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ARTICLE

Color degree and monochromatic degree conditions for short properly colored cycles in edge-colored graphs

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Abstract

For an edge-colored graph, its minimum color degree is defined as the minimum number of colors appearing on the edges incident to a vertex and its maximum monochromatic degree is defined as the maximum number of edges incident to a vertex with a same color. A cycle is called properly colored if every two of its adjacent edges have distinct colors. In this article, we first give a minimum color degree condition for the existence of properly colored cycles, then obtain the minimum color degree condition for an edge-colored complete graph to contain properly colored triangles. Afterwards, we characterize the structure of an edge-colored complete bipartite graph without containing properly colored cycles of length 4 and give the minimum color degree and maximum monochromatic degree conditions for an edge-colored complete bipartite graph to contain properly colored cycles of length 4, and those passing through a given vertex or edge, respectively.

KEYWORDS

complete (bipartite) graphs, edge-colored graphs, minimum color degree, maximum monochromatic degree, properly colored cycles

MATHEMATICS SUBJECT CLASSIFICATION:

05C15, 05C38

1 | INTRODUCTION

All graphs considered in this article are finite and simple. For terminology and notation not defined here, we refer the reader to [4].

Let G be a graph. We use $V(G)$ and $E(G)$ to denote the set of vertices and edges of G, respectively. For two disjoint subsets A and B of $V(G)$, denote by $E_G(A, B)$ the set of edges in G between A and B. If A contains only one vertex a, we write $E_G(a, B)$ instead of $E_G(\{a\}, B)$. An *edge-coloring* of G is a mapping $C_G : E(G) \to \mathbb{N}$, where $\mathbb{N}$ is the natural number set. We call G an *edge-colored graph* if it has such an edge-coloring and say that G is *properly edge-colored* (or briefly *PC*) if each pair of adjacent edges of G are in distinct colors, and *rainbow* if all edges of G are in distinct colors. Denote by $C(G)$ the set of colors appearing on the edges of G. For vertex-disjoint subgraphs F and H of G, we use $C(F, H)$ to denote the set of colors appearing on the edges between F and H. For a color $i \in C(G)$, we use G^i to denote the subgraph of G induced by $\{e \in E(G) : C_G(e) = i\}$. For a vertex v of G, the *color neighbor* of v, denote by $N_G^c(v)$, is the set of colors appearing on the edges incident to v. The *color degree* of v, denote by $d_G^c(v)$, is the cardinality of $N_G^c(v)$. For a subset S of $V(G)$, define the *minimum color degree* of vertices in S by $\delta_G^c(S) = \min\{d_G^c(v) : v \in S\}$ and the *maximum monochromatic degree* of vertices in S by $\Delta_G^{mon}(S) = \max\{d_{G^i}(v) : i \in C(G), v \in S\}$. For a subgraph H of G, define $\delta_G^c(H) = \delta_G^c(V(H))$ and $\Delta_G^{mon}(H) = \Delta_G^{mon}(V(H))$. If there is no ambiguity, we often write $C(e)$ for $C_G(e)$, $N^c(v)$ for $N_G^c(v)$, $d^c(v)$ for $d_G^c(v)$, $\delta^c(G)$ for $\delta_G^c(G)$ and $\Delta^{mon}(G)$ for $\Delta_G^{mon}(G)$. Throughout the article, we use C_ℓ to denote a cycle of length ℓ.

Subgraphs in edge-colored graphs have been well studied through the ages. For rainbow subgraphs, see the survey papers [8,12]. For PC subgraphs, especially, PC paths and cycles, see Chapter 16 of [2]. When considering the existence of PC cycles in an edge-colored graph, one often needs to know the structure of graphs containing no PC cycles. So we start with the following important structural result:

Theorem 1 (Grossman and Häggkvist [10], Yeo [22]). *Let G be an edge-colored graph containing no PC cycles. Then there is a vertex $z \in V(G)$ such that no component of $G - z$ is joint to z with edges of more than one color.*

There are lots of results and problems on the existence of PC Hamilton cycles and long cycles (see [1,6,7,16,17,19,20]). For short PC cycles, especially, a PC triangle (or a rainbow triangle), the well-known Gallai coloring theory gives a structural characterization of edge-colored complete graphs containing no rainbow triangles (see [9] and [11]). Conditions for the existence of rainbow triangles in edge-colored graphs (not necessarily complete) are given in [13] and [14]. In a variety of those work, minimum color degree conditions for edge-colored graphs to contain PC cycles with certain properties are often discussed. One natural problem in this area would be asking a sharp color degree condition for an edge-colored graph to contain a PC cycle. In this article, we first give a complete solution to this problem.

To state our answer, we construct an edge-colored graph G_D such that $\delta^c(G_D) = D$ and G_D contains no PC cycles. Let G_1 be an edge-colored graph that is isomorphic to K_2 with color $c_1^{G_1}$. For $1 \le i \le D - 1$, let G_{i+1} be an edge-colored graph obtained from $i + 1$ vertex-disjoint copies of G_i, say $H_1, \ldots, H_{i+1}$, and a new vertex v_{i+1} by joining v_{i+1} to each H_j for $1 \le j \le i + 1$ and coloring the edges from v_{i+1} to H_j with color $c_{i+1}^{H_j}$ (for $s \ne t$, $C(H_s)$ and $C(H_t)$ may be different as long as G_{i+1} satisfies the condition that $\delta^c(G_{i+1}) = i + 1$ and G_{i+1} has no PC cycles). By the construction, we can easily check that $|V(G_D)| = D! \sum_{i=0}^{D} \frac{1}{i!}$, $\delta^c(G_D) = D$ and G_D has no PC cycles.

