FINITE 2-DISTANCE TRANSITIVE GRAPHS

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ABSTRACT. A non-complete graph Γ is said to be $(G, 2)$-distance transitive if G is a subgroup of the automorphism group of Γ that is transitive on the vertex set of Γ, and for any vertex u of Γ, the stabilizer G_u is transitive on the sets of vertices at distance 1 and 2 from u. This paper investigates the family of $(G, 2)$-distance transitive graphs that are not $(G, 2)$-arc transitive. Our main result is the classification of such graphs of valency not greater than 5.

1. INTRODUCTION

Graphs that satisfy certain symmetry conditions have been a focus of research in algebraic graph theory. We usually measure the degree of symmetry of a graph by studying if the automorphism group is transitive on certain natural sets formed by combining vertices and edges. For instance, s-arc transitivity requires that the automorphism group should be transitive on the set of s-arcs (see Section 2 for precise definitions). The class of s-arc transitive graphs have been studied intensively, beginning with the seminal result of Tutte [13] that cubic s-arc transitive graphs must have $s \leqslant 5$. Later, in 1981, Weiss [15], using the finite simple group classification, showed that there are no 8-arc transitive graphs of valency at least 3. For a survey on s-arc transitive graphs, see [12].

Recently, several papers have considered conditions on undirected graphs that are similar to, but weaker than, s-arc transitivity. For examples of such conditions, we mention local s-arc transitivity, local s-distance transitivity, s-geodesic transitivity, and 2-path transitivity. Devillers et al. [4] studied the class of locally s-distance transitive graphs, using the normal quotient strategy developed for s-arc transitive graphs in [11]. The condition of s-geodesic transitivity was investigated in several papers [5, 6, 8]. A characterization of 2-path transitive, but not 2-arc transitive graphs was given by Li and Zhang [10].

In this paper we study the class of 2-distance transitive graphs. If G is a subgroup of the automorphism group of a graph Γ, then Γ is said to be $(G, 2)$-distance transitive if G acts transitively on the vertex set of Γ, and a vertex stabilizer G_u is transitive on the neighborhood $\Gamma(u)$ of u and on the second neighborhood $\Gamma_2(u)$ (see Section 2). The class of $(G, 2)$-distance transitive graphs is larger than the class of $(G, 2)$-arc transitive graphs, and in this paper we study the $(G, 2)$-distance transitive graphs that are not $(G, 2)$-arc transitive.

Our first theorem links the structure of $(G, 2)$-distance transitive, but not $(G, 2)$-arc transitive graphs to their valency and the value of the constant c_2 in the intersection array (see Definition 2.1).

Γ	valency	girth	G	Reference
$\overline{(2 \times 4)-\mathsf{grid}}$	3	4	satisfies Condition 3.1	Section 3.1
Octahedron	4	3	$G \leqslant S_2 \wr S_3$, $|S_2 \wr S_3 : G| \in \{1, 2\}$, G projects onto S_3	Lemma 3.4
H(2,3)	4	3	$G \leqslant S_3 \wr S_2$, $|S_3 \wr S_2 : G| \in \{1, 2\}$ G projects onto S_2	Proposition 5.2
the line graph of a connected $(G, 3)$-arc transitive graph	4	3		Proposition 5.2
$\overline{(2 \times 5)-\mathsf{grid}}$	4	4	satisfies Condition 3.1	Section 3.1
Icosahedron	5	3	$G \in \{A_5, A_5 \times C_2\}$	Lemma 3.5
$\overline{(2 \times 6)-\mathsf{grid}}$	5	4	satisfies Condition 3.1	Section 3.1

TABLE 1. $(G, 2)$-distance transitive, but not $(G, 2)$-arc transitive graphs of valency at most 5

Theorem 1.1. *Let Γ be a connected $(G, 2)$-distance transitive, but not $(G, 2)$-arc transitive graph of girth 4 and valency $k \geqslant 3$. Then $2 \leqslant c_2 \leqslant k-1$ and the following are valid.*

(i) *If $c_2 = k - 1$, then $\Gamma \cong \overline{(2 \times (k+1)) - \mathsf{grid}}$ and G satisfies Condition 3.1.*
(ii) *If $c_2 = 2$, then k is a prime-power such that $k \equiv 3 \pmod 4$ and G_u acts 2-homogeneously, but not 2-transitively on $\Gamma(u)$ for each $u \in V\Gamma$.*

The following corollary is a characterization of the family of connected $(G, 2)$-distance transitive, but not $(G, 2)$-arc transitive graphs of girth 4 and prime valency.

Corollary 1.2. *Let Γ be a connected $(G, 2)$-distance transitive, but not $(G, 2)$-arc transitive graph of girth 4 and prime valency p, and let $u \in V\Gamma$. Then the following are valid.*

(i) *Either $\Gamma \cong \overline{(2 \times (p+1)) - \mathsf{grid}}$, or $c_2 | p - 1$ and $2 \leqslant c_2 \leqslant (p-1)/2$.*
(ii) *If $c_2 = 2$, then $p \equiv 3 \pmod 4$ and G_u is 2-homogeneous, but not 2-transitive on $\Gamma(u)$.*
(iii) *If $c_2 = (p-1)/2$, then $|\Gamma_2(u)| = 2p$, and G_u is imprimitive on $\Gamma_2(u)$.*

Finally, our third main result determines all the possible $(G, 2)$-distance transitive, but not $(G, 2)$-arc transitive graphs of valency at most 5.

Theorem 1.3. *Let Γ be a connected $(G, 2)$-distance transitive, but not $(G, 2)$-arc transitive graph of valency $k \leqslant 5$. Then Γ and G must be as in one of the rows of Table 1.*

In Section 2 we state the most important definitions and some basic results related to 2-distance transitivity. In Section 3, we study some examples, such as grids, their complements, Hamming graphs, complete bipartite graphs, and platonic solids from the point of view of 2-distance transitivity. In Section 4, we consider 2-distance transitive graphs of girth 4. Finally the proofs of our main results are given in Section 5.

