PARTITE SATURATION PROBLEMS

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ABSTRACT. We look at several saturation problems in complete balanced blow-ups of graphs. We let H[n] denote the blow-up of H onto parts of size n and refer to a copy of H in H[n] as partite if it has one vertex in each part of H[n]. We then ask how few edges a subgraph G of H[n] can have such that G has no partite copy of H but such that the addition of any new edge from H[n] creates a partite H. When H is a triangle this value was determined by Ferrara, Jacobson, Pfender, and Wenger in [5]. Our main result is to calculate this value for $H = K_4$ when n is large. We also give exact results for paths and stars and show that for 2-connected graphs the answer is linear in n whilst for graphs which are not 2-connected the answer is quadratic in n. We also investigate a similar problem where G is permitted to contain partite copies of H but we require that the addition of any new edge from H[n] creates an extra partite copy of H. This problem turns out to be much simpler and we attain exact answers for all cliques and trees.

1. INTRODUCTION

The Turán problem of asking for the maximum number of edges a graph on a fixed number of vertices can have without containing some fixed subgraph H is one of the oldest and most famous questions in extremal graph theory, see [7],[8],[4].

Since the corresponding minimisation problem - asking how few edges an *H*-free graph can have - trivially gives the answer zero, if we want an interesting complementary question to the Turán problem we can require that our *H*-free graph *G* also has the property that it *nearly* contains a copy of *H*. By this we mean that the addition of any new edge to *G* creates an copy of *H* as a subgraph. Such a graph *G* is called *H*-saturated and over *H*-saturated graphs on *n* vertices the minimum number of edges is called the saturation number, sat(*H*, *n*). The study of saturation numbers was initiated by Erdős, Hajnal and Moon [3] when they proved that sat(K_r, n) = $(r-2)(n - \frac{1}{2}(r-1))$. It was later shown by Kászonyi and Tuza in [6] that cliques have the largest saturation number of any graph on *r* vertices which in particular implies that for any *H* the saturation number sat(*H*, *n*) grows linearly in *n*.

These saturation questions can be generalised to require our *H*-free graph *G* to be a subgraph of another fixed graph *F*. Here we insist that adding any new edge of *F* to *G* would create a copy of *H* in *G*. The minimum number of edges in such a *G* we denote by $\operatorname{sat}(H, F)$. One natural class of host graphs are complete *r*-partite graphs. In the bipartite case Bollobás [1, 2] and Wessel [9, 10] independently determined the saturation number $\operatorname{sat}(K_{a,b}, K_{c,d})$. Working in the *r*-partite setting with $r \ge 3$, Ferrara, Jacobson, Pfender, and Wenger determined in [5] the value of $\operatorname{sat}(K_3, K_r^n)$ for sufficiently large *n* and showed that $\operatorname{sat}(K_3, K_3^n) = 6n - 6$ for all *n*.

In this paper we consider the saturation problem when the host graph is a blow-up of the forbidden subgraph H. For any graph H and any $n \in \mathbb{N}$ let H[n] denote the graph obtained from H by replacing each vertex with an independent set of size n and each edge with a complete bipartite graph between the corresponding independent sets. A copy of Hin H[n] is called *partite* if it has exactly one vertex in each part of H[n]. For a subgraph G of H[n] we say G is H-partite-free if there is no partite copy of H in G. We say G is (H, H[n])-partite-saturated if G is H-partite-free but for any $uv \in E(H[n] \setminus G)$ the graph $G \cup uv$ is not H-partite-free. We consider the problem of determining the value

$$\operatorname{sat}_{p}(H, H[n]) := \min \{ e(G) : G \subseteq H[n] \text{ is } (H, H[n]) \text{-partite-saturated} \}$$

for graphs H.

Note that for a graph H with no homomorphism onto any proper subgraph of itself we have by definition $\operatorname{sat}_{p}(H, H[n]) = \operatorname{sat}(H, H[n])$. In this way we know that $\operatorname{sat}_{p}(K_{3}, K_{3}[n]) = 6n - 6$ from [5] and can drop the partite requirement when considering cliques. Our main result, Theorem 1, is to show that for sufficiently large n we have $\operatorname{sat}(K_{4}, K_{4}[n]) = 18n - 21$. In addition we calculate the partite-saturation numbers of stars and paths in Theorems 10 and 11 respectively.

In the original paper by Erdős, Hajnal and Moon they did not in fact require the graph G to be H-free but only required that the addition of any edge would create an extra copy of H. Interestingly for the problem they studied this did not have an effect as the extremal graphs were K_r -free even without requiring this restriction. We consider a similar notion in the partite setting. For $G \subseteq H[n]$ and $n \in \mathbb{N}$ we say G is (H, H[n])-partite-extra-saturated if for any $uv \in E(H[n] \setminus G)$ the graph $G \cup uv$ has more partite copies of H than G. We also ask, given a graph H and $n \in \mathbb{N}$, the value of

$$\operatorname{exsat}_{\mathbf{p}}(H, H[n]) := \min \left\{ e(G) : G \subseteq H[n] \text{ is } (H, H[n]) \text{-partite-extra-saturated} \right\}.$$

We observe some interesting differences in behaviour between these partite saturation numbers and the saturation numbers studied by Erdős, Hajnal and Moon. Whilst for graphs on r vertices cliques gave the largest values of $\operatorname{sat}(H, n)$ we find that cliques are not the graphs which maximise $\operatorname{sat}_p(H, H[n])$. In fact we find in Theorem 12 that $\operatorname{sat}_p(H, H[n])$ grows quadratically for graphs H which are not 2-connected whilst it grows linearly for those which are. On the other-hand we show in Theorem 13 that cliques do maximise the partite-extra-saturation numbers and that all partite-extra-saturation numbers are linear.

Notation. Most of the notation we use is standard. In a graph G for a vertex $v \in V(G)$ and a set $X \subseteq V(G)$ we let $\deg_G(v, X)$ denote the number of neighbours of v in X. Where X = V(G) we will abbreviate to $\deg_G(v)$ and when the graph G is clear from the context we will omit the subscript. For a vertex v we let N(v) denote the set of neighbours of v. **Organisation.** Section 2 is dedicated to determining the partite saturation number of K_4 . In Section 3 we then determine the partite saturation numbers of paths and stars. We look at the link between 2-connectivity and the order of magnitude of partite saturation numbers in Section 5. Finally in Section 6 we give some further remarks and open problems.

2. The Partite Saturation Number of K_4

Theorem 1. For all large enough $n \in \mathbb{N}$ we have

$$\operatorname{sat}(K_4, K_4[n]) = 18n - 21$$

Furthermore we determine the unique graph for which equality holds.

We first give a construction of a graph $G \subseteq K_4[n]$ that is $(K_4, K_4[n])$ -saturated and has 18n - 21 edges.

Let X_1, X_2, X_3, X_4 be the parts of $K_4[n]$. Choose vertices x_i and x'_i in each X_i . Let Z denote the set of these 8 vertices. Include in G the following 15 edges $x_1x_2, x_1x'_2, x_1x'_3, x_1x'_4, x_1x'_2, x_1x'_3, x_1x_4, x_2x_3, x_2x_4, x_2x'_4, x'_2x'_3, x'_2x_4, x_3x'_4, x'_3x_4, x'_3x'_4$. We now only add edges between Z and $V(G) \setminus Z$. Include all edges between $X_1 \setminus Z$ and each of x_2, x_3, x'_3 and x_4 . Attach all vertices in $X_2 \setminus Z$ to x'_1, x_3, x'_3, x_4 and x'_4 . Join all of $X_3 \setminus Z$ to each of x_1, x'_1, x_2 and x_4 and finally add all edges from $X_4 \setminus Z$ to x_1, x'_1, x_2, x'_2 and x_3 .

