# DEGREE CONDITIONS FOR MATCHABILITY IN 3-PARTITE HYPERGRAPHS

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ABSTRACT. We study conjectures relating degree conditions in 3-partite hypergraphs to the matching number of the hypergraph, and use topological methods to prove special cases. In particular, we prove a strong version of a theorem of Drisko [14] (as generalized by the first two authors [2]), that every family of 2n-1 matchings of size n in a bipartite graph has a partial rainbow matching of size n. We show that milder restrictions on the sizes of the matchings suffice. Another result that is strengthened is a theorem of Cameron and Wanless [11], that every Latin square has a diagonal (permutation submatrix) in which no symbol appears more than twice. We show that the same is true under the weaker condition that the square is row-Latin.

## 1. RAINBOW MATCHINGS AND MATCHINGS IN 3-PARTITE 3-UNIFORM HYPERGRAPHS

All conjectures and results mentioned in this paper can be traced back to a by now well known conjecture of Ryser [20], that for n odd every Latin square possesses a full transversal, namely a permutation submatrix with distinct symbols. Brualdi [10] and Stein [21] conjectured that for general n, every  $n \times n$  Latin square possesses a partial transversal of size n - 1. Stein [21] generalized this still further, replacing the Latinity condition by the milder requirement that each of the n symbols appears in n cells (implying, among other things, that each cell contains precisely one symbol). See [22] for a survey on these conjectures and related results.

These conjectures can be formulated in the terminology of 3-partite hypergraphs. We say that a pair (X, Y) of sides of a k-partite hypergraph H is simple if no pair (x, y) with  $x \in X, y \in Y$  appears in more than one edge of H. A hypergraph is called simple if no edge repeats more than once. In this terminology, an  $n \times n$  Latin square is an n-regular 3-partite hypergraph with all sides of size n, and all three pairs of sides being simple. For a hypergraph H denote by  $\nu(H)$  the maximal size of a matching in H. In this language the Brualdi-Stein conjecture is:

**Conjecture 1.1.** Let H be an n-regular 3-partite hypergraph with sides A, B, C, all of size n, and assume that all three pairs of sides, (A, B), (B, C) and (A, C), are simple. Then  $\nu(H) \ge n - 1$ .

While Stein's conjecture is:

**Conjecture 1.2.** Let H be an n-regular 3-partite hypergraph with sides A, B, C, all of size n, and assume that the pair (A, B) is simple. Then  $\nu(H) \ge n - 1$ .

In fact, we believe that an even stronger (and more simply stated) conjecture is true:

**Conjecture 1.3.** Let H be a simple n-regular 3-partite hypergraph with sides of size n. Then  $\nu(H) \ge n-1$ .

In the final section of this paper we shall present some yet more general conjectures. But let us first turn to a concept pertinent to these conjectures, that of *rainbow matchings*. Given a family  $F_1, \ldots, F_m$  of sets, a choice  $f_1 \in F_1, \ldots, f_m \in F_m$  is called a *rainbow set*. If the elements of the sets  $F_i$  are themselves sets, and if the sets  $f_i$  are disjoint, then the rainbow set is called a *rainbow matching*. If the elements of the sets in  $F_i$ are edges of a bipartite graph G, then a rainbow matching is a matching in a 3-partite hypergraph, in which the vertices of one side represent the sets  $F_i$ , and the other two sides are those of G.

A generalization of Conjecture 1.2 was proposed in [2]:

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**Conjecture 1.4.** Any n matchings of size n in a bipartite graph have a partial rainbow matching of size n-1.

Many partial results have been obtained on this conjecture, see, e.g., [23, 15, 5, 16, 8, 12, 13, 19].

What happens if we demand that  $\nu \ge n$ , rather than n-1, or in the terminology of rainbow matchings we want a rainbow matching of size n? Strangely, we need to almost double the requirement on the number of matchings. Drisko [14] proved a theorem which was later generalized in [2] to:

**Theorem 1.5.** A family of 2n-1 matchings of size n in a bipartite graph has a rainbow matching of size n.

This is sharp - repeating n-1 times each of the matchings consisting respectively of the even edges in  $C_{2n}$  and the odd edges in  $C_{2n}$  shows that 2n-2 matchings do not suffice. In [7] this was shown to be the only example demonstrating the sharpness of the theorem.