2. Basic Definitions and useful facts

In this paper, graphs are finite, simple, and undirected. For a graph Γ, let $V\Gamma$ and $\operatorname{Aut}\Gamma$ denote its vertex set and automorphism group, respectively. Let Γ be a graph and let u and v be vertices in Γ that belong to the same connected component. Then the *distance* between u and v is the length of a shortest path between u and v and is denoted by $d_\Gamma(u,v)$. We denote by $\Gamma_s(u)$ the set of vertices at distance s from u in Γ and we set $\Gamma(u) = \Gamma_1(u)$. The *diameter* $\operatorname{diam}\Gamma$ of Γ is the greatest distance between vertices in Γ. Let $G \leqslant \operatorname{Aut}\Gamma$ and let $s \leqslant \operatorname{diam}\Gamma$. We say that Γ is (G,s)-*distance transitive* if G is transitive on $V\Gamma$ and G_u is transitive on $\Gamma_i(u)$ for all $i \leqslant s$. If Γ is (G,s)-distance transitive for $s = \operatorname{diam}\Gamma$, then we simply say that it is G-*distance transitive*. By our definition, if $s > \operatorname{diam}\Gamma$, then Γ is not (G,s)-distance transitive. For instance, the complete graph is not $(G,2)$-distance transitive for any group G.

In the characterization of (G,s)-distance transitive graphs, the following constants are useful. Our definition is inspired by the concept of intersection arrays defined for the distance regular graphs (see [2]).

Definition 2.1. Let Γ be a (G,s)-distance transitive graph, $u \in V\Gamma$, and let $v \in \Gamma_i(u)$, $i \leqslant s$. Then the number of edges from v to $\Gamma_{i-1}(u)$, $\Gamma_i(u)$, and $\Gamma_{i+1}(u)$ does not depend on the choice of v and these numbers are denoted, respectively, by c_i, a_i, b_i.

Clearly we have that $a_i + b_i + c_i$ is equal to the valency of Γ whenever the constants are well-defined. Note that for $(G,2)$-distance transitive graphs, the constants are always well-defined for $i = 1,\ 2$.

A sequence $(v_0, \dots, v_s)$ of vertices of a graph is said to be an *s-arc* if v_i is connected to v_{i+1} for all $i \in \{0, \dots, s-1\}$ and $v_i \neq v_{i+2}$ for all $i \in \{0, \dots, s-2\}$. A graph Γ is called (G,s)-*arc transitive* if G acts transitively on the set of vertices and on the set of s-arcs of Γ. (We note that some authors define (G,s)-arc transitivity only requiring that G should be transitive on the set of s-arcs.) It is well-known, that Γ is $(G,2)$-arc transitive if and only if G is transitive on $V\Gamma$, and the stabilizer G_u is 2-transitive on $\Gamma(u)$ for some, and hence for all, $u \in V\Gamma$. We will use this fact without further reference in the rest of the paper.

The *girth* of a graph Γ is the length of a shortest cycle in Γ. Let Γ be a connected $(G,2)$-distance transitive graph. If Γ has girth at least 5, then for any two vertices u and v with $d_\Gamma(u,v) = 2$, there exists a unique 2-arc between u and v. Hence if Γ is $(G,2)$-distance transitive, then it is $(G,2)$-arc transitive. On the other hand, if the girth of Γ is 3, and Γ is not a complete graph, then some 2-arcs are contained in a triangle, while some are not. Hence Γ is not $(G,2)$-arc transitive. We record the conclusion of this argument in the following lemma.

Lemma 2.2. *Suppose that Γ is a $(G,2)$-distance transitive graph. If Γ has girth at least 5, then Γ is $(G,2)$-arc transitive. If Γ has girth 3, then Γ is not $(G,2)$-arc transitive.*

If Γ has girth 4, then Γ can be $(G,2)$-distance transitive, but not $(G,2)$-arc transitive. An infinite family of examples can be constructed using Lemma 3.2.

We close this section with two results on permutation group theory and another one on 2-geodesic transitive graphs. They will be needed in our analysis in Sections 4–5. Recall that a permutation group G acting on Ω is said to be 2-*homogeneous* if G is transitive on the set of 2-subsets of Ω.

Lemma 2.3 ([9])**.** *Let G be a 2-homogeneous permutation group of degree n which is not 2-transitive. Then the following statements are valid:*

  (i) $n = p^e \equiv 3 \pmod 4$ *where p is a prime;*
  (ii) $|G|$ *is odd and is divisible by $p^e(p^e - 1)/2$;*

Lemma 2.4. ([7, Theorem 1.51]) *If G is a primitive, but not 2-transitive permutation group on $2p$ letters where p is a prime, then $p = 5$ and $G \cong A_5$ or S_5.*

An s-*geodesic* in a graph Γ is a shortest path of length s between vertices in Γ. In particular, a vertex triple (u,v,w) with v adjacent to both u and w is called a 2-*geodesic* if u and w are not adjacent. A non-complete graph Γ is said to be $(G,2)$-*geodesic transitive* if G is transitive on both the arc set and on the set of 2-geodesics of Γ. Recall that the *line graph* $L(\Gamma)$ of a graph Γ is graph whose vertices are the edges of Γ and two vertices of $L(\Gamma)$ are adjacent if and only if they are adjacent to a common vertex of Γ. For a natural number n, we denote by K_n the *complete graph* on n vertices.

Lemma 2.5. ([5, Theorem 1.3]) *Let Γ be a connected, non-complete graph of valency 4 and girth 3. Then Γ is $(G,2)$-geodesic transitive if and only if, either $\Gamma = L(\mathrm{K}_4)$ or $\Gamma = L(\Sigma)$ where Σ is connected cubic $(G,3)$-arc transitive graph.*

We observe that the line graph of K_4 is precisely the octahedral graph (see Lemma 3.4).

3. Constructions, Examples & non-Examples

3.1. **Complements of grids and complete bipartite graphs.** For $n, m \geqslant 2$, we define the $(n \times m)$−grid as the graph having vertex set $\{(i,j) \mid 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m\}$, and two distinct vertices (i,j) and (r,s) are adjacent if and only if $i = r$ or $j = s$. The automorphism group of the $(n \times m)$−grid, when $n \neq m$, is the direct product $S_n \times S_m$; when $n = m$, it is $S_n \wr S_2$. The complement $\overline{\Gamma}$ of a graph Γ, is the graph with vertex set $V\Gamma$, and two vertices are adjacent in $\overline{\Gamma}$ if and only if they are not adjacent in Γ. Clearly, $\mathrm{Aut}\,\Gamma = \mathrm{Aut}\,\overline{\Gamma}$. Of particular interest to us is the complement graph $\overline{(2 \times m)}$−grid. The graph in Figure 1 is the $\overline{(2 \times 4)}$−grid. Observe that for $\Gamma = \overline{(2 \times m)}$−grid, we have $\mathrm{diam}\,\Gamma = 3$, and

$$c_1 = 1,\ a_1 = 0,\ b_1 = m - 2,\ c_2 = m - 2,\ a_2 = 0,\ b_2 = 1.$$

Condition 3.1. Let $m \geqslant 3$ and let $\pi : S_2 \times S_m \to S_2$ be the natural projection. We say that a subgroup G of $S_2 \times S_m$ satisfies Condition 3.1 if $G\pi = S_2$ and $G \cap S_m$ is a 2-transitive, but not 3-transitive subgroup of S_m.

