# DEGREE CONDITIONS FOR MATCHABILITY IN 3-PARTITE HYPERGRAPHS 

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#### Abstract

We study conjectures relating degree conditions in 3-partite hypergraphs to the matching number of the hypergraph, and use topological methods to prove special cases. In particular, we prove a strong version of a theorem of Drisko [14] (as generalized by the first two authors [2]), that every family of $2 n-1$ matchings of size $n$ in a bipartite graph has a partial rainbow matching of size $n$. We show that milder restrictions on the sizes of the matchings suffice. Another result that is strengthened is a theorem of Cameron and Wanless [11], that every Latin square has a diagonal (permutation submatrix) in which no symbol appears more than twice. We show that the same is true under the weaker condition that the square is row-Latin.


## 1. Rainbow matchings and matchings in 3-Partite 3-uniform hypergraphs

All conjectures and results mentioned in this paper can be traced back to a by now well known conjecture of Ryser 20, that for $n$ odd every Latin square possesses a full transversal, namely a permuatation submatrix with distinct symbols. Brualdi [10] and Stein 21 conjectured that for general $n$, every $n \times n$ Latin square possesses a partial transversal of size $n-1$. Stein [21] generalized this still further, replacing the Latinity condition by the milder requirement that each of the $n$ symbols appears in $n$ cells (implying, among other things, that each cell contains precisely one symbol). See [22] for a survey on these conjectures and related results.

These conjectures can be formulated in the terminology of 3-partite hypergraphs. We say that a pair $(X, Y)$ of sides of a $k$-partite hypergraph $H$ is simple if no pair $(x, y)$ with $x \in X, y \in Y$ appears in more than one edge of $H$. A hypergraph is called simple if no edge repeats more than once. In this terminology, an $n \times n$ Latin square is an $n$-regular 3-partite hypergraph with all sides of size $n$, and all three pairs of sides being simple. For a hypergraph $H$ denote by $\nu(H)$ the maximal size of a matching in $H$. In this language the Brualdi-Stein conjecture is:

Conjecture 1.1. Let $H$ be an n-regular 3-partite hypergraph with sides $A, B, C$, all of size $n$, and assume that all three pairs of sides, $(A, B),(B, C)$ and $(A, C)$, are simple. Then $\nu(H) \geq n-1$.

While Stein's conjecture is:
Conjecture 1.2. Let $H$ be an n-regular 3-partite hypergraph with sides $A, B, C$, all of size $n$, and assume that the pair $(A, B)$ is simple. Then $\nu(H) \geq n-1$.

In fact, we believe that an even stronger (and more simply stated) conjecture is true:
Conjecture 1.3. Let $H$ be a simple n-regular 3 -partite hypergraph with sides of size $n$. Then $\nu(H) \geq n-1$.
In the final section of this paper we shall present some yet more general conjectures. But let us first turn to a concept pertinent to these conjectures, that of rainbow matchings. Given a family $F_{1}, \ldots, F_{m}$ of sets, a choice $f_{1} \in F_{1}, \ldots, f_{m} \in F_{m}$ is called a rainbow set. If the elements of the sets $F_{i}$ are themselves sets, and if the sets $f_{i}$ are disjoint, then the rainbow set is called a rainbow matching. If the elements of the sets in $F_{i}$ are edges of a bipartite graph $G$, then a rainbow matching is a matching in a 3-partite hypergraph, in which the vertices of one side represent the sets $F_{i}$, and the other two sides are those of $G$.

A generalization of Conjecture 1.2 was proposed in [2]:

[^0]Conjecture 1.4. Any $n$ matchings of size $n$ in a bipartite graph have a partial rainbow matching of size $n-1$.