Theorem 2. *Let G be an edge-colored graph of order n with $\delta^c(G) \geq D$. Suppose that $n \leq D! \sum_{i=0}^{D} \frac{1}{i!}$ and G contains no PC cycles. Then the equality on n is attained, and moreover, G is isomorphic to $\boldsymbol{G}_D$, up to the edge-coloring structure.*

The problem of giving sharp minimum color degree conditions for short PC cycles seems more difficult. So far, we have some known results that give partial answers to this problem. Lo [16] showed that, for any constant number $\epsilon > 0$, if an edge-colored graph of order n (sufficiently large) has minimum color degree at least $\left(\frac{2}{3} + \epsilon\right) n$, then it contains a PC cycle of length ℓ for all $3 \leq \ell \leq n$. Li [14] proved that if an edge-colored graph of order n has minimum color degree at least $\frac{n+1}{2}$, then it contains a PC triangle. Čada et al. [5] proved that if an edge-colored triangle-free graph of order n has minimum color degree at least $\frac{n}{3} + 1$ then it contains a rainbow C_4. As seen in these results, it seems a reasonable approach for us to consider this problem for some specified graph classes. In this article, we shall restrict our considerations to this problem in edge-colored complete graphs and complete bipartite graphs.

In the study of PC Hamilton cycles and long cycles in edge-colored complete graphs, maximum monochromatic degree conditions are often involved (see [18] and [20] and the articles cited therein). Bollobás and Erdős [3] conjectured that every edge-colored complete graph K_n with $\Delta^{mon}(K_n) < \lfloor \frac{n}{2} \rfloor$ contains a PC Hamilton cycle. Recently, Lo [18] proved that this conjecture is true asymptotically. For short PC cycles, Wang and Zhou [21] showed that this upper bound of $\Delta^{mon}(K_n)$ can guarantee a PC triangle or a PC C_4 with two colors. Gyárfás and Simonyi [11] proved that each edge-colored complete graph K_n with $\Delta^{mon}(K_n) < \frac{2n}{5}$ contains a PC triangle and this bound is tight. Here, we give the color degree condition for PC triangles in edge-colored complete graphs.

Theorem 3. *If $\delta^c(K_n) > \log_2 n$ with $n \geq 3$, then K_n contains a PC C_3.*

Remark 1. *The bound of $\delta^c(K_n)$ in Theorem 3 is tight. The following construction due to Li and Wang [15] shows the sharpness. Let $G_1 = K_2$. For the unique edge $e \in E(G_1)$, let $C(e) = 1$. For $i = 2, 3, \ldots$, construct an edge-colored graph G_{i+1} by joining two disjoint copies of G_i completely with edges of color $i + 1$. The resulting edge-colored complete graph K_n satisfies $\delta^c(K_n) = \log_2 n$ but it has no PC C_3's.*

In what follows, we always consider an edge-colored complete bipartite graph $K_{m,n}$ with $m \geq 2$ and $n \geq 2$ (so we often omit lower bounds on m and n).

Observation 1. *If $\delta^c(K_{m,n}) \geq 2$, then $K_{m,n}$ contains a PC C_4 or a PC C_6.*

Proof. Since $\delta^c(K_{m,n}) \geq 2$, we have $m, n \geq 2$. Thus $K_{m,n} - x$ is connected for any $x \in V(K_{m,n})$. By Theorem 1, $K_{m,n}$ contains a PC cycle. Otherwise, there is a vertex $y \in V(K_{m,n})$ such that $d^c(y) = 1$, a contradiction.

Let $C = v_1 v_2 \cdots v_k v_1$ be a shortest PC cycle in $K_{m,n}$. If $k \leq 6$, then there is nothing to prove. Now, suppose that $k \geq 8$, $C(v_1 v_2) = p$ and $C(v_2 v_3) = q$. Then $C(v_3 v_4) \neq q$. If $C(v_3 v_4) = p$, then either $v_1 v_2 v_3 v_4 v_1$ or $v_1 v_4 v_5 \cdots v_k v_1$ is a shorter PC cycle than C, a contradiction. So we can assume that $C(v_3 v_4) = r$. Since $v_1 v_2 v_3 v_4 v_1$ is not a PC cycle, we have $C(v_1 v_4) = p$ or r. Without loss of generality, assume that $C(v_1 v_4) = p$. Since $v_1 v_4 v_5 \cdots v_k v_1$ is not a PC cycle, we have $C(v_4 v_5) = p$. Moreover, considering that neither $v_1 v_2 v_3 v_4 v_7 v_8 \cdots v_k v_1$ nor $v_1 v_4 v_7 v_8 \cdots v_k v_1$ is a PC cycle, we have $C(v_4 v_7) = C(v_7 v_8)$. Thus $C(v_4 v_7) \neq C(v_6 v_7)$. Since $v_4 v_5 v_6 v_7 v_4$ is not a PC C_4, we have $C(v_4 v_5) = C(v_4 v_7) = C(v_7 v_8) = p$. This implies that either $v_2 v_3 v_4 v_5 v_6 v_7 v_2$ or $v_1 v_2 v_7 v_8 \cdots v_k v_1$ is a PC cycle, a contradiction. ■

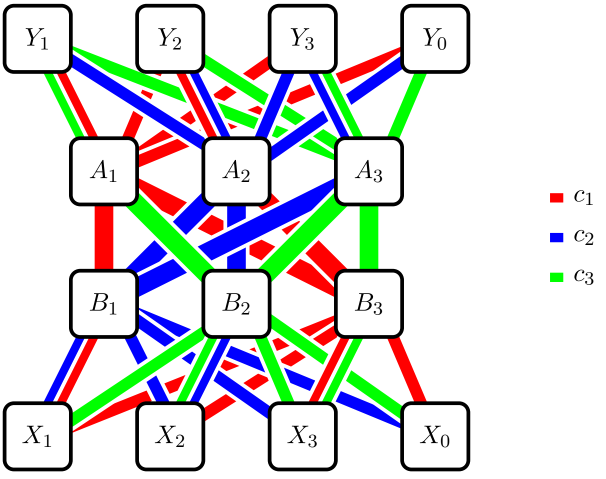

FIGURE 1 The structure of $K_{m,n}$ in Theorem 5

Theorem 4. *Let A and B be the partite sets of an edge-colored complete bipartite graph G with $|A| = m$, $|B| = n$,* and *$\delta^c(G) \geq 2$. Then G contains a PC C_4 if one of the following conditions holds.*

- **(i)** $\delta^c(G) \geq 3$;
- **(ii)** $\Delta_G^{mon}(A) < \frac{2n}{3}$;
- **(iii)** $\Delta_G^{mon}(B) < \frac{2m}{3}$.

