# On the size-Ramsey number of hypergraphs

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**Abstract**

The size-Ramsey number of a graph $G$ is the minimum number of edges in a graph $H$ such that every 2-edge-coloring of $H$ yields a monochromatic copy of $G$. Size-Ramsey numbers of graphs have been studied for almost 40 years with particular focus on the case of trees and bounded degree graphs.

We initiate the study of size-Ramsey numbers for $k$-uniform hypergraphs. Analogous to the graph case, we consider the size-Ramsey number of cliques, paths, trees, and bounded degree hypergraphs. Our results suggest that size-Ramsey numbers for hypergraphs are extremely difficult to determine, and many open problems remain.

## 1 Introduction

Given graphs $G$ and $H$, say that $H \to G$ if every 2-edge-coloring of $H$ results in a monochromatic copy of $G$ in $H$. Using this notation, the Ramsey number $R(G)$ of $G$ is the minimum $n$ such that $K_n \to G$. Instead of minimizing the number of vertices, one can minimize the number of edges. Define the *size-Ramsey number* $\hat{R}(G)$ of $G$ to be the minimum number of edges in a graph $H$ such that $H \to G$. More formally,

$$\hat{R}(G) = \min\{|E(H)| : H \to G\}.$$

The study of size-Ramsey numbers was proposed by Erdős, Faudree, Rousseau and Schelp [5] in 1978. By definition of $R(G)$, we have $K_{R(G)} \to G$. Since the complete graph on $R(G)$ vertices has $\binom{R(G)}{2}$ edges, we obtain the trivial bound

$$\hat{R}(G) \le \binom{R(G)}{2}. \tag{1}$$

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Chvátal (see, *e.g.*, [5]) showed that equality holds in (1) for complete graphs. In other words,

$$\hat{R}(K_n) = \binom{R(K_n)}{2}. \tag{2}$$

One of the first problems in this area was to determine the size-Ramsey number of the $n$ vertex path $P_n$. Answering a question of Erdős [4], Beck [1] showed that

$$\hat{R}(P_n) = O(n). \tag{3}$$

Since $\hat{R}(G) \geq |E(G)|$ for any graph, Beck's result is sharp in order of magnitude. The linearity of the size-Ramsey number of paths was generalized to bounded degree trees by Friedman and Pippenger [11] and to cycles by Haxell, Kohayakawa and Łuczak [12]. Beck [2] asked whether $\hat{R}(G)$ is always linear in the size of $G$ for graphs $G$ of bounded degree. This was settled in the negative by Rödl and Szemerédi [18], who proved that there are graphs of order $n$, maximum degree 3, and size-Ramsey number $\Omega(n(\log n)^{1/60})$. They also conjectured that for a fixed integer $\Delta$ there is an $\varepsilon > 0$ such that

$$\Omega(n^{1+\varepsilon}) = \max_G \hat{R}(G) = O(n^{2-\varepsilon}),$$

where the maximum is taken over all graphs $G$ of order $n$ with maximum degree at most $\Delta$. The upper bound was recently proved by Kohayakawa, Rödl, Schacht, and Szemerédi [15]. For further results about the size-Ramsey number see, *e.g*, the survey paper of Faudree and Schelp [8].

Somewhat surprisingly the size-Ramsey numbers have not been studied for hypergraphs, even though classical Ramsey numbers for hypergraphs have been studied extensively since the 1950's (see, e.g., [7, 6]), and more recently [3]. In this paper we initiate this study for $k$-uniform hypergraphs. A *$k$-uniform hypergraph* $\mathcal{G}$ (*$k$-graph* for short) on a vertex set $V(\mathcal{G})$ is a family of $k$-element subsets (called edges) of $V(\mathcal{G})$. We write $E(\mathcal{G})$ for its edge set. Given $k$-graphs $\mathcal{G}$ and $\mathcal{H}$, say that $\mathcal{H} \to \mathcal{G}$ if every 2-edge-coloring of $\mathcal{H}$ results in a monochromatic copy of $\mathcal{G}$ in $\mathcal{H}$. Define the *size-Ramsey number* $\hat{R}(\mathcal{G})$ of a $k$-graph $\mathcal{G}$ as

$$\hat{R}(\mathcal{G}) = \min\{|E(\mathcal{H})| : \mathcal{H} \to \mathcal{G}\}.$$

## 2 Results and open problems

Motivated by extending the basic theory from graphs to hypergraphs, we prove results for cliques, trees, paths, and bounded degree hypergraphs.

### 2.1 Cliques

For every $k$-graph $\mathcal{G}$, we trivially have

$$\hat{R}(\mathcal{G}) \leq \binom{R(\mathcal{G})}{k},$$

where $R(\mathcal{G})$ is the ordinary Ramsey number of $\mathcal{G}$. Our first objective was to generalize (2) to 3-graphs, which shows that equality holds for graphs. It is fairly easy to obtain a lower bound for $\hat{R}(\mathcal{K}_n^{(3)})$ that is quadratic in $R(\mathcal{K}_n^{(3)})$, but we were only able to do slightly better.

**Theorem 2.1** $\hat{R}(\mathcal{K}_n^{(3)}) \geq \frac{n^2}{96}\binom{R(\mathcal{K}_n^{(3)})}{2}$.

The following basic questions remain open.

**Question 2.2** *Is* $\hat{R}(\mathcal{K}_n^{(k)}) = \binom{R(\mathcal{K}_n^{(k)})}{k}$*?*

**Question 2.3** *For* $k \geq 3$ *let* $N = R(\mathcal{K}_n^{(k)})$*. Define* $\mathcal{K}_N^{(k)-}$ *to be the hypergraph obtained from* $\mathcal{K}_N^{(k)}$ *by removing one edge. Is it true that* $\mathcal{K}_N^{(k)-} \to \mathcal{K}_n^{(k)}$*?*

Clearly, the affirmative answer to the latter gives a negative answer to Question 2.2.

### 2.2 Trees

Given integers $1 \leq \ell < k$ and $n$, a $k$-graph $\mathcal{T}_{n,\ell}^{(k)}$ of order $n$ with edge set $\{e_1, \ldots, e_m\}$ is an *$\ell$-tree*, if for each $2 \leq j \leq m$ we have $|e_j \cap \bigcup_{1 \leq i < j} e_i| \leq \ell$ and $e_j \cap \bigcup_{1 \leq i < j} e_i \subseteq e_{i_0}$ for some $1 \leq i_0 < j$. We are able to give the following general upper bound for trees.

**Theorem 2.4** *Let* $1 \leq \ell < k$ *be fixed integers. Then*

$$\hat{R}(\mathcal{T}_{n,\ell}^{(k)}) = O(n^{\ell+1}).$$

One can easily show that this bound is tight in order of magnitude when $\ell = 1$ (see Section 4 for details). The situation for $\ell \geq 2$ is much less clear.

**Question 2.5** *Let* $2 \leq \ell < k$ *be fixed integers. Is it true that for every* $n$ *there exists a* $k$*-uniform* $\ell$*-tree* $\mathcal{T}$ *of order at most* $n$ *such that*

$$\hat{R}(\mathcal{T}) = \Omega(n^{\ell+1}).$$

Here is another related question pointed out by Fox [9]. Let us weaken the restriction on the edge intersection in the definition of $\mathcal{T}_{n,\ell}^{(k)}$. Let $\bar{\mathcal{T}}_{n,\ell}^{(k)}$ be a $k$-graph of order $n$ with edge set $\{e_1, \ldots, e_m\}$ such that for each $2 \leq j \leq m$ we have $|e_j \cap \bigcup_{1 \leq i < j} e_i| \leq \ell$.