Many partial results have been obtained on this conjecture, see, e.g., [23, 15, 5, 16, 8, 12, 13, 19 .
What happens if we demand that $\nu \geq n$, rather than $n-1$, or in the terminology of rainbow matchings we want a rainbow matching of size $n$ ? Strangely, we need to almost double the requirement on the number of matchings. Drisko [14] proved a theorem which was later generalized in [2] to:
Theorem 1.5. A family of $2 n-1$ matchings of size $n$ in a bipartite graph has a rainbow matching of size $n$.
This is sharp - repeating $n-1$ times each of the matchings consisting respectively of the even edges in $C_{2 n}$ and the odd edges in $C_{2 n}$ shows that $2 n-2$ matchings do not suffice. In [7] this was shown to be the only example demonstrating the sharpness of the theorem.

In 3-partite hypergraphs terminology, Theorem 1.5 reads:
Theorem 1.6. Let $H$ be a 3-partite hypergraph with sides $A, B, C$, and assume that
(1) $|A|=2 n-1$.
(2) $\operatorname{deg}(a)=n$ for every $a \in A$.
(3) The pairs $(A, B)$ and $(A, C)$ are simple.

Then $\nu(H) \geq n$.
It is plausible that condition (3) in this theorem is too restrictive.
Conjecture 1.7. Let $H$ be a 3-partite hypergraph with sides $A, B, C$, and assume that
(1) $|A| \geq 2 n-1$.
(2) $\operatorname{deg}(a) \geq n$ for every $a \in A$.
(3) $\operatorname{deg}(v) \leq 2 n-1$ for every $v \in B \cup C$.

Then $\nu(H) \geq n$.
The proofs of Theorem 1.5 given in [14, 2] were combinatorial. A topological proof yields a stronger version:

Theorem 1.8. Let $\mathcal{G}=(V, E)$ be a bipartite graph and let $\mathcal{F}=\left\{F_{1}, F_{2}, \ldots, F_{2 n-1}\right\}$ be a family of matchings in $\mathcal{G}$ so that $\left|F_{i}\right| \geq i$ for $i=1, \ldots, n-1$ and $\left|F_{i}\right|=n$ for $i=n, \ldots, 2 n-1$. Then $\mathcal{F}$ has a rainbow matching of size $n$.

In fact, this condition is also necessary. Call a sequence of numbers $a_{1} \leq a_{2} \leq \ldots \leq a_{2 n-1}$ accommodating if every family $\mathcal{F}=\left\{F_{1}, F_{2}, \ldots, F_{2 n-1}\right\}$ of matchings in a bipartite graph satisfying $\left|F_{i}\right| \geq a_{i}$ has a rainbow matching of size $n$.

Theorem 1.9. A sequence $a_{1} \leq a_{2} \leq \ldots \leq a_{2 n-1}$ is accommodating if and only if $a_{i} \geq \min (i, n)$ for all $i \leq 2 n-1$.

We call a pair $(X, Y)$ of sides in a $k$-partite hypergraph $H$ p-simple if no pair $(x, y)$, for $x \in X, y \in Y$ is contained in more than $p$ edges of $H$.

The following is a special case of Conjecture 1.7
Theorem 1.10. Let $H$ be a 3-partite hypergraph with sides $A, B, C$, and assume that
(1) $|A| \geq 2 n-1$ and $|B|=|C|=n$,
(2) $\operatorname{deg}(a)=n$ for every $a \in A$, and
(3) The pair $(A, C)$ is simple and the pair $(B, C)$ is 2-simple.

Then $\nu(H) \geq n$.
This theorem will yield a strengthening of the following result of Cameron and Wanless [11], which can be regarded as "half" of Conjecture 1.2.

Theorem 1.11. Every Latin square contains a permutation submatrix in which no symbol appears more than twice.

For a set $S$ of vertices in a hypergraph $H$ let $\delta(S)$ (resp. $\Delta(S)$ ) be the minimal (resp. maximal) degree of a vertex in $S$. In this terminology, Theorem 1.11 is:

Theorem 1.12. Let $H$ be an n-regular 3 -partite hypergraph with sides $A, B, C$ such that $|A|=|B|=|C|=n$, and all pairs of sides are simple. Then there exists a set $F$ of $n$ edges, satisfying
(1) $\Delta_{F}(B \cup C)=\delta_{F}(B \cup C)=1$
(2) $\Delta_{F}(A) \leq 2$.