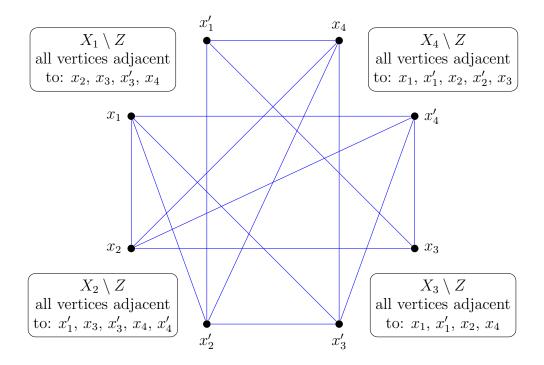


FIGURE 1. K_4 -Partite-Saturation Construction

Proposition 2. G is a $(K_4, K_4[n])$ -saturated graph with 18n - 21 edges.

Proof. To see that this graph is K_4 -free note that the graph induced on $V(G) \setminus Z$ has no edges so any K_4 would have to come from a triangle in Z extended to a vertex outside of Z. There are just six triangles induced on Z and none of them extend to a K_4 .

To see that G is $(K_4, K_4[n])$ -saturated we first observe that for any pair i, j there is an edge in Z such that $(X_i \cup X_j) \setminus Z$ is contained in the common neighbourhood of the ends of that edge. Therefore we could only add an edge with at least one end in Z.

For a vertex $v \in X_1 \setminus Z$ the only incident edges we could add are vx'_2 or vx'_4 . These additional edges would create a K_4 on $vx'_2x'_3x_4$ or $vx_2x_3x'_4$ respectively. For a vertex $v \in X_2 \setminus Z$ the only incident edge we could add is vx_1 but this would create a K_4 on $x_1vx'_3x'_4$. Similar arguments show we cannot add edges incident to $X_3 \setminus Z$ and $X_4 \setminus Z$. Adding any edge to Z that has either x_1 or x'_3 as an endpoint will create a K_4 in Z. Adding any other edge of Z will create a triangle on Z that extends to a K_4 with a vertex outside of Z. That G has 18n - 21 edges is easy to check.

Before proving a matching lower bound we need the following lemmas.

Lemma 3. Any $(K_4, K_4[n])$ -saturated graph G with $n \ge 2$ has minimum degree at least 4.

Proof. Let G be a $(K_4, K_4[n])$ -saturated graph on $X_1 \cup \cdots \cup X_4$. Suppose for contradiction that there exists $a_1 \in X_1$ with at most 3 neighbours. If a_1 has no neighbours in one part, say X_2 , then by saturation it must be adjacent to all vertices in the other parts, which for $n \ge 2$ contradicts the fact that $\deg(a_1) \le 3$. So a_1 must have exactly three neighbours with one in each of the parts. Call these $x_i \in X_i$ for i = 2, 3, 4. Then for any i = 2, 3, 4adding the edge a_1y_i for some $y_i \in X_i \setminus x_i$ must create a K_4 . This implies that x_2x_3, x_2x_4 and x_3x_4 are all edges of G but along with a_1 this gives a K_4 .

We can also say more about the neighbourhoods of vertices with degree exactly 4.

Lemma 4. Let G be a $(K_4, K_4[n])$ -saturated graph on $X_1 \cup \cdots \cup X_4$ with $n \ge 3$ and let v be a vertex of degree exactly 4. Then v has one neighbour in each of two parts and two neighbours in one part. The neighbourhood of v induces a path beginning and ending with the vertices in the same part. All neighbours of v have degree at least n - 2.

Proof. Suppose $v \in X_1$. If v had no neighbour in some X_i $(i \neq 1)$ it would be adjacent to all vertices in other parts meaning it would have degree greater than 4. Suppose without loss of generality that the neighbours of v are x_2, x_3, x'_3 and x_4 with the subscripts denoting the parts containing each vertex. By considering the effect of adding the edge vy_3 for some $y_3 \in X_3 \setminus \{x_3, x'_3\}$ we see that the edge x_2x_4 is present. We also see that all vertices in $X_3 \setminus \{x_3, x'_3\}$ are adjacent to x_2 and x_4 . Similarly by considering a vertex in $X_2 \setminus \{x_2\}$ we see that there must be an edge between x_4 and one of x_3 or x'_3 . Without loss of generality assume $x_4x'_3$ is present. Finally by considering a vertex in $X_4 \setminus \{x_4\}$ we see that x_2 is adjacent to either x_3 or x'_3 . In order not to create a K_4 it must be that x_2x_3 is present. We now cannot have the edges x_4x_3 or $x_2x'_3$. We then see that all vertices in $X_4 \setminus \{x_4\}$ are adjacent to x_3 and all vertices in $X_2 \setminus \{x_2\}$ are adjacent to x'_3 . Hence the neighbours of vall have degree at least n-2.

It follows that when n > 6 vertices of degree exactly 4 cannot be adjacent.

The following lemma gives us minimum degree conditions that more reflect those of the upper bound construction.

Lemma 5. Let G be a $(K_4, K_4[n])$ -saturated graph with $n \ge 22$ on $X_1 \cup \cdots \cup X_4$. There cannot be two degree 4 vertices, $a_i \in X_i$ and $a_j \in X_j$ with $i \ne j$ such that a_i has just one neighbour in X_j . Furthermore there are at most two parts with minimum degree 4.

Proof. Suppose for contradiction that $a_1 \in X_1$ and $a_2 \in X_2$ are degree 4 vertices such that a_1 has just one neighbour in X_2 and let x_2, x_3, x'_3, x_4 denote the neighbours of a_1 . Then (up to switching between x'_3 and x_3) the edges x_2x_3, x_2x_4, x'_3x_4 are all present. We also know that x_2 is adjacent to all of $(X_3 \cup X_4) \setminus x'_3$, that x_3 is adjacent to all of $X_4 \setminus x_4$, that x'_3 is adjacent to all of $X_2 \setminus x_2$, and x_4 is adjacent to all of $(X_2 \cup X_3) \setminus x_3$. In particular this implies we have the edges $a_2x'_3$ and a_2x_4 . The vertex a_2 also has some neighbour $x_1 \in X_1 \setminus a_1$. As a_2 has degree 4 it must have one more neighbour. We split into cases depending on where this final neighbour is and show that each case leads to a contradiction. The possible cases are:

- (i) a_2 has another neighbour $v \in (X_1 \cup X_3) \setminus \{a_1, x_1, x_3, x'_3\}$.
- (ii) a_2 is adjacent to x_3 .
- (iii) a_2 has another neighbour $x'_4 \in X_4 \setminus x_4$.

Case i) Since x_3 is not adjacent to a_2 it must be adjacent to x_4 as $X_4 \cap N(y_2) = \{x_4\}$ and hence $x_1x_2x_3x_4$ forms a K_4 .

Case ii) By considering vertices in $X_3 \setminus N(a_2)$ we must have the edge x_1x_4 and we see that x_1 is adjacent to all of $X_3 \setminus \{x_3, x'_3\}$. We also see that all vertices in $X_1 \setminus N(a_2)$ are adjacent to x'_3 and x_4 . This means that in fact all vertices in $(X_1 \cup X_2) \setminus \{x_1, x_2\}$ are adjacent to x'_3 and x_4 and hence all edges in $X_1 \cup X_2$ have one end in $\{x_1, x_2\}$. In fact all edges in $X_1 \cup X_2$ have exactly one end in $\{x_1, x_2\}$ as if the edge x_1x_2 were present this would create a K_4 with x_4 and any vertex in $X_3 \setminus \{x_3, x'_3\}$.