**Question 2.6** *Let* $2 \leq \ell < k$ *be fixed integers. Is* $\hat{R}(\bar{\mathcal{T}}_{n,\ell}^{(k)})$ *polynomial in* $n$*?*

### 2.3 Paths

Given integers $1 \leq \ell < k$ and $n \equiv \ell \pmod{k-\ell}$, we define an *$\ell$-path* $\mathcal{P}_{n,\ell}^{(k)}$ to be the $k$-uniform hypergraph with vertex set $[n]$ and edge set $\{e_1, \ldots, e_m\}$, where $e_i = \{(i-1)(k-\ell)+1, (i-1)(k-\ell)+2, \ldots, (i-1)(k-\ell)+k\}$ and $m = \frac{n-\ell}{k-\ell}$. In other words, the edges are intervals of length $k$ in $[n]$ and consecutive edges intersect in precisely $\ell$ vertices. The two extreme cases of $\ell = 1$ and $\ell = k-1$ are referred to as, respectively, *loose* and *tight* paths. Clearly every $\ell$-path is also an $\ell$-tree. Thus, by Theorem 2.4 we obtain the following result.

$$\hat{R}(\mathcal{P}_{n,\ell}^{(k)}) = O(n^{\ell+1}). \tag{4}$$

Our first result shows that determining the size-Ramsey number of a path $\mathcal{P}_{n,\ell}^{(k)}$ for $\ell \leq \frac{k}{2}$ can easily be reduced to the graph case.

**Proposition 2.7** *Let $1 \leq \ell \leq \frac{k}{2}$. Then,*

$$\hat{R}(\mathcal{P}_{n,\ell}^{(k)}) \leq \hat{R}(P_n) = O(n).$$

Clearly, this result is optimal.

Determining the size-Ramsey number of a path $\mathcal{P}_{n,\ell}^{(k)}$ for $\ell > \frac{k}{2}$ seems to be a much harder problem. Here we will only consider tight paths ($\ell = k-1$). By (4) we get

$$\hat{R}(\mathcal{P}_{n,k-1}^{(k)}) = O(n^k). \tag{5}$$

The most complicated result of this paper is the following improvement of (5).

**Theorem 2.8** *Fix $k \geq 3$ and let $\alpha = (k-2)/(\binom{k-1}{2}+1)$. Then*

$$\hat{R}(\mathcal{P}_{n,k-1}^{(k)}) = O(n^{k-1-\alpha}(\log n)^{1+\alpha}).$$

The gap in the exponent of $n$ between the upper and lower bounds for this problem remains quite large (between 1 and $k-1-\alpha$). We believe that the lower bound is much closer to the truth. Indeed, the following question still remains open.

**Question 2.9** *Is $\hat{R}(\mathcal{P}_{n,k-1}^{(k)}) = O(n)$?*

If true, then since $\hat{R}(\mathcal{P}_{n,\ell}^{(k)}) \leq \hat{R}(\mathcal{P}_{n,k-1}^{(k)})$, this would imply the linearity of the size-Ramsey number of all $\ell$-paths.

### 2.4 Bounded degree hypergraphs

Our main result about bounded degree hypergraphs is that their size-Ramsey numbers can be superlinear. This is proved by extending the methods of Rödl and Szémerédi [18] to the hypergraph case.

**Theorem 2.10** *Let $k \geq 3$ be an integer. Then there is a positive constant $c = c(k)$ such that for every $n$ there is a $k$-graph $\mathcal{G}$ of order at most $n$ with maximum degree $k+1$ such that*

$$\hat{R}(\mathcal{G}) = \Omega(n(\log n)^c).$$

There are several other problems to consider such as finding the asymptotic of the size-Ramsey number of cycles and many other classes of hypergraphs. In general, they seem to be very difficult. Therefore, this paper is the first step towards a better understanding of this concept.

In the next sections we prove these result for cliques (Section 3), trees (Section 4), paths (Section 5), and hypergraphs with bounded degree (Section 6).

## 3 Cliques

**Proof of Theorem 2.1.** We show that if $\mathcal{H}$ is a 3-graph with $|E(\mathcal{H})| < \frac{n^2}{96}\binom{R(\mathcal{K}_n^{(3)})}{2}$ for $n \geq 4$, then $\mathcal{H} \nrightarrow \mathcal{K}_n^{(3)}$.

Induction on $N = |V(\mathcal{H})|$. If $N < R(\mathcal{K}_n^{(3)})$, then there is a 2-coloring of $K_N^{(3)}$ with no monochromatic $K_n^{(3)}$. Since $\mathcal{H} \subseteq K_N^{(3)}$, this coloring yields a 2-coloring of $\mathcal{H}$ with no monochromatic $K_n^{(3)}$.

Suppose that $N \geq R(\mathcal{K}_n^{(3)})$. Since $|E(\mathcal{H})| < \frac{n^2}{96}\binom{R(\mathcal{K}_n^{(3)})}{2}$, there are $u$ and $v$ in $V(\mathcal{H})$ with $\deg(u,v) = |\{e \in E(\mathcal{H}) : \{u,v\} \subseteq e\}| < \frac{n^2}{32}$. Otherwise,

$$|E(\mathcal{H})| = \frac{1}{3} \sum_{\{u,v\} \in \binom{V(\mathcal{H})}{2}} \deg(u,v) \geq \frac{1}{3}\binom{N}{2}\frac{n^2}{32} > |E(\mathcal{H})|,$$

a contradiction.

Let $u$ and $v$ be such that $\deg(u,v) < \frac{n^2}{32}$. Define $\mathcal{H}_u$ as follows:

$$V(\mathcal{H}_u) = V(\mathcal{H}) \setminus \{v\}$$

and

$$E(\mathcal{H}_u) = \{e : v \notin e \in E(\mathcal{H})\} \cup \{\{u,x,y\} : \{v,x,y\} \in E(\mathcal{H}) \text{ and } \{u,x,y\} \notin E(\mathcal{H})\}.$$

Clearly, $|V(\mathcal{H}_u)| = N-1$ and $|E(\mathcal{H}_u)| \leq |E(\mathcal{H})| < \frac{n^2}{96}\binom{R(\mathcal{K}_n^{(3)})}{2}$. By the inductive hypothesis there is a 2-coloring $\chi_u$ of the edges of $\mathcal{H}_u$ with no monochromatic $\mathcal{K}_n^{(3)}$. Let $T = T_1 = N_{\mathcal{H}}(u,v) = \{w \in V(\mathcal{H}) : \{u,v,w\} \in E(\mathcal{H})\}$. Thus, $T_1 \subseteq V(\mathcal{H}_u)$ and $|T_1| < \frac{n^2}{32}$. If there exists $S_1 \subseteq T_1$ such that $|S_1| \geq \frac{n}{4}$ and $\mathcal{H}_u[S_1 \cup \{u\}]$ is monochromatic, then set $T_2 = T_1 \setminus S_1$. If there exists $S_2 \subseteq T_2$ such that $|S_2| \geq \frac{n}{4}$ and $\mathcal{H}_u[S_2 \cup \{u\}]$ is monochromatic, then set $T_3 = T_2 \setminus S_2$. We continue this process obtaining

$$T = S_1 \cup S_2 \cup \cdots \cup S_m \cup U,$$

where $\mathcal{H}_u[S_i \cup \{u\}]$ is monochromatic, $|S_i| \geq \frac{n}{4}$, and $\mathcal{H}_u[U \cup \{u\}]$ contains only monochromatic cliques of order at most $\frac{n}{4}$.