Here the vertices in $A$ represent symbols, those in $B$ represent columns, and those in $C$ represent rows. Theorem 1.12 follows quite directly from Theorem 1.5

Proof. For each $a \in A$ let $M_{a}=N_{a}=\{(b, c)|b \in B, c \in C|(a, b, c) \in H\}$. By the simplicity assumption, these are matchings of size $n$. By Theorem [1.5, the set of matchings $\left\{M_{a} \mid a \in A\right\} \cup\left\{N_{a} \mid a \in A\right\}$ possesses a rainbow matching $R$ of size $n$. Defining $F=\{(a, b, c) \mid(b, c) \in R\}$ yields the desired result.

The topological tools will allow us to strengthen Theorem 1.11 to row-Latin squares, namely squares in which no symbol appears twice in the same row (but may appear more than once in the same column). In 3 -partite hypergraph terminology:
Theorem 1.13. For the conclusion of Theorem 1.12 to hold it suffices that the two pairs $(A, C)$ and $(B, C)$ are simple.

Of course, by symmetry, it suffices also to assume that $(A, B)$ and $(B, C)$ are simple.

## 2. A TOPOLOGICAL TOOL

For a graph $G$ denote by $\mathcal{I}(G)$ the complex (closed down hypergraph) of independent sets in $G$. If $G=L(H)$, the line graph of a hypergraph $H$, then $\mathcal{I}(G)$ is the complex of matchings in $H$. A simplicial complex $\mathcal{C}$ is called (homologically) $k$-connected if for every $-1 \leq j \leq k$, the $j$-th reduced simplicial homology group of $\mathcal{C}$ with rational coefficients $\tilde{H}_{j}(\mathcal{C})$ vanishes. The (homological) connectivity $\eta_{H}(\mathcal{C})$ is the largest $k$ for which $\mathcal{C}$ is $k$-connected, plus 2 .

## Remark 2.1.

(a) This is a shifted (by 2) version of the usual definition of connectivity. The shift simplifies the statements below, as well as the statements of basic properties of the connectivity parameter.
(b) If $\tilde{H}_{j}(\mathcal{C})=0$ for all $j$ then we define $\eta_{H}(\mathcal{C})=\infty$.
(c) There exists also a homotopical notion of connectivity, $\eta_{h}(\mathcal{C})$ : it is the minimal dimension of a "hole" in the complex. The first topological version of Hall's theorem [6] used that notion. The relationship between the two parameters is that $\eta_{H} \geq \eta_{h}$ for all complexes, and if $\eta_{h}(\mathcal{C}) \geq 3$, meaning that the complex is simply connected, then $\eta_{H}(\mathcal{C})=\eta_{h}(\mathcal{C})$. All facts mentioned in this article (in particular, the main tool we use, the Meshulam game) apply also to $\eta_{h}$.

Notation 2.2. Given sets $\left(V_{i}\right)_{i=1}^{n}$ and a subset $I$ of $[n]=\{1, \ldots, n\}$, we write $V_{I}$ for $\bigcup_{i \in I} V_{i}$. For $A \subseteq V(G)$ we denote by $\mathcal{I}(G) \upharpoonright A$ the complex of independent sets in the graph induced by $G$ on $A$.

Given sets $\left(V_{i}\right)_{i=1}^{n}$ of vertices in a graph $G$, an independent transversal is an independent set containing at least one vertex from each $V_{i}$. Note that in this definition the transversal needs not be injective, namely a set $V_{i}$ may be represented twice, which makes a difference if the sets are not disjoint. In our application (as is the case in the most common applications of the theorem) the sets $V_{i}$ are disjoint, in which case the transversal may well be assumed to be injective.