In fact, we obtained Theorem 4 as a corollary of the following stronger structural result.

Theorem 5. *Let A and B be the partite sets of an edge-colored complete bipartite graph G with $|A| = m$ and $|B| = n$. If $\delta^c(G) \geq 2$ and G contains no PC C_4, then A and B, respectively, can be partitioned into $\{A_1, A_2, A_3, X_1, X_2, X_3, X_0\}$ and $\{B_1, B_2, B_3, Y_1, Y_2, Y_3, Y_0\}$ (see Fig. 1) such that the following properties hold for $i = 1, 2, 3$ (indices are taken modulo 3).*

- **(i)** $A_i, B_i \neq \emptyset$;
- **(ii)** $C(A_i, B_{i-1} \cup B_i) = \{c_i\}$ *and* $C(A_i, B_{i+1}) = \{c_{i-1}\}$; *s*
- **(iii)** $C(A_i, \bigcup_{0 \leq j \leq 3, j \neq i} Y_j) \subseteq \{c_i\}$ *and* $C(A_i, y_i) = \{c_{i-1}, c_i\}$ *for each vertex* $y_i \in Y_i$;
- **(iv)** $C(B_i, \bigcup_{0 \leq j \leq 3, j \neq i} X_j) \subseteq \{c_{i+1}\}$ *and* $C(B_i, x_i) = \{c_i, c_{i+1}\}$ *for each vertex* $x_i \in X_i$.

Remark 2. *The bounds in Theorem 4 are sharp. Let G be an edge-colored $K_{m,n}$ admitting a partition as that in Figure 1 with $X_i, Y_i = \emptyset$, $|A_j| = \frac{m}{3}$ and $|B_j| = \frac{n}{3}$ for all $i \in [0, 3]$ and $j \in [1, 3]$. Then $\delta^c(G) = 2$, $\Delta_G^{mon}(A) = \frac{2n}{3}$ and $\Delta_G^{mon}(B) = \frac{2m}{3}$. But G contains no PC C_4.*

By applying Theorem 4 (i), we obtain the following corollaries on vertex-disjoint PC cycles in edge-colored complete bipartite graphs.

Corollary 1. *For $k \geq 2$, if $\delta^c(K_{m,n}) \geq 2k$, then $K_{m,n}$ contains k vertex-disjoint PC cycles $H_1, \ldots, H_k$ such that $4 \leq |V(H_1)| \leq 6$ and $|V(H_i)| = 4$ for every $2 \leq i \leq k$.*

Corollary 2. *For $k \geq 2$, if $\delta^c(K_{m,n}) \geq 2k + 1$, then $K_{m,n}$ contains k vertex-disjoint PC C_4's.*

In connection with Theorem 4, we would like to consider a sharp color degree condition for an edge-colored $K_{m,n}$ to satisfy the property that each vertex is contained in a PC C_4. In fact, this was originally discussed for edge-colored complete graphs by Fujita and Magnant in [7]; they conjectured that, if $\delta^c(K_n) \geq \frac{n+1}{2}$, then this edge-colored K_n is properly *vertex pancyclic* (i.e. each vertex of the K_n is contained in a PC cycle of length ℓ for every $3 \leq \ell \leq n$), and they showed that if $\delta^c(K_n) \geq \frac{n+1}{2}$ then each vertex is contained in a PC C_3, a PC C_4 and a PC cycle of length at least five when $n \geq 13$. In this article, we propose the complete bipartite version of their conjecture.

Conjecture 1. *If $\delta^c(K_{m,n}) \geq \frac{m+n}{4} + 1$ then each vertex of the $K_{m,n}$ is contained in a PC cycle of length ℓ, where ℓ is any even integer with $4 \leq \ell \leq \min\{2m, 2n\}$.*

Regarding this conjecture, we could manage to show that if $\delta^c(K_{m,n}) \geq \frac{m+n}{4} + 1$ then each vertex of the $K_{m,n}$ is contained in a PC C_4. In fact, we can prove the following maximum monochromatic condition for PC C_4's passing given vertices, with this minimum color degree result as a corollary.

Theorem 6. *Let A and B be the partite sets of an edge-colored complete bipartite graph G with $|A| = m$ and $|B| = n$. Then every vertex of G is contained in a PC C_4 if one of the following conditions holds:*

(i) $\Delta_G^{mon}(A) \leq \frac{3n-m}{4}$ *and* $\Delta_G^{mon}(B) \leq \frac{3m-n}{4}$*;*
(ii) $\Delta_G^{mon}(A) \leq \frac{n}{2}$ *and* $\Delta_G^{mon}(B) \leq \frac{m}{2}$.

Corollary 3. *Let A and B be the partite sets of an edge-colored complete bipartite graph G with $|A| = m$ and $|B| = n$. Then every vertex of G is contained in a PC C_4 if one of the following conditions holds:*

(i) $\delta^c(G) \geq \frac{m+n}{4} + 1$*;*
(ii) $\delta_G^c(A) \geq \frac{n}{2} + 1$ *and* $\delta_G^c(B) \geq \frac{m}{2} + 1$.