Now we define a 2-coloring $\chi$ of $\mathcal{H}$.

(i) If $v \notin e$, then $\chi(e) = \chi_u(e)$.

(ii) If $v \in e = \{v,x,y\}$ and $u \notin e$, then $\chi(e) = \chi_u(\{u,x,y\})$.

(iii) If $\{u,v\} \subseteq e = \{u,v,x\}$ and $x \in S_i$, then $e$ takes the opposite color to the color of $\mathcal{H}_u[S_i \cup \{u\}]$.

(iv) If $\{u,v\} \subseteq e = \{u,v,x\}$ and $x \in U$, then color $e$ arbitrarily.

Now suppose that there is a monochromatic clique $\mathcal{K} = \mathcal{K}_n^{(3)}$ in $\mathcal{H}$. Such a clique must contain $v$. Now there are two cases to consider. If $u \notin V(\mathcal{K})$, then the subgraph of $\mathcal{H}_u$ induced by $V(\mathcal{K}) \cup \{u\} \setminus \{v\}$ is also a monochromatic copy of $\mathcal{K}_n^{(3)}$, a contradiction. Otherwise, $u \in V(\mathcal{K})$. Thus, $V(\mathcal{K}) \setminus \{u,v\} \subseteq T$ and $|V(\mathcal{K}) \setminus \{u,v\}| = n-2$. Observe that $|V(\mathcal{K}) \cap S_i| \leq 2$ and $|V(\mathcal{K}) \cap U| < \frac{n}{4}$. But this yields a contradiction

$$n - 2 = |V(\mathcal{K}) \setminus \{u,v\}| < 2m + \frac{n}{4} < 2\frac{\frac{n^2}{32}}{\frac{n}{4}} + \frac{n}{4} = \frac{n}{2} \leq n-2,$$

for $n \geq 4$. $\square$

## 4 Trees

First for convenience we recall the definition of a hypertree. Given integers $1 \le \ell < k$ and $n$, recall that a $k$-graph $\mathcal{T}_{n,\ell}^{(k)}$ of order $n$ with edge set $\{e_1, \dots, e_m\}$ is an *$\ell$-tree*, if for each $2 \le j \le m$ we have $|e_j \cap \bigcup_{1\le i<j} e_i| \le \ell$ and $e_j \cap \bigcup_{1\le i<j} e_i \subseteq e_{i_0}$ for some $1 \le i_0 < j$.

**Proof of Theorem 2.4.** Fix $1 \le \ell \le k$. We are to show that $\hat{R}(\mathcal{T}_{n,\ell}^{(k)}) = O(n^{\ell+1})$. Recall that a *partial Steiner system* $S(t, k, N)$ is a $k$-graph of order $N$ such that each $t$-tuple is contained in at most one edge. Due to a result of Rödl [17] it is known that there is a constant $N_0 = N_0(t, k)$ such that for every $N \ge N_0$ there is an $\mathcal{S} = S(t, k, N)$ with the number of edges satisfying

$$\frac{9}{10} \cdot \frac{\binom{N}{t}}{\binom{k}{t}} \le |E(\mathcal{S})| \le \frac{\binom{N}{t}}{\binom{k}{t}} \tag{6}$$

(see also [14, 19, 20, 21] for similar results). It is easy to observe that for $1 \le s \le t$ every $s$-tuple is contained in at most $\frac{\binom{N-s}{t-s}}{\binom{k-s}{t-s}}$ edges.

Fix $1 \le \ell < k$. Let $N = \lceil cn \rceil + \ell$, where the constant $c$ is defined as

$$c = \max\left\{ N_0(\ell+1, k), \frac{20}{9}(\ell+1)\binom{k}{\ell+1} \right\}.$$

Let $\mathcal{H}$ be a $S(\ell+1, k, N)$ satisfying (6). Observe that if $\ell + 1 = k$, then $\mathcal{H}$ can be viewed as a complete $k$-graph of order $N$. Clearly, $|E(\mathcal{H})| = O(n^{\ell+1})$. It remains to show that for any $\mathcal{T} = \mathcal{T}_{n,\ell}^{(k)}$ tree, $\mathcal{H} \to \mathcal{T}$.

Define a *degree* of a set $U \subseteq V(\mathcal{H})$ $(1 \le |U| < k)$ by

$$\deg(U) = |\{e \in E(\mathcal{H}) : e \supseteq U\}|$$

and for $E(\mathcal{H}) \neq \emptyset$ a *minimum (non-zero) $\ell$-degree* by

$$\delta_\ell(\mathcal{H}) = \min\{\deg(U) : |U| = \ell \text{ and } U \subseteq e \text{ for some } e \in E(\mathcal{H})\}.$$

First observe that for any 2-coloring of the edges of $\mathcal{H}$, there is a monochromatic sub-hypergraph $\mathcal{F}$ with $\delta_\ell(\mathcal{F}) \ge n$. Indeed, suppose that $\mathcal{H}$ is colored with blue and red colors. Assume by symmetry that the red hypergraph $\mathcal{R}$ has at least $\frac{1}{2}|E(\mathcal{H})|$ edges. Set $\mathcal{R}_0 = \mathcal{R}$. If there exists $U_0 \subseteq V(\mathcal{R}_0)$ with $\deg_{\mathcal{R}_0}(U_0) < n$, then let $\mathcal{R}_1 = \mathcal{R}_0 - U_0$ (we remove $U_0$ and all incident to $U_0$ edges). Now we repeat the process. If there exists $U_1 \subseteq V(\mathcal{R}_1)$ with $\deg_{\mathcal{R}_1}(U_1) < n$, then let $\mathcal{R}_2 = \mathcal{R}_1 - U_1$. Continue this way to obtain hypergraphs

$$\mathcal{R} = \mathcal{R}_0 \supseteq \mathcal{R}_1 \supseteq \mathcal{R}_2 \supseteq \cdots \supseteq \mathcal{R}_m,$$

where either $\delta_\ell(\mathcal{R}_m) \ge n$ or $\mathcal{R}_m$ is empty hypergraph. But the latter cannot happen, since the number of removed edges from $\mathcal{R}$ is less than

$$\binom{N}{\ell} n = \binom{N}{\ell+1}\frac{\ell+1}{N-\ell} n \le \binom{N}{\ell+1}\frac{\ell+1}{c} \le \frac{9}{20} \cdot \frac{\binom{N}{\ell+1}}{\binom{k}{\ell+1}} < \frac{1}{2}|E(\mathcal{H})|.$$

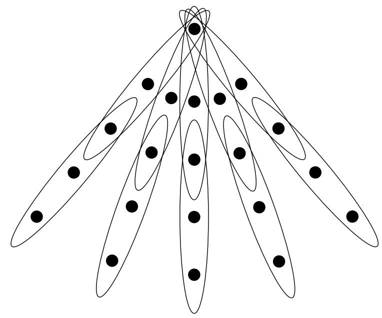

Figure 1: A star of order $n$ with $\frac{n-1}{4}$ arms each of length 2.