FIGURE 1. The grid complement $\overline{(2 \times 4)-\mathsf{grid}}$; and on the right represented according to a distance-partition.

Lemma 3.2. *Let* $\Gamma = \overline{(2 \times m)-\mathsf{grid}}$ *with* $m \geqslant 4$, *and let* $G \leqslant \operatorname{Aut}\Gamma = S_2 \times S_m$. *Then* Γ *is* $(G, 2)$-*distance transitive, but not* $(G, 2)$-*arc transitive if and only if* G *satisfies Condition 3.1.*

Proof. Let $\Delta_1 = \{(1, i) \mid i = 1, 2, \ldots, m\}$ and $\Delta_2 = \{(2, i) \mid i = 1, 2, \ldots, m\}$ be the two biparts of $V\Gamma$. Let $u = (1, 1) \in \Delta_1$. Suppose first that Γ is $(G, 2)$-distance transitive, but not $(G, 2)$-arc transitive. Since G is transitive on $V\Gamma$, G projects onto S_2, that is, $G\pi = S_2$. Let $H = G \cap S_m$. Then $G_u = H_1$, $\Delta_2 = \Gamma(u) \cup \{(2, 1)\}$ and $\Gamma_2(u) = \Delta_1 \setminus \{u\}$. Since Γ is $(G, 2)$-distance transitive, $G_u = H_1$ is transitive on both $\Gamma(u)$ and $\Gamma_2(u)$. Hence H_1 is transitive on $\{2, \ldots, m\}$, and so H is a 2-transitive subgroup of S_m. Since Γ is not $(G, 2)$-arc transitive, $G_u = H_1$ is not 2-transitive on $\{2, 3, \ldots, m\}$, so H is not 3-transitive. Thus G satisfies Condition 3.1.

Conversely, suppose that G satisfies Condition 3.1. Then $H = G \cap S_m$ is transitive on Δ_1 and Δ_2, and G swaps these two sets. Thus G is transitive on $V\Gamma$. As H is a 2-transitive, but not 3-transitive subgroup of S_m, H_1 is transitive, but not 2-transitive on $\Gamma(u) = \{(2, i) \mid i = 2, \ldots, m\}$ and on $\Gamma_2(u) = \{(1, i) \mid i = 2, \ldots, m\}$. Hence Γ is $(G, 2)$-distance transitive, but not $(G, 2)$-arc transitive. $\square$

A list of 2-transitive, but not 3-transitive permutation groups can be found in [3, pp. 194-197].

Complete bipartite graphs appear frequently in this paper. Since $K_{m,n}$ with $m \neq n$ is not regular, we study $K_{m,m}$. The full automorphism group of $K_{m,m}$ is $S_m \wr S_2$, and this automorphism group acts 2-arc transitively on $K_{m,m}$. In the lemma below, we show that there is no 2-distance transitive action on $K_{m,m}$ which is not 2-arc transitive.

Lemma 3.3. *Let* $\Gamma \cong K_{m,m}$ *with* $m \geqslant 2$ *and let* $G \leqslant \operatorname{Aut}\Gamma$. *Then* Γ *is* $(G, 2)$-*distance transitive if and only if it is* $(G, 2)$-*arc transitive.*

Proof. If Γ is $(G, 2)$-arc transitive, then, by definition, it is $(G, 2)$-distance transitive. Conversely, suppose that Γ is $(G, 2)$-distance transitive with some $G \leqslant \operatorname{Aut}\Gamma$. Let $V\Gamma = \Delta_1 \cup \Delta_2$ be the bipartition of $V\Gamma$ where $\Delta_1 = \{(1, i) \mid i = 1, \ldots, m\}$ and $\Delta_2 = \{(2, i) \mid i = 1, \ldots, m\}$. The full automorphism group of Γ is $S_m \wr S_2$. Since $G \leqslant \operatorname{Aut}\Gamma$ is assumed to be vertex transitive, $G_{\Delta_1} = G_{\Delta_2}$ is transitive on both Δ_1 and Δ_2. Set $G_0 = G_{\Delta_1}$. Thus G_0 is a subdirect subgroup in $M^{(1)} \times M^{(2)}$ where $M^{(i)} \leqslant S_m$ and $M^{(i)}$ is the image of G_0 under the i-th coordinate projection $S_m \times S_m \to S_m$. Further, G projects onto S_2 under the natural projection $\operatorname{Aut}\Gamma \to$

S_2. If $x = (x_1, x_2)\sigma \in G$ with $x_i \in S_m$ and $\sigma = (1, 2) \in S_2$, then $(M^{(1)})^{x_1} = M^{(2)}$, and so $M^{(1)}$ and $M^{(2)}$ are conjugate subgroups of S_m. Hence possibly replacing G with its conjugate $G^{(x_1, 1)}$, we may assume without loss of generality that $M^{(1)} = M^{(2)} = M$.

Let $u = (1, 1) \in V\Gamma$. Then $\Gamma(u) = \Delta_2$ and $\Gamma_2(u) = \Delta_1 \setminus \{u\}$. Further, G_u stabilizes Δ_1, and hence $G_u \leqslant G_0$. Since Γ is $(G, 2)$-distance transitive, it follows that G_u is transitive on both Δ_2 and $\Delta_1 \setminus \{u\}$. Set $H = M_1$. Since $G_u \leqslant H \times M$, the stabilizer H must be transitive on $\{2, \ldots, m\}$, and hence M is a 2-transitive subgroup of S_m. In particular M contains a unique minimal normal subgroups N and this minimal normal subgroup is either elementary abelian or simple. Since N is transitive, we can write $M = NH$. We have that G_0 contains $1 \times N$ if and only if it contains $N \times 1$. Hence we need to consider two cases: the first is when G_0 contains $N \times N$ and the second is when it does not.