If all vertices in $X_3 \setminus \{x_3, x'_3\}$ were adjacent to all of $X_4 \setminus x_4$ this would give at least (n-2)(n-1) edges which is greater than 18*n* for $n \ge 22$. Therefore consider some vertex $v_3 \in X_3 \setminus \{x_3, x'_3\}$ which is non-adjacent to some $v_4 \in X_4 \setminus x_4$. As v_4 is non-adjacent to v_3 it must be adjacent to both ends of an edge in $N(v_3) \cap (X_1 \cup X_2)$. We know that this edge has exactly one end in $\{x_1, x_2\}$ but this creates a K_4 with v_3 and x_4 .

Case iii) As x'_4 is not adjacent to a_1 it is adjacent to x_2 and x_3 . By considering vertices in $X_4 \setminus N(a_2)$ we see that $x_1x'_3$ is an edge of G and all vertices in $X_4 \setminus N(a_2)$ are adjacent to x_1 and x'_3 . By considering vertices in $X_3 \setminus N(a_2)$ we see that $x_1x'_4$ is an edge of G(as x_1x_4 would create a K_4) and all vertices in $X_3 \setminus N(a_2)$ are adjacent to x_1 and x'_4 . Finally by considering vertices in $X_1 \setminus N(a_2)$ we observe that all vertices in $X_1 \setminus x_1$ are adjacent to x'_3 and x_4 (as x'_4 cannot be adjacent to x'_3). Now we know that all vertices in $(X_1 \cup X_2) \setminus \{x_1, x_2\}$ are adjacent to both ends of the edge x'_3x_4 and so there are no edges in $(X_1 \cup X_2) \setminus \{x_1, x_2\}$. Furthermore $x_1x_2 \notin E(G)$ as this would create a K_4 with x_4 and any vertex in $X_3 \setminus \{x_3, x'_3\}$. If all vertices in $X_3 \setminus \{x_3, x'_3\}$ were adjacent to all of $X_4 \setminus \{x_4, x'_4\}$ there would be at least $(n-2)^2$ edges in G which is more than 18n edges for $n \ge 22$. Therefore we can assume there is a vertex $v_3 \in X_3 \setminus \{x_3, x'_3\}$ and a vertex $v_4 \in X_4 \setminus \{x_4, x'_4\}$ which is not adjacent to v_3 . Then v_4 must be adjacent to both ends of an edge e in $N(v_3) \cap (X_1 \cup X_2)$. This edge has exactly one end in $\{x_1, x_2\}$. If the edge e

is incident to x_2 but not x_1 then it forms a K_4 with v_3 and x_4 . If instead e is incident to x_1 but not x_2 it forms a K_4 with x'_3 and v_4 .

It follows from Lemma 4 and the above that there can be at most two parts with minimum degree exactly 4 otherwise we would have a degree 4 vertex with just one neighbour in the part containing another degree 4 vertex. \Box

Another distinctive feature of the upper bound construction is that low degree vertices are not adjacent to other low degree vertices. In proving the lower bound it is helpful to prove that at most a constant number of low degree vertices are adjacent to other low degree vertices. We do that in the following lemma.

Lemma 6. For any $k \ge 5$ suppose G is a $(K_4, K_4[n])$ -saturated graph on $X_1 \cup \cdots \cup X_4$. Then there are at most $24k^2(2k^2)^{2k^2}$ vertices v such that $5 \le \deg(v) \le k$ and v is adjacent to another vertex of degree between 5 and k.

Proof. Call a vertex bad if it satisfies $5 \leq \deg(v) \leq k$ and is adjacent to another vertex with degree between 5 and k. Let $K = 24k^2(2k^2)^{2k^2}$ and suppose for contradiction that there are more than K bad vertices in G. Without loss of generality assume there are at least $\frac{K}{4}$ such vertices in X_1 . Call the set of these vertices A_0 and let B_0 denote the set of bad vertices in $X_2 \cup X_3 \cup X_4$ which are adjacent to a bad vertex in A_0 . By counting $e(A_0, B_0)$ from each side we see that $|A_0| \leq e(A_0, B_0) \leq k|B_0|$ and hence $|B_0| \geq \frac{K}{4k}$. By averaging we may assume without loss of generality that there are at least $\frac{K}{12k}$ bad vertices in X_2 adjacent to vertices in A_0 . Let B_1 denote $B_0 \cap X_2$ and let A_1 be the vertices of A_0 which have a neighbour in B_1 . Then every vertex in A_1 and B_1 has a neighbour in the other. By double counting we see that $|B_1| \leq e(A_1, B_1) \leq k|A_1|$ and so we know that both A_1 and B_1 contain at least $\frac{K}{12k^2}$ vertices.

For $i = 0, ..., k^2 + 1$ we construct a collection of sets $U_i \subseteq X_1, V_i \subseteq X_2$ such that $U_{i+1} \subseteq U_i$ and $V_{i+1} \subseteq U_i$. We also select vertices $u_i \in U_i$ and edges $e_i \in E(X_3, X_4)$ such that the following properties are satisfied for all $i = 0, ..., k^2 + 1$.

- (i) All vertices in V_{i+1} are adjacent to both endpoints of e_{i+1} .
- (ii) The vertex u_i is adjacent to both endpoints of e_{i+1} .
- (iii) $|V_i| \ge \frac{K}{12k} (2k^2)^{-i} = 2k(2k^2)^{2k^2 i}.$
- (iv) Each vertex in U_i has a neighbour in V_i .
- (v) Each vertex in V_i has a neighbour in U_i .

(vi)
$$|U_i| \ge \frac{K}{12k^2} (2k^2)^{-i} = 2(2k^2)^{2k^2-i}$$

Before constructing these objects we show how they prove the lemma. Since $|V_i| \ge 2k(2k^2)^{2k^2-i}$ we see that the set V_{k^2+1} is non-empty. Any vertex in V_{k^2+1} is adjacent to both ends of all the edges e_1, \ldots, e_{k^2} . As vertices in V_{k^2+1} have at most k neighbours it must be that two of these edges are the same. If $e_s = e_t$ for some $s < t \le k^2$ then we have that u_t is adjacent to some vertex v in V_s . As v is in V_s it is adjacent to both ends of e_s and so forms a K_4 along with u_t . This gives our contradiction.

We begin constructing these objects by letting $U_0 = A_1$ and $V_0 = B_1$. Given U_i and V_i satisfying the above properties we choose any $u_i \in U_i$ and will find U_{i+1} , V_{i+1} , e_i and P_i

satisfying the properties above. By saturation for any vertex v in $V_i \setminus N(u_i)$ there exists an edge $e \in E(X_3, X_4)$ such that both v and u_i are adjacent to both of the endpoints of e. Since u_i has at most k neighbours there are fewer than k^2 such candidates for e and hence at least $\frac{1}{k^2}|V_i \setminus N(u_i)|$ vertices of $V_i \setminus N(u_i)$ are adjacent to the endpoints of the same edge $e \in E(X_3, X_4)$. Let e_{i+1} be this edge and let V_{i+1} be the vertices of $V_i \setminus N(u_i)$ that are adjacent to both ends of e_{i+1} . From this we see that properties (i) and (ii) hold.

Using $|V_i| \ge \frac{K}{12k} (2k^2)^{-i} \ge 2k$ we then have

$$|V_{i+1}| \ge \frac{1}{k^2} |V_i \setminus N(u_i)| \ge \frac{1}{k^2} (|V_i| - k)$$
$$\ge \frac{1}{2k^2} |V_i| \ge \frac{K}{12k} (2k^2)^{-(i+1)}.$$

This gives property (iii). We let $U_{i+1} = U_i \cap N(V_{i+1})$ which ensures (iv) and (v). Therefore $|V_{i+1}| \leq e(U_{i+1}, V_{i+1}) \leq k|U_{i+1}|$ and we see that $|U_{i+1}| \geq \frac{1}{k}|V_{i+1}| \geq \frac{K}{12k^2}(2k^2)^{-(i+1)}$ giving (vi).

With these lemmas we are now ready to prove Theorem 1.

Proof of Theorem 1. Let G be a $(K_4, K_4[n])$ -saturated graph.