In 3-partite hypergraphs terminology, Theorem 1.5 reads:

**Theorem 1.6.** Let H be a 3-partite hypergraph with sides A, B, C, and assume that

- (1) |A| = 2n 1.
- (2) deg(a) = n for every  $a \in A$ .
- (3) The pairs (A, B) and (A, C) are simple.

Then  $\nu(H) \geq n$ .

It is plausible that condition (3) in this theorem is too restrictive.

**Conjecture 1.7.** Let H be a 3-partite hypergraph with sides A, B, C, and assume that

- (1) |A| > 2n 1.
- (2)  $deg(a) \ge n$  for every  $a \in A$ .
- (3)  $deg(v) \leq 2n-1$  for every  $v \in B \cup C$ .

Then  $\nu(H) \geq n$ .

The proofs of Theorem 1.5 given in [14, 2] were combinatorial. A topological proof yields a stronger version:

**Theorem 1.8.** Let  $\mathcal{G} = (V, E)$  be a bipartite graph and let  $\mathcal{F} = \{F_1, F_2, \ldots, F_{2n-1}\}$  be a family of matchings in  $\mathcal{G}$  so that  $|F_i| \ge i$  for  $i = 1, \ldots, n-1$  and  $|F_i| = n$  for  $i = n, \ldots, 2n-1$ . Then  $\mathcal{F}$  has a rainbow matching of size n.

In fact, this condition is also necessary. Call a sequence of numbers  $a_1 \leq a_2 \leq \ldots \leq a_{2n-1}$  accommodating if every family  $\mathcal{F} = \{F_1, F_2, \ldots, F_{2n-1}\}$  of matchings in a bipartite graph satisfying  $|F_i| \geq a_i$  has a rainbow matching of size n.

**Theorem 1.9.** A sequence  $a_1 \leq a_2 \leq \ldots \leq a_{2n-1}$  is accommodating if and only if  $a_i \geq \min(i, n)$  for all  $i \leq 2n-1$ .

We call a pair (X, Y) of sides in a k-partite hypergraph H p-simple if no pair (x, y), for  $x \in X, y \in Y$  is contained in more than p edges of H.

The following is a special case of Conjecture 1.7:

**Theorem 1.10.** Let H be a 3-partite hypergraph with sides A, B, C, and assume that

(1)  $|A| \ge 2n - 1$  and |B| = |C| = n,

- (2) deg(a) = n for every  $a \in A$ , and
- (3) The pair (A, C) is simple and the pair (B, C) is 2-simple.

Then  $\nu(H) \geq n$ .

This theorem will yield a strengthening of the following result of Cameron and Wanless [11], which can be regarded as "half" of Conjecture 1.2:

**Theorem 1.11.** Every Latin square contains a permutation submatrix in which no symbol appears more than twice.

For a set S of vertices in a hypergraph H let  $\delta(S)$  (resp.  $\Delta(S)$ ) be the minimal (resp. maximal) degree of a vertex in S. In this terminology, Theorem 1.11 is:

**Theorem 1.12.** Let H be an n-regular 3-partite hypergraph with sides A, B, C such that |A| = |B| = |C| = n, and all pairs of sides are simple. Then there exists a set F of n edges, satisfying

(1)  $\Delta_F(B \cup C) = \delta_F(B \cup C) = 1$ (2)  $\Delta_F(A) \le 2.$ 

Here the vertices in A represent symbols, those in B represent columns, and those in C represent rows. Theorem 1.12 follows quite directly from Theorem 1.5:

*Proof.* For each  $a \in A$  let  $M_a = N_a = \{(b,c) \mid b \in B, c \in C \mid (a,b,c) \in H\}$ . By the simplicity assumption, these are matchings of size n. By Theorem 1.5, the set of matchings  $\{M_a \mid a \in A\} \cup \{N_a \mid a \in A\}$  possesses a rainbow matching R of size n. Defining  $F = \{(a,b,c) \mid (b,c) \in R\}$  yields the desired result.  $\Box$ 

The topological tools will allow us to strengthen Theorem 1.11 to row-Latin squares, namely squares in which no symbol appears twice in the same row (but may appear more than once in the same column). In 3-partite hypergraph terminology:

**Theorem 1.13.** For the conclusion of Theorem 1.12 to hold it suffices that the two pairs (A, C) and (B, C) are simple.