Suppose first that G_0 contains $N \times N$. In particular, $1 \times N \leqslant G_u$. For all $h_2 \in H$, there is some $n_1 h_1 \in M$ with $n_1 \in N$ and $h_1 \in H$ such that $(n_1 h_1, h_2) \in G_0$. Since $N \times 1 \leqslant G_0$, this implies that $(h_1, h_2) \in G_0$ and also $(h_1, h_2) \in G_u$. Thus G_u projects onto $NH = M$ by the second projection. Hence G_u is 2-transitive on $\Delta_2 = \Gamma(u)$, which shows that Γ is $(G, 2)$-arc transitive.

Suppose now that $N \times N$ is not contained in G_0. Since $G_0 \cap (1 \times M)$ is a normal subgroup of M and N is the unique minimal normal subgroup of M, we find that $G_0 \cap (1 \times M) = 1$ and, similarly, that $G_0 \cap (M \times 1) = 1$. Therefore G_0 is a diagonal subgroup; that is,

$$G_0 = \{(t, \alpha(t)) \mid t \in M\}$$

with some $\alpha \in \operatorname{Aut} M$. As H is the stabilizer of 1 in M, we have that $G_u = \{(t, \alpha(t)) \mid t \in H\}$. On the other hand, G_u is transitive on Δ_2, and hence $\alpha(H)$ is a transitive subgroup of M. Thus we obtain the factorization $M = H\alpha(H)$. The following possibilities are listed in [1, Theorem 1.1].

(a) Either M is affine and is isomorphic to $[(\mathbb{F}_2)^3 \rtimes \operatorname{PSL}(3, 2)] \wr X$ where X is a transitive permutation group;
(b) or $\operatorname{Soc} M \cong \operatorname{P\Omega}_8^+(q)$, $\operatorname{Sp}(4, q)$ (q even with $q \geqslant 4$), A_6, M_{12}.

In case (a), if $X \neq 1$, then M is contained in a wreath product in product action, and such a wreath product is never 2-transitive. Thus $X = 1$, $m = 8$, $M = (\mathbb{F}_2)^3 \rtimes \operatorname{PSL}(3, 2)$, and $G_u \cong \operatorname{PSL}(3, 2)$ acting transitively on Δ_2. However, this transitive action of $\operatorname{PSL}(3, 2)$ is 2-transitive, which gives that Γ is $(G, 2)$-arc transitive.

In case (b), inspecting the list of almost simple 2-transitive groups in [3], we find that there are no 2-transitive groups with socle $\operatorname{P\Omega}_8^+(q)$ or $\operatorname{Sp}(4, q)$ with q even and $q \geqslant 4$. Hence $\operatorname{Soc} M = A_6$ or M_{12}. Then G_u is either A_5, S_5 or M_{11} acting transitively on Δ_2. These actions are all 2-transitive, which implies that Γ is $(G, 2)$-arc transitive. $\qquad\square$

3.2. **Hamming graphs and platonic solids.** For d, $q \geqslant 2$, the vertex set of the *Hamming graph* $\operatorname{H}(d, q)$ is the set $\{1, \ldots, q\}^d$ and two vertices $u = (\alpha_1, \ldots, \alpha_d)$ and $v = (\beta_1, \ldots, \beta_d)$ are adjacent if and only if their Hamming distance is one; that is, they differ in precisely one coordinate. The Hamming graph has diameter d and has girth 4 when $q = 2$ and girth 3 when $q > 2$. The wreath product $W = S_q \wr S_d$ is the full automorphism group of Γ, acting distance transitively, see [2, Section 9.2]. The Hamming graphs are well studied, due in part to their applications to coding theory. Hamming graphs arise in two cases of our research. The first case

is the cube $\Gamma = H(3,2)$. The standard construction of the cube graph is precisely the same as for the Hamming graphs with $d = 3$ and $q = 2$, and so this graph is the 'standard' cube with 8 vertices (the cube $H(3,2)$ is also isomorphic to the grid complement $\overline{(2 \times 4) - \mathsf{grid}}$). The second case is $\Gamma = H(d,2)$ when $d > 2$; see Lemma 4.3.

Some platonic solids (cube, octahedron and icosahedron) appear in some form in our investigation. The cube appears as the $\overline{(2 \times 4) - \mathsf{grid}}$. We discuss in more detail the octahedron and the icosahedron. The octahedron (see Figure 2) has 6 vertices and diameter 2. Its automorphism group $S_2 \wr S_3$ acts imprimitively preserving the partition of vertices into antipodal pairs. We denote by π the natural projection $S_2 \wr S_3 \to S_3$.

FIGURE 2. The octahedron, displayed according to its distance-partition.

Lemma 3.4. *Let Γ be the octahedron, and let $G \leqslant \operatorname{Aut}\Gamma$. Then Γ is not $(G,2)$-arc transitive. Further, Γ is $(G,2)$-distance transitive, if and only if either $G = S_2 \wr S_3$, or G is an index 2 subgroup of $S_2 \wr S_3$ and $G\pi = S_3$.*

Proof. Since Γ is non-complete of girth 3, Γ is not $(G,2)$-arc transitive. Now assume that Γ is $(G,2)$-distance transitive. Let $u = a$ be the vertex in the graph of Figure 2. Since G_u is transitive on $\Gamma(u)$ and $|\Gamma(u)| = 4$, $|G_u|$ is divisible by 4. Further, $|G : G_u| = 6$, and so $|G|$ is divisible by 24. Suppose that G is a proper subgroup of $\operatorname{Aut}\Gamma = S_2 \wr S_3$. Then $|G| = 24$. As $|\operatorname{Aut}\Gamma| = 48$, G is an index 2 subgroup of $S_2 \wr S_3$. The three antipodal blocks of $V\Gamma$ in the graph of Figure 2 are $\Delta_1 = \{a, a'\}$, $\Delta_2 = \{b, b'\}$ and $\Delta_3 = \{c, c'\}$. Since G is transitive on $V\Gamma$, G is transitive on the three antipodal blocks. Thus the image $G\pi$ of G in S_3 is $\mathbb{Z}_3$ or S_3. Assume $G\pi = \mathbb{Z}_3$. Then G_u acts on the three antipodal blocks trivially. Hence G_u does not map Δ_2 to Δ_3, contradicting that G_u is transitive on $\Gamma(u)$. Therefore $G\pi = S_3$. Simple calculation shows that the conditions stated in the lemma are sufficient for 2-distance transitivity. $\square$

The icosahedron has automorphism group $S_2 \times A_5$ acting arc transitively.