We first make the following claim, the proof of which we postpone, about the minimum degree conditions of the parts of G.

Claim 7. If G has at most 18n - 21 edges then G has precisely two parts of minimum degree exactly 4 and two parts of minimum degree exactly 5.

From Lemma 5 we know that all degree 4 vertices in the two minimum degree 4 parts have two neighbours in the other minimum degree 4 part. We can now assume we have degree 4 vertices $a_1 \in X_1$ and $a_3 \in X_3$. Let the neighbours of a_1 be x_2, x_3, x'_3 and x_4 . We see that all vertices in $X_3 \setminus \{x_3, x'_3\}$ (including a_3) are adjacent to x_2 and x_4 and that x_2 and x_4 are adjacent. Let the other two neighbours of a_3 be x_1 and x'_1 . Since any vertex v in $X_2 \setminus x_2$ is not adjacent to a_1 , adding the edge a_1v must create a K_4 using v and a_1 . Similarly, since any vertex v in $X_2 \setminus x_2$ is not adjacent to a_3 , adding the edge a_3v must create a K_4 using v and a_3 . This implies that v is adjacent to x_4 and that x_4 is adjacent to one of x_1 or x'_1 and also one of x_3 or x'_3 . Without loss of generality assume we have the edges x'_1x_4 and x'_3x_4 . Similar arguments with a vertex in $X_4 \setminus x_4$ show that all vertices in X_4 are adjacent to x_2 and also that we have the edges x_1x_2 and x_2x_3 .

We further see that by saturation every vertex of $(X_1 \cup X_3) \setminus \{x_1, x'_1, x_3, x'_3\}$ is adjacent to x_2 and x_4 . This means there are no edges with both ends lying in $(X_1 \cup X_3) \setminus \{x_1, x'_1, x_3, x'_3\}$. All vertices in $X_2 \setminus x_2$ are adjacent to x'_1, x'_3 and x_4 . All vertices of $X_4 \setminus x_4$ are adjacent to x_1, x_3 and x_2 .

We now have that all vertices in $(X_1 \cup X_3) \setminus \{x_1, x'_1, x_3, x'_3\}$ are adjacent to x_2 and x_4 . All vertices in $X_2 \setminus x_2$ are adjacent to x'_1 , x'_3 and x_4 whilst all vertices in $X_4 \setminus x_4$ are adjacent to all of x_1 , x_2 and x_3 .

The following claim, for which we again postpone the proof, gives us more conditions on the neighbourhoods of various vertices.

Claim 8. All vertices in $X_1 \setminus \{x_1, x'_1\}$ are adjacent to x_3 and x'_3 . All vertices in $X_3 \setminus \{x_3, x'_3\}$ are adjacent to x_1 and x'_1 . All vertices in $(X_2 \cup X_4) \setminus \{x_2, x_4\}$ are adjacent to at least 3 of $\{x_1, x'_1, x_3, x'_3\}$. Both $x_1x'_3$ and x'_1x_3 are edges of G.

Under the assumption of Claim 8 we now see that all vertices in $X_2 \setminus x_2$ are adjacent to x'_1 , x'_3 , x_4 and one of x_1 or x_3 . Let A^1 denote the set of vertices in $X_2 \setminus x_2$ which are adjacent to x_1 but not x_3 and let A^3 denote the set of vertices in $X_2 \setminus x_2$ which are adjacent to x_3 but not x_1 .

Similarly all vertices in $X_4 \setminus x_4$ are adjacent to x_1, x_3, x_2 and one of x'_1 or x'_3 . Let B^1 denote the set of vertices in $X_4 \setminus x_4$ which are adjacent to x'_1 but not x'_3 and let B^3 denote the set of vertices in $X_4 \setminus x_4$ which are adjacent to x'_3 but not x'_3 .

Adding any edge between A^1 and B^1 (likewise between A^3 and B^3) cannot create a K_4 so by saturation the induced graphs on (A^1, B^1) and (A^3, B^3) are complete. Any edge between A^1 and B^3 would create a K_4 with $x_1x'_3$ whilst any edge between A^3 and B^1 would give a K_4 using x'_1x_3 therefore the bipartite graphs on (A^1, B^3) and (A^3, B^1) are empty.

Hence we see that there are at least

$$5(2n-2-|A^{1}|-|A^{3}|-|B^{1}|-|B^{3}|) +4(|A^{1}|+|A^{3}|+|B^{1}|+|B^{3}|) +|A^{1}||B^{1}|+|A^{3}||B^{3}|+4n-4+1$$

edges with at least one end in $X_2 \cup X_4$. The +1 term comes from the edge x_2x_4 and the +4n-4 term comes from the edges with one end in $\{x_2, x_4\}$ and the other end in $X_1 \cup X_3$. Along with the 4n-6 edges between X_1 and X_3 this gives a total of at least

(1)
$$18n - 21 + (|A^1| - 1)(|B^1| - 1) + (|A^3| - 1)(|B^3| - 1)$$

edges. We argue that either A^1 or B^1 being non-empty implies the other is non-empty.

Suppose there were a vertex in A^1 . Then because it has degree at least 5 but is not adjacent to x_3 it has a neighbour v in $X_4 \setminus x_4$. This neighbour v cannot be adjacent to x'_3 or we would have a K_4 . Therefore $v \in B^1$. Similarly for a vertex in B^1 . Likewise either of A^3 or B^3 being non-empty implies the other is also non-empty.

This now means we have at least 18n - 21 edges. Furthermore, since $A^1 \cup A^3 = X_2 \setminus x_2$ and $B^1 \cup B^3 = X_4 \setminus x_4$, equality in (1) is attained only if either $|A^1| = |B^3| = 1$ or $|A^3| = |B^1| = 1$. Letting x'_2 and x'_4 be the vertices in the sets of size 1 we have our extremal construction.

It remains to prove Claims 7 and 8.

Proof of Claim 7. We use Lemma 6 applied with k = 180. As in Lemma 6 we refer to vertices of degree between 5 and k which are adjacent to another such vertex as *bad*.

We now split our vertices into groups by their degrees and whether or not they are bad, and then count edges of G by counting edges between these groups.

We label our groups as follows

- V_{bad} is the set of bad vertices.
- $A := \{v : \deg(v) \ge k+1\}.$

•
$$B := \{v : 5 \leq \deg(v) \leq k\} \setminus V_{\text{bad}}.$$

•
$$C := \{v : \deg(v) = 4\}.$$

We note that vertices in $B \cup C$ only have neighbours in A.

Now $e(G) \ge e(B, A) + e(C, A) \ge 5|B| + 4|C|$. We also have $e(G) \ge e(A, V(G)) \ge \frac{k+1}{2}|A|$. If $|A| \ge \frac{36n}{k+1}$ this gives at least 18n edges so we may assume $|A| < \frac{36n}{k+1}$.

Along with the fact that $|V_{bad}| \leq K = 24k^2(2k^2)^{(2k^2)}$ we see that $|B| \geq 4n - |C| - K - \frac{36n}{k+1}$. Since $e(G) \geq 5|B| + 4|C|$ we have at least $20n - |C| - 5K - \frac{180n}{k+1}$ edges.

If we have at most one X_i with minimum degree 4 we know $|C| \leq n$. This implies that G has at least $19n - 5K - \frac{180n}{k+1}$ edges. For k = 180 and large enough n this is at least 18n.

