A RELAXATION OF THE STRONG BORDEAUX CONJECTURE

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Abstract. Let $c_1, c_2, \cdots, c_k$ be k non-negative integers. A graph G is $(c_1, c_2, \cdots, c_k)$-colorable if the vertex set can be partitioned into k sets $V_1, V_2, \ldots, V_k$, such that the subgraph $G[V_i]$, induced by V_i, has maximum degree at most c_i for $i = 1, 2, \ldots, k$. Let $\mathcal{F}$ denote the family of plane graphs with neither adjacent 3-cycles nor 5-cycle. Borodin and Raspaud (2003) conjectured that each graph in $\mathcal{F}$ is $(0,0,0)$-colorable. In this paper, we prove that each graph in $\mathcal{F}$ is $(1,1,0)$-colorable, which improves the results by Xu (2009) and Liu-Li-Yu (2014+).

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1. Introduction

All graphs considered in this paper are finite, simple, and undirected. Call a graph G *planar* if it can be embedded into the plane so that its edges meet only at their ends. As proved by Graey et al [8], the problem of deciding whether a planar graph is properly 3-colorable is NP-complete. In 1959, Grötzsch [9] showed that every triangle-free planar graph is 3-colorable. A lot of research was devoted to find sufficient conditions for a planar graph to be 3-colorable, by allowing a triangle together with some other conditions. The well-known Steinberg's conjecture [15] stated below is one of such numerous efforts.

Conjecture 1.1 (Steinberg, [15]). *All planar graphs without* 4*-cycles and* 5*-cycles are* 3*-colorable.*

Towards this conjecture, Erdős suggested to find a constant c such that a planar graph without cycles of length from 4 to c is 3-colorable. The best constant people so far is $c = 7$, found by Borodin, Glebov, Raspaud, and Salavatipour [4]. For more results, see the recent nice survey by Borodin [1].

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Another relaxation of the conjecture is to allow some defects in the color classes. A graph is $(c_1, c_2, \cdots, c_k)$-*colorable* if the vertex set can be partitioned into k sets $V_1, V_2, \ldots, V_k$, such that for every $i \in [k] := \{1, 2, \ldots, k\}$ the subgraph $G[V_i]$ has maximum degree at most c_i. With this notion, a properly 3-colorable graph is $(0,0,0)$-colorable. Chang, Havet, Montassier, and Raspaud [6] proved that all planar graphs without 4-cycles or 5-cycles are $(2,1,0)$-colorable and $(4,0,0)$-colorable. It is shown in [11, 12, 18] that planar graphs without 4-cycles or 5-cycles are $(3,0,0)$- and $(1,1,0)$-colorable. Wang and his coauthors (private communication) further proved such graphs are $(2,0,0)$-colorable.

As usual, a 3-cycle is also called a *triangle*. Havel (1969, [10]) asked if each planar graph with large enough distances between triangles is $(0,0,0)$-colorable. This was resolved by Dvořák, Král and Thomas [7]. We say that two cycles are *adjacent* if they have at least one edge in common and *intersecting* if they have at least one common vertex. Borodin and Raspaud in 2003 made the following two conjectures, which have common features with Havel's and Steinberg's 3-color problems.

Conjecture 1.2 (Bordeaux Conjecture, [5])**.** *Every planar graph without intersecting* 3*-cycles and without* 5*-cycles is* 3*-colorable.*

Conjecture 1.3 (Strong Bordeaux Conjecture, [5])**.** *Every planar graph without adjacent* 3*-cycles and without* 5*-cycles is* 3*-colorable.*

Let $d^{\triangledown}$ denote the minimal distance between triangles. Towards the conjectures, Borodin and Raspaud [5] showed that planar graphs with $d^{\triangledown} \geq 4$ and without 5-cycles are $(0,0,0)$-colorable. This result was later improved to $d^{\triangledown} \geq 3$ by Borodin and Glebov [2], and independently by Xu [16]. Borodin and Glebov [3] further improved this result to $d^{\triangledown} \geq 2$.

With the relaxed coloring notation, Xu [17] showed that all planar graphs without adjacent 3-cycles and without 5-cycles are $(1,1,1)$-colorable. Recently, Liu, Li and Yu [13, 14] proved that every planar graph without intersecting 3-cycles and without 5-cycles is $(2,0,0)$-colorable and $(1,1,0)$-colorable. In this paper, we improve the results by Xu [17] and by Liu-Li-Yu [13], and prove the following result.

Theorem 1.4. *Every planar graph without* 5*-cycles and adjacent 3-cycles is* $(1,1,0)$*-colorable.*

We actually prove a stronger result. To state it, we introduce the following notion. Let H be a subgraph of G. Then (G, H) is *superextendable* if every $(1,1,0)$-coloring of H can be extended to a $(1,1,0)$-coloring of G such that each vertex $u \in G - H$ is coloured differently from its neighbors in H. If (G, H) is superextendable, then we call H a *superextendable sugraph* of G. Let $\mathcal{F}$ be the family of planar graphs without 5-cycles and adjacent 3-cycles.

Theorem 1.5. *Every triangle or* 7*-cycle of a graph in* $\mathcal{F}$ *is superextendable.*

Proof of Theorem 1.4 from Theorem 1.5: Let G be a graph in $\mathcal{F}$. If G is triangle-free, then G is 3-colorable by the Gröztch Theorem, and is naturally $(1,1,0)$-colorable; if G has a triangle, then every $(1,1,0)$-coloring of this triangle can be superextended to the whole graph G by Theorem 1.5. So Theorem 1.4 follows. ■

Like many of the results of this kind, we also use a discharging argument to prove Theorem 1.5. The main difficulty still lies on the cases when a 4-vertex or a 5-vertex is incident with many triangles or many 4-faces. Fortunately, we could utilize many of the lemmas from Xu [17] and Liu-Li-Yu [13] to handle those difficult situations.

We use $G = (V, E, F)$ to denote a plane graph with vertex set $V(G)$, edge set $E(G)$, and face set $F(G)$. For a face $f \in F(G)$, let $b(f)$ denote the boundary of a face f. A k-vertex (k^+-vertex, k^--vertex) is a vertex of degree k (at least k, at most k). The same notation will apply to faces and cycles. An $(l_1, l_2, \ldots, l_k)$-face is a k-face $v_1v_2\ldots v_k$ with $d(v_i) = l_i$, respectively. If a 3-vertex is incident with a triangle, then its neighbor not on the triangle is called its *outer neighbor*, and the 3-face is a *pendant 3-face* of its outer neighbor. Let C be a cycle of a plane graph G. We use $int(C)$ and $ext(C)$ to denote the sets of vertices located inside and outside C, respectively. A cycle C is called a *separating cycle* if $int(C) \neq \emptyset \neq ext(C)$, and is called a *nonseparating cycle* otherwise. We also use C to denote the set of vertices of C.

Let $S_1, S_2, \ldots, S_l$ be pairwise disjoint subsets of $V(G)$. We use $G[S_1, S_2, \ldots, , S_l]$ to denote the graph obtained from G by contracting all the vertices in S_i to a single vertex for each $i \in \{1, 2, \ldots, l\}$. Let $x(y)$ be the resulting vertex by identifying x and y in G.

The paper is organized as follows. In Section 2, we show the reducible structures useful in our proof. In Section 3, we are devoted to the proof of Theorem 1.5 by a discharging procedure.

2. Reducible configurations

Suppose that (G, C_0) is a counterexample to Theorem 1.5 with minimum $\sigma(G) = |V(G)|+|E(G)|$, where C_0 is a triangle or a 7-cycle in G.

If C_0 is a separating cycle, then C_0 is superextendable in both $G \setminus ext(C_0)$ and $G \setminus int(C_0)$. Hence, C_0 is superextendable in G, contrary to the choice of C_0. Thus we assume that C_0 is the boundary of the outer face of G.

