3 n_{1} / 16-1 / 8-30 / 16+2$, which completes the proof of Theorem 3.

## $4 \quad L \subseteq P$

The fact that $L \subseteq P$ is an immediate consequence of Theorem 5, which we prove in this section.

Suppose that $\left(x_{3}, x_{2}, x_{1}\right) \in L$, so there is some real number $K$ such that

$$
\begin{equation*}
\nu(G) \geq x_{3} n_{3}(G)+x_{2} n_{2}(G)+x_{1} n_{1}(G)-K \tag{6}
\end{equation*}
$$

for every connected subcubic graph $G$ (where $n_{i}(G)$ denotes the number of vertices of $G$ of degree $i$ ). We fix a choice of $\left(x_{3}, x_{2}, x_{1}\right)$ and $K$ for the rest of this section.

We will consider six special families of graphs: each family will show that $\left(x_{3}, x_{2}, x_{1}\right)$ must satisfy one of the inequalities in the definition of $P$. An example from each family is shown in the figures.

Example 1. Let $t$ be an odd positive integer. The graph $G_{1}(t)$ is the tree with a root plus $t+1$ levels, indexed by $i=0, \ldots, t$, in which level $i$ contains $3 \cdot 2^{i}$ vertices, and all vertices except the leaves have degree 3 . Thus $G_{1}(t)$ is (internally) a cubic tree and has depth $t+1$. Then $n_{1}=3 \cdot 2^{t}$, $n_{2}=0$ and $n_{3}=1+3\left(2^{t}-1\right)=3 \cdot 2^{t}-2$. Since $G_{1}(t)$ is bipartite with one partition class $S$ formed by the vertices at levels $0,2, \ldots, t-1$ we see $\nu\left(G_{1}(t)\right) \leq|S|=3\left(4^{(t+1) / 2}-1\right) / 3=2^{t+1}-1$. By (6) we must have

$$
\left(3 \cdot 2^{t}-2\right) x_{3}+3 \cdot 2^{t} x_{1}-K \leq 2 \cdot 2^{t}-1,
$$

and so, dividing by $3 \cdot 2^{t}$ and letting $t \rightarrow \infty$, we see that

$$
x_{3}+x_{1} \leq 2 / 3 .
$$

Example 2. Let $J$ denote the graph obtained by subdividing one edge of $K_{4}$, and let $x$ denote the single vertex of degree 2 in $J$. We define the graph $G_{2}(t)$, again for odd $t$, by identifying each leaf in $G_{1}(t)$ with the vertex $x$ in a copy of the graph $J$, such that all copies are disjoint from each other and the rest of the graph. Then for this graph $n_{1}=n_{2}=0$, and $n_{3}=$ $3 \cdot 2^{t}-2+15 \cdot 2^{t}=9 \cdot 2^{t+1}-2$. The same set $S$ as before now has the property that removing it leaves $1+3\left(2+2^{3}+\ldots+2^{t}\right)=1+6\left(4^{(t+1) / 2}-\right.$


1) $/ 3=2^{t+2}-1$ odd components. Therefore any maximum matching in $G$ must leave exposed at least $2^{t+2}-1-|S|=2^{t+1}$ vertices. This tells us $\nu\left(G_{2}(t)\right) \leq\left(9 \cdot 2^{t+1}-2-2^{t+1}\right) / 2=\left(2^{t+4}-2\right) / 2=2^{t+3}-1$. So (6) implies that

$$
\left(9 \cdot 2^{t+1}-2\right) x_{3}-K \leq 2^{t+3}-1
$$

Dividing by $9 \cdot 2^{t+1}$ and taking a limit gives

$$
x_{3} \leq 4 / 9
$$

Example 3. Let $t \geq 2$ be a positive integer. The graph $G_{3}(t)$ is obtained from the cycle with $2 t$ vertices by attaching a pendant edge to every second vertex. Then $n_{1}=n_{2}=n_{3}=t$. The graph is bipartite with one vertex class consisting of the vertices of degree 3 , so $\nu\left(G_{3}(t)\right) \leq n_{3}=t$. Thus

$$
x_{3} t+x_{2} t+x_{1} t-K \leq t .
$$

Dividing by $t$ and taking a limit gives

$$
x_{3}+x_{2}+x_{1} \leq 1 .
$$

Example 4. The graph $G_{4}(t)$ is obtained from $G_{3}(t)$ by adding $t$ disjoint copies of $J$, identifying the vertex $x$ in each copy with the leaf of a pendant edge. Then $n_{1}=0, n_{2}=t$ and $n_{3}=6 t$. The set of degree- 3 vertices on the cycle has size $t$ and leaves $2 t$ odd components when deleted, showing $\nu\left(G_{4}(t)\right) \leq(7 t-t) / 2=3 t$. Thus

$$
6 x_{3} t+x_{2} t-K \leq 3 t
$$



Dividing by $6 t$ and taking a limit gives

$$
x_{3}+x_{2} / 6 \leq 1 / 2 .
$$

Example 5. For each even integer $t \geq 4$, let $G_{5}(t)$ be obtained from a cubic graph $H$ on $t$ vertices by subdividing every edge of $H$ exactly once (for sake of definiteness, we may may take $H$ to be a cycle of length $t$ with opposite vertices joined). Then $n_{1}=0, n_{2}=e(H)=3 t / 2$ and $n_{3}=t$. Then $G_{5}(t)$ is bipartite with one vertex class $V(H)$ of size $t$, so $\nu(G) \leq t$. Thus

$$
x_{3} t+3 x_{2} t / 2-K \leq t
$$

Dividing by $t$ and taking a limit gives

$$
x_{3}+3 x_{2} / 2 \leq 1
$$

Example 6. Finally, for odd integers $t \geq 3$, we let $G_{6}(t)$ be the odd cycle of length $t$. Then $n_{1}=n_{3}=0$ and $n_{2}=t$, while $\nu=(t-1) / 2$. Thus

$$
x_{2} t / 2-K \leq t / 2-1 / 2
$$

Dividing by $t / 2$ and taking a limit gives

$$
x_{2} \leq 1 / 2
$$

The proof of Theorem 5 is now immediate.
Proof of Theorem 5. If $\left(x_{3}, x_{2}, x_{1}\right) \notin P$ then it fails to satisfy one of the inequalities used to define $P$. Therefore, taking the example above that corresponds to this inequality (and noting that all the examples are connected) we see that by taking $t$ large we can force $K$ to be arbitrarily large.

In fact it is easy to see that equality holds in each expression bounding $\nu\left(G_{i}(t)\right)$, but we do not need this fact. Finally, we note that Example 1 is sharp for (2) and (5); Example 2 is sharp for (5); and Example 6 is sharp for (1) and (3).

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