Now we greedily embed $\mathcal{T}$ into $\mathcal{F} = \mathcal{R}_m$. At every step we have a connected sub-tree $\mathcal{T}_i \subseteq \mathcal{T}$. Assume that we already embedded $i$ edges of $\mathcal{T}$ obtaining $\mathcal{T}_i$. Let $|U| \leq \ell$ be such that $U \subseteq e$ for some $e \in E(\mathcal{T}_i)$. Observe that there is always an edge $f \in E(\mathcal{F}) \setminus E(\mathcal{T}_i)$ such that $f \cap V(\mathcal{T}_i) = U$. Indeed, if $|U| = \ell$, then this is true since $\deg_{\mathcal{F}}(U) \geq n$ and $|V(\mathcal{T}_i)| < n$ and every $(\ell+1)$-tuple of vertices of $\mathcal{F}$ is contained in at most one edge in $\mathcal{F}$. Otherwise, if $|U| < \ell$, first we find a set $W \subseteq V(\mathcal{F}) \setminus V(\mathcal{T}_i)$ such that $|W| = \ell - |U|$ and $U \cup W$ is contained in an edge of $\mathcal{F}$, and next apply the previous argument to $U \cup W$. Thus, we can extend $\mathcal{T}_i$ to $\mathcal{T}_{i+1}$, as required. $\square$

As mentioned in the introduction, it would be interesting to decide whether Theorem 2.4 is tight up to the hidden constant. This is definitely the case for $\ell = 1$. Indeed, let $\mathcal{T}$ be a $k$-uniform star-like tree of order $n$ defined as follows. Assume that $2k-2$ divides $n-1$. $\mathcal{T}$ consists of $\frac{n-1}{2k-2}$ arms $\mathcal{P}_i$ (each with two edges): $E(\mathcal{P}_i) = \{\{v, w_1^i, w_2^i, \ldots, w_{k-1}^i\}, \{w_{k-1}^i, w_k^i, \ldots, w_{2k-2}^i\}\}$, where $1 \leq i \leq \frac{n-1}{2k-2}$ and all $w_j^i$ vertices are pairwise different (see Figure 1).

Assume that $\mathcal{H} \to \mathcal{T}$ and color $e \in \mathcal{H}$ by red if degree (in $\mathcal{H}$) of every vertex in $e$ is less than $\frac{n-1}{2k-2}$; otherwise $e$ is blue. Since $\mathcal{H} \to (\mathcal{T})_2^e$ and there is no red copy of $\mathcal{T}$, there must be a blue copy of $\mathcal{T}$. Every edge in such a copy has at least one vertex of degree at least $\frac{n-1}{2k-2}$ (in $\mathcal{H}$). Since $\mathcal{T}$ has $\frac{n-1}{2k-2}$ vertex disjoint edges and every edge (in $\mathcal{H}$) can intersect at most 3 of those disjoint edges,

$$\hat{R}(\mathcal{T}) \geq \frac{1}{3} \cdot \frac{n-1}{2k-2} \cdot \frac{n-1}{2k-2} = \Omega(n^2).$$

# 5 Paths

In this section we prove Proposition 2.7 and Theorem 2.8.

**Proof of Proposition 2.7.** Let $H$ be a graph satisfying $H \to P_n$ and $|E(H)| = O(n)$ (*cf.* (3)). We construct a $k$-graph $\mathcal{H}$ as follows. Replace every vertex $v \in V(H)$ by an $\ell$-tuple $\{v_1, v_2, \ldots, v_\ell\}$ (different for every $v$) and each $e = \{v, w\} \in E(H)$ by

$$\{v_1, \ldots, v_\ell, w_1, \ldots, w_\ell, x_1, \ldots, x_{k-2\ell}\},$$

where $x_1, \ldots, x_{k-2\ell}$ are different for every edge $e$, too. Thus, $\mathcal{H}$ is a $k$-graph with $|V(\mathcal{H})| = \ell|V(H)| + (k-2\ell)|E(H)|$ and $|E(\mathcal{H})| = |E(H)|$. Now color $E(\mathcal{H})$. This coloring (uniquely) defines a coloring of $E(H)$. Since $H$ contains a monochromatic copy of $P_n$, $\mathcal{H}$ also contains a monochromatic copy of $\mathcal{P}^{(k)}_{n,\ell}$. Consequently, $\mathcal{H} \to \mathcal{P}^{(k)}_{n,\ell}$ and the proof is complete. $\square$

We now turn to the main result of this section which we restate for convenience.

**Theorem 2.8** *Fix $k \geq 3$ and let $\alpha = (k-2)/(\binom{k-1}{2}+1)$. Then*

$$\hat{R}(\mathcal{P}^{(k)}_{n,k-1}) = O(n^{k-1-\alpha}(\log n)^{1+\alpha}).$$

First we prove an auxiliary result. In order to do it we state some necessary notation. Set

$$\beta = \frac{1}{\binom{k-1}{2}+1}.$$

For a graph $G = (V, E)$ let $\mathcal{T}_\ell(G)$ be the set of all cliques of order $\ell$ and let $t_\ell = |\mathcal{T}_\ell(G)|$. Let $A \subseteq V$ and $\mathcal{B} \subseteq \mathcal{T}_{k-1}(G)$ be a family of pairwise vertex-disjoint cliques. Define $x_{A,\mathcal{B}}$ as the number of $k$-cliques of $G$ which $k-1$ vertices form a vertex set of some $B \in \mathcal{B}$ and the remaining vertex is from $V \setminus (A \cup \bigcup_{B\in\mathcal{B}} V(B))$. Similarly, let $y_{A,\mathcal{B}}$ be the number of $k$-cliques in $G$ which $k-1$ vertices form a vertex set of some $B \in \mathcal{B}$ and the remaining vertex is from $A \cup \bigcup_{B\in\mathcal{B}} V(B)$. Finally, let $z_C$ (for $C \subseteq V$) be the number of $k$-cliques containing at least one vertex from $C$.

**Proposition 5.1** *Let $k \geq 3$ be an integer and let $c = \frac{1}{3^{3k}}$. Then there exists a graph $G = (V, E)$ of order $n$ (for sufficiently large $n$) satisfying the following:*

*(i) For every $A \subseteq V$, $|A| \leq cn$, and every $\mathcal{B} \subseteq \mathcal{T}_{k-1}(G)$, $|\mathcal{B}| = cn$, vertex disjoint $(k-1)$-cliques such that $A \cap \bigcup_{B\in\mathcal{B}} V(B) = \emptyset$ we have*

$$y_{A,\mathcal{B}} \leq \frac{1}{k+1} x_{A,\mathcal{B}}.$$

*(ii) For every $C \subseteq V$, $|C| \leq (k-1)cn$,*

$$z_C \leq \frac{t_k}{4k}.$$

*(iii) The total number of $k$-cliques satisfies*

$$t_k \leq \nu n^{k-1-\alpha}(\log n)^{1+\alpha},$$

*where $\nu = (3/2)^k \frac{d^{\binom{k}{2}}}{(k-1)(k-2)}$.*

*Proof.* It suffices to show that the random graph $G \in \mathbb{G}(n,p)$ with $p = d(\log n/n)^\beta$ and $d = 3000$ satisfies *a.a.s.*[1] (i) - (iii).