Let $F_k = \{f : f \text{ is a } k\text{-face and } b(f) \cap C_0 = \emptyset\}$, $F'_k = \{f : f \text{ is a } k\text{-face and } |b(f) \cap C_0| = 1\}$, and $F''_k = \{f : f \text{ is a } k\text{-face and } |b(f) \cap C_0| = 2\}$.

Since $G \in \mathcal{F}$, the following is immediate.

Proposition 2.1. *Every vertex not on C_0 has degree at least 3, and no 3-face shares an edge with a 4-face in G.*

The following is a summary of some basic properties of G when we consider superextendablity of a 3-cycle or a 7-cycle. The proofs of those results can be found, for example, in [17] or [14].

Lemma 2.2 (Xu, [17]; Liu-Li-Yu, [14])**.** *The following are true about G:*

(1) The graph G contains neither separating triangles nor separating 7-cycles.
(2) If G has a separating 4-cycle $C_1 = v_1v_2v_3v_4v_1$, then $ext(C_1) = \{b, c\}$ such that v_1bc is a 3-cycle. Furthermore, the 4-cycle is the unique separating 4-cycle.
(3) Let x, y be two nonadjacent vertices on C_0. Then $xy \notin E(G)$ and $N(x) \cap N(y) \subseteq C_0$.
(4) Let f be a 4-face with $b(f) = v_1v_2v_3v_4v_1$ and let $v_1 \in C_0$. Then, $v_3 \notin C_0$. Moreover, $|N(v_3) \cap C_0| = 1$ if $f \in F''_4$, and $|N(v_3) \cap C_0| = 0$ if $f \in F'_4$.
(5) Let u, w be a pair of diagonal vertices on a 4-face. Then $G[\{u, w\}] \in \mathcal{F}$.

The following holds for minimum graphs that are not $(1, 1, 0)$-colorable.

Lemma 2.3. *The following are true in G.*

(1) *(Lemma 2.5 from* [12]*) No* 3*-vertex* $v \notin C_0$ *is adjacent to two* 3*-vertices in* $int(C_0)$.
(2) *(Lemma 2.3 from* [12]*)* G *has no* $(3,3,4^-)$*-face* $f \in F_3$.
(3) *(Lemma 2.8 from* [12]*) If* $v \in int(C_0)$ *be a* 4*-vertex incident with exactly one* 3*-face that is a* $(3,4,4)$*-face in* F_3*, then a neighbor of* v *not on the face is either in* C_0 *or a* 4^+*-vertex.*
(4) *(Lemma 2.6 from* [12]*) Let* $v \in int(C_0)$ *be the* 3*-vertex on a* $(3,4,4)$*-face* $f \in F_3$*. Then the neighbor of* v *not on* f *is either on* C_0 *or a* 4^+*-vertex.*
(5) *(Lemma 3(3) from* [18]*) Suppose that* $v \in int(C_0)$ *is a* 4*-vertex incident with two faces from* F_3*. If one of the faces is a* $(3,4,4)$*-face, then* v *has a* 5^+*-neighbor on the other face.*

A 4-vertex $v \in int(C_0)$ is *bad* if it is incident with a $(3,4,4)$-face from F_3, A $(3,4,5^+)$-face from F_3 is *bad* if the 4-vertex on it is bad. A 5-vertex *bad* if it is incident with a bad $(3,4,5)$-face or a $(3,3,5)$-face.

Lemma 2.4. *Suppose that* $v \in int(C_0)$ *is a* 5*-vertex incident with two* 3*-faces* f_1 *and* f_3 *from* F_3*. Let* v_5 *be the remaining neighbor of* v*. Then each of the followings holds.*
(1) *(Lemma 5 from* [18]*) If both* f_1 *and* f_3 *are* $(3,4^-,5)$*-faces, then* v_5 *is either on* C_0 *or a* 4^+*-vertex.*
(2) *(Lemma 4(1) from* [18]*) At most one of* f_1 *and* f_3 *is bad.*
(3) *(Lemma 4(2) from* [18]*) If* f_1 *is a bad* $(3,4,5)$*-face and* f_3 *is a* $(3,4,5)$*-face, then the outer neighbor of the* 3*-vertex on* $b(f_3)$ *is either on* C_0 *or a* 4^+*-vertex.*
(4) *If* f_1 *is a bad* $(3,4,5)$*-face and* f_3 *is a* $(4,4,5)$*-face, then at most one* 4*-vertex on* $b(f_3)$ *is bad.*
(5) *(Lemma 8 from* [18]*) No* 6*-vertex in* $int(C_0)$ *is incident with three* $(3,4^-,6)$*-faces from* F_3.

Proof. We only prove (4). Let $f_1 = vv_1v_2$ with $d(v_1) = 4$ and $d(v_2) = 3$, and $f_3 = vv_3v_4$ with $d(v_3) = d(v_4) = 4$. And let v_i' and v_i''(if any) be the other neighbors of v_i for $i = 1,2,3,4$. Suppose that both 4-vertices on $b(f_3)$ are bad.

Let $S = \{v, v_1, v_2, v_3, v_4, v_1', v_1'', v_3', v_3'', v_4', v_4''\}$, where $d(v_1'') = d(v_3'') = d(v_4'') = 4$, and let $H = G \setminus S$. Since $\sigma(H) < \sigma(G)$, C_0 has a superextension c on H. Based on c, we properly color $\{v_1'', v_1', v_3'', v_3', v_3, v_4'', v_4', v, v_2\}$ in order. Now v_4 can be colored as it has four properly colored neighbors. If v, v_4 are colored differently, then v_1 can also be colored, as it has four properly colored neighbors as well. Thus, $c(v) = c(v_4) = 1$, and v_1 cannot be colored. It follows that $\{c(v_1'), c(v_1'')\} = \{c(v_4'), c(v_4'')\} = \{2,3\}$. If $c(v_3) = 3$, then we can recolor v_4 with 2, and color v_1, so let $c(v_3) = 2$. If $c(v_5) = 2$, then we recolor v with 3 and color v_1 with 1 and recolor v_2 accordingly, so let $c(v_5) = 3$. Recolor v with 2 and color v_1 with 1. Now we can recolor v_2 with 1 (if $c(v_2') = 3$) or 3 (if $c(v_2') \neq 3$). □

For a 3-vertex in a 3-face $f \in F_3$, it is *weak* if it is adjacent to a 3-vertex not on f or C_0, and *strong* if it is adjacent to a vertex on C_0 or a 4^+-vertex not on f. For a vertex $v \in int(C_0)$ with $d(v) \in \{5,6\}$, v is *weak* if v is incident with two $(5,5^-,3)$-faces from F_3 one of which is bad and adjacent to a pendant 3-face in F_3 when $d(v) = 5$, or v is incident to two bad $(6,4,3)$-faces and one $(3,5^+,6)$-face from F_3 when $d(v) = 6$.

Lemma 2.5. *(1) There is no* $(3,5^+,5^+)$*-face with three weak vertices.*
(2) *(Lemma 11 in* [12]*) There is no* $(3,5^+,5)$*-face* $f = uvw$ *such that* u, v *are weak and* w *is incident with a* $(5,3,3)$*-face.*

Proof. We only give the proof of (1) here. Suppose that a $(3,5^+,5^+)$-face $f = uvw$ contains three weak vertices. When $d(v) = 5$, we label $N(v) - \{u, w\}$ as v_1, v_2, v_3 such that $d(v_2) = d(v_3) = 3$ and v_1

is a bad 4-vertex whose neighbors are v_2, v_1', v_1'' with $d(v_1') = 4$; when $d(v) = 6$, we label $N(v) - \{u, w\}$ as v_1, v_2, v_4, v_5 such that v_1, v_4 are bad 4-vertices with $N(v_1) = \{v_2, v_1', v_1''\}$, $N(v_4) = \{v_5, v_4', v_4''\}$ and $d(v_1') = d(v_4') = 4$. Similarly, label $N(w) - \{u, v\}$ as w_1, w_2, w_3. Let $S_1 = N(v) \cup \{v_1', v_1''\}$ if $d(v) = 5$, and $S_1 = N(v) \cup \{v_1', v_1'', v_4', v_4''\}$ if $d(v) = 6$.