Lemma 3.5. *Let Γ be the icosahedron, and let $G \leqslant \operatorname{Aut}\Gamma$. The graph Γ is $(G,2)$-distance transitive if and only if $G = S_2 \times A_5$ or $G = A_5$. In particular, Γ is not $(G,2)$-arc transitive.*

Proof. By [6, Theorem 1.5], $\operatorname{Aut}\Gamma \cong S_2 \times A_5$. It is easy to see that for $G \in \{S_2 \times A_5, A_5\}$, Γ is $(G,2)$-distance transitive. Suppose that Γ is $(G,2)$-distance transitive. Then G is transitive on $V\Gamma$ and G_u is transitive on $\Gamma(u)$, and so $12 = |V\Gamma|$ divides

FIGURE 3. The icosahedron, displayed according to its distance-partition.

$|G|$ and $|\Gamma(u)| = 5$ divides $|G_u|$. Thus 60 divides $|G|$. Since $G \leqslant \operatorname{Aut} \Gamma \cong S_2 \times A_5$, it follows that $G = S_2 \times A_5$ or $G = A_5$. Finally, as Γ is a non-complete graph of girth 3, Γ is not $(G, 2)$-arc transitive. $\qquad \square$

4. GRAPHS OF GIRTH 4

By the assertion of Lemma 2.2, to study the family of $(G, 2)$-distance transitive, but not $(G, 2)$-arc transitive graphs, we only need to consider the graphs with girth 3 or 4. This section is devoted to the girth 4 case, and the structure of such graphs depends strongly upon the value of the constant c_2 as in Definition 2.1. We begin with a simple combinatorial result:

Lemma 4.1. *Let Γ be a $(G, 2)$-distance transitive graph with valency k and girth at least 4. Let $u \in V\Gamma$. Then there are $k(k-1)$ edges between $\Gamma(u)$ and $\Gamma_2(u)$, and $k(k-1) = c_2|\Gamma_2(u)|$.*

Proof. Consider a vertex $v \in \Gamma(u)$. Since Γ has girth more than 3, all of the neighbors of v, except for u, lie in $\Gamma_2(u)$. Thus, there are $k - 1$ edges from v to $\Gamma_2(u)$. Since there are k such vertices v, there are $k(k-1)$ edges between $\Gamma(u)$ and $\Gamma_2(u)$. As Γ is $(G, 2)$-distance transitive, the equation $k(k-1) = c_2|\Gamma_2(u)|$ follows by counting the same quantity from the other side: each vertex in $\Gamma_2(u)$ is incident with exactly c_2 edges between $\Gamma_2(u)$ and $\Gamma(u)$. $\qquad \square$

For a vertex $u \in V\Gamma$, we denote by $G_u^{\Gamma_i(u)}$ the permutation group induced by G_u on $\Gamma_i(u)$.

Lemma 4.2. *Let Γ be a $(G, 2)$-distance transitive, but not $(G, 2)$-arc transitive graph with valency k and girth 4, and suppose that $c_2 = 2$. Then G_u acts 2-homogeneously, but not 2-transitively on $\Gamma(u)$ for each $u \in V\Gamma$. Further, $k = p^e \equiv 3$ (mod 4) where p is a prime.*

Proof. Since $c_2 = 2$, each vertex $w \in \Gamma_2(u)$ uniquely determines a 2-subset in $\Gamma(u)$, namely the intersection $\Gamma(w) \cap \Gamma(u)$. We claim that the map $\psi : w \mapsto \Gamma(w) \cap \Gamma(u)$ is a bijection between $\Gamma_2(u)$ and the set of 2-subsets of $\Gamma(u)$. Suppose that $\psi(w_1) = \psi(w_2) = \{v_1, v_2\}$. Then $u, w_1, w_2 \in \Gamma(v_1)$ and $v_2 \in \Gamma_2(v_1)$. On the other hand, as v_2 is adjacent to u, w_1, w_2, there are three edges from v_2 to $\Gamma(v_1)$, which is

impossible, as $c_2 = 2$. Hence ψ is injective. Since Γ has girth 4, it follows from Lemma 4.1 that $|\Gamma_2(u)| = k(k-1)/2 = \binom{k}{2}$, and so the map ψ is a bijection. Hence G_u is transitive on $\Gamma_2(u)$ if and only if it is transitive on the set of 2-subsets in $\Gamma(u)$, that is, $G_u^{\Gamma(u)}$ acts 2-homogeneously on $\Gamma(u)$. Since Γ is not $(G, 2)$-arc transitive, $G_u^{\Gamma(u)}$ is not 2-transitive on $\Gamma(u)$. Thus by Lemma 2.3, $k = p^e \equiv 3 \pmod 4$ where p is a prime. $\qquad \square$

In the following lemma we characterize $(G, 2)$-distance transitive, but not $(G, 2)$-arc transitive Hamming graphs over an alphabet of size 2.

Lemma 4.3. *Let $\Gamma = \mathrm{H}(d, 2)$ with $d > 2$, and let $G \leqslant \operatorname{Aut} \Gamma \cong S_2 \wr S_d$. Then Γ is $(G, 2)$-distance transitive, but not $(G, 2)$-arc transitive if and only if $G = S_2 \wr H$ where H is a 2-homogeneous, but not 2-transitive subgroup of S_d. Further, in this case, $d = p^e \equiv 3 \pmod 4$.*

Proof. By [2, p. 222], Γ is $\operatorname{Aut} \Gamma$-distance transitive of girth 4, valency d, and $c_2 = 2$. Assume that the action of G on Γ is 2-distance transitive, but not 2-arc transitive. Then by Lemma 4.2, G_u is 2-homogeneous, but not 2-transitive on $\Gamma(u)$, for all u. Further, $d = p^e \equiv 3 \pmod 4$ where p is a prime. Let $A = \operatorname{Aut} \Gamma = M \rtimes S_d$ where $M = (S_2)^d$. Let u be the vertex $(1, \ldots, 1)$ and set $H = G_u$. If $g \in G$, then $g = mh$ where $m \in M$ and $h \in S_d$, and so $h \in H$. Hence $G \leqslant MH$. Then, by Dedekind's Modular Law, $(G \cap M)H = G \cap (MH) = G$. Thus $G \cap M$ is a transitive subgroup of G. Since M is regular, $G \cap M = M$, and so $M \leqslant G$. Thus $G = M \rtimes H = S_2 \wr H$. As the action of H on $\Gamma(u)$ is faithful, $H = G_u^{\Gamma(u)}$.