Below we will use the following bounds on the tails of the binomial distribution $\text{Bin}\,(n,p)$ (for details, see, *e.g.*, [13]):

$$\Pr(\text{Bin}\,(n,p) \le (1-\gamma)\mathbb{E}(X)) \le \exp\left(-\frac{\gamma^2}{2}\mathbb{E}(X)\right), \tag{7}$$

$$\Pr(\text{Bin}\,(n,p) \ge (1+\gamma)\mathbb{E}(X)) \le \exp\left(-\frac{\gamma^2}{3}\mathbb{E}(X)\right). \tag{8}$$

First we show that $G$ *a.a.s.* satisfies (i). Fix an $A \subseteq V$ and $\mathcal{B} \subseteq \mathcal{T}_{k-1}$ with $|\mathcal{B}| = cn$. Observe that without loss of generality we may assume that $|A| = cn$. Note that $x_{A,\mathcal{B}} \sim \text{Bin}\left(cn(n - cn - (k-1)cn), p^{k-1}\right)$. Thus,

$$\mathbb{E}(x_{A,\mathcal{B}}) = c(1-kc)n^2p^{k-1} = d^{k-1}c(1-kc)n^{2-(k-1)\beta}(\log n)^{(k-1)\beta}$$

and (7) (applied with $\gamma = 1/2$) implies

$$\begin{aligned}\Pr\left(x_{A,\mathcal{B}} \le \frac{\mathbb{E}(x_{A,\mathcal{B}})}{2}\right) &\le \exp\left(-\frac{1}{8}\mathbb{E}(x_{A,\mathcal{B}})\right)\\ &= \exp\left(-\frac{d^{k-1}}{8}c(1-kc)n^{2-(k-1)\beta}(\log n)^{(k-1)\beta}\right).\end{aligned} \tag{9}$$

Now we bound from above the number of all possible choices for $A$ and $\mathcal{B}$. Clearly we have at most $n^{cn}$ choices for $A$. Observe that the number of choices for $\mathcal{B}$ can be bounded from above by the number of ways of choosing an ordered subset of vertices of size $(k-1)cn$. Indeed, suppose that $v_1, \dots, v_{(k-1)cn}$ is such a choice. Then $\mathcal{B}$ can be defined as $\big\{\{v_1,\dots,v_{k-1}\},\{v_k,\dots,v_{2k-2}\},\dots,\{v_{(k-1)cn-k+1},\dots,v_{(k-1)cn}\}\big\}$. Thus we conclude that there are at most $n^{kcn}$ ways to choose $A$ and $\mathcal{B}$. Hence, by (9)

$$\begin{aligned}\Pr\left(\bigcup_{A,\mathcal{B}}\left\{x_{A,\mathcal{B}} \le \frac{\mathbb{E}(x_{A,\mathcal{B}})}{2}\right\}\right) &\le n^{kcn}\Pr\left(x_{A,\mathcal{B}} \le \frac{\mathbb{E}(x_{A,\mathcal{B}})}{2}\right)\\ &\le \exp\left(kcn\log n - \frac{d^{k-1}}{8}c(1-kc)n^{2-(k-1)\beta}(\log n)^{(k-1)\beta}\right)\\ &= o(1).\end{aligned} \tag{10}$$

Similarly, since $y_{A,\mathcal{B}} \sim \text{Bin}\left(cn \cdot kcn, p^{k-1}\right)$,

$$\mathbb{E}(y_{A,\mathcal{B}}) = kc^2n^2p^{k-1} = d^{k-1}kc^2n^{2-(k-1)\beta}(\log n)^{(k-1)\beta}.$$

and since $c = \frac{1}{3^{3k}} \le \frac{1}{k(3k+4)}$,

$$\begin{aligned}\frac{\mathbb{E}(x_{A,\mathcal{B}})}{2(k+1)} &= \frac{c(1-kc)}{2(k+1)}d^{k-1}n^{2-(k-1)\beta}(\log n)^{(k-1)\beta}\\ &\ge \frac{3}{2}d^{k-1}kc^2n^{2-(k-1)\beta}(\log n)^{(k-1)\beta}\\ &= \frac{3}{2}\mathbb{E}(y_{A,\mathcal{B}}).\end{aligned}$$

[1]An event $E_n$ occurs *asymptotically almost surely*, or *a.a.s.* for brevity, if $\lim_{n\to\infty}\Pr(E_n) = 1$.

Inequality (8) (applied with $\gamma = 1/2$) yields

$$\Pr\left(y_{A,\mathcal{B}} \geq \frac{\mathbb{E}(x_{A,\mathcal{B}})}{2(k+1)}\right) \leq \Pr\left(y_{A,\mathcal{B}} \geq \frac{3}{2}\mathbb{E}(y_{A,\mathcal{B}})\right) \leq \exp\left(-\frac{1}{12}\mathbb{E}(y_{A,\mathcal{B}})\right).$$

Therefore, we deduce that

$$\Pr\left(\bigcup_{A,\mathcal{B}}\left\{y_{A,\mathcal{B}} \geq \frac{\mathbb{E}(x_{A,\mathcal{B}})}{2(k+1)}\right\}\right) \leq n^{kcn}\exp\left(-\frac{1}{12}\mathbb{E}(y_{A,\mathcal{B}})\right) = o(1). \tag{11}$$

Consequently, by (10) and (11) we get that *a.a.s.*

$$y_{A,\mathcal{B}} \leq \frac{\mathbb{E}(x_{A,\mathcal{B}})}{2(k+1)} \leq \frac{x_{A,\mathcal{B}}}{k+1}$$

for any choice of $A$ and $\mathcal{B}$. This finishes the proof of (i).

For each vertex $v \in V$, let $\deg_k(v)$ denote the number of $k$-cliques of $G$ which contain $v$. In order to show that *a.a.s.* $G$ also satisfies (ii), we will first estimate $\deg_k(v)$ for each $v \in V$.

The standard application of (8) (applied with $\mathrm{Bin}\,(n-1,p)$ and $\gamma = 1/2$) with the union bound imply that *a.a.s.* the degree of every vertex $v \in V(G)$ satisfies

$$\deg(v) \leq \frac{3}{2}dn^{1-\beta}(\log n)^{\beta}.$$

The number of $k$-cliques which contain $v$ is equal to the number of $(k-1)$-cliques in the neighborhood of $v$. Therefore, in order to show (ii) it suffices to bound the number of $(k-1)$-cliques in any set of size at most $\frac{3}{2}dn^{1-\beta}(\log n)^{\beta}$.

Let $S \subseteq V$ with $s = |S| = \frac{3}{2}dn^{1-\beta}(\log n)^{\beta}$. First we will decompose all $(k-1)$-tuples of $S$ into linear $(k-1)$-uniform hypergraphs $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_m$ with

$$m = (1+o(1))\binom{s}{k-1}\binom{k-1}{2}/\binom{s}{2}$$

and

$$|\mathcal{S}_i| = (1+o(1))\frac{\binom{s}{2}}{\binom{k-1}{2}}$$

for every $1 \leq i \leq m$. That means that each $(k-1)$-tuple of $S$ belongs to exactly one $\mathcal{S}_i$ and each pair of elements of $S$ appears in at most one $(k-1)$-tuple in $\mathcal{S}_i$. The existence of such a decomposition follows from a more general result of Pippenger and Spencer [16] (see also [10]).