Remark 3. *The bounds for Δ^{mon} and δ^c in Theorem 6 and Corollary 3 are tight. Let A and B be the partite sets of an edge-colored complete bipartite graph G with $|A| = 2t - 2$ and $|B| = 2s - 1$. Partition A and B into A_1, A_2 and $B_1, B_2, \{v\}$, respectively, such that $|A_1| = |A_2| = t - 1$ and $|B_1| = |B_2| = s - 1$. For vertices $a \in A$ and $b \in B$, color the edge ab with*

$$C(ab) = \begin{cases} b, & \text{if } a \in A_i, b \in B_i, i = 1, 2; \\ a, & \text{if } a \in A_i, b \notin B_i, i = 1, 2. \end{cases}$$

Then $\delta_G^c(A) = \frac{|B|+1}{2}$ and $\delta_G^c(B) = \frac{|A|}{2} + 1$, $\Delta_G^{mon}(A) = \frac{|B|+1}{2}$, and $\Delta_G^{mon}(B) = \frac{|A|}{2}$, but the vertex v is not contained in any PC C_4's.

Moreover, we have the following analogous results:

Theorem 7. *Let A and B be the partite sets of an edge-colored complete bipartite graph G with $|A| = m$ and $|B| = n$. If $\Delta_G^{mon}(A) \leq \frac{n}{3}$ and $\Delta_G^{mon}(B) \leq \frac{m}{3}$, then every edge of G is contained in a PC C_4.*

Corollary 4. *Let A and B be the partite sets of an edge-colored complete bipartite graph G with $|A| = m$ and $|B| = n$. If $\delta_G^c(A) \geq \frac{2n}{3} + 1$ and $\delta_G^c(B) \geq \frac{2m}{3} + 1$, then each edge of G is contained in a PC C_4.*

Remark 4. *The bounds of Δ^{mon} in Theorem 7 are tight. Let A and B be the partite sets of an edge-colored complete bipartite graph G with $|A| = 3s + 1$ and $|B| = 3t + 1$. Partition A into*

$A_1, A_2, A_3, \{x\}$ and B into $B_1, B_2, B_3, \{y\}$ such that $|A_i| = s$ and $|B_i| = t$ ($i = 1, 2, 3$). For vertices $a \in A$ and $b \in B$, color the edge ab with

$$C(ab) = \begin{cases} c_0, & \text{if } a = x, b \in B_3 \cup \{y\} \text{ or } a \in A_3, b = y; \\ b, & \text{if } a \in A_i \cup \{x\}, b \in B_i, i = 1, 2; \\ a, & \text{if } a \in A_i, b \notin B_i \cup B_3, i = 1, 2. \end{cases}$$

Use new colors to give a proper coloring to the graph induced by the rest of the edges. Then $\Delta_G^{mon}(A) = t + 1 = \lceil \frac{|B|}{3} \rceil$ and $\Delta_G^{mon}(B) = s + 1 = \lceil \frac{|A|}{3} \rceil$, but there is no PC C_4 containing the edge xy.

We give the proofs of our results in the rest of this article.

2 | PROOF OF THEOREM 2

For convenience, put $z(D) = D! \sum_{i=0}^{D} \frac{1}{i!}$. It is easy to check that the theorem holds for $D = 2$. Suppose that $n \le z(D)$ ($D \ge 3$) and G has no PC cycles. By Theorem 1, there exists a vertex $v \in V(G)$ such that, for every component $H_1, H_2, \dots, H_t$ of $G \setminus \{v\}$, all the edges in $E_G(v, H_i)$ has a common color. This implies that $D \le t$. We may assume that $|V(H_1)| \le \frac{n-1}{t}$. Note that $\delta^c(G[H_1]) \ge D - 1$. Thus, by the induction hypothesis,

$$(D-1)! \sum_{i=0}^{D-1} \frac{1}{i!} \le |V(H_1)| \le \frac{n-1}{t} \le \frac{n-1}{D}$$

holds. This implies that $n = z(D)$ and hence all the equalities are attained in above. Thus, $t = D$ and $|V(H_i)| = \frac{z(D)-1}{D} = z(D-1)$ holds for every $1 \le i \le D$. For every $1 \le i \le D$, applying the induction hypothesis to $G[H_i]$, we know that $G[H_i]$ is isomorphic to G_{D-1}, up to the edge-coloring structure because $G[H_i]$ has no PC cycles. This implies that G is isomorphic to G_D, up to the edge-coloring structure.

3 | PROOF OF THEOREM 3

To prove Theorem 3[1], we need the following result due to Gallai [9].

Theorem 8 (Gallai [9]). *For an edge-colored complete graph K_n, if it does not contain a PC C_3 then $V(K_n)$ can be partitioned into several (at least two) parts such that between the parts, there are a total of at most two colors and, between every pair of parts, there is only one color on the edges.*

For convenience we name the partition in the above theorem as *Gallai partition*.

Proof of Theorem 3. We prove the theorem by induction on n. It is easy to check that the theorem holds for small n. So we assume that $n \ge 5$.

Suppose that K_n contains no PC C_3. Then it follows from Theorem 8 that K_n has a Gallai partition $S_1, S_2, \dots, S_k$ with $k \ge 2$. If $k \ge 4$ then there exists S_j such that $|S_j| \le \frac{n}{4}$. Since $\delta^c(K_n[S_j]) > \log_2 |S_j|$, applying induction hypothesis to $K_n[S_j]$, we can find a PC C_3, a contradiction. Thus we may assume that $2 \le k \le 3$. In this case, we can take S_j so that only one color is used on edges between S_j and $V(K_n) \setminus S_j$. Again, applying induction hypothesis to the smaller part of $\{S_j, V(K_n) \setminus S_j\}$, we can find a PC C_3, a contradiction. Hence the theorem holds. ∎