We first have the following claim:

In a $(1,1,0)$-coloring of $G - S_1$, w can be properly colored.

Proof of the claim: Consider a $(1,1,0)$-coloring c of $G - S_1$.

First let $d(w) = 5$. We may assume that w_1, w_2, w_3 are colored differently. Note that we may recolor $w_3, w_1', w_1'', w_1, w_2$ in the order so that they are all properly colored. If $c(w_3) = 3$, then $\{c(w_1), c(w_2)\} = \{1, 2\}$, thus we can recolor w_2 so that it has the same color with w_1. Then w can be properly colored. If $c(w_3) = 1$ (or 2 by symmetry), then $\{c(w_1), c(w_2)\} = \{2, 3\}$; when $c(w_1) = 2$, we can recolor w_2 with 2, and color w properly; when $c(w_1) = 3$, w_1', w_2'' are colored 1 and 2, respectively, and we can recolor w_1 with 1, then color w properly.

Now assume that $d(w) = 6$. Again, we may recolor $w_1', w_1'', w_1, w_2, w_4', w_4'', w_4, w_5$ properly in the order. If there are only two colors on w_1, w_2, w_4, w_5, then w can be properly colored. If w_2 (or w_5) is colored with 3, then we can recolor it with 1 or 2; if $c(w_1) = 3$, then we can recolor w_1 with 1 or 2 so that it is different from the color of w_2. By doing this, we may assume that 3 is not on the four neighbors of w, so w can be properly color with 3. Thus we have the claim.

The following claim now gives a contradiction:

A $(1,1,0)$-coloring of $G - S_1$ with w being properly colored can be extended to a $(1,1,0)$-coloring of G.

Proof of the claim: Let c be a $(1,1,0)$-coloring of $G - S_1$ in which w is properly colored.

First assume that $d(v) = 5$. We color $u, , v_3$ properly in the order. If v can be properly colored, then we color v_2, v_1', v_1'' properly in the order, and finally color v_1, which can be color as it has only four properly colored neighbors. If v cannot be properly colored, then w, u, v_3 have different colors, thus v can be colored 1 or 2. Color v with 1 for a moment. Color v_2, v_1', v_1'' properly in the order. Now try to color v_1. If v_1 is not colorable, then it must be $(c(v_1'), c(v_1''), c(v_2)) = (3, 2, 2)$, in which case, we can color v_1 with 1 and color v with 2.

Now assume that $d(v) = 6$. We color $u, v, v_2, v_5, v_1', v_1'', v_4', v_4''$ properly in the order. Now v_1, v_4 can be colored, unless that both of them have the same color, say 1, with v. In the bad case, we can recolor v_2, v_5 with 1 or 3, then color v with 2. Thus the claim is true and we have a contraction. $\Box$

Now we discuss the configurations about 4-faces from F_4. Some of Lemmas 2.6- 2.10 have their initial forms in [17, 14].

Lemma 2.6. *(Adapted from Lemma 3.6 in* [13]*)*

(1) No 4*-face is from* F_4' *in* G.
(2) Let $f \in F_4$ *and let* v, x *be a pair of diagonal vertices on* $b(f)$. *Then* $d(v) \geq 4$ *or* $d(x) \geq 4$.

Lemma 2.7. *Let* $v \in int(C_0)$ *be a bad* 4*-vertex, or a* 5*-vertex incident with a bad* $(5,4,3)$*-face, or a* 5*-vertex incident with a* $(5,3,3)$*-face from* F_3. *If* v *is incident to a* 4*-face* f, *then its diagonal vertex on* $b(f)$ *is a* 4^+*-vertex.*

Proof. We consider the case when $d(v) = 5$. The other cases are very similar and simpler. Let $f_1 = vv_1v_2$ be a bad $(5, 4, 3)$-face with $d(v_1) = 4$, and let $f_3 = vv_3u_3v_4$ be a 4-face with $d(u_3) = 3$ in G. Let v_1', v_1'' be the two other neighbors of v_1 with $d(v_1') = 4$ and $d(v_1'') = 3$. Let $G' = G \setminus S$ and $H = G'[\{v_3, v_4\}]$, where $S = \{v, v_1, v_2, v_1', v_1''\}$. Let v_3^* be the resulting vertex by identifying v_3 with v_4. By Lemma 2.2 (5), $H \in \mathcal{F}$. Since $\sigma(H) < \sigma(G)$, C_0 has a superextension ϕ_H on H. Based on ϕ_H, we color v_3, v_4 with the color $\phi_H(v_3^*)$ and recolor properly u_3 with a color in $\{1, 2, 3\} \setminus \{\phi_H(v_3^*), \phi_H(u_3')\}$, where u_3' is the other neighbor of u_3 in G. Next, properly color v with a color in $\{1, 2, 3\} \setminus \{\phi_H(v_3^*), \phi_H(v_5)\}$, and properly color v_2, v_1', v_1'' in order, and finally color v_1 as it has four properly colored neighbors. Thus, C_0 has a superextension ϕ_G on G, a contradiction. □

For a positive integer n, let $[n] = \{1, 2, \dots, n\}$. For a vertex $v \in int(C_0)$ with $d(v) = k$, let $v_1, v_1, \dots, v_k$ denote the neighbors of v in a cyclic order. Let f_i be the face with vv_i and vv_{i+1} as two boundary edges for $i = 1, 2, \dots, k$, where the subscripts are taken modulo k. A k-vertex $v \in int(C_0)$ is *poor* if it is incident with k 4-faces from F_4, and *rich* otherwise.

The following is a very useful lemma in the remaining proofs.

Lemma 2.8. *((Lemma 3. 10 from [14]) Let $v \in int(C_0)$ be a 4-vertex with $N(v) = \{v_i : i \in [4]\}$. If v is incident with two 4-faces that share exactly an edge, then no t-path joins v_i and v_{i+2} in G for $t \in \{1, 2, 3, 5\}$, where the subscripts of v are taken modulo 4.*

Lemma 2.9. *Let $v \in int(C_0)$ be a 4-vertex incident with a 4-face $f_i = vv_iu_iv_{i+1}$. Then each of the followings holds, where the subscripts are taken modulo 4.*

(1) If $d(v_i) = d(u_i) = 3$, then f_{i-1} and f_{i+1} are 6^+-faces. Consequently, if v is poor, then v is not incident with $(3, 3, 4, 4^+)$-faces.
(2) (Lemma 3.11 (1) in [14]) If $f_{i+1} = vv_{i+1}u_{i+1}v_{i+2}$, then $d(u_i) \geq 4$ or $d(u_{i+1}) \geq 4$.
(3) (Lemma 3.11 (2) in [14]) If $f_{i+2} = vv_{i+2}u_{i+2}v_{i+3}$, then $d(u_i) \geq 4$ or $d(u_{i+2}) \geq 4$.
(4) (Lemma 3.12 from [14]) If v is a poor 4-vertex, then either $d(v_i) \geq 5$ or $d(v_{i+2}) \geq 5$.