We can also rule out the possibility of there being a part with minimum degree greater than 5. With V_{bad} , A, and C defined as above let $B^{(5)} := \{v \in B : \deg(v) = 5\}$ and let $B^{(6+)} := \{v \in B : \deg(v) \ge 6\}$. We still have that $|B| = |B^{(5)}| + |B^{(6+)}| \ge 4n - |C| - K - \frac{36n}{k+1}$ and $|C| \le 2n$. If one part had minimum degree at least 6 that would imply that $|B^{(5)}| \le n$ and so we would have

$$\begin{aligned} e(G) \ge 6|B^{(6+)}| + 5|B^{(5)}| + 4|C| \\ = 6|B| - |B^{(5)}| + 4|C| \\ \ge 6(4n - |C| - K - \frac{36n}{k+1}) - |B^{(5)}| + 4|C| \\ = 24n - 2|C| - |B^{(5)}| - 6K - \frac{216n}{k+1} \\ \ge 19n - 6K - \frac{216n}{k+1}. \end{aligned}$$

For k = 216 and n large enough this is more than 18n.

e

Proof of Claim 8. We first consider a degree 5 vertex, a_2 , in $X_2 \setminus x_2$. We consider separately the cases of whether a_2 is adjacent to neither, one, or both of x_1 and x_3 .

Firstly we suppose the vertex a_2 is not adjacent either of x_1 or x_3 . Adding the edge a_2x_1 must create a K_4 using a_2 , x_1 and a vertex in X_4 . Since x_4 is not adjacent to x_1 it must be the case that a_2 has a neighbour $x'_4 \in X_4 \setminus x_4$. If a_2 had two no neighbours in $(X_1 \cup X_3) \setminus \{x'_1, x'_3\}$ there would have to be an edge from x'_1 to x'_3 but this would create a K_4 . Assume, without loss of generality, that a_2 has a neighbour $x''_1 \in X_1 \setminus \{x_1, x'_1\}$. By considering vertices in $X_4 \setminus N(a_2)$ we see that x'_3 is adjacent to x''_1 . This means we now have a K_4 on the vertices x''_1, a_2, x'_3, x_4 .

If instead a_2 had exactly one neighbour from $\{x_1, x_3\}$ then by symmetry we may assume it is adjacent to x_1 but not x_3 . By saturation the addition of the edge a_2x_3 must create a K_4 . Since x_3 is not adjacent to x_4 the vertex a_2 must have a neighbour x'_4 in $X_4 \setminus x_4$. Now a_2 is adjacent to x_1, x'_1, x'_3, x_4 and x'_4 and because a_2 has degree 5 these are all of its neighbours. As the only neighbour of a_2 in X_3 is x'_3 it must be the case that all vertices in $(X_1 \cup X_4) \setminus N(a_2)$ are adjacent to x'_3 . We also see that if any vertex v in $X_3 \setminus \{x_3, x_3\}$ were not adjacent to x'_1 then, since adding the edge a_2v must create a K_4 , we must have that v is adjacent to x_1 and x'_4 which would create a K_4 on $\{x_1, x_2, v, x'_4\}$. Therefore every vertex in $X_3 \setminus \{x_3, x'_3\}$ is adjacent to x'_1 . By considering vertices on $X_4 \setminus N(a_2)$ it must

also be the case that x'_3 is adjacent to x_1 . From the fact that x_3 is not adjacent to a_2 we can see that x_3 must be adjacent to x'_1 and that x'_4 is also adjacent to x'_1 . Now consider a degree 5 vertex, a_4 in $X_4 \setminus \{x_4, x'_4\}$. We know that a_4 is adjacent to x'_3 and we split into the case of when a_4 is adjacent to x'_1 or not.

If a_4 is not adjacent to x'_1 then a_4 has a neighbour $x'_2 \in X_2 \setminus x_2$. We know that x'_2 is adjacent to x'_1 . In order to create a K_4 if $a_4x'_1$ were added it must be the case that x'_2 is adjacent to x_3 . As x_1 is the only neighbour of a_4 in X_1 is must be the case that all vertices in $(X_2 \cup X_3) \setminus N(a_4)$ are adjacent to x_1 . Now all vertices in $(X_3 \cup X_4) \setminus \{x_3, x'_3, x_4\}$ are adjacent to both x_1 and x_2 which are themselves adjacent to each other. Therefore there are no edges between $X_3 \setminus \{x_3, x'_3\}$ and $X_4 \setminus x_4$. We also know that all vertices in $(X_2 \cup X_4) \setminus \{x_2, x'_2, x_4, x'_4\}$ are adjacent to both ends of the edge $x_1x'_3$. Hence there are no edges between $X_2 \setminus \{x_2, x'_2\}$ and $X_4 \setminus \{x_4, x'_4\}$. Since all vertices in $(X_1 \cup X_2) \setminus \{x_1, x'_1, x_2\}$ are adjacent to x'_3 and x_4 there are no edges between $X_1 \setminus \{x_1, x'_1\}$ and $X_2 \setminus x_2$. In particular any vertex v in $X_1 \setminus \{x_1, x'_1\}$ is not adjacent to x'_2 and by considering the K_4 created if a_4v were added we see that v is adjacent to x_3 . Since v was arbitrary all vertices in $X_1 \setminus \{x_1, x'_1\}$ are adjacent to x_3 . This proves the lemma for this case.

If instead a_4 is adjacent to x'_1 then as a_4 is of degree 5 and is adjacent to x_1, x'_1, x_2, x_3 , and x'_3 these are all of its neighbours. Any vertex in $X_1 \setminus \{x_1, x'_1\}$ is non-adjacent to a_4 and so must be adjacent to both ends of some edge in $N(a_4)$. This edge must be x_2x_3 and so all vertices in $X_1 \setminus \{x_1, x'_1\}$ are adjacent to x_3 . Similarly vertices in $X_3 \setminus \{x_3, x'_3\}$ are non-adjacent to a_4 and so must be adjacent to x_1 . All vertices in $X_2 \setminus x_2$ are non-adjacent to a_4 and hence must be adjacent to an edge in $N(a_4)$ implying each vertex in $X_2 \setminus x_2$ is adjacent to at least one of x_1 or x_3 .

Finally we consider the case where a_2 is adjacent to both x_1 and x_3 . We can assume all degree 5 vertices in X_4 are adjacent to both x'_1 and x'_3 or we would be in a situation symmetric to the last case we considered. Let a_4 be such a degree 5 vertex in X_4 . Since all vertices in $X_1 \setminus \{x_1, x'_1\}$ and $X_3 \setminus \{x_3, x'_3\}$ are not adjacent to either a_2 or a_4 they must be adjacent to both ends of an edge in $N(a_2)$ and both ends of an edge in $N(a_4)$. This implies that vertices in $X_1 \setminus \{x_1, x'_1\}$ are adjacent to x_3 and x'_3 and that vertices in $X_3 \setminus \{x_3, x'_3\}$ are adjacent to x_1 and x'_1 . Similarly we see that vertices in $X_4 \setminus x_4$ are non-adjacent to a_2 and hence must be adjacent to an edge in $N(a_2)$. Therefore all vertices in $X_4 \setminus x_4$ are adjacent to one of x'_1 or x'_3 . Similarly all vertices in $X_2 \setminus x_2$ are adjacent to one of x_1 or x_3 . This also shows that at least one of the edges $x_1x'_3$ or x'_1x_3 exists. If one of them is not present, say $x_1x'_3 \notin E(G)$ then by saturation there is some adjacent pair $b_2 \in X_2 \setminus x_2$, $b_4 \in X_4 \setminus x_4$ which are both adjacent to x_1 and x'_3 . We also know, however, that b_2 and b_4 are both adjacent to x'_1 and x_3 but this gives a K_4 on x'_1, b_2, x_3, b_4 . Therefore both $x_1x'_3$ and x'_1x_3 exist.

This completes the proof.

PARTITE SATURATION PROBLEMS

3. SATURATION NUMBERS OF PATHS AND STARS

We begin this section by determining the partite saturation numbers of stars on at least three vertices.

Lemma 9. For any $r \ge 2$, $n \in \mathbb{N}$ and any connected graph H which contains a vertex v such that $H \setminus v$ has r components we have $\operatorname{sat}_{p}(H, H[n]) \ge (r-1)n^{2}$.