Conversely, assume that $G = S_2 \wr H$ and H is a 2-homogeneous, but not 2-transitive subgroup of S_d. Then G is transitive on $V\Gamma$. Since $G_u^{\Gamma(u)} = G_u = H$, $G_u^{\Gamma(u)}$ acts 2-homogeneously, but not 2-transitively on $\Gamma(u)$ for each $u \in V\Gamma$. Hence Γ is not $(G, 2)$-arc transitive and $G_u^{\Gamma(u)}$ is transitive on the set of 2-subsets of $\Gamma(u)$. Since Γ has girth 4 and $c_2 = 2$, we can construct a one-to-one correspondence between the 2-subsets of $\Gamma(u)$ and vertices of $\Gamma_2(u)$ as in the proof of Lemma 4.2. Thus G_u is transitive on $\Gamma_2(u)$, so Γ is $(G, 2)$-distance transitive. $\qquad \square$

We have treated the case where $c_2 = 2$. When c_2 is 'large' (that is, close to the valency) we can say a lot about the structure of Γ.

Lemma 4.4. *If Γ is a connected $(G, 2)$-distance transitive graph with valency k and girth 4, then the following are valid.*

(i) *If $c_2 = k$, then $\Gamma = \mathrm{K}_{k,k}$.*
(ii) *If $k \geqslant 3$ and $c_2 = k - 1$, then $\Gamma = \overline{(2 \times (k+1))-\mathsf{grid}}$.*

Proof. (i) Let (u, v, w) be a 2-arc. Since Γ has girth 4, u and w are nonadjacent, so w has k neighbors in $\Gamma(u)$, as $c_2 = k$. Since the valency of Γ is k, this forces $\Gamma(u) = \Gamma(w)$. By the $(G, 2)$-distance transitivity of Γ, every vertex in $\Gamma_2(u)$ has all its neighbors in $\Gamma(u)$, and this implies that $\Gamma_3(u)$ is empty and there are no edges in $\Gamma_2(u)$. Thus Γ is a bipartite graph and the two biparts are $\Gamma(u)$ and $\{u\} \cup \Gamma_2(u)$. Every edge between the two biparts is present, so Γ is a complete bipartite graph. Since Γ is regular of valency k, we have $\Gamma = \mathrm{K}_{k,k}$.

(ii) Let (u, v, w) be a 2-arc. Since Γ has girth 4 and $c_2 = k - 1$, by Lemma 4.1, we have $|\Gamma_2(u)| = k$. Let w' be the unique vertex in $\Gamma_2(u)$ that is not adjacent to v. Assume that the induced subgraph $[\Gamma_2(u)]$ contains an edge. As G_u is transitive on

$\Gamma_2(u)$, every vertex of $\Gamma_2(u)$ is adjacent to some vertex of $\Gamma_2(u)$. Since Γ has girth 4, the $k - 1$ vertices in $\Gamma_2(u) \cap \Gamma(v)$ are pairwise nonadjacent, so every vertex of $\Gamma_2(u) \cap \Gamma(v)$ is adjacent to w', which is impossible, as $|\Gamma(u) \cap \Gamma(w')| = k - 1$. Thus there are no edges in $[\Gamma_2(u)]$. Thus each vertex in $\Gamma_2(u)$ is adjacent to a unique vertex in $\Gamma_3(u)$.

Let $z \in \Gamma_3(u) \cap \Gamma(w)$. Since $c_2 = k - 1$, every pair of vertices at distance 2 have $k - 1$ common neighbors, so $|\Gamma(v) \cap \Gamma(z)| = k - 1$. Hence z is adjacent to all vertices of $\Gamma_2(u)$ that are adjacent to v. If for all $v' \in \Gamma(u)$, $\Gamma_2(u) \cap \Gamma(v) = \Gamma_2(u) \cap \Gamma(v')$, then $|\Gamma_2(u)| = k - 1$, which is a contradiction. Thus $\Gamma(u)$ contains a vertex v' such that $\Gamma_2(u) \cap \Gamma(v) \neq \Gamma_2(u) \cap \Gamma(v')$. In particular, $\Gamma_2(u) = \Gamma_2(u) \cap (\Gamma(v) \cup \Gamma(v'))$. Now v' and z must have a common neighbor in $\Gamma_2(u)$, and so v' and z are at distance 2. Thus, as $c_2 = k - 1$, z is adjacent to all vertices of $\Gamma_2(u)$ that are adjacent to v'. Thus z is adjacent to all vertices of $\Gamma_2(u)$. Since $|\Gamma_2(u)| = k$, we find that there are no more vertices in Γ. Therefore, we have determined Γ completely, and $\Gamma = \overline{(2 \times (k + 1))-\mathsf{grid}}$. $\qquad\square$

5. Proof of Main Results

We first prove Theorem 1.1.

Proof of Theorem 1.1. Since Γ has girth 4, it follows that $2 \leqslant c_2 \leqslant k$. If $c_2 = k$, then, by Lemma 4.4, $\Gamma = \mathrm{K}_{k,k}$. However, by Lemma 3.3, Γ is $(G, 2)$-arc transitive, whenever it is $(G, 2)$-distance transitive, and hence this case cannot arise. Thus $2 \leqslant c_2 \leqslant k - 1$. Statement (i) now follows from Lemmas 4.4(ii) and 3.2, while statement (ii) follows from Lemma 4.2 $\qquad\square$

Next we prove Corollary 1.2.

Proof of Corollary 1.2. If $p = 2$, then Γ is a cycle graph, so Γ is $(G, 2)$-distance transitive if and only if it is $(G, 2)$-arc transitive, which is a contradiction. Thus $p \geqslant 3$. Then by Theorem 1.1, either $\Gamma \cong \overline{(2 \times (p + 1))-\mathsf{grid}}$, or $2 \leqslant c_2 \leqslant p - 2$. Assume that $2 \leqslant c_2 \leqslant p - 2$. It follows from Lemma 4.1 that $p(p - 1) = c_2|\Gamma_2(u)|$. Since $2 \leqslant c_2 \leqslant p - 2$, p and c_2 are coprime, so c_2 divides $p - 1$. As $c_2 < p - 1$, we get $2 \leqslant c_2 \leqslant (p - 1)/2$ and this proves (i). Statement (ii) follows from Theorem 1.1(ii). Assume that $c_2 = (p - 1)/2$. By Lemma 4.1, $|\Gamma_2(u)| = 2p$. If G_u were primitive on $\Gamma_2(u)$, then by Lemma 2.4, we would have, $p = 5$, and hence $c_2 = 2$. However, In this case $p \equiv 3 \pmod 4$, which is a contradiction. Thus G_u is imprimitive on $\Gamma_2(u)$ and this shows (iii). $\qquad\square$

One can form an infinite family of examples that satisfy the conditions of Corollary 1.2 from Hamming graphs $\mathrm{H}(p, 2)$ using Lemma 4.3.