Proof. (1) Suppose that f_{i-1} and f_i are 4-faces with $d(u_i) = d(v_i) = 3$, where v_i's are neighbors of 4-vertex v. Identify v_{i-1}, v_i, and v_{i+1} into one vertex, we get a new graph in $\mathcal{F}$, so the new graph is $(1, 1, 0)$-colorable. Now the original graph has a $(1, 1, 0)$-coloring, unless u_{i-1} has the same color (1 or 2), which by symmetry we assume to be 1, as v_{i-1}, v_i and v_{i+1}. We uncolor v_i and v, and then color u_i and v properly. Clearly, u_i and v are colored 2 or 3. If u_i and v are colored differently, color v_i with 2; if u_i and v are colored the same, color v_i with an available color. □

Let v be a 5^+-vertex in $int(C_0)$. For convenience, we use $Q_4(v)$ to denote the set of poor 4-vertices in $N(v) \setminus C_0$ that are incident with $(3, 4, 4, 4)$-faces from F_4.

Lemma 2.10. *Let v be a poor 5-vertex in G. Then*

(1) (Lemma 3.13(2) from [14]) At most two vertices in $\{u_i : i \in [5]\}$ are 3-vertices.
(2) (Lemma 3.13(3) from [14]) If $d(u_i) = 3$, then either $d(v_{i-1}) \geq 5$ or $d(v_{i+2}) \geq 5$.
(3) (Lemma 3.13(1) from [14]) If $d(u_i) = d(v_i) = 3$, then $d(u_j) \geq 4$ for $j \in [5] \setminus \{i\}$.
(4) If f_i is a $(5, 4, 3, 4)$-face, then at most one of v_i, v_{i+1} is in $Q_4(v)$.
(5) If $d(v_i) = d(u_i) = d(v_{i+2}) = 3$, then $d(v_j) \geq 5$ for $j \in [5] \setminus \{i, i + 2\}$.
(6) If f_i is a $(5, 3, 4, 4)$-face such that u_i and v_{i+1} are poor 4-vertices, then $v_{i+1} \notin Q_4(v)$.

Proof. (4) Without loss of generality, assume that $f_1 = vv_1u_1v_2$ is a $(5,4,3,4)$-face. Assume further that v_1, v_2 are in $Q_4(v)$. By Lemma 2.9(4),$d(u_2) \geq 5$ and $d(u_5) \geq 5$ as $d(u_1) = 3$. Let $N(u_1) = \{v_1, v_2, w\}$ and $N(w) \cap N(v_1) = \{v_1'\}$ and $N(w) \cap N(v_2) = \{v_2'\}$. It implies that $u_1v_1v_1w$ and $u_1v_2v_2'w$ are $(3,4,4,4)$-faces. So $d(v_1') = d(v_2') = d(w) = d(v_1) = d(v_2) = 4$. Let $H = G[\{u_1, v_1', v_2', v\}]$. By Lemmas 2.2 (5) and 2.8, $H \in \mathcal{F}$. Let c be a coloring of (H, C_0). Let v' be the resulting vertex of the identification. In G, color u_1, u_2, u_3 with $c(v')$, and properly color v_1, v_2, w in the order, and finally color u_1. Thus, (G, C_0) is superextendable, a contradiction.

(5) By symmetry, assume that $d(v_1) = d(u_1) = d(v_3) = 3$. By Lemma 2.7 (2), $d(v_2) \geq 4$, and furthermore, by Lemma 2.9 (2), $d(v_2) \geq 5$.

We show that $d(v_5) \geq 5$. Suppose otherwise that $d(v_5) \leq 4$. Then by Lemma 2.6(2), $d(v_5) = 4$. Let $G' = G - \{v, v_5\}$ and $H = G'[\{u_4, u_5\}, \{v_2, v_4\}]$. By Lemmas 2.2 (5) and 2.8, $H \in \mathcal{F}$. Then (H, C_0) is superextendable, and let c be a coloring of (H, C_0). In G, color v_2, v_4 and u_4, u_5 with the colors of the identified vertices, respectively, then properly recolor v_5, u_1, v_1, v_3 in the order. Let c' be the resulting coloring of $G - v$. Now we color v. If $c'(v_2) = c'(v_4) = 3$, then v can be colored, as the other three colored neighbors are all properly colored, so we may assume that $c'(v_2) = c'(v_4) = 1$. If $c'(v_1) = 1$, then clearly v can be colored; If $c'(v_1) = 3$, then we uncolor v_1 and color v with 3 (if $c'(v_3), c'(v_5) \neq 3$) or with 2 (if $c'(v_3) = 3$ or $c'(v_5) = 3$), and now v_1 can be colored, as $c'(u_1) \in \{1, 2\}$ and u_1 is properly colored, a contradiction.

Similarly, we have $d(v_4) \geq 5$.

(6) As v_{i+1} is a poor 4-vertex and $d(u_i) = 4$, by Lemma 2.9(4), $d(u') \geq 5$, where u' is the diagonal vertex to v on the 4-face incident with v_{i+1}; similarly, u_i is a poor 4-vertex and $d(v_i) = 3$, by Lemma 2.9(4), $d(u_1') \geq 5$, where u_1' is the diagonal vertex to v_2 on the 4-face incident with u_1. It follows that no 4-face incident with v_{i+1} is a $(3,4,4,4)$-face, so $v_{i+1} \notin Q_4(v)$. $\square$

3. Discharging procedure

In this section, we prove the main theorem by a discharging argument.

Let the initial charge of a vertex v be $\mu(v) = 2d(v) - 6$, the initial charge of a face $f \neq C_0$ be $\mu(f) = d(f) - 6$, and $\mu(C_0) = d(C_0) + 6$. By Euler's formula, $\sum_{x \in V \cup F} \mu(x) = 0$.

A $(3,4,4,5^+)$- or $(3,4,5^+,4)$-face $f \in F_4$ is *superlight* if both 4-vertices on $b(f)$ are poor and *light* otherwise.

The following are the discharging rules:

(R1) Let $v \in int(C_0)$ with $d(v) = 4$ and $f \in F_3 \cup F_4$ be a face incident with v.

- (R1.1) When $f \in F_3$, f gets 1 from v, unless v is incident with f and a $(3,4,4)$-face $f' \in F_3$, in which case, v gives $\frac{5}{4}$ to f' and $\frac{3}{4}$ to f.
- (R1.2) When $f \in F_4$, f gets 1 from v if it is a $(4,3,3,4^+)$-face, $\frac{2}{3}$ if it is a $(4,4,4,3)$-face or v is rich, and $\frac{1}{2}$ otherwise.

(R2) Let $v \in int(C_0)$ with $d(v) \geq 5$.

- (R2.1) Let $f = vuw$ be a 3-face in F_3 incident with v.
  - (R2a) Let f be a $(5^+,3,3)$-face. Then v gives 2 to f.
  - (R2b) Let f be a $(5^+,4,3)$-face. Then v gives $\frac{9}{4}$ to f if u is bad and w is weak; 2 to f if u is not bad and w is weak; $\frac{7}{4}$ to f if u is bad and w is strong; $\frac{3}{2}$ to f if u is not bad and w is strong.

(R2c) Let f be a $(5^+, 5^+, 3)$-face. Then v gives $\frac{5}{4}$ to f if v is weak, and $\frac{7}{4}$ otherwise.

(R2d) Let f be a $(5^+, 4^+, 4^+)$-face. Then v gives $\frac{3}{2}$ to f if both u, w are bad 4-vertices; $\frac{5}{4}$ to f if exactly one of u and w is a bad 4-vertex; 1 to other $(5^+, 4^+, 4^+)$-faces.

(R2.2) For each 4-face $f \in F_4$ incident with v, v gives 1 to f if f is a $(5^+, 4^+, 3, 3)$-face or a superlight $(3, 4, 4, 5^+)$- or $(3, 4, 5^+, 4)$-face; $\frac{5}{6}$ to f if f is a light $(3, 4, 4, 5^+)$- or $(3, 4, 5^+, 4)$-face; $\frac{3}{4}$ to f if f is a $(3, 4, 5^+, 5^+)$-face or a $(3, 5^+, 4, 5^+)$-face; $\frac{1}{2}$ to f if f is a $(4^+, 4^+, 4^+, 5^+)$-face.