4 | PROOFS OF THEOREMS 4 AND 5

Proof of Theorem 4. Let $G \cong K_{m,n}$ with partite sets A and B satisfies that $\delta^c(G) \geq 2$ and contains no PC C_4. Proving Theorem 4 is equivalent to show that $\delta^c(G) = 2$, $\Delta_G^{mon}(A) \geq \frac{2n}{3}$ and $\Delta_G^{mon}(B) \geq \frac{2m}{3}$. By Theorem 5, the partite sets A and B can be partitioned into sets

$$\{A_1, A_2, A_3, X_1, X_2, X_3, X_0\} \quad \textit{and} \quad \{B_1, B_2, B_3, Y_1, Y_2, Y_3, Y_0\},$$

respectively, with properties in the statement of Theorem 5. Choose a vertex $a_1 \in A_1$. Since $C(a_1, B) = \{c_1, c_3\}$, there holds $d_G^c(a_1) = 2$. Recall that $\delta^c(G) \geq 2$. We have $\delta^c(G) = 2$. Now consider the number of vertices that are adjacent to a_1 with the color c_1. We have $d_{G^{c_1}}(a_1) \geq n - |B_2| - |Y_1|$. Choose vertices $a_2 \in A_2$ and $a_3 \in A_3$. Similarly, we have $d_{G^{c_2}}(a_2) \geq n - |B_3| - |Y_2|$ and $d_{G^{c_3}}(a_3) \geq n - |B_1| - |Y_3|$. Thus

$$\sum_{1 \leq i \leq 3} d_{G^{c_i}}(a_i) \geq 3n - \sum_{1 \leq i \leq 3} |B_i| - \sum_{1 \leq j \leq 3} |Y_j| \geq 3n - \sum_{1 \leq i \leq 3} |B_i| - \sum_{0 \leq j \leq 3} |Y_j| = 2n.$$

There must exist a vertex a_i for some i with $1 \leq i \leq 3$ such that $d_{G^{c_i}}(a_i) \geq \frac{2n}{3}$. Thus $\Delta_G^{mon}(A) \geq \frac{2n}{3}$. Similarly, we can obtain $\Delta_G^{mon}(B) \geq \frac{2m}{3}$. ∎

Proof of Theorem 5. By contradiction. Let $G \cong K_{m,n}$ be a counterexample to Theorem 5 with $m + n$ as small as possible. Since $\delta^c(G) \geq 2$, by Observation 1, G must contain a PC cycle of length 6. This implies that $m, n \geq 3$. If $m = n = 3$, then G contains a PC Hamilton cycle of length 6. Note that G contains no PC C_4. It is easy to check that G satisfies Theorem 5, a contradiction. So we have $m \geq 3, n \geq 3$ and $m + n \geq 7$. Now, we proceed by proving the following Claims.

Claim 1. *There exists a vertex $v \in V(G)$ such that $\delta^c(G - v) \geq 2$.*

Proof. By contradiction. Suppose to the contrary that $\delta^c(G - v) = 1$ for every vertex $v \in V(G)$. For vertices $u, v \in V(G)$, we say *v dominates u* if $d_{G-v}^c(u) = 1$. Thus, for each vertex $v \in V(G)$, there exists a vertex u such that v dominates u. If a vertex $u \in V(G)$ is dominated by two vertices, then $d_G(u) = d_G^c(u) = 2$. Thus $m = 2$ or $n = 2$, a contradiction. So each vertex $u \in V(G)$ is dominated by at most one vertex. Based on these conclusions, we can construct a directed graph D such that $V(D) = V(G)$ and $uv \in A(D)$ if and only if u dominates v. Then we have $\sum_{v \in V(D)} d_D^+(v) \geq |V(D)|$ and $\sum_{v \in V(D)} d_D^-(v) \leq |V(D)|$. This implies that $d_D^+(v) = d_D^-(v) = 1$ for each vertex $v \in V(D)$. Thus D is composed of disjoint cycles. Let $D_1, D_2, \ldots, D_k$ be the components of D. Then $D_i (1 \leq i \leq k)$ is either a cycle of length 2 or an even cycle of length at least 6. In the later case, D_i corresponds to a PC cycle in G.

If D contains two components of order 2, then we can obtain a PC C_4 in G by combining them together, a contradiction. So D contains at most one component of order 2. Note that $|V(D)| = |V(G)| \geq 7$. Then D must contain a component (say D_1) that is an even cycle of length at least six. Let $D_1 = u_1u_2 \cdots u_tu_1$. For each i with $1 \leq i \leq t$, since $u_iu_{i+1}u_{i+2}u_{i+3}u_i$ (indices are taken modulo t) is not a PC cycle and u_{i+2} dominates u_{i+3} in G, we have $C(u_iu_{i+3}) = C(u_iu_{i+1}) = C(u_{i+3}u_{i+4})$. This implies that $C(u_iu_{i+1}), C(u_{i+1}u_{i+2})$, and $C(u_{i+2}u_{i+3})$ are three distinct colors and $|D_1| = 6p$ for some integer $p \geq 1$. Without loss of generality, assume that $C(u_iu_{i+1}) = c_i$ for $i = 1, 2, 3$. If $p \geq 2$, then consider the color $C(u_1u_6)$. Since u_t dominates u_1, $C(u_1, V(G) - u_t) = \{c_1\}$, in particular, $C(u_1u_6) = c_1$. Note that $C(u_5u_6) = C(u_2u_3) = c_2$ and $C(u_6u_7) = C(u_3u_4) = c_3$. We have $d_G^c(u_6) \geq 3$. This contradicts that u_5 dominates u_6. Hence, $p = 1$, $D_1 = u_1u_2 \cdots u_6u_1$ and $C(u_1u_2), C(u_3u_4)$, and $C(u_5u_6)$ are three distinct colors. Without loss generality, assume that $u_1, u_3, u_5 \in A$. Note that $|V(D)| = |V(G)| \geq 7$. Then D must contain a component (say D_2) that is different from D_1. Let uv be an arc in D_2 with

$v \in B$. Since u_{i-1} dominates u_i (indices are taken modulo 6) for $i = 1, 3, 5$, we have $C(vu_1) = C(u_1u_2)$, $C(vu_3) = C(u_3u_4)$, and $C(vu_5) = C(u_5u_6)$. Thus $d^c_G(v) \geq 3$. This contradicts that u dominates v.