The following is a topological version of Hall's theorem:
Theorem 2.3. If $\eta_{H}\left(\mathcal{I}(G) \upharpoonright V_{I}\right) \geq|I|$ for every $I \subseteq[n]$ then there exists an independent transversal.

Variants of this theorem appeared implicitly in [6] and [17, and the theorem is stated and proved explicitly as Proposition 1.6 in [18].

A standard argument of adding dummy vertices yields the deficiency version of Theorem 2.3.
Theorem 2.4. If $\eta_{H}\left(\mathcal{I}\left(G\left[\bigcup_{i \in I} V_{i}\right]\right)\right) \geq|I|-d$ for every $I \subseteq[m]$ then the system has a partial independent transversal of size $m-d$.

In order to apply these theorems, combinatorially formulated lower bounds on $\eta_{H}(\mathcal{I}(G))$ are needed. One such bound is due to Meshulam [18] and is conveniently expressed in terms of a game between two players, CON and NON, on the graph $G$. CON wants to show high connectivity, NON wants to thwart her attempt. At each step, CON chooses an edge $e$ from the graph remaining at this stage, where in the first step the graph is $G$. NON can then either
(1) Delete $e$ from the graph. We call such a step a "deletion", and denote the resulting graph by " $G-e$ ". or
(2) Remove the two endpoints of $e$, together with all neighbors of these vertices and the edges incident to them, from the graph. We call such a step an "explosion", and denote the resulting graph by " $G * e$ ".

The result of the game (payoff to CON) is defined as follows: if at some point there remains an isolated vertex, the result is $\infty$. Otherwise, at some point all vertices have disappeared, in which case the result of the game is the number of explosion steps. We define $\Psi(G)$ as the value of the game, i.e., the maximum payoff to CON in an optimal play of both players.

Theorem 2.5. $18 \eta_{H}(\mathcal{I}(G)) \geq \Psi(G)$.
Remark 2.6. This formulation of $\Psi$ is equivalent to a recursive definition of $\Psi(G)$ as the maximum over all edges of $G$, of $\min (\Psi(G-e), \Psi(G * e)+1)$. For an explicit proof of Theorem 2.5 using the recursive definition of $\Psi$, see Theorem 1 in [1]. The "game" formulation first appeared in (4].

We shall use the Meshulam bound on line graphs. If $G=L(H)$ then playing the game means that CON offers NON a pair of edges in $H$ having a common vertex. Deletion in $G$ corresponds to separating the two edges at the vertex where they meet. Explosion corresponds to removing the three endpoints of these edges.

## 3. Proof of Theorem 1.9

Lemma 3.1. Let $G$ be a bipartite graph with sides $U, W$, and assume that there exist $u_{1}, \ldots, u_{2 \ell-1} \in U$ satisfying deg $\left(u_{i}\right) \geq \min (i, \ell)$ for all $i \leq 2 \ell-1$. Then $\Psi(L(G)) \geq \ell$.

Proof. By induction on $\ell$. For $\ell=1$ the statement is obvious. For $\ell>1$ CON chooses an edge $e=u_{1} w$ for some $w \in W$, and offers NON in any order all pairs $e, f$ for edges $f=y w \neq e$ (here $y \in U$ ). Assume first that $N O N$ explodes one of these pairs. Since $G$ is simple, the degree of each vertex in $U \cap V\left(G^{\prime}\right)$ is reduced by the explosion by at most 1 . This implies that the remaining graph $G^{\prime}$ satisfies the hypothesis of the theorem with $\ell-1$ replacing $\ell$. By the induction hypothesis, we have $\Psi\left(L\left(G^{\prime}\right)\right) \geq \ell-1$ and hence $\Psi(L(G)) \geq \ell$.