Let $s_i$ be the random variable that counts the number of $(k-1)$-tuples of $\mathcal{S}_i$ which appear as $(k-1)$-cliques of $G$. Observe that $s_i \sim \mathrm{Bin}\left(|\mathcal{S}_i|, p^{\binom{k-1}{2}}\right)$. Therefore for each $i$,

$$\begin{aligned}
\mathbb{E}(s_i) &= (1+o(1))\frac{\binom{s}{2}}{\binom{k-1}{2}}p^{\binom{k-1}{2}} \\
&= (1+o(1))\frac{s^2}{(k-1)(k-2)}p^{\binom{k-1}{2}} \\
&= (1+o(1))\frac{9}{4(k-1)(k-2)}d^{2+\binom{k-1}{2}}n^{1-\beta}(\log n)^{1+\beta}
\end{aligned}$$

and by (8) (with $\gamma = 1/2$)

$$\Pr\left(s_i \geq \frac{3}{2}\mathbb{E}(s_i)\right) \leq \exp\left(-\frac{1}{12}\mathbb{E}(s_i)\right) \leq \exp\left(-\frac{3}{16k^2}d^{2+\binom{k-1}{2}}n^{1-\beta}(\log n)^{1+\beta}\right).$$

Consequently, the union bound over all subsets $S \subseteq V$ of size $s$ and over all $i$ for each $1 \leq i \leq m$ implies

$$\begin{aligned}
\Pr\left(\bigcup_{S,\, i}\left\{s_i \geq \frac{3}{2}\mathbb{E}(s_i)\right\}\right) &\leq \binom{n}{s} \cdot m \cdot \exp\left(-\frac{3}{16k^2}d^{2+\binom{k-1}{2}}n^{1-\beta}(\log n)^{1+\beta}\right) \\
&\leq n^s \cdot s^{k-3} \cdot \exp\left(-\frac{3}{16k^2}d^{2+\binom{k-1}{2}}n^{1-\beta}(\log n)^{1+\beta}\right) \\
&= s^{k-3} \cdot \exp\left(s\log n - \frac{3}{16k^2}d^{2+\binom{k-1}{2}}n^{1-\beta}(\log n)^{1+\beta}\right) \\
&= s^{k-3} \cdot \exp\left(n^{1-\beta}(\log n)^{1+\beta}\left(\frac{3}{2}d - \frac{3}{16k^2}d^{2+\binom{k-1}{2}}\right)\right) \\
&= o(1),
\end{aligned}$$

since $s^{k-3}$ grows like a polynomial in $n$. Therefore it follows that *a.a.s.*

$$\deg_k(v) = \sum_{i=1}^{m} s_i \leq m \cdot \frac{3}{2}\mathbb{E}(s_i) \leq s^{k-3} \cdot \frac{3}{2}\mathbb{E}(s_i) = \nu n^{(k-2)(1-\beta)}(\log n)^{1+\alpha}, \tag{12}$$

where

$$\nu = \left(\frac{3}{2}\right)^k \frac{d^{\binom{k}{2}}}{(k-1)(k-2)}. \tag{13}$$

In a similar way one can show that

$$\deg_k(v) \geq \lambda n^{(k-2)(1-\beta)}(\log n)^{1+\alpha},$$

where

$$\lambda = \left(\frac{1}{2}\right)^{k-1} \frac{d^{\binom{k}{2}}}{(k-1)(k-2)}. \tag{14}$$

Note that equation (12) gives the bound

$$t_k \leq \nu n^{(k-2)(1-\beta)+1}(\log n)^{1+\alpha} = \nu n^{k-1-\alpha}(\log n)^{1+\alpha},$$

which proves part (iii).

Now we finish the proof of (ii). Since each $k$-clique is counted exactly $k$ times, the number of $k$-cliques is *a.a.s.* at least

$$t_k \geq \frac{n}{k} \cdot \lambda n^{(k-2)(1-\beta)}(\log n)^{1+\alpha} = \frac{\lambda}{k}n^{k-1-\alpha}(\log n)^{1+\alpha}. \tag{15}$$

It follows now from (12) and (15) that given a set $C \subseteq V$, $|C| \leq (k-1)cn$, the number of $k$-cliques of $G$ which intersect $C$ is *a.a.s.* at most

$$z_C \leq (k-1)cn \cdot \nu n^{(k-2)(1-\beta)}(\log n)^{1+\alpha} = \frac{c(k-1)k\nu}{\lambda} \cdot \frac{\lambda}{k} n^{k-1-\alpha}(\log n)^{1+\alpha} \leq \frac{c(k-1)k\nu}{\lambda} t_k.$$

Finally observe that (13), (14) together with the choice of $c$ yield that

$$\frac{c(k-1)k\nu}{\lambda} \leq \frac{1}{4k}$$

implying condition (ii), as required. □

Now we are ready to prove main result of this section.

**Proof of Theorem 2.8**. We show that there exists a $k$-graph $\mathcal{H}$ with $|\mathcal{H}| = O(n^{k-1-\alpha}(\log n)^{1+\alpha})$ such that any two-coloring of the edges of $\mathcal{H}$ yields a monochromatic copy of $\mathcal{P}^{(k)}_{n,k-1}$.

Let $G$ be a graph from Proposition 5.1. Set $V(\mathcal{H}) = V(G)$ and let $E(\mathcal{H})$ be the set of $k$-cliques in $G$. We prove that such $\mathcal{H}$ is a Ramsey $k$-graph for $\mathcal{P}^{(k)}_{m,k-1}$ with $m = cn$, where $c = \frac{1}{3^{3k}}$.

Take an arbitrary red-blue coloring of the edges of $\mathcal{H}_0 = \mathcal{H}$ and assume that there is no monochromatic $\mathcal{P}^{(k)}_{m,k-1}$. We will consider the following greedy *procedure* which at each step finds a blue tight path of length $i$ labeled as $v_1, v_2, \ldots, v_i$.

(1) Let $\mathcal{B} = \emptyset$ be the *trash* set of $(k-1)$-tuples and $U = V(\mathcal{H})$ be the set of *unused* vertices and set $i := 0$. At any point in the process, if $|\mathcal{B}| = m$, then stop.

(2) (In this step $i = 0$.) If possible, then choose a blue edge from $U$ and label its vertices by $v_1, \ldots, v_k$ and then set $i := k$. Otherwise, if not possible, stop.

(3) (In this step $i \geq k$.) Let $v_{i-k+1}, \ldots, v_{i-1}, v_i$ be the labels of the last $k-1$ vertices of the constructed blue path. If possible, select a vertex $u \in U$ for which $v_{i-k+1}, \ldots, v_{i-1}, v_i, u$ form a blue edge. Label $u$ as $v_{i+1}$, set $U := U \setminus \{u\}$ and $i := i + 1$. Repeat this step until no such $u$ can be found.

(4) (In this step also $i \geq k$.) Let $v_{i-k+1}, \ldots, v_{i-1}, v_i$ be the labels of the last $k-1$ vertices of the constructed blue path which cannot be extended in a sense described in step (3). Remove these $k-1$ vertices from the path and set $\mathcal{B} := \mathcal{B} \cup \{\{v_{i-k+1}, \ldots, v_{i-1}, v_i\}\}$ and $i := i - k + 1$. After this removal there are two possibilities:

   (i) if $i < k$, then put back $v_1, \ldots, v_i$ to $U$ (i.e. $U := U \cup \{v_1, \ldots, v_i\}$), set $i := 0$, and return to step (2);

   (ii) otherwise, return to step (3).