(R2.3) If $Q_4(v) \neq \emptyset$, then v gives $\frac{1}{6}$ to each 4-vertex in $Q_4(v)$.

(R3) Each 4^+-vertex sends $\frac{1}{2}$ to each of its pendant 3-faces in F_3.

(R4) Let $v \in C_0$. Then v gives 3 to each incident 3-face from F_3'; $\frac{3}{2}$ to each incident face from F_3''; 1 to each incident 4-face from F_4''.

(R5) C_0 gives 2 to each 2-vertex on C_0; $\frac{3}{2}$ to each 3-vertex on C_0; 1 to each 4-vertex on C_0; and $\frac{1}{2}$ to each 5-vertex on C_0. In addition, if C_0 is a 7-face with six 2-vertices, then it gets 1 from the incident face.

We will show that each $x \in F(G) \cup V(G)$ has final charge $\mu^*(x) \geq 0$ and at least one face has positive charge, to reach a contradiction.

As G contains no 5-faces, and 6^+-faces other than C_0 are not involved in the discharging procedure, we will check the final charge of the 3- and 4-faces other than C_0 first.

Lemma 3.1. *Let f be a i-face in $F(G) \setminus C_0$ for $i = 3, 4$. Then $\mu^*(f) \geq 0$.*

Proof. Suppose that $d(f) = 3$ and $f = vuw$ with $d(v) \leq d(u) \leq d(w)$. By Lemma 2.2 (3), $|b(f) \cap C_0| \leq 2$. If $|b(f) \cap C_0| = 2$, then $f \in F_3''$, by (R4), $\mu^*(f) \geq -3 + 2 \times \frac{3}{2} = 0$; if $|b(f) \cap C_0| = 1$, then $f \in F_3'$, by (R4), $\mu^*(f) \geq -3 + 3 = 0$. Hence, let $|b(f) \cap C_0| = 0$. By Proposition 2.1, $d(v) \geq 3$.

Assume first that $d(v) = 3$. If f is a $(3, 3, a)$-face, by Lemma 2.3 (2), $a \geq 5$ and the outer neighbors of u, v are of degree at least 4 or on C_0, then by (R2a) and (R3), $\mu^*(f) \geq -3 + 2 \times \frac{1}{2} + 2 = 0$. If f is a $(3, 4, 4)$-face, by Lemma 2.3 (4), the third neighbor of v is a 4^+-vertex or on C_0, then by (R1.1) and (R3), $\mu^*(f) \geq -3 + 2 \times \frac{5}{4} + \frac{1}{2} = 0$. Now let f be a $(3, 4, 5^+)$-face. Then by (R1.1) and (R2b), $\mu^*(f) \geq -3 + \frac{9}{4} + \frac{3}{4} = 0$ if v is weak and u is bad; $\mu^*(f) \geq -3 + \frac{7}{4} + \frac{3}{4} + \frac{1}{2} = 0$ if v is strong and u is bad; $\mu^*(f) \geq -3 + 2 + 1 = 0$ if v is weak and u is not bad; $\mu^*(f) \geq -3 + \frac{3}{2} + 1 + \frac{1}{2} = 0$ if v is strong and u is not bad.

Assume that $d(v) = 4$. Then $d(w) \geq d(u) \geq 4$. If f is a $(4, 4, 4)$-face, then by Lemma 2.3 (5), none of the 4-vertices on f can be bad, thus by (R1.1), $\mu^*(f) \geq -3 + 3 \times 1 = 0$. Now assume that f is a $(4, 4^+, 5^+)$-face. In this case, if v, u are two bad 4-vertices, then by (R1.1) and (R2c), f receives at least $\frac{3}{2}$ from w and at least $\frac{3}{4}$ from each of v and u, thus $\mu^*(f) \geq -3 + 2 \times \frac{3}{4} + \frac{3}{2} = 0$; if one of v, u is not bad, then by (R1.1) and (R2c), w gives at least $\frac{5}{4}$ to f and u, v give at least $(\frac{3}{4} + 1)$ to f, then $\mu^*(f) \geq -3 + (\frac{3}{4} + 1) + \frac{5}{4} = 0$; for other cases, by (R1.1), and (R2c), f receives at least 1 from each vertex on $b(f)$, thus $\mu^*(f) \geq -3 + 3 \times 1 = 0$.

Finally, let $d(v) \geq 5$. It follows that f is a $(5^+, 5^+, 3)$-face, then by Lemma 2.5 (1), at most two of the three vertices are weak, so by (R2c) and (R3), $\mu^*(f) \geq -3 + \min\{\frac{5}{4} + \frac{7}{4}, 2 \times \frac{7}{4}, 2 \times \frac{5}{4} + \frac{1}{2}\} \geq 0$.

Suppose that $d(f) = 4$ and $f = vuwx$. By Lemma 2.2 (4), $|b(f) \cap C_0| \leq 2$. If $|b(f) \cap C_0| = 2$, then $f \in F_4''$, by (R4), $\mu^*(f) \geq -2 + 2 \times 1 = 0$. By Lemma 2.6(1) $F_4' = \emptyset$. Hence, assume that $|b(f) \cap C_0| = 0$. By Proposition 2.1, $d(z) \geq 3$ for each $z \in b(f)$. By Lemma 2.6 (2), if $d(z) = 3$ for some $z \in b(f)$, then its diagonal vertex on $b(f)$ is a 4^+-vertex.

If f is a $(3,3,4^+,4^+)$-face, then by (R1.2) and (R2.2), $\mu^*(f) \geq -2+2\times 1 = 0$. If f is a $(3,4,4,4)$-face, then by (R1.2), $\mu^*(f) \geq -2+3\times\frac{2}{3} = 0$. If f is a $(3,4^+,5^+,5^+)$-face or $(3,5^+,4^+,5^+)$-face, then by (R2.2), $\mu^*(f) \geq -2+2\cdot\frac{3}{4}+\frac{1}{2} = 0$. If f is a $(4^+,4^+,4^+,4^+)$-face, then by (R1.2) and (R2.2), $\mu^*(f) \geq -2+4\times\frac{1}{2} = 0$. Finally, let f be a $(3,4,4,5^+)$-face or $(3,4,5^+,4)$-face. If f is superlight, then f gets 1 from the 5^+-vertex on $b(f)$ and at least $\frac{1}{2}$ from each 4-vertex on $b(f)$, thus $\mu^*(f) \geq -2+1+2\times\frac{1}{2} = 0$. Otherwise, f is light, by (R2.2), f receives $\frac{5}{6}$ from the 5^+-vertex, $\frac{2}{3}$ from a rich 4-vertex and at least $\frac{1}{2}$ from the other 4-vertex on $b(f)$, then $\mu^*(f) \geq -2+\frac{5}{6}+\frac{2}{3}+\frac{1}{2} = 0$. $\square$

Let v be a k-vertex in $int(C_0)$. Let t_i be the number of i-faces incident with v in F_i for $i \in \{3,4\}$. Let t_p be the number of pendant 3-faces adjacent to v. By Proposition 2.1,

$$(1) \qquad t_3 \leq \lfloor\frac{k}{2}\rfloor, \text{ and } \; t_4 \leq \max\{0, k-2t_3-t_p-1\} \text{ if } t_4 > 0.$$

Lemma 3.2. *Let $v \in int(C_0)$ be a 4-vertex. Then $\mu^*(v) \geq 0$.*

Proof. If $N(v)\cap C_0 \neq \emptyset$, then $t_3 \leq 1$, thus $\mu^*(v) \geq 2-\max\{\frac{5}{4}+\frac{1}{2}, 2\cdot 1, 3\cdot\frac{1}{2}\} \geq 0$. So, let $N(v)\cap C_0 = \emptyset$. Clearly, $t_3 \leq 2$.