Next consider the case that NON separates $e$ from $f$ for all $f=y w \neq e$. Then CON offers all pairs $e, g$ for $g=u_{1} z \neq e$ (here $z \in W$ ). NON has to explode one of these pairs, say $e, g$ for $g=u_{1} z$, so as not to render $e$ isolated. Since all pairs $e, f$ for $f=y w \neq e$ have been separated, this explosion preserves the vertex $w$, removing at it only the edge $u_{1} w$. Thus, again, the degrees of all $u_{i}, i>1$ are reduced by only 1 , and the condition of the lemma holds with $\ell-1$ replacing $\ell$. Again, the desired conclusion follows by induction.

Proof of Theorem 1.9. For the sufficiency part, let $a_{1} \leq a_{2} \leq \ldots \leq a_{2 n-1}$ be a sequence of natural numbers satisfying $a_{i} \geq \min (i, n)$ for all $i \leq 2 n-1$ and let $\mathcal{F}=\left\{F_{1}, F_{2}, \ldots, F_{2 n-1}\right\}$ be a family of matchings in a bipartite graph $\mathcal{G}=(V, E)$ satisfying $\left|F_{i}\right| \geq a_{i}$ for all $i \leq 2 n-1$. We need to show that $\mathcal{F}$ has a rainbow matching of size $n$. Let $A$ and $B$ be the two sides of $V$ and let $m=|A| \geq n$. By considering a third side $C$ of size $2 n-1$ whose vertices correspond to the matchings $F_{i}$ we obtain a 3 -partite 3 -hypergraph $H$. We need to show that $H$ has a matching of size $n$.

Consider the bipartite graph $G$ induced on $B$ and $C$. Since the sets $F_{i}$ are matchings, $G$ is simple. The vertices in $A$ induce a partition $V_{1}, \ldots, V_{m}$ on $E(G)$. By the hypothesis of the theorem, the vertices of $C$
can be ordered as $c_{i}, i \leq 2 n-1$, where $\operatorname{deg}_{G}\left(c_{i}\right) \geq \min (i, n)$. For $I \subseteq A$ let $G^{I}$ be the graph with sides $B, C$ induced by $I$, namely having as edges pairs $(b, c)$ completed by some $a \in I$ to an edge of $H$. Write $k=|A|-|I|$. Since the sets $F_{i}$ are matchings, meaning that the pair of sides $(A, C)$ in $H$ is simple, for every $c \in C$ we have $d e g_{G^{I}}(c) \geq d e g_{G}(c)-k$. This entails that the conditions of Lemma 3.1 are valid for $G^{I}$ with $\ell=n-k$. By the lemma, we have $\eta_{H}\left(\mathcal{I}\left(L\left(G^{I}\right)\right)\right) \geq n-k$. By Theorem 2.4 this suffices to show that $\nu(H) \geq n$, namely there is a rainbow matching of size $n$.

For the other direction of the theorem, let $a_{1}, \ldots, a_{n}$ be an ascending sequence of natural numbers such that $a_{k} \leq k-1$ for some $k \leq n$. Let $C_{2 n}$ be a cycle of size $2 n$ and let $M \cup N$ be a partition of its edges into two matchings, each of size $n$. Consider the following family $\mathcal{F}=\left\{F_{1}, F_{2}, \ldots, F_{2 n-1}\right\}$ of matchings. Each of $F_{n+1}, \ldots, F_{2 n-1}$ is a copy of $M$. Let $N=\left\{e_{1}, \ldots, e_{n}\right\}$ and for each $i=1, \ldots, n$ let $F_{i}$ be a copy of the matching $\left\{e_{1}, \ldots, e_{\min \left(n, a_{i}\right)}\right\}$. Clearly, these matchings satisfy the condition of the theorem. We claim that they do not possess a rainbow matching of size $n$. Clearly, a matching of size $n$ is contained either in $M$ or in $N$, and since there are only $n-1$ matchings $F_{i}$ that meet $M$, a rainbow matching of size $n$ must represent the matchings $F_{1}, \ldots, F_{n}$. But this is impossible, since the union of the matchings $F_{1}, \ldots, F_{k}$ contain together fewer than $k$ edges.