This procedure will terminate under two circumstances: either $|\mathcal{B}| = m$ or no blue edge can be found in step (2).

First let us consider the case when $|\mathcal{B}| = m$, that means, there are $m$ vertex disjoint $(k-1)$-tuples in $\mathcal{B}$. Denote by $A$ the vertex set of the blue path which was obtained when $|\mathcal{B}| = m$. Clearly, $|A| < m$, otherwise there would be a blue $\mathcal{P}^{(k)}_{m,k-1}$. We are going to apply Proposition 5.1 with sets $A$ and $\mathcal{B}$. Notice that every edge of $\mathcal{H}$ which contains a $(k-1)$-tuple from $\mathcal{B}$ and the remaining vertex from $V(\mathcal{H}) \setminus (A \cup \bigcup_{B\in\mathcal{B}} B)$ must be colored red. (This is because for a $(k-1)$-tuple to end up in $\mathcal{B}$, there must have been no vertex $u$ in step (3) that could extend the blue path.) It also follows from step (3) that each $(k-1)$-tuple in $\mathcal{B}$ is contained in at least one blue edge. Thus, Proposition 5.1 (i) implies that $y_{A,\mathcal{B}} \leq \frac{1}{k+1} x_{A,\mathcal{B}}$. That means that the number of red edges which contain a $(k-1)$-tuple from $\mathcal{B}$ and the remaining vertex from $U$ is at least $k+1$ times the number of blue edges with a $(k-1)$-tuple from $\mathcal{B}$.

Now remove all the blue edges from $\mathcal{H}$ which contain a $(k-1)$-tuple from $\mathcal{B}$ and denote such $k$-graph by $\mathcal{H}_1$. Perform the above procedure on $\mathcal{H}_1$. This will generate a new trash set $\mathcal{B}_1$. Observe that $\mathcal{B}_1 \cap \mathcal{B} = \emptyset$, since every edge of $\mathcal{H}_1$ which contains a $(k-1)$-tuple from $\mathcal{B}$ must be red. Again, if $|\mathcal{B}_1| = m$, then we use the same argument as above to find that the number of red edges in $\mathcal{H}_1$ which contain a $(k-1)$-tuple from $\mathcal{B}_1$ and the remaining vertex from $U$ is at least $k+1$ times the number of blue edges in $\mathcal{H}_1$ with a $(k-1)$-tuple from $\mathcal{B}_1$. Indeed, we can again apply the inequality from Proposition (i). This is because $y_{A,\mathcal{B}_1}$ is smaller than the number of all blue edges in $\mathcal{H}$ containing a $(k-1)$-tuple from $\mathcal{B}_1$, while (since we do not remove red edges) $x_{A,\mathcal{B}_1}$ remains same in both $\mathcal{H}_1$ and $\mathcal{H}$. Now remove the blue edges from $\mathcal{H}_1$ which contain a $(k-1)$-tuple from $\mathcal{B}_1$ obtaining a $k$-graph $\mathcal{H}_2$. Keep repeating the procedure until it is no longer possible.

At some point, we will run out of blue edges in $\mathcal{H}_j$ for some $j \geq 1$, and the procedure will terminate prematurely in step (2). In this case $|\mathcal{B}_j| < m$, $|A| = 0$ and $U$ has no blue edges. However, there still may be some blue edges which contain a vertex from $\bigcup_{B\in\mathcal{B}_j} V(B)$. Proposition 5.1 (ii) (applied for $C = \bigcup_{B\in\mathcal{B}_j} V(B)$) implies that the number of such edges is at most

$$z_C \leq \frac{t_k}{4k}.$$

Let $x^i_{A,\mathcal{B}}$ and $y^i_{A,\mathcal{B}}$ be the numbers corresponding to $x_{A,\mathcal{B}}$ and $y_{A,\mathcal{B}}$ obtained at the end of the procedure applied to $\mathcal{H}_i$. Thus,

$$y^i_{A,\mathcal{B}} \leq \frac{1}{k+1} x^i_{A,\mathcal{B}}$$

for each $0 \leq i \leq j-1$.

Let $t_R$ and $t_B$ denote the number of red and blue edges in $\mathcal{H}$. Observe that

$$t_B \leq \sum_{0\leq i\leq j-1} y^i_{A,\mathcal{B}} + z_C \leq \frac{1}{k+1} \sum_{0\leq i\leq j-1} x^i_{A,\mathcal{B}} + \frac{t_k}{4k}. \tag{16}$$

Furthermore, since all sets $\mathcal{B}_i$ are mutually disjoint, each red edge in $\mathcal{H}$ containing a $(k-1)$-tuple from some $\mathcal{B}_i$ can be only counted at most $k$ times. Thus,

$$\sum_{0\leq i\leq j-1} x^i_{A,\mathcal{B}} \leq k \cdot t_R. \tag{17}$$

Consequently, by (16) and (17), we get

$$t_k = t_R + t_B \le t_R + \frac{k}{k+1} t_R + \frac{t_k}{4k}$$

and so

$$t_R \ge \frac{4k-1}{4k} \cdot \frac{k+1}{2k+1} t_k > \frac{1}{2} t_k.$$

The conclusion is that there are more red edges than there are blue edges in $\mathcal{H}$. If we reverse the procedure and look for a red path instead of a blue one, we will conclude that there are more blue edges than red edges. Since these two statements contradict each other, the only way to avoid both statements is if a monochromatic path exists. □

# 6 Hypergraphs with bounded degree

In this section we prove Theorem 2.10, which states that hypergraphs with bounded degree can have nonlinear size-Ramsey numbers.

**Proof of Theorem 2.10.** We modify an idea from Rödl and Szemerédi [18]. For simplicity we only present a proof for $k = 3$, which can easily be generalized to $k \ge 3$. The hypergraph $\mathcal{G}$ will be constructed as the vertex disjoint union of graphs $\mathcal{G}_i$ each of which is a tree with a path added on its leaves. Next we will describe the details of such construction.

Set $c = \frac{1}{5}$. We make no effort to optimize $c$ and always assume that $n$ is sufficiently large.

Let

$$t = \left\lfloor \log_2 \left( \frac{2 \log_2 n}{\log_2 \log_2 n} \right) \right\rfloor .$$

Consider a binary 3-tree $\mathcal{B} = (V, E)$ on $1 + 2 + 4 + \cdots + 2^t$ vertices rooted at vertex $z$ (see Figure 2). Denote by $L(\mathcal{B})$ the set of all its leafs. Call the edge containing $z$ the *root edge*. Observe that

$$|V(\mathcal{B})| = 1 + 2 + 4 + \cdots + 2^t = 2^{t+1} - 1 < \log_2 n \tag{18}$$

(recall that $n$ is large enough) and

$$|L(\mathcal{B})| = 2^t.$$

Let $\varphi$ by an automorphism of $\mathcal{B}$. Since the root edge $e$ is the unique edge with exactly one vertex of degree 1, $\varphi(z) = z$. The other two vertices of $e$ are permuted by $\varphi$. Consequently, $\varphi$ permutes two vertices of every other edge. Hence, it is easy to observe that the order of the automorphism group of $\mathcal{B}$ satisfies

$$|Aut(\mathcal{B})| = 2^{1+2+4+\cdots+2^{t-1}} = 2^{2^t - 1} < 2^{2^t}.$$

Now consider a tight path $\mathcal{P}$ of length $|L(\mathcal{B})|$ placed on the leaves $L(\mathcal{B})$ in an arbitrary order. Considering labeled vertices of $L(\mathcal{B})$ there are clearly $|L(\mathcal{B})|!$ such paths. Label them by $\mathcal{P}_i$ for $i = 1, 2, \ldots, |L(\mathcal{T})|!$. Let $\mathcal{B}_i$ be vertex disjoint copies of $\mathcal{B}$ and $\mathcal{G}_i = \mathcal{B}_i \cup \mathcal{P}_i$, where $V(\mathcal{P}_i) = L(\mathcal{B}_i)$.