Of course, by symmetry, it suffices also to assume that (A, B) and (B, C) are simple.

## 2. A topological tool

For a graph G denote by  $\mathcal{I}(G)$  the complex (closed down hypergraph) of independent sets in G. If G = L(H), the line graph of a hypergraph H, then  $\mathcal{I}(G)$  is the complex of matchings in H. A simplicial complex C is called (homologically) k-connected if for every  $-1 \leq j \leq k$ , the j-th reduced simplicial homology group of C with rational coefficients  $\tilde{H}_j(C)$  vanishes. The (homological) connectivity  $\eta_H(C)$  is the largest k for which C is k-connected, plus 2.

Remark 2.1.

- (a) This is a shifted (by 2) version of the usual definition of connectivity. The shift simplifies the statements below, as well as the statements of basic properties of the connectivity parameter.
- (b) If  $\tilde{H}_j(\mathcal{C}) = 0$  for all j then we define  $\eta_H(\mathcal{C}) = \infty$ .
- (c) There exists also a homotopical notion of connectivity,  $\eta_h(\mathcal{C})$ : it is the minimal dimension of a "hole" in the complex. The first topological version of Hall's theorem [6] used that notion. The relationship between the two parameters is that  $\eta_H \ge \eta_h$  for all complexes, and if  $\eta_h(\mathcal{C}) \ge 3$ , meaning that the complex is simply connected, then  $\eta_H(\mathcal{C}) = \eta_h(\mathcal{C})$ . All facts mentioned in this article (in particular, the main tool we use, the Meshulam game) apply also to  $\eta_h$ .

Notation 2.2. Given sets  $(V_i)_{i=1}^n$  and a subset I of  $[n] = \{1, \ldots, n\}$ , we write  $V_I$  for  $\bigcup_{i \in I} V_i$ . For  $A \subseteq V(G)$  we denote by  $\mathcal{I}(G) \upharpoonright A$  the complex of independent sets in the graph induced by G on A.

Given sets  $(V_i)_{i=1}^n$  of vertices in a graph G, an *independent transversal* is an independent set containing at least one vertex from each  $V_i$ . Note that in this definition the transversal needs not be injective, namely a set  $V_i$  may be represented twice, which makes a difference if the sets are not disjoint. In our application (as is the case in the most common applications of the theorem) the sets  $V_i$  are disjoint, in which case the transversal may well be assumed to be injective.

The following is a topological version of Hall's theorem:

**Theorem 2.3.** If  $\eta_H(\mathcal{I}(G) \upharpoonright V_I) \geq |I|$  for every  $I \subseteq [n]$  then there exists an independent transversal.

Variants of this theorem appeared implicitly in [6] and [17], and the theorem is stated and proved explicitly as Proposition 1.6 in [18].

A standard argument of adding dummy vertices yields the deficiency version of Theorem 2.3:

**Theorem 2.4.** If  $\eta_H(\mathcal{I}(G[\bigcup_{i \in I} V_i])) \ge |I| - d$  for every  $I \subseteq [m]$  then the system has a partial independent transversal of size m - d.

In order to apply these theorems, combinatorially formulated lower bounds on  $\eta_H(\mathcal{I}(G))$  are needed. One such bound is due to Meshulam [18] and is conveniently expressed in terms of a game between two players, CON and NON, on the graph G. CON wants to show high connectivity, NON wants to thwart her attempt. At each step, CON chooses an edge e from the graph remaining at this stage, where in the first step the graph is G. NON can then either

- (1) Delete e from the graph. We call such a step a "deletion", and denote the resulting graph by "G-e". or
- (2) Remove the two endpoints of e, together with all neighbors of these vertices and the edges incident to them, from the graph. We call such a step an "explosion", and denote the resulting graph by "G \* e".