It may be worth noting that an anaolgous version of Conjecture 1.4 fails. Let $Q^{1}, \ldots, Q^{2 k}$ be disjoint copies of $P_{3}$, the path with three edges, let $O_{i}$ be the set of two odd edges in $Q^{i}(i \leq 2 k)$, let $e_{i}$ be the middle edge in $Q^{i}$, let $F_{i}=\left\{e_{1}, \ldots, e_{k}\right\}$ for $i \leq k$, and let $F_{i}=\bigcup\left\{O_{j} \mid 1 \leq j \leq i-k\right\} \cup\left\{e_{j} \mid j>i-k\right\}$ for $i>k$. Then $\left|F_{i}\right| \geq i$ for all $i$, and the largest rainbow matching is of size $\left\lfloor\frac{3 k}{2}\right\rfloor$.

## 4. Proof of Theorems 1.10 and 1.13

For the convenience of the reader, let us repeat Theorem 1.10
Theorem 1.10, Let $H$ be a 3 -partite hypergraph with sides $A, B, C$, and assume that
(1) $|A| \geq 2 n-1$ and $|B|=|C|=n$,
(2) $\operatorname{deg}(a)=n$ for every $a \in A$, and
(3) The pair $(A, C)$ is simple and the pair $(B, C)$ is 2-simple.

Then $\nu(H) \geq n$.
Proof. Since the pair $(A, C)$ is simple, it follows from (2) that the degree of each vertex in $C$ is $|A|$. Since $(B, C)$ is 2 -simple, we have

$$
\begin{equation*}
\Delta(B) \leq 2 n \tag{1}
\end{equation*}
$$

For $b \in B$ let $V_{b}=\{(a, c) \mid(a, b, c) \in E(H)\}$ be a set of edges in $A \times C$. We need to show that the sets $V_{b}$ have a full rainbow matching. For $K \subseteq B$ let $G_{K}$ be the bipartite graph with sides $A$ and $C$, and with edge set $\bigcup_{b \in K} V_{b}$. By Theorems 2.3 and 2.5 it suffices to show that

$$
\begin{equation*}
\Psi\left(\mathcal{I}\left(L\left(G_{K}\right)\right)\right) \geq|K| \tag{2}
\end{equation*}
$$

for every set $K \subseteq B$. Write $k=|K|$. The minimal value of $\left|E\left(G_{K}\right)\right|$ occurs when all the vertices in $B \backslash K$ have maximal degree, which by (1) means:

$$
\begin{equation*}
\left|E\left(G_{K}\right)\right| \geq n(2 n-1)-2 n(n-k)=2 n k-n \tag{3}
\end{equation*}
$$

and by the 2-simplicity of the pair $(B, C)$ we have

$$
\begin{equation*}
\Delta_{G_{K}}(C) \leq 2 k \tag{4}
\end{equation*}
$$

We play Meshulam's game on $G_{K}$ as follows. Let $u_{1}, u_{2}, \ldots, u_{n}$ be the vertices of $C$. For each $i \leq n$ let $d_{i}$ be the degree of $u_{i}$ in $G_{K}$ and assume that $d_{1} \leq d_{2} \leq \cdots \leq d_{n}$. CON goes over the vertices in this order. Let $\ell_{i}$ be the degree of $u_{i}$ at the time it is handled. For each $i$, if $\ell_{i} \geq 2$ then CON offers all pairs of edges
meeting at $u_{i}$, in any order. This he does until NON explodes a pair, or until all edges meeting at $u_{i}$ are separated from each other. If $\ell_{i}<2$ then $u_{i}$ is skipped and CON handles the next vertex in the list.