The proof of Claim 1 is complete. ■

Let v be the vertex in the statement of Claim 1 and let $H = G - v$. Then H is an edge-colored complete bipartite graph with $\delta^c(H) \geq 2$. By symmetry, without loss of generality, assume that $v \in B$. Recall the assumption that G is a minimum counterexample to Theorem 5. The partite sets (A and $B - v$) of H can be partitioned into $\{A_1, A_2, A_3, X_1, X_2, X_3, X_0\}$ and $\{B_1, B_2, B_3, Y_1, Y_2, Y_3, Y_0\}$, respectively, with the properties in the statement of Theorem 5. Now we continue the proof by analyzing the colors that are appearing between v and $\bigcup_{1\leq i\leq 3} A_i$. Since X_i and Y_i ($0 \leq i \leq 3$) can possibly be empty sets, for $S \subseteq \bigcup_{i=0}^{3}(X_i \cup Y_i)$, we sometimes write $C(v, S) \subseteq \{c\}$ to say that $C(v, S) = \{c\}$ if S is nonempty. When there are no emphases, in the following, indices are always taken modulo 3.

Claim 2. $|C(v, \bigcup_{1\leq i\leq 3} A_i)| \geq 2$.

Proof. Suppose to the contrary that $|C(v, \bigcup_{1\leq i\leq 3} A_i)| = 1$. Let $C(v, \bigcup_{1\leq i\leq 3} A_i) = \{\alpha\}$. Since $d^c_G(v) \geq 2$, there exists a vertex $x \in \bigcup_{0\leq i\leq 3} X_i$ such that $C(vx) \neq \alpha$. Let $a_i \in A_i$ and $b_j \in B_j$ be arbitrarily chosen vertices for $i, j = 1, 2, 3$. If $x \in X_0$, then one of the cycles $\{xva_ib_ix : i = 1, 2, 3\}$ must be a PC C_4, a contradiction. So $x \notin X_0$. If $x \in X_1$, then consider cycles $\{xva_2b_2x, xva_3b_3x\}$ and that $C(vx) \neq \alpha$. We have $C(vx) = c_1$ and $\alpha = c_2$ (this can be verified by firstly proving that $C(vx) \neq c_3$ and $\alpha \neq c_3$). Note that $C(x_1, B_1) = \{c_1, c_2\}$ for each vertex $x_1 \in X_1$. There exists a vertex $b'_1 \in B_1$ such that $C(xb'_1) = c_2$. This implies that vxb'_1a_1v is a PC C_4, a contradiction. Thus $x \notin X_1$. Similarly, we can prove that $x \notin X_2 \cup X_3$. This contradiction completes the proof of Claim 2. ■

Claim 3. $C(v, A_i) \subseteq \{c_{i-1}, c_i\}$ *for* $i = 1, 2, 3$.

Proof. We firstly prove that $C(v, A_1) \subseteq \{c_1, c_2, c_3\}$. Suppose to the contrary that there exists a vertex $a \in A_1$ such that $C(va) = \alpha \notin \{c_1, c_2, c_3\}$. Let a_2, a_3, b_1, b_2 and b_3 be arbitrary vertices in A_2, A_3, B_1, B_2, and B_3, respectively. Consider cycles vab_1a_3v and vab_3a_3v. We have $C(v, A_3) = \{\alpha\}$. Then consider cycles $va_3b_3a_2v$ and $va_3b_2a_2v$. We have $C(v, A_2) = \{\alpha\}$. By Claim 2, there must exist a vertex $a' \in A_1$ such that $C(va') \neq \alpha$. This implies that either $a'b_2a_2va'$ or $a'b_3a_3va'$ is a PC C_4, a contradiction.

Now, we will show that $c_2 \notin C(v, A_1)$. Suppose the contrary. Let $a \in A_1$ be a vertex satisfying $C(va) = c_2$. Let a_2, a_3, b_1, b_2, and b_3 be arbitrary vertices in A_2, A_3, B_1, B_2, and B_3, respectively. Consider cycles ab_1a_2va and ab_1a_3va. We have $C(v, A_2 \cup A_3) = \{c_2\}$. By Claim 2, there must exist a vertex $a' \in A_1$ such that $C(va') \neq c_2$. We assert that $C(va') = c_1$. Otherwise, $va'b_3a_3v$ is a PC C_4, a contradiction. So we have $C(v, A_1) = \{c_1, c_2\}$. Let $A'_1 = \{u \in A_1 : C(uv) = c_1\}$ and $A''_1 = A_1 \setminus A'_1$. Then $A'_1, A''_1 \neq \emptyset$. For each vertex $x \in X_2 \cup X_3 \cup X_0$, by considering the cycle vab_1xv, we have $C(v, X_2 \cup X_3 \cup X_0) \subseteq \{c_2\}$. Define $B'_1 = B_1 \cup \{v\}$, $X'_1 = X_1 \cup A''_1$. Partition Y_1 into three sets Y'_1, Y''_1, and Y'''_1 such that

$$Y'_1 = \{y \in Y_1 : C(y, A'_1) = \{c_1, c_3\}\},$$

$$Y''_1 = \{y \in Y_1 : C(y, A'_1) = \{c_1\}\}$$

and

$$Y'''_1 = \{y \in Y_1 : C(y, A'_1) = \{c_3\}\}.$$

If $Y_1''' = \emptyset$, then let $Y_0' = Y_0 \cup Y_1''$. Thus the partite sets of G can be partitioned into

$$\{A_1', A_2, A_3, X_1', X_2, X_3, X_0\} \quad \textit{and} \quad \{B_1', B_2, B_3, Y_1', Y_2, Y_3, Y_0'\},$$

respectively, with properties in the statement of Theorem 5, a contradiction. If $Y_1''' \neq \emptyset$, then choose a vertex $y \in Y_1'''$. Since $C(y, A_1) = \{c_1, c_3\}$ and $C(y, A_1') = \{c_3\}$, there exists a vertex $a'' \in A_1''$ such that $C(a''y) = c_1$. Arbitrarily choose a vertex $a' \in A_1'$. Then $ya'va''y$ is a PC C_4, a contradiction.