The result of the game (payoff to CON) is defined as follows: if at some point there remains an isolated vertex, the result is  $\infty$ . Otherwise, at some point all vertices have disappeared, in which case the result of the game is the number of explosion steps. We define  $\Psi(G)$  as the value of the game, i.e., the maximum payoff to CON in an optimal play of both players.

**Theorem 2.5.** [18]  $\eta_H(\mathcal{I}(G)) \ge \Psi(G)$ .

Remark 2.6. This formulation of  $\Psi$  is equivalent to a recursive definition of  $\Psi(G)$  as the maximum over all edges of G, of  $\min(\Psi(G-e), \Psi(G*e)+1)$ . For an explicit proof of Theorem 2.5 using the recursive definition of  $\Psi$ , see Theorem 1 in [1]. The "game" formulation first appeared in [4].

We shall use the Meshulam bound on line graphs. If G = L(H) then playing the game means that CON offers NON a pair of edges in H having a common vertex. Deletion in G corresponds to separating the two edges at the vertex where they meet. Explosion corresponds to removing the three endpoints of these edges.

## 3. Proof of Theorem 1.9

**Lemma 3.1.** Let G be a bipartite graph with sides U, W, and assume that there exist  $u_1, \ldots, u_{2\ell-1} \in U$ satisfying  $deg(u_i) \ge \min(i, \ell)$  for all  $i \le 2\ell - 1$ . Then  $\Psi(L(G)) \ge \ell$ .

Proof. By induction on  $\ell$ . For  $\ell = 1$  the statement is obvious. For  $\ell > 1$  CON chooses an edge  $e = u_1 w$  for some  $w \in W$ , and offers NON in any order all pairs e, f for edges  $f = yw \neq e$  (here  $y \in U$ ). Assume first that NON explodes one of these pairs. Since G is simple, the degree of each vertex in  $U \cap V(G')$  is reduced by the explosion by at most 1. This implies that the remaining graph G' satisfies the hypothesis of the theorem with  $\ell - 1$  replacing  $\ell$ . By the induction hypothesis, we have  $\Psi(L(G')) \geq \ell - 1$  and hence  $\Psi(L(G)) \geq \ell$ .

Next consider the case that NON separates e from f for all  $f = yw \neq e$ . Then CON offers all pairs e, g for  $g = u_1 z \neq e$  (here  $z \in W$ ). NON has to explode one of these pairs, say e, g for  $g = u_1 z$ , so as not to render e isolated. Since all pairs e, f for  $f = yw \neq e$  have been separated, this explosion preserves the vertex w, removing at it only the edge  $u_1w$ . Thus, again, the degrees of all  $u_i, i > 1$  are reduced by only 1, and the condition of the lemma holds with  $\ell - 1$  replacing  $\ell$ . Again, the desired conclusion follows by induction.  $\Box$ 

Proof of Theorem 1.9. For the sufficiency part, let  $a_1 \leq a_2 \leq \ldots \leq a_{2n-1}$  be a sequence of natural numbers satisfying  $a_i \geq \min(i, n)$  for all  $i \leq 2n-1$  and let  $\mathcal{F} = \{F_1, F_2, \ldots, F_{2n-1}\}$  be a family of matchings in a bipartite graph  $\mathcal{G} = (V, E)$  satisfying  $|F_i| \geq a_i$  for all  $i \leq 2n-1$ . We need to show that  $\mathcal{F}$  has a rainbow matching of size n. Let A and B be the two sides of V and let  $m = |A| \geq n$ . By considering a third side C of size 2n-1 whose vertices correspond to the matchings  $F_i$  we obtain a 3-partite 3-hypergraph H. We need to show that H has a matching of size n.