If $t_3 = 2$, then by Lemma 2.3 (5), at most one of the triangles is a $(3,4,4)$-face, thus by (R1.1), $\mu^*(v) \geq 2-\max\{\frac{5}{4}+\frac{3}{4}, 2\cdot 1\} = 0$. If $(t_3,t_4) = (1,1)$, then when v is not bad, by (R1.1) and (R1.2), v gives at most one to each of the incident faces, thus $\mu^*(v) \geq 2-2\cdot 1 = 0$, and when v is bad, v cannot be incident with a $(3,3,4,4^+)$-face by Lemma 2.7, then by (R1.1) and (R1.2), $\mu^*(v) \geq 2-\frac{5}{4}-\frac{2}{3} = \frac{1}{12} > 0$. Let $(t_3,t_4) = (1,0)$. Then $0 \leq t_p \leq 2$. By Lemma 2.3 (3), at least one of the other neighbors of v is a 4^+-vertex or in C_0 when v is bad, thus by (R1.1) and (R3), $\mu^*(v) \geq 2-\max\{\frac{5}{4}+\frac{1}{2}, 1+2\times\frac{1}{2}\} = 0$.

Now, we assume that $t_3 = 0$. If $t_p \geq 2$, then $t_4 \leq 1$, so by (R1.2) and (R3), $\mu^*(v) \geq 2-\max\{4\cdot\frac{1}{2}, 1+2\times\frac{1}{2}\} = 0$. Assume that $t_p = 1$ and $t_4 = 2$. Let v be incident with 4-faces $f_3 = vv_2u_2v_3$ and $f_4 = vv_3u_3v_4$ in F_4. By Lemmas 2.6 (2) and 2.9, at most two of the vertices in $\{v_2,u_2,v_3,u_3,v_4\}$ are 3-vertices, and when $d(v_3) = 3$, none of the other vertices is a 3-vertex, then by (R1.2) and (R3), v gives at most $\max\{2\cdot\frac{2}{3}, 1+\frac{1}{2}\} = \frac{3}{2}$ to f_3 and f_4, thus $\mu^*(v) \geq 2-\frac{3}{2}-\frac{1}{2} = 0$.

Lastly, let $t_3 = t_p = 0$. If $t_4 \leq 2$, by (R1.2), $\mu^*(v) \geq 2-2\times 1 = 0$. Let $t_4 = 3$. If v is not incident with a $(4,3,3,4^+)$-face, then by (R1.2), $\mu^*(v) \geq 2-3\times\frac{2}{3} = 0$; If v is incident with a $(4,3,3,4^+)$-face, then by Lemma 2.9, the other incident 4-faces are $(4,4^+,4^+,4^+)$-faces, so by (R1.2), $\mu^*(v) \geq 2-1-2\times\frac{1}{2} = 0$. Hence assume that $t_4 = 4$, that is, v is poor. By Lemma 2.9 (4), v is adjacent to at least two 5^+-vertices, and without loss of generality, let $d(v_3), d(v_4) \geq 5$. By Lemma 2.9 (1), v is not incident with $(3,3,4,4^+)$-faces. Thus, if v is not incident with $(3,4,4,4)$-faces, then by (R1), $\mu^*(v) \geq 2-4\times\frac{1}{2} = 0$, and if v is incident with a $(3,4,4,4)$-face, then by (R2.3), v gets $\frac{1}{6}$ from each of its 5^+-neighbors, so by (R1) and (R2.3), $\mu^*(v) \geq 2-3\times\frac{1}{2}-\frac{2}{3}+2\cdot\frac{1}{6} > 0$. $\square$

Lemma 3.3. *Let $v \in int(C_0)$ be a k-vertex with $k \geq 5$. If $u \in Q_4(v)$, then one of the 4-faces that contain uv as an edge contains no 3-vertices or is a $(3,5^+,4,5^+)$-face.*

Proof. As $u \in Q_4(v)$, u is a poor 4-vertex and incident with one $(3,4,4,4)$-face. Suppose that $f_i = uv_iu_iv_{i+1}$ for $i \in [4]$, where $v = v_4$ and the subscripts are taken modulo 4. We show that f_3 or f_4 contains no 3-vertices or is a $(3,5^+,4,5^+)$.

If $d(u_3) \geq 4$ and $d(u_4) \geq 4$, then by Lemma 2.9(4), either $d(v_1) \geq 5$ or $d(v_3) \geq 5$, so f_3 or f_4 cannot contain 3-vertices. Thus, by symmetry, let $d(u_3) = 3$. By Lemma 2.9 (1)-(3), $d(v_3) \geq 4$ and

$d(u_j) \geq 4$ for $j \in [4] \setminus \{3\}$. So f_4 contains no 3-vertices if $d(v_1) \neq 3$. Let $d(v_1) = 3$. Then $v_1uv_2u_1$ is the $(3,4,4,4)$-face. By Lemma 2.9(4), $d(v_3) \geq 5$, so f_3 is a $(3,5^+,4,5^+)$-face, as desired. $\square$

Let $v \in int(C_0)$ with $d(v) = k \geq 5$. By Lemma 3.3, a vertex in $Q_4(v)$ must either share a 4-face without 3-vertices with v, or is on a $(3,5^+,4,5^+)$-face. In the former case, the 4-face could contain at most two vertices from $Q_4(v)$, then the charges from v to the vertices and the 4-face are at most $\frac{1}{2} + 2 \cdot \frac{1}{6} < 1$. In the latter case, the face contains exactly one vertex from $Q_4(v)$, then by (R2), the charges from v to the vertex and the 4-face are at most $\frac{3}{4} + \frac{1}{6} < 1$. Thus, by (R2),

$$\mu^*(v) \geq 2k - 6 - \frac{9}{4}t_3 - t_4 - \frac{1}{2}t_p \tag{2}$$

$$\geq (k - 2t_3 - t_4 - t_p) + (\frac{7}{8}k - 6) \geq \frac{7}{8}k - 6 \ \ (\text{as } t_3 \leq \lfloor \frac{k}{2} \rfloor). \tag{3}$$

Lemma 3.4. *Suppose that $v \in int(C_0)$ is a 5-vertex with $t_3 > 0$. Then $\mu^*(v) \geq 0$.*

Proof. If $|N(v) \cap C_0| \geq 2$, then $t_3 + t_p \leq 2$ and $t_4 = 0$, as $t_3 > 0$, so by (R1)-(R5), $\mu^*(v) \geq 4 - 9/4 - 1/2 > 0$. If $|N(v) \cap C_0| = 1$, then v cannot be incident with two bad 3-faces in F_3 by Lemma 2.4 (2), and when v is incident with a bad 3-face and a $(3,4,5)$-face, the 3-vertex is strong on the $(3,4,5)$-face by Lemma 2.4 (3), so $\mu^*(v) \geq 4 - \max\{\frac{9}{4} + \frac{1}{2}, \frac{9}{4} + \frac{7}{4}, 2 \times 2\} = 0$ by (R2.1)-(R2.3) and (R3). Therefore, we assume that $|N(v) \cap C_0| = 0$.