Theorem 10. For any $r \ge 2$ and $n \in \mathbb{N}$ all $(K_{1,r}, K_{1,r}[n])$ -partite-saturated graphs have exactly $(r-1)n^2$ edges.

We show how Theorem 10 follows from Lemma 9 before proving Lemma 9 itself.

Proof of Theorem 10. The star $K_{1,r}$ has a vertex v such that $K_{1,r} \setminus v$ has r connected components and hence $\operatorname{sat}_p(K_{1,r}, K_{1,r}[n]) \ge (r-1)n^2$. For any $(K_{1,r}, K_{1,r}[n])$ -partitesaturated graph G any vertex in the part corresponding to the centre of the star must have degree at most (r-1)n or by the pigeonhole principle it would have a neighbour in each remaining part giving a partite copy of $K_{1,r}$. This maximum degree condition implies at most $(r-1)n^2$ edges.

Proof of Lemma 9. Let v_1 be the cut-vertex of H and let v_2, \ldots, v_{r+1} be neighbours of v_1 which are in distinct components of $H \setminus \{v_1\}$. Let X_i denote the part of H[n] corresponding to v_i and let H_i denote the component of v_i in $H \setminus \{v_1\}$. Consider a $(K_{1,r}, K_{1,r}[n])$ -partitesaturated graph G and an arbitrary vertex $x_1 \in X_1$. If x_1 has fewer than (r-1)n neighbours then there are two parts, say X_2 and X_3 , such that each has a vertex non-adjacent to x_1 . Call these vertices x_2 and x_3 . Since G is saturated adding the edge x_1x_2 must create a copy of H using x_1 and hence there must be a copy of $H \setminus H_2$ in G using x_1 . Similarly adding the edge x_1x_3 must create a copy of H implying the existence of a copy of $H \setminus H_3$ at x_1 . The union of these two subgraphs contains a partite copy of H which contradicts Gbeing H-free. Hence each vertex in X_1 has at least (r-1)n neighbours and so G has at least $(r-1)n^2$ edges.

We now determine the partite saturation numbers of paths on at least 4 vertices.

Theorem 11. For any $r \ge 4$ and $n \ge 2r$ we have the following.

(2)
$$\operatorname{sat}_{p}(P_{r}, P_{r}[n]) = \begin{cases} (\frac{r}{2} - 1)n^{2} + (r - 2)n + 3 - r, \text{ for } r \text{ even} \\ (\frac{r}{2} - \frac{1}{2})n^{2} + (r - 4)n + 5 - r, \text{ for } r \text{ odd} \end{cases}$$

Proof. Let X_1, \ldots, X_r be the parts of $P_r[n]$ with X_i adjacent to X_{i+1} for each *i*.

We first give an upper bound construction. Given subsets $A_i \subseteq X_i$ define the graph Gon $\bigcup_i X_i$ to be the graph with precisely the edges that lie in (A_i, A_{i+1}) or $(X_i \setminus A_i, X_{i+1})$ for some $i \leq r-1$. For the upper bound if r is even consider the graph G created as above with $A_1 := X_1$, $A_r := \emptyset$, $|A_i| = 1$ for all even $i \leq r-2$ and $|A_i| = n-1$ for all odd $3 \leq i \leq r-1$. If r is odd consider the construction G given as above but with the A_i satisfying $A_1 := X_1$, $A_r = \emptyset$, $|A_{r-1}| = n-1$, $|A_i| = 1$ for all even $i \leq r-3$ and $|A_i| = n-1$ for all odd $3 \leq i \leq r-2$.

For the lower bound we assume that for some $r \ge 4$ and some $n \ge 2r$ equation (2) does not hold. Then consider the least such r and some $n \ge 2r$ for which (2) fails. In particular by this minimality and Theorem 10 (which gives the partite saturation of $K_{1,2} = P_3$) we see that

(3)
$$\operatorname{sat}_{p}(P_{r-1}, P_{r-1}[n]) \ge \left(\frac{r-1}{2} - 1\right)n^{2}.$$

Now consider a $(P_r, P_r[n])$ -partite-saturated graph G on $X_1 \cup \cdots \cup X_r$. Let N_2 denote the set of vertices in X_2 which are adjacent to at least one vertex of X_1 . For each $i \ge 3$ let N_i denote the set of vertices of X_i which are adjacent to at least one vertex of N_{i-1} . Since there can be no partite path on r vertices it must be the case that $N_r = \emptyset$. If $N_{r-1} = \emptyset$ then (X_{r-1}, X_r) must be complete in G as adding an edge to this pair cannot create a partite copy of P_r . If (X_{r-1}, X_r) is complete then $X_1 \cup \cdots \cup X_{r-1}$ is $(P_{r-1}, P_{r-1}[n])$ -partitesaturated so by (3) there are at least $\frac{r}{2}n^2$ edges in G. This is at least as many as required. Therefore we may assume $N_i \neq \emptyset$ for all $2 \leq i \leq r-1$. If $N_i = X_i$ for some $i \geq 2$ then the pairs (X_j, X_{j+1}) are complete for all $1 \leq j \leq i-1$. Then $X_i \cup \cdots \cup X_r$ is $(P_{r-i+1}, P_{r-i+1}[n])$ partite-saturated so by (3) there are at least $(\frac{r-1}{2})n^2$ edges in G. This is at least as many as required. We now assume $N_i \neq X_i$ for all $2 \leq i \leq r$ so for all $i = 2, \ldots, r-1$ we have $1 \leq |N_i| \leq n-1$. For each $i \geq 2$ let $\overline{N_i}$ denote $X_i \setminus N_i$. We observe that (X_1, N_1) and $(\overline{N_{r-1}}, X_r)$ must be complete. As are (N_i, N_{i+1}) and $(\overline{N_i}, X_{i+1})$ for $2 \leq i \leq r-2$ because adding edges to either of these pairs cannot create a partite copy of P_r . Therefore we find that G has all possible edges except those in pairs $(X_1, \overline{N_2})$ or $(N_i, \overline{N_{i+1}})$ for $2 \leq i \leq r-1$ and so e(G) is at least

(4)
$$(r-1)n^2 - n|\overline{N_2}| - \sum_{i=2}^{r-1} |N_i||\overline{N_{i+1}}| = (r-2)n^2 + n|N_2| - n\sum_{i=2}^{r-1} |N_i| + \sum_{i=2}^{r-2} |N_i||N_{i+1}|.$$

Suppose $N_2, ..., N_{r-1}$ have been chosen to minimise the above expression under the assumption that each $|N_i|$ is between 1 and n-1. The contribution to (4) from terms that include N_2 is exactly $|N_2||N_3|$ which (regardless of the value of $|N_3|$) is minimised by taking $|N_2| = 1$. For $3 \leq i \leq r-2$ the contribution to (4) from terms that include N_i is

$$|N_i|(|N_{i-1}| + |N_{i+1}| - n)$$
 .

When $|N_{i-1}| = 1$ the above expression is at most zero and so minimised by taking $|N_i| = n - 1$. If $|N_{i-1}| = n - 1$ it is at least zero and so minimised by taking $|N_i| = 1$. In this way using $|N_2| = 1$ we can see that for $2 \leq i \leq r - 2$ we have the following.

$$|N_i| = \begin{cases} 1, \text{ for } i \text{ even} \\ n-1, \text{ for } i \text{ odd} \end{cases}$$

The contribution to (4) from the N_{r-1} terms is $|N_{r-1}|(|N_{r-2}|-n)$ which is always negative and so the expression is minimised when $|N_{r-1}| = n - 1$. The graph given with the N_i taking these sizes is the same as our upper bound construction completing the proof. \Box

PARTITE SATURATION PROBLEMS

4. 2-Connectivity and the Growth of Saturation Numbers

Recall that a graph is 2-connected if after the removal of any single vertex it is still connected. Observe that if H' can be obtained from H by adding or removing isolated vertices then $\operatorname{sat}_{p}(H, H[n]) = \operatorname{sat}_{p}(H', H'[n])$. It is also clear that $\operatorname{sat}_{p}(K_{2}, K_{2}[n]) = 0$.