Consider the bipartite graph G induced on B and C. Since the sets  $F_i$  are matchings, G is simple. The vertices in A induce a partition  $V_1, \ldots, V_m$  on E(G). By the hypothesis of the theorem, the vertices of C

can be ordered as  $c_i$ ,  $i \leq 2n-1$ , where  $deg_G(c_i) \geq \min(i, n)$ . For  $I \subseteq A$  let  $G^I$  be the graph with sides B, C induced by I, namely having as edges pairs (b, c) completed by some  $a \in I$  to an edge of H. Write k = |A| - |I|. Since the sets  $F_i$  are matchings, meaning that the pair of sides (A, C) in H is simple, for every  $c \in C$  we have  $deg_{G^I}(c) \geq deg_G(c) - k$ . This entails that the conditions of Lemma 3.1 are valid for  $G^I$  with  $\ell = n - k$ . By the lemma, we have  $\eta_H(\mathcal{I}(L(G^I))) \geq n - k$ . By Theorem 2.4 this suffices to show that  $\nu(H) \geq n$ , namely there is a rainbow matching of size n.

For the other direction of the theorem, let  $a_1, \ldots, a_n$  be an ascending sequence of natural numbers such that  $a_k \leq k-1$  for some  $k \leq n$ . Let  $C_{2n}$  be a cycle of size 2n and let  $M \cup N$  be a partition of its edges into two matchings, each of size n. Consider the following family  $\mathcal{F} = \{F_1, F_2, \ldots, F_{2n-1}\}$  of matchings. Each of  $F_{n+1}, \ldots, F_{2n-1}$  is a copy of M. Let  $N = \{e_1, \ldots, e_n\}$  and for each  $i = 1, \ldots, n$  let  $F_i$  be a copy of the matching  $\{e_1, \ldots, e_{\min(n,a_i)}\}$ . Clearly, these matchings satisfy the condition of the theorem. We claim that they do not possess a rainbow matching of size n. Clearly, a matching of size n is contained either in M or in N, and since there are only n-1 matchings  $F_i$  that meet M, a rainbow matching of size n must represent the matchings  $F_1, \ldots, F_n$ . But this is impossible, since the union of the matchings  $F_1, \ldots, F_k$  contain together fewer than k edges.

It may be worth noting that an analogous version of Conjecture 1.4 fails. Let  $Q^1, \ldots, Q^{2k}$  be disjoint copies of  $P_3$ , the path with three edges, let  $O_i$  be the set of two odd edges in  $Q^i$   $(i \le 2k)$ , let  $e_i$  be the middle edge in  $Q^i$ , let  $F_i = \{e_1, \ldots, e_k\}$  for  $i \le k$ , and let  $F_i = \bigcup \{O_j \mid 1 \le j \le i - k\} \cup \{e_j \mid j > i - k\}$  for i > k. Then  $|F_i| \ge i$  for all i, and the largest rainbow matching is of size  $\lfloor \frac{3k}{2} \rfloor$ .

4. Proof of Theorems 1.10 and 1.13

For the convenience of the reader, let us repeat Theorem 1.10:

**Theorem 1.10.** Let H be a 3-partite hypergraph with sides A, B, C, and assume that

(1)  $|A| \ge 2n - 1$  and |B| = |C| = n,

(2) deg(a) = n for every  $a \in A$ , and

(3) The pair (A, C) is simple and the pair (B, C) is 2-simple.

Then  $\nu(H) \geq n$ .

*Proof.* Since the pair (A, C) is simple, it follows from (2) that the degree of each vertex in C is |A|. Since (B, C) is 2-simple, we have

(1) 
$$\Delta(B) \le 2n$$

For  $b \in B$  let  $V_b = \{(a, c) \mid (a, b, c) \in E(H)\}$  be a set of edges in  $A \times C$ . We need to show that the sets  $V_b$  have a full rainbow matching. For  $K \subseteq B$  let  $G_K$  be the bipartite graph with sides A and C, and with edge set  $\bigcup_{b \in K} V_b$ . By Theorems 2.3 and 2.5 it suffices to show that

(2) 
$$\Psi(\mathcal{I}(L(G_K))) \ge |K|$$

for every set  $K \subseteq B$ . Write k = |K|. The minimal value of  $|E(G_K)|$  occurs when all the vertices in  $B \setminus K$  have maximal degree, which by (1) means:

(3) 
$$|E(G_K)| \ge n(2n-1) - 2n(n-k) = 2nk - n,$$

and by the 2-simplicity of the pair (B, C) we have

(4) 
$$\Delta_{G_K}(C) \le 2k$$

We play Meshulam's game on  $G_K$  as follows. Let  $u_1, u_2, \ldots, u_n$  be the vertices of C. For each  $i \leq n$  let  $d_i$  be the degree of  $u_i$  in  $G_K$  and assume that  $d_1 \leq d_2 \leq \cdots \leq d_n$ . CON goes over the vertices in this order. Let  $\ell_i$  be the degree of  $u_i$  at the time it is handled. For each i, if  $\ell_i \geq 2$  then CON offers all pairs of edges

meeting at  $u_i$ , in any order. This he does until NON explodes a pair, or until all edges meeting at  $u_i$  are separated from each other. If  $\ell_i < 2$  then  $u_i$  is skipped and CON handles the next vertex in the list.

Let  $p_i$  be the number of explosions performed until  $u_i$  was handled, including the possible explosion at  $u_i$  itself. Assume first that for some i NON separated all pairs of edges meeting at  $u_i$  and  $p_i + \ell_i \ge k$ . Let  $e_1, \ldots, e_{\ell_i}$  be the edges meeting at  $u_i$  at the time it is handled and let  $w_1, \ldots, w_{\ell_i}$  be their corresponding endpoints at A. For each  $j = 1, \ldots, \ell_i$ , CON offers the pairs of edges  $(e_j, f)$  for all edges f meeting  $e_j$  at  $w_j$ . Note that at least one such f exists for each  $w_j$ , otherwise  $e_j$  is isolated, meaning that the score of the game is  $\infty$ . Also note that NON must explode a pair in each  $w_j$ , otherwise the corresponding  $e_j$  will become isolated. Thus, CON scores  $\ell_i$  points at  $u_i$ , which together with the  $p_i$  already scored, the score of the game is at least k.

Hence, if we make the negation assumption that the score of the game is less than k, then for each  $u_i$  one of the following two occurs:

POS1: NON exploded a pair at  $u_i$ , or POS2: NON separated all the edges meeting at  $u_i$  and  $p_i + \ell_i < k$ .

In each explosion two vertices from A are removed along with their incident edges. Thus, as  $G_K$  is simple, in each explosion the degree of each vertex in C decreases by at most two. Hence,

(5) 
$$\ell_i \ge d_i - 2p_i \text{ for all } i.$$

Suppose NON separated all the edges meeting at  $u_i$ . Then, by POS2 we have  $p_i + \ell_i < k$ . This together with (5) yield,

Claim 1: If NON separated all the edges meeting at  $u_i$  then  $d_i < k + p_i$ .

For each i = 1, ..., n let  $\pi_i = p_i + (n - i)$ . Note that by the negation assumption  $\pi_n = p_n < k$ , so there exists a minimal index t for which  $\pi_t < k$ .

Claim 2: NON separated all the edges meeting at  $u_t$ .

Proof of Claim 2. Consider first the case t = 1. If NON exploded a pair at  $u_1$  then  $p_1 = 1$  and we have  $\pi_1 = n \ge k$ , contradicting the definition of t. So we may assume that t > 1. If NON exploded a pair at  $u_t$  then  $p_{t-1} = p_t - 1$  and thus  $\pi_{t-1} = p_{t-1} + (n - (t-1)) = p_t - (n-t) < k$ , contradicting the minimality of t.

Now, by the minimality of t we have  $\pi_{t-1} = p_{t-1} + (n - (t-1)) \ge k$  and  $\pi_t = p_t + (n-t) < k$ . By Claim 2,  $p_{t-1} = p_t$ . Hence  $p_{t-1} + (n - (t-1)) = k$  and  $p_t + (n-t) = k - 1$ . Thus

Claim 3:  $t = n - k + p_t + 1$ .