Assume first that $t_3 = 2$. Let $f_1 = vv_1v_2$ and $f_3 = vv_3v_4$ be the incident 3-faces and v_5 be the fifth neighbor of v. By Lemma 2.4 (2), at most one of f_1, f_3 is bad. If both f_1 and f_3 are $(3,4^-,5)$-faces, then by Lemma 2.4 (1), $d(v_5) \geq 4$ or $v_5 \in C_0$, , and by Lemma 2.4 (3), if one is bad, then the 3-vertex on the other one is strong, thus by (R2b), $\mu^*(v) \geq 4 - \max\{\frac{9}{4} + \frac{7}{4}, 2 \times 2\} = 0$. If f_1 is a $(3,4,5)$-face and f_3 is a $(3,5,5^+)$-face, then by (R2) and (R3), $\mu^*(v) \geq 4 - (\frac{9}{4} + \frac{5}{4} + \frac{1}{2}) = 0$ if v is weak, and $\mu^*(v) \geq 4 - \max\{\frac{9}{4} + \frac{7}{4}, \frac{7}{4} + \frac{7}{4} + \frac{1}{2}\} \geq 0$ if v is not weak. If none of f_1, f_3 is a $(3,4,5)$-face, then by (R2), $\mu^*(v) \geq 4 - 2 \cdot \frac{7}{4} - \frac{1}{2} = 0$.

Finally, let $t_3 = 1$. Then $t_4 \leq 2$. If $t_4 \leq 1$, then $|Q_4(v)| = 0$, thus, by (R2a), (R2.2) and (R3), $\mu^*(v) \geq 4 - \frac{9}{4} - 1 - \frac{1}{2} = \frac{1}{4} > 0$. Thus assume that $t_4 = 2$ and let $f_1 = vv_1v_2$, $f_3 = vv_3u_3v_4$ and $f_4 = vv_4u_4v_5$ be the incident faces. Note that v_3, v_5 are rich and $|Q_4(v)| \leq 1$. If f_1 is not bad, then by Lemma 3.3 and by (R2.1), (R2.2) and (R2.3), $\mu^*(v) \geq 4 - 2 - \max\{1 + \frac{3}{4} + \frac{1}{6}, 2 \cdot 1\} = 0$. Therefore, let f_1 be a bad $(5,4,3)$-face. By Lemma 2.7, $d(u_3) \geq 4$ and $d(u_4) \geq 4$. Consider $d(v_4) = 3$ first. Then $|Q_4(v)| = 0$, and by Lemma 2.6, $d(v_3) \geq 4$ and $d(v_5) \geq 4$. Therefore, if $d(v_3) = d(v_5) = 4$, then as v_3, v_5 are rich, by (R2a) and (R2.2), v gives at most $\frac{5}{6}$ to each of f_3, f_4, thus, $\mu^*(v) \geq 4 - \frac{9}{4} - 2 \times \frac{5}{6} = \frac{1}{12} > 0$; if $d(v_3) \geq 5$ or $d(v_5) \geq 5$, then there are at least two 5^+-vertices in $b(f_3)$ or $b(f_4)$, thus, by (R2a) and (R2.2), $\mu^*(v) \geq 4 - \frac{9}{4} - 1 - \frac{3}{4} = 0$. Assume next that $d(v_4) = 4$. Then $|Q_4(v)| \leq 1$. By Lemma 2.9 (2), either $d(v_3) \geq 4$ or $d(v_5) \geq 4$. It means that f_3 or f_4 is a $(5,4,4^+,4^+)$-face, by (R2a), (R2.2) and (R2.3), $\mu^*(v) \geq 4 - \frac{9}{4} - 1 - \frac{1}{2} - \frac{1}{6} = \frac{1}{12} > 0$. Assume that $d(v_4) \geq 5$. Then $|Q_4(v)| = 0$, and f_3, f_4 are $(5,5^+,4^+,3^+)$-faces, by (R2a) and (R2.2), v gives at most $\frac{3}{4}$ to each of f_3, f_4, then $\mu^*(v) \geq 4 - \frac{9}{4} - 2 \times \frac{3}{4} = \frac{1}{4} > 0$. $\square$

For a poor 5-vertex $v \in int(C_0)$, let $f(v) = (f_1, f_2, f_3, f_4, f_5)$, where $f_i = vv_iu_iv_{i+1}$ with $i \in \mathcal{Z}_5$, the cyclic group of order 5. We say that v gives a *charge sequence* $(a_1, a_2, a_3, a_4, a_5)$ *to* $f(v)$ if v gives at most a_i to f_i by (R2.2).

Lemma 3.5. *For each* 5*-vertex* $v \in int(C_0)$, $\mu^*(v) \geq 0$.

Proof. By Lemma 3.4, we may assume that $t_3 = 0$. By (1), $\mu^*(v) \geq 4 - t_4 - t_p/2 \geq 0$ if $t_4 \leq 4$. Thus, we let $t_4 = 5$, that is, v is poor. Let $M(v) = \{u_1, u_2, u_3, u_4, u_5\}$. By Lemma 2.10(1), $M(v)$ has at most two 3-vertices, and by Lemma 2.6 (2), there are at most two 3-vertices in $N(v)$.

Case 1. $N(v)$ has exactly two 3-vertices.

By symmetry and Lemma 2.6 (2), we may assume that $d(v_1) = d(v_3) = 3$. By Lemma 2.6 (2), $d(v_2), d(v_4), d(v_5) \geq 4$. Furthermore, by Lemma 2.9 (2), $d(v_2) \neq 4$, thus $d(v_2) \geq 5$.

Assume that some vertex, say u_1, in $M(v)$ has degree 3. Then by Lemma 2.10 (3), $d(u_j) \geq 4$ for $j \in [5] \setminus \{1\}$, and by Lemma 2.10(5), $d(v_j) \geq 5$ for $j \in [5] \setminus \{1, 3\}$, thus $|Q_4(v)| = 0$, and f_2, f_3, f_4, f_5 are $(5, 5^+, 4^+, 3)$- $(5, 3, 4^+, 5^+)$-, $(5, 4^+, 4^+, 5^+)$-, and $(5, 5^+, 4^+, 3)$-faces, respectively. By (R2.2) and (R2.3), v gives a charge sequence $(1, \frac{3}{4}, \frac{3}{4}, \frac{1}{2}, \frac{3}{4})$ to $f(v)$, thus $\mu^*(v) \geq 4 - 1 - 3 \times \frac{3}{4} - \frac{1}{2} = \frac{1}{4} > 0$. Hence we assume that $M(v)$ has no 3-vertex.

As f_1, f_2 are $(3, 4^+, 5^+, 5)$-faces and f_4 is a $(4^+, 4^+, 4^+, 5)$-face, by (R2.2), v gives $2 \cdot \frac{3}{4} + \frac{1}{2} = 2$ to them. Consider f_3 (and similarly f_5), which is a $(3, 4^+, 4^+, 5)$-face. We claim that

$$v \text{ gives at most 1 to the face and the vertex in } Q_4(v) \cap b(f_3),$$

which shows that $\mu^*(v) \geq 4 - 2 - 2 \cdot 1 = 0$. In fact, if it contains another 5^+-vertex, then by (R2.2) and (R2.3), v gives at most $\frac{3}{4} + \frac{1}{6} < 1$, as desired. So let it be a $(3, 4, 4, 5)$-face. If it contains two poor 4-vertices, then it is superlight and by Lemma 2.10(4), it contains no vertex in $Q_4(v)$, thus by (R2.2), v gives 1 to it; otherwise, it is light, thus by (R2.2) and (R2.3), v gives $\frac{5}{6} + \frac{1}{6} = 1$ to it.

Case 2. $N(v)$ has exactly one 3-vertex. By symmetry, we assume that $d(v_1) = 3$.