Hence, $C(v, A_1) \subseteq \{c_1, c_3\}$. Similarly, we can prove that $C(v, A_2) \subseteq \{c_1, c_2\}$ and $C(v, A_3) \subseteq \{c_2, c_3\}$. This completes the proof of Claim 3. ■

Claim 4. *If $c_{i-1} \in C(v, A_i)$ for some i with $1 \leq i \leq 3$, then $C(v, A_{i-1}) = \{c_{i-1}\}$ and $C(v, A_{i+1}) = \{c_{i+1}\}$.*

Proof. Assume that there exists a vertex $a \in A_1$ such that $C(va) = c_3$. Then let a_2, a_3, b_1, b_2, and b_3 be arbitrary vertices in A_2, A_3, B_1, B_2, and B_3, respectively. Consider the cycle vab_1a_2v. We have $C(v, A_2) \subseteq \{c_2, c_3\}$. By Claim 3, $C(v, A_2) \subseteq \{c_1, c_2\}$. This implies that $C(v, A_2) = \{c_2\}$. Consider the cycle vab_3a_3v. We have $C(v, A_3) = \{c_3\}$. The left cases can be verified by similar arguments. The proof of Claim 4 is complete. ■

Claim 5. $C(v, A_i) = \{c_{i-1}\}$ *for some i with* $1 \leq i \leq 3$.

Proof. By Claim 3, $C(v, A_i) \subseteq \{c_{i-1}, c_i\}$ for all i with $1 \leq i \leq 3$.

If $C(v, A_i) = \{c_{i-1}, c_i\}$ for some i with $1 \leq i \leq 3$, then by Claim 4, $C(v, A_{i-1}) = \{c_{i-1}\}$ and $C(v, A_{i+1}) = \{c_{i+1}\}$. Without loss of generality, assume that $i = 1$. Let $Y_1' = Y_1 \cup \{v\}$. Thus the partite sets of G can be partitioned into

$$\{A_1, A_2, A_3, X_1, X_2, X_3, X_0\} \quad \textit{and} \quad \{B_1, B_2, B_3, Y_1', Y_2, Y_3, Y_0\},$$

respectively, with properties in the statement of Theorem 5, a contradiction.

If $C(v, A_i) = \{c_i\}$ for all i with $1 \leq i \leq 3$, then let $Y_0' = Y_0 \cup \{v\}$. Thus the partite sets of G can be partitioned into

$$\{A_1, A_2, A_3, X_1, X_2, X_3, X_0\} \quad \textit{and} \quad \{B_1, B_2, B_3, Y_1, Y_2, Y_3, Y_0'\},$$

respectively, with properties in the statement of Theorem 5, a contradiction.

So we have $C(v, A_i) = \{c_{i-1}\}$ for some i with $1 \leq i \leq 3$. ■

According to Claim 5, without loss of generality, assume that $C(v, A_1) = \{c_3\}$. Then by Claim 4, $C(v, A_2) = \{c_2\}$ and $C(v, A_3) = \{c_3\}$. Let a_1, a_3, and b_2 be arbitrary vertices in A_1, A_3, and B_2, respectively. For each vertex $x \in X_2 \cup X_0$, by considering the cycle va_1b_1xv, we have $C(v, X_2 \cup X_0) \subseteq \{c_2, c_3\}$. Now we will prove that $C(v, X_1 \cup X_3) \subseteq \{c_3\}$. For each vertex $x_1 \in X_1$, by the definition of X_1, there exist vertices b_1 and b_1' in B_1 such that $C(x_1b_1) = c_1$ and $C(x_1b_1') = c_2$. Consider the cycle $a_1b_1'x_1va_1$. We have $C(vx_1) \in \{c_2, c_3\}$. If $C(vx_1) = c_2$, then $vx_1b_1a_3v$ is a PC C_4, a contradiction. So we have $C(v, X_1) \subseteq \{c_3\}$. For each vertex $x_3 \in X_3$, there exists a vertex $b \in B_3$ such that $C(x_3b) = c_3$. Consider the cycle va_1bx_3v. We have $C(vx_3) = c_3$. Thus $C(v, X_3) \subseteq \{c_3\}$.

Let $X_0' = \{x \in X_0 : C(vx) = c_3\}$ and $X_0'' = X_0 \setminus X_0'$. Then $C(v, X_0'') \subseteq \{c_2\}$. Let $B_2' = B_2 \cup \{v\}$ and $X_2' = X_2 \cup X_0''$. Thus the partite sets of G can be partitioned into

$$\{A_1, A_2, A_3, X_1, X_2', X_3, X_0'\} \quad \textit{and} \quad \{B_1, B_2', B_3, Y_1, Y_2, Y_3, Y_0\},$$

respectively, with properties in the statement of Theorem 5, a contradiction. This completes the proof of Theorem 5. ■

5 | PROOF OF THEOREM 6

By contradiction. Suppose that there exists a vertex $v \in V(G)$ such that any PC C_4 does not contain v. Without loss of generality, let $v \in B$ and $d = d^c(v)$. Now, partition A into d nonempty subsets $A_1, A_2, \dots, A_d$ such that the color c_i is assigned to all edges between v and A_i for each $1 \le i \le d$, where $c_i \ne c_j$ for $1 \le i < j \le d$.