In the following, we prove Theorem 1.3, that is, we determine all $(G, 2)$-distance transitive, but not $(G, 2)$-arc transitive graphs of valency at most 5. We split the proof into two parts, as we consider the girth 4 and 3 cases separately in Propositions 5.1 and 5.2, respectively.

Proposition 5.1. *Let Γ be a connected $(G, 2)$-distance transitive, but not $(G, 2)$-arc transitive graph of girth 4 and valency $k \in \{3, 4, 5\}$. Then $\Gamma \cong \overline{(2 \times k + 1)-\mathsf{grid}}$, and G satisfies Condition 3.1.*

Proof. We claim that $c_2 = k - 1$ in all cases. By Theorem 1.1, $c_2 \leqslant k - 1$. If $k = 3$, then $c_2 \geqslant 2 = k - 1$ follows from the girth condition, and so $c_2 = k - 1$. If $k \in \{4, 5\}$

and $c_2 \leqslant k - 2$, then we must have that $c_2 = 2$ (use Corollary 1.2 for $k = 5$). Hence, by Lemma 4.2, $k \equiv 3 \pmod 4$: a contradiction, as $k \in \{4, 5\}$. Now the rest follows from Theorem 1.1(i). □

Proposition 5.2. *Let Γ be a connected $(G, 2)$-distance transitive graph of girth 3 and valency 4 or 5, and let $u \in V\Gamma$. Then one of the following is valid.*

(i) *Γ is the octahedron and either $G = S_2 \wr S_3$ or G is an index 2 subgroup of $S_2 \wr S_3$ and G projects onto S_3;*

(ii) *$\Gamma \cong \mathrm{H}(2, 3)$ and either $G = S_3 \wr S_2$ or G is an index 2 subgroup of $S_3 \wr S_2$ and G projects onto S_2;*

(iii) *$|\Gamma_2(u)| = 8$ and Γ is the line graph of a connected cubic $(G, 3)$-arc transitive graph;*

(iv) *Γ is the icosahedron and $G = A_5$ or $A_5 \times S_2$.*

In cases (i)–(iii), the valency of Γ is 4, while in case (iv), the valency is 5.

Proof. Suppose first that the valency is 4. Since Γ is $(G, 2)$-distance transitive of valency 4 and girth 3, it follows that the induced graph $[\Gamma(u)]$ is a vertex transitive graph with 4 vertices of valency k where $1 \leqslant k \leqslant 3$. If $[\Gamma(u)]$ has valency 3, then $[\Gamma(u)]$ is complete, and so Γ is complete, which is a contradiction. If $[\Gamma(u)]$ has valency 2, then $[\Gamma(u)] \cong C_4$. Hence $|\Gamma_2(u) \cap \Gamma(v)| = 1$ for any arc (u, v), so $G_{u,v}$ is transitive on $\Gamma_2(u) \cap \Gamma(v)$, that is, Γ is $(G, 2)$-geodesic transitive. Thus by [5, Corollary 1.4], Γ is the octahedron. It follows from Lemma 3.4 that either $G = S_2 \wr S_3$, or G is an index 2 subgroup of $S_2 \wr S_3$ and G projects onto S_3. Hence, case (i) is valid.

Now suppose that $[\Gamma(u)]$ has valency 1. Then $[\Gamma(u)] \cong 2\,\mathrm{K}_2$ and there are 8 edges between $\Gamma(u)$ and $\Gamma_2(u)$. Further, each arc lies in a unique triangle. Let $\Gamma(u) = \{v_1, v_2, v_3, v_4\}$ be such that (v_1, v_2) and (v_3, v_4) are two arcs. Then $|\Gamma_2(u) \cap \Gamma(v_1)| = 2$, say $\Gamma_2(u) \cap \Gamma(v_1) = \{w_1, w_2\}$. Since $[\Gamma(v_1)] \cong 2\,\mathrm{K}_2$, it follows that v_2 is adjacent to neither w_1 nor w_2. As $|\Gamma_2(u) \cap \Gamma(v_2)| = 2$, we have $|\Gamma_2(u)| \geqslant 4$. Since there are 8 edges between $\Gamma(u)$ and $\Gamma_2(u)$ and since G_u is transitive on $\Gamma_2(u)$, we obtain that $8 \mid |\Gamma_2(u)|$, and so $|\Gamma_2(u)| \in \{4, 8\}$.

Suppose first that $|\Gamma_2(u)| = 4$. As noted above, v_2 is not adjacent to w_1 or w_2. Set $\Gamma_2(u) \cap \Gamma(v_2) = \{w_3, w_4\}$. Then $\Gamma_2(u) = \{w_1, w_2, w_3, w_4\}$. Since $[\Gamma(v_1)] \cong [\Gamma(v_2)] \cong 2\,\mathrm{K}_2$, it follows that w_1, w_2 are adjacent and, similarly, w_3, w_4 are adjacent. Since $|\Gamma_2(u)| = 4$ and there are 8 edges between $\Gamma(u)$ and $\Gamma_2(u)$, we must have $|\Gamma(u) \cap \Gamma(w_i)| = 2$. Since v_2, w_1 are nonadjacent, w_1 is adjacent either to v_3 or to v_4, say v_3. Then $\Gamma(u) \cap \Gamma(w_1) = \{v_1, v_3\}$. As each arc lies in a unique triangle and (v_1, w_1, w_2) is a triangle, it follows that v_3 is not adjacent to w_2. Hence v_3 is adjacent to either w_3 or w_4, say w_3. Then $\Gamma(v_3) = \{u, v_4, w_1, w_3\}$. Since $[\Gamma(v_3)] \cong 2\,\mathrm{K}_2$ and u, v_4 are adjacent, it follows that w_1, w_3 are adjacent. Thus, $\Gamma(w_1) = \{v_1, w_2, v_3, w_3\}$. Finally, as $|\Gamma_2(u) \cap \Gamma(v_4)| = 2$ and v_4 is adjacent to neither w_1 nor w_3, v_4 is adjacent to both w_2 and w_4. Since $[\Gamma(v_4)] \cong 2\,\mathrm{K}_2$ and (v_3, u, v_4) is a triangle, it follows that w_2, w_4 are adjacent. Now, the graph Γ is completely determined and $\Gamma \cong \mathrm{H}(2, 3)$. By [2, Theorem 9.2.1], Γ is $(\mathrm{Aut}\,\Gamma, 2)$-distance transitive where $\mathrm{Aut}\,\Gamma \cong S_3 \wr S_2$. Suppose that G is a proper subgroup of $\mathrm{Aut}\,\Gamma$. Since G_u is transitive on $\Gamma(u)$ and $|\Gamma(u)| = 4$, $|G_u|$ is divisible by 4, so $|G|$ is divisible by $4|V\Gamma| = 36$. It follows that $|G| = 36$, so G is an index 2 subgroup of $S_3 \wr S_2$. Finally, as G_u is transitive on $\Gamma(u)$, G_u projects onto S_2. Thus (ii) is valid.