Theorem 12. For any graph H with $e(H) \ge 2$ and no isolated vertices, if H is 2-connected then $\operatorname{sat}_{p}(H, H[n]) = \Theta(n)$ and if H is not 2-connected then $\operatorname{sat}_{p}(H, H[n]) = \Theta(n^{2})$.

Proof. If H is connected but not 2-connected then there must be a cut vertex, v, of H such that $H \setminus v$ has at least two components. Then by Lemma 9 we have $\operatorname{sat}_{p}(H, H[n]) \ge n^{2}$.

We now consider the case when H is disconnected but has no isolated vertices. Let H_1 and H_2 be two connected components of H. $G \subseteq H[n]$ is (H, H[n])-partite-saturated then by saturation the induced graph of G onto at least one of $H_1[n]$ or $H_2[n]$ must be complete. Since each H_i contains an edge this means G has at least n^2 edges.

Finally we consider the case when H is 2-connected. The fact that $\operatorname{sat}_{p}(H, H[n]) = \Omega(n)$ comes from the fact that in an (H, H[n])-saturated graph G every vertex, x, has degree at least one. If not adding an edge incident to x would not create a copy of H since H has minimum degree at least two by 2-connectivity.

We now give an upper bound construction. For each edge ij of H we define H_{ij} to be the graph obtained from H be removing all edges incident to i or j including the edge ij. We define $V_i(H_{ij})$ to be the vertices of $H_{ij} \setminus \{i, j\}$ which were incident to i in H. Similarly $V_j(H_{ij})$. For $n \ge e(H)$ we let $G_1 \subseteq H[n]$ be the disjoint union of a copy of H_{ij} for each edge ij of H. Create G_2 from G_1 by adjoining each vertex of $V_i(H_{ij})$ (in the copy of H_{ij} in G_1) to every vertex in $X_i \setminus V(G_1)$, and by adjoining each vertex of $V_j(H_{ij})$ to every vertex in $X_j \setminus V(G_1)$ for each edge ij of H. We then create G_3 from G_2 by arbitrarily adding edges until the graph is (H, H[n])-partite-saturated.

We claim that G_3 is (H, H[n])-partite-saturated and has at most $2e(H)^2n - e(H)^3$ edges. To prove this it is sufficient to show that G_2 has no partite copy of H and that G_3 has at most $2e(H)^2n - e(H)^3$ edges. We first note that there are no edges of G_2 or G_3 with both end points in $V(G_3) \setminus V(G_1)$ since any such edge $x_i x_j$ would form a copy of H with the H_{ij} . We can then bound the number of edges of G_3 by $E(H[n]) - E(H[n - e(H)]) = n^2 e(H) - (n - e(H))^2 e(H) = 2e(H)^2 n - e(H)^3$.

Suppose now for contradiction that G_2 has a partite copy of H. Denote the vertices of this copy of H by x_i for i = 1, ..., |H|. Since G_1 is H-free at least one of the x_i 's lies in $V(G_2) \setminus V(G_1)$. Suppose without loss of generality that $x_1 \notin V(G_1)$. Let x_2 be a neighbour of x_1 in the partite copy of H. Since there are no edges with both end points in $V(G_2) \setminus V(G_1)$ it must be the case that $x_2 \in V(G_1)$. Since x_1x_2 is an edge of G_2 it must be the case that $x_2 \in V_1(H_{1i})$ for some i adjacent to 1 in H. Suppose $x_2 \in V_1(H_{13})$. Then similarly $x_3 \in V_1(H_{1k})$ for some $k \neq 3$. Therefore x_2 and x_3 are in different H_{ij} 's and hence different connected components of G_1 . Our copy of H is separated by following the set

$$\{x_i : x_i \notin V(G_1) \text{ and } x_i \text{ is adjacent to a vertex in } H_{13}\}$$
.

Since H is 2-connected this set must contain at least two vertices, one of which is x_1 . The only x_i 's that vertices in H_{13} can be adjacent to outside of H_{13} are x_1 and x_3 but $x_3 \in V(G_1)$ which gives a contradiction.

5. Extra-Saturation Numbers

In this section we determine the partite extra-saturation numbers of cliques and trees, and show that of graphs on r vertices the cliques have the largest partite extra-saturation numbers.

Since it follows from the proof of $\operatorname{sat}(K_3, K_3[n]) = 6n - 6$ in [5] that $\operatorname{exsat}_p(K_3, K_3[n]) = 6n - 6$ we look only at cliques on at least 4 vertices. The proof of the following Theorem uses ideas from [5].

Theorem 13. For any integer $r \ge 4$ and all large enough $n \in \mathbb{N}$ we have

$$\operatorname{exsat}_{p}(K_{r}, K_{r}[n]) = (2n-1)\binom{r}{2}.$$

Proof. For the upper bound consider the graph G consisting of a copy of K_r with each vertex of this clique adjacent to all vertices in adjacent parts of $K_r[n]$. For the lower bound consider a $(K_r, K_r[n])$ -partite-extra-saturated graph G on $X_1 \cup \cdots \cup X_r$.

For all i = 1, ..., r let $\delta_i := \min\{d(x) : x \in X_i\}$. Since for any i we have $e(G) \ge \delta_i n$ we must have $\delta_i < r^2$ or G would have more than $(2n-1)\binom{r}{2}$ edges. By the fact that any vertex which is not adjacent to some part must be incident to all vertices in the other parts we see that $\delta_i \ge r-1$ for all i.

Claim 14. All vertices of degree r - 1 are in a K_r .

Proof. If $v \in X_1$ is a vertex of degree r-1 it must have a neighbour in each adjacent part. Denote these by $x_i \in X_i$ for i = 2, ..., r. For any $y_2 \in X_2 \setminus x_2$ adding the edge vy_2 must create a new K_r . This new clique must be on $\{v, y_2, x_3, x_4, ..., x_r\}$ so $x_3, ..., x_r$ must all be pairwise adjacent. Similarly for any $y_3 \in X_3 \setminus x_3$ adding the edge vy_3 must create a new K_r (which must be $\{v, x_2, y_3, x_4, x_5, ..., x_r\}$) so $x_2, x_4, x_5, ..., x_r$ must all be pairwise adjacent. This gives a K_r on $v, x_2, x_3, ..., x_r$.

Let x_i be a vertex of degree δ_i for each i. For each i let $Y_i := \bigcup_{j \neq i} (N(x_j) \cap X_i)$ and let $Y := \bigcup_i Y_i = \bigcup_i N(x_i)$. Observe that $|Y| \leq r^3$.

Claim 15. For all $i \neq j$, each vertex in $X_i \setminus Y_i$ has a neighbour in Y_j .

Proof. Given some $i \neq j$ and a vertex $v \in X_i \setminus Y_i$ consider any $k \in \{1, ..., r\} \setminus \{i, j\}$. As v is not in Y_i it must be that v is not adjacent to x_k . Therefore, by saturation, adding vx_k creates a new K_r . This K_r must use a neighbour of x_k in X_j and hence this neighbour is both in Y_j and also adjacent to v.