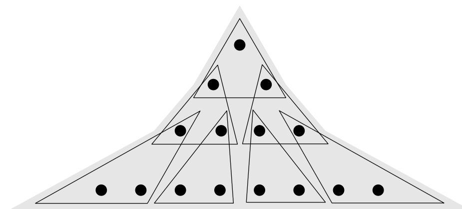

Figure 2: Binary 3-tree $\mathcal{B}$ on $1+2+4+8$ vertices and rooted at vertex $z$.

Let $\varphi$ be an isomorphism between $\mathcal{G}_i$ and $\mathcal{G}_j$. Since the only vertices of degree 4 are on paths $\mathcal{P}_i$ and $\mathcal{P}_j$, $\varphi(\mathcal{P}_i) = \mathcal{P}_j$. Thus,

$$\varphi(E(\mathcal{B}_i)) = \varphi(E(\mathcal{G}_i) \setminus E(\mathcal{P}_i)) = E(\mathcal{G}_j) \setminus E(\mathcal{P}_j) = E(\mathcal{B}_j)$$

and so $\mathcal{B}_i$ and $\mathcal{B}_j$ are isomorphic. Thus, the number of pairwise non-isomorphic $\mathcal{G}_i$'s is at least

$$\frac{|L(\mathcal{B})|!}{|Aut(\mathcal{B})|} \geq \frac{(2^t)!}{2^{2^t}} \geq \frac{\left(\frac{2^t}{e}\right)^{2^t}}{2^{2^t}} \geq \frac{\left(2^{t-2}\right)^{2^t}}{2^{2^t}} = 2^{(t-3)2^t} > n.$$

Set

$$q = \left\lfloor \frac{n}{|V(\mathcal{B})|} \right\rfloor$$

and let $\mathcal{G} = \mathcal{G}_1 \cup \cdots \cup \mathcal{G}_q$, where all $\mathcal{G}_1, \ldots, \mathcal{G}_q$ are pairwise non-isomorphic. We show that $\mathcal{G}$ is a desired hypergraph.

Clearly, $|V(\mathcal{G})| \leq n$. Furthermore, by (18), we get

$$|V(\mathcal{G})| = q|V(\mathcal{B})| \geq \left(\frac{n}{|V(\mathcal{B})|} - 1\right)|V(\mathcal{B})| > n - \log_2 n.$$

Moreover, $\Delta(\mathcal{H}) = 4$ and the independence number of $\mathcal{G}$ satisfies

$$\alpha(\mathcal{G}) \leq \frac{8}{9}n. \tag{19}$$

Indeed, let $I \subseteq V = V(\mathcal{G})$ be an independent set of size $\alpha = \alpha(\mathcal{G})$. We estimate the number of edges $e(I, V \setminus I)$ between sets $I$ and $V \setminus I$. First observe that

$$e(I, V \setminus I) \leq \Delta(\mathcal{G}) \cdot |V \setminus I| \leq 4(n - \alpha).$$

Next, since each triple between $I$ and $V \setminus I$ intersects one of the partition classes on 2 vertices and $\delta(\mathcal{G}) = 1$,

$$e(I, V \setminus I) \geq \frac{\delta(\mathcal{G}) \cdot |I|}{2} = \frac{\alpha}{2}.$$

This implies that

$$\frac{\alpha}{2} \leq 4(n - \alpha)$$

and so (19).

Now we are ready to finish the proof and show that for any 3-graph with

$$|E(\mathcal{H})| \le \frac{1}{30} n (\log_2 n)^{\frac{1}{5}}$$

we have $\mathcal{H} \nrightarrow \mathcal{G}$.

Set $d = (\log_2 n)^{\frac{1}{5}}$ and define $V_{high} \subseteq V(\mathcal{H})$ as

$$V_{high} = \{v \in V(\mathcal{H}) : \deg(v) \ge d\}$$

and

$$V_{low} = V(\mathcal{H}) \setminus V_{high}.$$

Clearly, $|V_{high}| \le \frac{n}{10}$; for otherwise, $|E(\mathcal{H})| > \frac{n}{10} \cdot d \cdot \frac{1}{3} \ge |E(\mathcal{H})|$, a contradiction.

Recall that $\mathcal{G}$ consists of $q$ pairwise non-isomorphic copies of $\mathcal{G}_i$. We estimate the number of copies of $\mathcal{G}_i$'s contained in a sub-hypergraph induced by $V_{low}$. First fix an edge $e$ in $V_{low}[\mathcal{H}]$ and count the number of copies of $\mathcal{G}_i$'s for which $e$ is a root edge. Since $\deg(v) \le d$ for each $v \in V_{low}$, we get that this number is at most

$$3 \cdot d^{2+4+\cdots+2^{t-1}} \cdot d^{2^t} \le d^{2 \cdot 2^t} \le (\log_2 n)^{\frac{1}{5} \cdot 2 \cdot \frac{2 \log_2 n}{\log_2 \log_2 n}} = n^{\frac{4}{5}},$$

where the factor 3 counts the number of choices for the root vertex, the next factors count the number of possible $\mathcal{B}_i$'s with $e$ as a root, and the last factor counts the number of paths on the set of leafs. Thus, there is an $i_0$ such that $\mathcal{G}_{i_0}$ appears in $V_{low}[\mathcal{H}]$ at most

$$\frac{n^{\frac{4}{5}} \cdot |E(\mathcal{H})|}{q} < \frac{n^{\frac{4}{5}} \cdot n (\log_2 n)^{\frac{1}{5}}}{\frac{n}{\log_2 n}} = n^{\frac{4}{5}} (\log_2 n)^{\frac{6}{5}}$$

times.

Denote by $\mathcal{F}$ the sub-hypergraph consisting of root edges from all copies of $\mathcal{G}_{i_0}$ in $V_{low}[\mathcal{H}]$. Thus,

$$|V(\mathcal{F})| \le 3n^{\frac{4}{5}} (\log_2 n)^{\frac{6}{5}}.$$

Color edges in $\mathcal{F}$ together with edges incident to $V_{high}$ blue; otherwise red. Clearly, there is no red copy of $\mathcal{G}$, since there is no red copy of $\mathcal{G}_{i_0}$. Moreover, there is no blue copy of $\mathcal{G}$, since every blue sub-hypergraph of order $|V(\mathcal{G})|$ has an independent set of size at least

$$|V(\mathcal{G})| - |V_{high}| - |V(\mathcal{F})| > (n - \log_2 n) - \frac{n}{10} - 3n^{\frac{4}{5}} (\log_2 n)^{\frac{6}{5}} = \frac{9}{10} n - \log_2 n - 3n^{\frac{4}{5}} (\log_2 n)^{\frac{6}{5}},$$

which is strictly bigger than $\alpha(\mathcal{G})$ (*cf.* (19)). □

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