Consider a spanning subgraph G' of G with

$$E(G') = \{ab : a \in A, b \in B, C(ab) \ne C(av)\}.$$

Let

$$B' = \{x \in B : \text{There exist } i, j \text{ with } i \ne j, u \in A_i, w \in A_j \text{ such that} ux, wx \in E(G')\}.$$

For a vertex $x \in B'$, since v is not contained in any PC C_4's, we have $C(ux) = C(wx)$, where u, w are as that in the definition of B'. By this observation, we obtain the following claim.

Claim 6. *For a vertex $x \in B'$, all edges of $E_{G'}(x, A)$ has a same color, i.e. $d^c_{G'}(x) = 1$.*

For simplicity, in the following, we use Δ_A and Δ_B respectively to denote $\Delta^{mon}_G(A)$ and $\Delta^{mon}_G(B)$.

Claim 7. *The following statements hold:*

(1) $mn - m\Delta_A - (n-1)\Delta_B > 0$;

(2) $|E_{G'}(A, (B - v)\backslash B')| \le (n - 1 - |B'|)\Delta_B$;

(3) $B' \ne \emptyset$.

Proof.

(1) If $\Delta^{mon}_G(A) \le \frac{3n-m}{4}$ and $\Delta^{mon}_G(B) \le \frac{3m-n}{4}$, then we have $3m > n$ (since $\frac{3m-n}{4} > 0$). Thus,

$$mn - m\Delta_A - (n-1)\Delta_B \ge \frac{1}{4}[(m-n)^2 + 3m - n] > 0.$$

If $\Delta^{mon}_G(A) \le \frac{n}{2}$ and $\Delta^{mon}_G(B) \le \frac{m}{2}$, then

$$mn - m\Delta_A - (n-1)\Delta_B \ge \frac{m}{2} > 0.$$

(2) For a vertex $b \in (B - v)\backslash B'$, by the definition of B', b can be adjacent to at most one set of $A_1, A_2, \dots, A_d$ in G'. Since all edges between v and A_i are colored in c_i, we have $|A_i| \le \Delta_B$ for all $1 \le i \le d$. Thus $|E_{G'}(A, (B - v)\backslash B')| \le (n - 1 - |B'|)\Delta_B$.

(3) Suppose that $B' = \emptyset$. Then by the definition of G' and Claim 7 (2)

$$m(n - \Delta_A) \le |E_{G'}(A, B - v)| \le (n-1)\Delta_B.$$

So we have

$$m\Delta_A + (n-1)\Delta_B \ge mn.$$

This contradicts Claim 7 (1). ■

Utilizing Claim 7, we obtain

$$\frac{|E_{G'}(A,B')|}{|B'|} = \frac{|E_{G'}(A,B-v)| - |E_{G'}(A,(B-v)\backslash B')|}{|B'|} \geq \frac{m(n-\Delta_A)-(n-1-|B'|)\Delta_B}{|B'|}$$

$$= \Delta_B + \frac{mn - m\Delta_A - (n-1)\Delta_B}{|B'|} > \Delta_B.$$

Hence, there must exist a vertex $x \in B'$ such that $d_{G'}(x) > \Delta_B$. By Claim 6, we know that there exists a color appearing at least $\Delta_B + 1$ times on the edges incident to x, a contradiction.

6 | PROOF OF THEOREM 7

By contradiction. Suppose that $e = xy \in E(G)$ is an edge satisfying $x \in A, y \in B$ but not contained in any PC C_4's. Let $A' = \{a \in A : C(ay) = C(xy)\}$ and $B' = \{b \in B : C(xb) = C(xy)\}$.

Obviously, $x \in A'$,$y \in B'$, $|A'| \leq \Delta_G^{mon}(A) \leq \frac{m}{3}$, $|B'| \leq \Delta_G^{mon}(B) \leq \frac{n}{3}$ and $C(ab) \in \{C(ay), C(xb)\}$ for all $a \in A\backslash A'$ and $b \in B\backslash B'$. Now construct an oriented graph D with

$$V(D) = (A\backslash A') \cup (B\backslash B')$$

and

$$A(D) = \{ab : C(ab) = C(xb), a \in A\backslash A', b \in B\backslash B'\}$$

$$\cup\{ba : C(ab) \neq C(xb), a \in A\backslash A', b \in B\backslash B'\}.$$

Clearly, for each vertex $v \in V(D)$, all edges between v and $N_D^-(v) \cup \{x,y\}$ have a same color. Thus

$$d_D^-(a) \leq \frac{n}{3} - 1 < \frac{n}{3}$$

and

$$d_D^-(b) \leq \frac{m}{3} - 1 < \frac{m}{3}$$

for all $a \in A\backslash A'$ and $b \in B\backslash B'$. Then we have

$$\frac{n(m-|A'|)}{3} + \frac{m(n-|B'|)}{3} > \sum_{v \in V(D)} d_D^-(v) = |A(D)| = (m-|A'|)(n-|B'|).$$

This implies that

$$0 > mn + 3|A'||B'| - 2n|A'| - 2m|B'| = \left(\frac{2m}{\sqrt{3}} - \sqrt{3}|A'|\right)\left(\frac{2n}{\sqrt{3}} - \sqrt{3}|B'|\right) - \frac{mn}{3}$$

$$\geq \left(\frac{2m}{\sqrt{3}} - \frac{m}{\sqrt{3}}\right)\left(\frac{2n}{\sqrt{3}} - \frac{n}{\sqrt{3}}\right)$$

$$- \frac{mn}{3}\left(|A'| \leq \frac{m}{3}, |B'| \leq \frac{n}{3}.\right) = 0,$$

a contradiction, which completes the proof.

ENDNOTE

[1] The second and third authors of this article first gave a proof of Theorem 3 in a manuscript without using Theorem 8. The proof we present here is more simple than that one and was suggested by two referees of that article. We include it here for completeness.

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