We can now lower bound the edges of G by

$$e(G) \ge e(Y, X \setminus Y) + e(X \setminus Y)$$

$$\ge \sum_{v \in X \setminus Y} \left(\deg(v, Y) + \frac{1}{2} \left(\deg(v, X \setminus Y) \right) \right)$$

$$\ge \sum_{i} |X_{i} \setminus Y| \left(r - 1 + \frac{1}{2} \left(\delta_{i} - (r - 1) \right) \right)$$

$$= \frac{1}{2} (r - 1) |X \setminus Y| + \frac{1}{2} \sum_{i} |X_{i} \setminus Y| \delta_{i}$$

$$\ge \frac{1}{2} n \left(r(r - 1) + \sum_{i} \delta_{i} \right) - \frac{1}{2} r^{3} \left(r - 1 + \sum_{i} \delta_{i} \right)$$

$$\ge \frac{1}{2} n \left(r(r - 1) + \sum_{i} \delta_{i} \right) - r^{6}$$

$$= (2n - 1) \binom{r}{2} + \frac{1}{2} n \sum_{i} \left(\delta_{i} - (r - 1) \right) + \binom{r}{2} - r^{6}$$

Therefore for $n > 2r^6$ we have $\delta_i = r - 1$ for all *i*. Each of the x_i 's has one neighbour in each adjacent part and is in a copy of K_r . We see that by saturation for a vertex vof degree r - 1 every vertex w in a different part from v which is not adjacent to v is incident to all neighbours of v outside of the part of w. Therefore vertices of degree r - 1are not adjacent. We also see that for any $i \neq j$ the vertices x_i and x_j have r - 2 common neighbours and so with the sets Y_i and Y as before we find that $|Y_i| = 1$ for all i, so Y = r.

Using (5) we get

$$e(G) \ge e(Y, X \setminus Y) + e(X \setminus Y) + e(Y)$$

$$\ge \frac{1}{2}(r-1)|X \setminus Y| + \frac{1}{2}\sum_{i}|X_i \setminus Y|\delta_i + e(Y)$$

$$\ge 2(n-1)\binom{r}{2} + e(Y).$$

Since there is a K_r on Y we have $e(Y) = \binom{r}{2}$ and the result follows.

The upper bound construction can be generalised to any H by letting G consist of a copy of H with each vertex of this H adjacent to all vertices in adjacent parts of H[n]. This gives an upper bound of

$$\operatorname{exsat}_{p}(H, H[n]) \leq (2n-1)e(H)$$
.

In particular this shows that over graphs H on r vertices the cliques give rise to the largest value of $\operatorname{exsat}_{p}(H, H[n])$ and also that all partite extra-saturation numbers of graphs with at least two edges are linear.

Next we determine the partite extra-saturation number of trees.

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Theorem 16. For any tree T on at least 3 vertices and any natural number $n \ge 4$ we have $\operatorname{exsat}_p(T, T[n]) = (|T| - 1)n$

Proof. For an upper bound construction let G be the union of n disjoint partite copies of T.

Turning our attention to the lower bound we let L denote the set of leaves of T and call the vertices in $C = V(T) \setminus L$ core vertices.

Now suppose G is a (T, T[n])-extra-saturated graph with $n \ge 4$. Let x be a vertex of G lying in a part associated to a core vertex $v \in C$. In G the vertex x must either have a neighbour in each adjacent part of T[n] or it must be that $\deg_G(x) \ge n(\deg_T(v) - 1) \ge 2 \deg_T(v)$. This is because if x had no neighbour in some adjacent part it must be adjacent to all vertices in the other adjacent parts. Since $\deg_T(v) \ge 2$ and $n \ge 4$ this means x has at least $2 \deg_T(v)$ neighbours. We let L[n] and C[n] denote the set of vertices in T[n] that lie in parts corresponding to L and C respectively.

We have

(6)
$$e(G) = \sum_{x \in C[n]} \left(\frac{1}{2} \deg_G(x, C[n]) + \deg_G(x, L[n]) \right)$$
$$= \frac{1}{2} \sum_{x \in C[n]} \left(\deg_G(x) + \deg_G(x, L[n]) \right).$$

Let $x \in C[n]$ be a vertex associated in the part associated to a vertex $v \in C$. If x is adjacent to a vertex in each adjacent part then

$$\deg_G(x) + \deg_G(x, L[n]) \ge \deg_T(v) + \deg_T(v, L)$$

otherwise we also obtain

$$\deg_G(x) + \deg_G(x, L[n]) \ge \deg_G(x) \ge 2 \deg_T(v) \ge \deg_T(v) + \deg_T(v, L).$$

Using these and (6), we see that

$$e(G) \ge \frac{n}{2} \sum_{v \in C} \left(\deg_T(v) + \deg_T(v, L) \right)$$
$$= n \cdot e(T) = n \left(|T| - 1 \right)$$

completing the proof.

6. Concluding Remarks

It would be very nice to be able to determine the value of $\operatorname{sat}(K_r, K_r[n])$ for $r \ge 5$. Exact answers here would probably be very difficult though it may be possible to determine up to an error term of o(n) or even O(1). It would be helpful to be able to determine the following value in order to make progress on this problem.

For integers $r \ge s \ge 3$ let m(r, s) denote the fewest vertices an r-partite graph G can have such that G is K_s -free but every set of s - 1 parts contains a K_{s-1} .

We can use m(r, r-1) and m(r-1, r-1) to get upper and lower bounds respectively on sat $(K_r, K_r[n])$.

For the upper bound let $F \subseteq K_r[n]$ be a K_{r-1} -free graph on m(r, r-1) vertices such that any r-2 parts contain a K_{r-2} . Create a $(K_r, K_r[n])$ -saturated graph $G \subseteq K_r[n]$ by attaching all vertices of F to all vertices outside of F which lie in a different part. Then if necessary add edges between vertices of F until the graph is $(K_r, K_r[n])$ -saturated. This implies that sat $(K_r, K_r[n])$ is less than $m(r, r-1) \cdot (r-1)n$. Using the fact that m(4, 3) = 6this shows that sat $(K_4, K_4[n]) \leq 18n$ which we know from Theorem 1 to be close to the correct answer.

For the lower bound we prove a minimum degree condition in all $(K_r, K_r[n])$ -saturated graphs. If G is a $(K_r, K_r[n])$ -saturated graph note that any vertex in G is either adjacent to all vertices in one part of $K_r[n]$ or its neighbourhood induces an (r-1)-partite graph which is K_{r-1} -free but where there is a K_{r-2} on any r-2 parts. Therefore, for $n \ge m(r-1, r-1)$ we have $\delta(G) \ge m(r-1, r-1)$ and hence $\operatorname{sat}(K_r, K_r[n]) \ge m(r-1, r-1) \cdot rn/2$. When r = 4 this gives the minimum degree condition of $\delta(G) \ge m(3, 3) = 4$.

References

- B. Bollobás, Determination of extremal graphs by using weights, Wiss. Z. Techn. Hochsch. Ilmenau 13 (1967), 419–421.
- 2. ____, On a conjecture of Erdős, Hajnal and Moon, Amer. Math. Monthly 74 (1967), 178–179.
- P. Erdős, A. Hajnal, and J. W. Moon, A problem in graph theory, Amer. Math. Monthly 71 (1964), 1107–1110.
- P. Erdős and A. H. Stone, On the structure of linear graphs, Bull. Amer. Math. Soc. 52 (1946), 1087–1091.
- 5. M Ferrara, M Jacobson, F Pfender, and P Wenger, *Graph saturation in multipartite graphs*, arXiv preprint arXiv:1408.3137 (2014).
- L. Kászonyi and Zs. Tuza, Saturated graphs with minimal number of edges, J. Graph Theory 10 (1986), no. 2, 203–210.
- 7. W Mantel, Problem 28, Wiskundige Opgaven (1907).
- 8. P Turán, On an extremal problem in graph theory, Matematikai s Fizikai Lapok (in Hungarian) (1941).
- W. Wessel, Über eine Klasse paarer Graphen. I. Beweis einer Vermutung von Erdős, Hajnal und Moon, Wiss. Z. Techn. Hochsch. Ilmenau 12 (1966), 253–256.
- 10. ____, Über eine Klasse paarer Graphen. II. Bestimmung der Minimalgraphen, Wiss. Z. Techn. Hochsch. Ilmenau **13** (1967), 423–426.

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