Let $p_{i}$ be the number of explosions performed until $u_{i}$ was handled, including the possible explosion at $u_{i}$ itself. Assume first that for some $i$ NON separated all pairs of edges meeting at $u_{i}$ and $p_{i}+\ell_{i} \geq k$. Let $e_{1}, \ldots, e_{\ell_{i}}$ be the edges meeting at $u_{i}$ at the time it is handled and let $w_{1}, \ldots, w_{\ell_{i}}$ be their corresponding endpoints at $A$. For each $j=1, \ldots, \ell_{i}$, CON offers the pairs of edges $\left(e_{j}, f\right)$ for all edges $f$ meeting $e_{j}$ at $w_{j}$. Note that at least one such $f$ exists for each $w_{j}$, otherwise $e_{j}$ is isolated, meaning that the score of the game is $\infty$. Also note that NON must explode a pair in each $w_{j}$, otherwise the corresponding $e_{j}$ will become isolated. Thus, CON scores $\ell_{i}$ points at $u_{i}$, which together with the $p_{i}$ already scored, the score of the game is at least $k$.

Hence, if we make the negation assumption that the score of the game is less than $k$, then for each $u_{i}$ one of the following two occurs:

POS1: NON exploded a pair at $u_{i}$, or
POS2: NON separated all the edges meeting at $u_{i}$ and $p_{i}+\ell_{i}<k$.
In each explosion two vertices from $A$ are removed along with their incident edges. Thus, as $G_{K}$ is simple, in each explosion the degree of each vertex in $C$ decreases by at most two. Hence,

$$
\begin{equation*}
\ell_{i} \geq d_{i}-2 p_{i} \text { for all } i \tag{5}
\end{equation*}
$$

Suppose NON separated all the edges meeting at $u_{i}$. Then, by POS2 we have $p_{i}+\ell_{i}<k$. This together with (5) yield,

Claim 1: If NON separated all the edges meeting at $u_{i}$ then $d_{i}<k+p_{i}$.
For each $i=1, \ldots, n$ let $\pi_{i}=p_{i}+(n-i)$. Note that by the negation assumption $\pi_{n}=p_{n}<k$, so there exists a minimal index $t$ for which $\pi_{t}<k$.

Claim 2: NON separated all the edges meeting at $u_{t}$.
Proof of Claim 2. Consider first the case $t=1$. If NON exploded a pair at $u_{1}$ then $p_{1}=1$ and we have $\pi_{1}=n \geq k$, contradicting the definition of $t$. So we may assume that $t>1$. If NON exploded a pair at $u_{t}$ then $p_{t-1}=p_{t}-1$ and thus $\pi_{t-1}=p_{t-1}+(n-(t-1))=p_{t}-(n-t)<k$, contradicting the minimality of $t$.

Now, by the minimality of $t$ we have $\pi_{t-1}=p_{t-1}+(n-(t-1)) \geq k$ and $\pi_{t}=p_{t}+(n-t)<k$. By Claim $2, p_{t-1}=p_{t}$. Hence $p_{t-1}+(n-(t-1))=k$ and $p_{t}+(n-t)=k-1$. Thus

Claim 3: $t=n-k+p_{t}+1$.
We calculate an upper bound on $\left|E\left(G_{K}\right)\right|$. By Claims 1 and 2 and the fact that the $d_{i} \mathrm{~s}$ are ascending we have $d_{i}<k+p_{t}$ for all $i=1, \ldots, t$. From this and Claim 3, we conclude that the first $t$ vertices are incident to less than $\left(n-k+p_{t}+1\right)\left(k+p_{t}\right)$ edges in $\left|E\left(G_{K}\right)\right|$. By (4) and Claim 3 , the remaining $n-t$ edges are incident to at most $\left(k-p_{t}-1\right) 2 k$ edges. So, we have,

$$
\begin{equation*}
\left|E\left(G_{K}\right)\right|<\left(n-k+p_{t}+1\right)\left(k+p_{t}\right)+\left(k-p_{t}-1\right) 2 k \tag{6}
\end{equation*}
$$