Let us now consider the case when $|\Gamma_2(u)| = 8$. Then for each $z \in \Gamma_2(u)$, there is a unique 2-geodesic between u and z. Hence there is a one-to-one correspondence between the set of 2-geodesics starting from u and the set of vertices in $\Gamma_2(u)$. Since G_u is transitive on $\Gamma_2(u)$, it follows that G_u is transitive on the set of 2-geodesics starting from u, so Γ is $(G, 2)$-geodesic transitive. Therefore by Lemma 2.5, Γ is the line graph of a connected cubic $(G, 3)$-arc transitive graph. Therefore (iii) is valid.

Assume now that the valency is 5. Let (u, v) be an arc. Since Γ is G-arc transitive, the induced subgraph $[\Gamma(u)]$ is vertex transitive. As Γ has girth 3 and non-complete, the valency k of $[\Gamma(u)]$ is at most 3. Since $[\Gamma(u)]$ is undirected, it follows that $[\Gamma(u)]$ has $5k/2$ edges, and so k is even; that is, $k = 2$. Thus $[\Gamma(u)] \cong C_5$.

Set $\Gamma(u) = \{v_1, v_2, v_3, v_4, v_5\}$ with $v_1 = v$ and assume $(v_1, \ldots, v_5)$ is a 5-cycle. Then $|\Gamma_2(u) \cap \Gamma(v_1)| = 2$ and say $\Gamma_2(u) \cap \Gamma(v_1) = \{w_1, w_2\}$. Then $\Gamma(v_1) = \{u, v_2, v_5, w_1, w_2\}$. As $[\Gamma(v_1)] \cong C_5$ and (v_2, u, v_5) is a 2-arc, it follows that w_1, w_2 are adjacent, v_2 is adjacent to one of w_1 and w_2 and v_5 is adjacent to the other. Without loss of generality, assume v_2 is adjacent to w_1 and v_5 is adjacent to w_2. In particular, v_2 and w_2 are not adjacent. Moreover, $2 \leqslant c_2 \leqslant 4$. Since there are 10 edges between $\Gamma(u)$ and $\Gamma_2(u)$, we have $10 = c_2|\Gamma_2(u)|$, so $c_2 = 2$ and $|\Gamma_2(u)| = 5$.

Since $|\Gamma_2(u) \cap \Gamma(v_2)| = 2$, there exists w_3 in $\Gamma_2(u)$ which is adjacent to v_2, and so $\Gamma(v_2) = \{u, v_1, v_3, w_1, w_3\}$. Note that (w_1, v_1, u, v_3) is a 3-arc, and as $[\Gamma(v_2)] \cong C_5$, it follows that w_3 is adjacent to both v_3 and w_1. Since G_u is transitive on $\Gamma_2(u)$, $[\Gamma_2(u)]$ is a vertex transitive graph. Recall that w_1 is adjacent to w_2 and w_3. It follows that $[\Gamma_2(u)] \cong C_5$. Thus $|\Gamma_3(u) \cap \Gamma(w_1)| = 1$, say $\Gamma_3(u) \cap \Gamma(w_1) = \{e\}$. Then (v_1, w_1, e) and (v_2, w_1, e) are two 2-geodesics. As $c_2 = 2$, $|\Gamma(v_1) \cap \Gamma(e)| = |\Gamma(v_2) \cap \Gamma(e)| = 2$. Hence $\{w_1, w_2, w_3\} \subseteq \Gamma_2(u) \cap \Gamma(e)$.

Since $|\Gamma_2(u) \cap \Gamma(v_3)| = 2$, there exists $w_4 (\neq w_3) \in \Gamma_2(u)$ such that v_3, w_4 are adjacent. Noting that $\Gamma(u) \cap \Gamma(w_1) = \{v_1, v_2\}$ and $\Gamma(u) \cap \Gamma(w_2) = \{v_1, v_5\}$, we find $w_4 \notin \{w_1, w_2, w_3\}$. Since $[\Gamma(v_3)] \cong C_5$ and (w_3, v_2, u, v_4) is a 3-arc, it follows that w_4 is adjacent to both v_4 and w_3. As (v_3, w_3, e) is a 2-geodesic, $|\Gamma(v_3) \cap \Gamma(e)| = 2$, so $w_4 \in \Gamma_2(u) \cap \Gamma(e)$. Now (v_4, w_4, e) is a 2-geodesic, so $|\Gamma(v_4) \cap \Gamma(e)| = 2$, hence $\Gamma_2(u) \cap \Gamma(v_4) \subset \Gamma(e)$. Let the remaining vertex of $\Gamma_2(u)$ be w_5. Since $|\Gamma(u) \cap \Gamma(w_5)| = 2$, it follows that w_5 is adjacent to both v_4, v_5. Hence $\Gamma_2(u) \cap \Gamma(v_4) = \{w_4, w_5\} \subset \Gamma(e)$. Thus $\Gamma_2(u) = \Gamma(e)$, so $\Gamma_3(u) = \{e\}$. Now we have completely determined the graph Γ, and this graph is the icosahedron. Finally, by Lemma 3.5, $G \cong S_2 \times A_5$ or A_5. $\square$

The proof of Theorem 1.3. If the valency of Γ is 2 or the girth is greater than 4, then Γ cannot be $(G, 2)$-distance transitive, but not $(G, 2)$-arc transitive. Hence the valency is at least 3. If the valency and the girth are both equal to 3, then $\Gamma = K_4$. Hence Theorem 1.3 follows from Proposition 5.1, in the case of girth 4, and from Proposition 5.2 in the case of girth 4. $\square$

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