We calculate an upper bound on  $|E(G_K)|$ . By Claims 1 and 2 and the fact that the  $d_i$ s are ascending we have  $d_i < k + p_t$  for all i = 1, ..., t. From this and Claim 3, we conclude that the first t vertices are incident to less than  $(n - k + p_t + 1)(k + p_t)$  edges in  $|E(G_K)|$ . By (4) and Claim 3, the remaining n - t edges are incident to at most  $(k - p_t - 1)2k$  edges. So, we have,

(6) 
$$|E(G_K)| < (n-k+p_t+1)(k+p_t) + (k-p_t-1)2k.$$

Let  $s = k - p_t$ . Then (6) can be written in a somewhat simpler form:

(7) 
$$|E(G_K)| < (n-s+1)(2k-s) + (s-1)2k.$$

Let m = s(n - s + 1). Rearranging terms in (7) we obtain

$$|E(G_K)| < 2nk - m_i$$

By the negation assumption  $s \ge 1$  implying  $m \ge n$ . By (8) it follows that  $|E(G_K)| < 2nk - n$ , contradicting (3). $\square$ 

Proof of Theorem 1.13. Let A' be the union of two identical copies of A, that is,  $A' = A \cup A^{\dagger}$ , where  $A^{\dagger} = \{a^{\dagger} \mid a \in A\}$  and let H' be the hypergraph with sides A', B, C, defined by  $E(H') = E(H) \cup \{(a^{\dagger}, b, c) \mid b \in A\}$  $(a, b, c) \in H$ . We have |A'| = 2n. Also, since (A, C) is simple so is (A', C), and since (B, C) is simple, the pair (B, C) is 2-simple in H'. By Theorem 1.10, we have  $\nu(H') = n$ , which implies the desired result.  $\square$ 

### 5. Possible generalizations

In [8] the following conjecture was proposed:

**Conjecture 5.1.** Let H be a simple 3-partite d-regular hypergraph with sides of size n.

(1) If  $d \le n$  then  $\nu(H) \ge \frac{d-1}{d}n$ . (2) If  $d \ge 2n - 1$  then  $\nu(H) = n$ .

Part (1) would imply Conjecture 1.3. Part (2) is sharp. To see this, let a, b, c be vertices in the respective sides A, B, C of a hypergraph H with |A| = |B| = |C| = n, put in E(H) the set  $\{(a, b, x) \mid x \in C \setminus \{c\}\} \cup$  $\{(a, y, c) \mid y \in B \setminus \{b\}\} \cup \{(z, b, c) \mid z \in A \setminus \{a\}\}$ , and complete it to a 2n - 2 regular hypergraph by adding edges not containing any of a, b, c. In such a hypergraph  $\nu \leq n-1$ , since a, b, c cannot be covered by the same matching.

An asymmetric formulation of the conjecture may better capture its essence:

**Conjecture 5.2.** Let H be a simple 3-partite hypergraph with sides A, B, C.

(1) If  $d = \delta(A) \ge \Delta(B \cup C)$  then  $\nu(H) \ge \frac{d-1}{d}|A|$ . (2) If  $\delta(A) \ge \max(\Delta(B \cup C), 2|A| - 1)$  then  $\nu(H) = |A|$ .

Remark 5.3. Theorem 2.3 can be used to prove that if  $\delta(A) > 2\Delta(B \cup C) - 1$  then  $\nu(H) = |A|$ .

Note that in item (2) there is a jump by a factor of 2 with respect to (1), similar to that between Conjecture 1.4 and Theorem 1.5. By (1) to get  $\nu(H) \geq |A| - 1$  we only (conjecturally) need  $\delta(A) \geq \max(\Delta(B \cup C), |A|)$ .

A conjecture generalizing Theorem 1.5 in the same spirit is:

**Conjecture 5.4.** Let H be a simple 3-partite hypergraph with sides A, B, C, and suppose that |A| = 2n - 1.  $deg(a) \ge n$  for all  $a \in A$ , and  $deg(v) \le 2n-1$  for all  $v \in B \cup C$ . Then  $\nu(H) \ge n$ .

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