Let $s=k-p_{t}$. Then (6) can be written in a somewhat simpler form:

$$
\begin{equation*}
\left|E\left(G_{K}\right)\right|<(n-s+1)(2 k-s)+(s-1) 2 k \tag{7}
\end{equation*}
$$

Let $m=s(n-s+1)$. Rearranging terms in (7) we obtain

$$
\begin{equation*}
\left|E\left(G_{K}\right)\right|<2 n k-m \tag{8}
\end{equation*}
$$

By the negation assumption $s \geq 1$ implying $m \geq n$. By (8) it follows that $\left|E\left(G_{K}\right)\right|<2 n k-n$, contradicting (3).

Proof of Theorem 1.13. Let $A^{\prime}$ be the union of two identical copies of $A$, that is, $A^{\prime}=A \cup A^{\dagger}$, where $A^{\dagger}=\left\{a^{\dagger} \mid a \in A\right\}$ and let $H^{\prime}$ be the hypergraph with sides $A^{\prime}, B, C$, defined by $E\left(H^{\prime}\right)=E(H) \cup\left\{\left(a^{\dagger}, b, c\right) \mid\right.$ $(a, b, c) \in H\}$. We have $\left|A^{\prime}\right|=2 n$. Also, since $(A, C)$ is simple so is $\left(A^{\prime}, C\right)$, and since $(B, C)$ is simple, the pair $(B, C)$ is 2 -simple in $H^{\prime}$. By Theorem 1.10 , we have $\nu\left(H^{\prime}\right)=n$, which implies the desired result.

## 5. Possible generalizations

In [8] the following conjecture was proposed:
Conjecture 5.1. Let $H$ be a simple 3-partite d-regular hypergraph with sides of size $n$.
(1) If $d \leq n$ then $\nu(H) \geq \frac{d-1}{d} n$.
(2) If $d \geq 2 n-1$ then $\nu(H)=n$.

Part (1) would imply Conjecture 1.3. Part (2) is sharp. To see this, let $a, b, c$ be vertices in the respective sides $A, B, C$ of a hypergraph $H$ with $|A|=|B|=|C|=n$, put in $E(H)$ the set $\{(a, b, x) \mid x \in C \backslash\{c\}\} \cup$ $\{(a, y, c) \mid y \in B \backslash\{b\}\} \cup\{(z, b, c) \mid z \in A \backslash\{a\}\}$, and complete it to a $2 n-2$ regular hypergraph by adding edges not containing any of $a, b, c$. In such a hypergraph $\nu \leq n-1$, since $a, b, c$ cannot be covered by the same matching.

An asymmetric formulation of the conjecture may better capture its essence:
Conjecture 5.2. Let $H$ be a simple 3-partite hypergraph with sides $A, B, C$.
(1) If $d=\delta(A) \geq \Delta(B \cup C)$ then $\nu(H) \geq \frac{d-1}{d}|A|$.
(2) If $\delta(A) \geq \max (\Delta(B \cup C), 2|A|-1)$ then $\nu(H)=|A|$.

Remark 5.3. Theorem 2.3 can be used to prove that if $\delta(A) \geq 2 \Delta(B \cup C)-1$ then $\nu(H)=|A|$.
Note that in item (2) there is a jump by a factor of 2 with respect to (1), similar to that between Conjecture 1.4 and Theorem 1.5. By (1) to get $\nu(H) \geq|A|-1$ we only (conjecturally) need $\delta(A) \geq \max (\Delta(B \cup C),|A|)$.

A conjecture generalizing Theorem 1.5 in the same spirit is:
Conjecture 5.4. Let $H$ be a simple 3 -partite hypergraph with sides $A, B, C$, and suppose that $|A|=2 n-1$, $\operatorname{deg}(a) \geq n$ for all $a \in A$, and $\operatorname{deg}(v) \leq 2 n-1$ for all $v \in B \cup C$. Then $\nu(H) \geq n$.

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