Assume first that $M(v)$ contains no 3-vertex. Then each of f_2, f_3, f_4 is a $(5, 4^+, 4^+, 4^+)$-face, and f_1 is a $(5, 3, 4^+, 4^+)$-face and f_5 is a $(5, 4^+, 4^+, 3)$-face. By (R2.2) and (R2.3), v gives a charge sequence $(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1)$ to $f(v)$, thus $\mu^*(v) \geq 4 - 2 \times 1 - 3 \times \frac{1}{2} - 3 \times \frac{1}{6} = 0$ if $|Q_4(v)| \leq 3$. Thus, assume that $v_j \in Q_4(v)$ for $j \in [5] \setminus \{1\}$. If u_1 is a poor 4-vertex, then $f_1 = vv_1u_1v_2$ is a $(5, 3, 4, 4)$-face such that u_1, v_2 are poor and $v_2 \in Q_4(v)$, a contradiction to Lemma 2.10(6). Thus, assume that u_1 is not a poor 4-vertex. In this case, f_1 is light. By (R2.2), v gives $\frac{5}{6}$ to f_1. Thus, $\mu^*(v) \geq 4 - 1 - \frac{5}{6} - 3 \times \frac{1}{2} - 4 \times \frac{1}{6} = 0$.

Next, assume that $M(v)$ contains exactly one 3-vertex. Let $d(u_1) = 3$ (or by symmetry $d(u_5) = 3$). By our assumption, $d(u_j) \geq 4$ for $j \neq 1$ and $j \in [5]$. Thus, each of f_2, f_3, f_4 is a $(5, 4^+, 4^+, 4^+)$-face, f_5 is a $(5, 4^+, 4^+, 3)$-face. Note that if $d(v_2) \geq 5$, then $|Q_4(v)| \leq 3$; if $d(v_2) = 4$, then by Lemma 2.9(1), v_2 is rich, which implies that $v_2 \notin Q_4(v)$, then $|Q_4(v)| \leq 3$. By (R2.2) and (R2.3), v gives a charge sequence $(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1)$ to $f(v)$, and $\mu^*(v) \geq 4 - 2 \times 1 - 3 \times \frac{1}{2} - 3 \times \frac{1}{6} = 0$. Hence, we may assume, by symmetry, that either $d(u_3) = 3$ or $d(u_2) = 3$.

Let $d(u_3) = 3$. By Lemma 2.10 (2), either $d(v_2) \geq 5$ or $d(v_5) \geq 5$, and by symmetry we may assume that $d(v_5) \geq 5$. This implies that f_5 is a $(3, 5, 4^+, 5^+)$-face and $v_5 \notin Q_4(v)$. In this case, both f_2, f_4 are two $(5, 4^+, 4^+, 4^+)$-faces, and f_1 is a $(5, 3, 4^+, 4^+)$-face. If $d(v_3) \geq 5$ or $d(v_4) \geq 5$, then $|Q_4(v)| \leq 2$, and by (R2.2) and (R2.3), v gives a charge sequence $(1, \frac{1}{2}, \frac{3}{4}, \frac{1}{2}, \frac{3}{4})$ to $f(v)$, thus, $\mu^*(v) \geq 4 - 1 - 2 \times \frac{3}{4} - 2 \times \frac{1}{2} - 2 \times \frac{1}{6} = \frac{1}{6} > 0$. Then assume that $d(v_3) = d(v_4) = 4$. By Lemma 2.10(4), either $v_3 \notin Q_4(v)$ or $v_4 \notin Q_4(v)$. If f_1 is a $(5, 3, 4, 4)$-face with two poor 4-vertices, then by Lemma 2.10(6), $v_2 \notin Q_4(v)$, it follows that $|Q_4(v)| \leq 1$, so by (R2.2) and (R2.3), v gives a charge sequence $(1, \frac{1}{2}, 1, \frac{1}{2}, \frac{3}{4})$ to $f(v)$, thus, $\mu^*(v) \geq 4 - 1 - \frac{1}{2} - 1 - \frac{1}{2} - \frac{3}{4} - \frac{1}{6} = \frac{1}{12} > 0$; otherwise, f_1 is a light $(5, 3, 4^+, 4^+)$-face, so by (R2.2) and (R2.3), $\mu^*(v) \geq 4 - \frac{5}{6} - \frac{1}{2} - 1 - \frac{1}{2} - \frac{3}{4} - 2 \cdot \frac{1}{6} > 0$.

Let $d(u_2) = 3$ now. By Lemma 2.10 (2), $d(v_4) \geq 5$. Then both f_3, f_4 are $(5, 4^+, 4^+, 4^+)$-faces. If v_2 is a rich 4-vertex or 5^+-vertex, then $v_2 \notin Q_4(v)$ and $|Q_4(v)| \leq 2$, so by (R2.2) and (R2.3), v gives at most $\frac{5}{6}$ to each of f_1, f_2, and v gives a charge sequence $(\frac{5}{6}, \frac{5}{6}, \frac{1}{2}, \frac{1}{2}, 1)$ to $f(v)$, it follows that $\mu^*(v) \geq 4 - 2 \times \frac{5}{6} - 2 \times \frac{1}{2} - 1 - 2 \times \frac{1}{6} = 0$. Therefore, we may assume that v_2 is a poor 4-vertex. Then by Lemmas 2.9 (4), $d(u_1) \geq 5$ as $d(u_2) = 3$, and by Lemma 2.10(4), and $v_2 \notin Q_4(v)$ or $v_3 \notin Q_4(v)$. Consider f_5. By Lemma 2.10(6), it is either a light $(5, 3, 4^+, 4^+)$-face, or $(5, 3, 4, 4)$-face with two poor 4-vertices but $v_5 \notin Q_4(v)$. By (R2.2) and (R2.3), v gives 1 to f_2, $\frac{3}{4}$ to f_1, and $\frac{5}{6}$ or 1 to f_5 (depend on whether it is light or superlight). Thus, $\mu^*(v) \geq 4 - \frac{3}{4} - 1 - 2 \times \frac{1}{2} - \max\{1 + \frac{1}{6}, \frac{5}{6} + 2 \cdot \frac{1}{6}\} = \frac{1}{12} > 0$.

Assume finally that $M(v)$ contains exactly two 3-vertices. If $d(u_1) = 3$ or $d(u_5) = 3$, then by Lemma 2.10 (3), $M(v)$ contains exactly one 3-vertex, contrary to our assumption, so by symmetry, $d(u_2) = d(u_3) = 3$ or $d(u_2) = d(u_4) = 3$.

Let $d(u_2) = d(u_3) = 3$. By Lemma 2.10 (2), $d(v_4) \geq 5$ as $d(u_2) = 3$, and either $d(v_2) \geq 5$ or $d(v_5) \geq 5$ as $d(u_3) = 3$. By Lemma 2.9(4), v_3 is not a poor 4-vertex as $d(u_2), d(u_3) < 4$. It follows that $|Q_4(v)| \leq 1$. If $d(v_2) = 4$, then $d(v_5) \geq 5$ and f_2 is a light $(5, 4, 3, 4^+)$-face, so by (R2.2) and (R2.3), v gives a charge sequence $(1, \frac{5}{6}, \frac{3}{4}, \frac{1}{2}, \frac{3}{4})$ to $f(v)$, and $\mu^*(v) \geq 4 - 1 - \frac{5}{6} - 2 \cdot \frac{3}{4} - \frac{1}{2} - \frac{1}{6} = 0$; if $d(v_2) \geq 5$, then f_2, f_3 are $(5, 5^+, 3, 4)$-faces, so by (R2.2) and (R2.3), v gives a charge sequence $(\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{1}{2}, 1)$ to $f(v)$, and $\mu^*(v) \geq 4 - 1 - 3 \cdot \frac{3}{4} - \frac{1}{2} - \frac{1}{6} > 0$.

Let $d(u_2) = d(u_4) = 3$. By Lemma 2.10 (2), $d(v_3) \geq 5$ and $d(v_4) \geq 5$. If both v_2, v_5 are poor 4-vertices, then by applying Lemma 2.9(4) to v_2 and v_5, respectively, $d(u_1) \geq 5$ and $d(u_5) \geq 5$ as $d(u_2) = d(u_4) = 3$, so by (R2.2) and (R2.3), v gives a charge sequence $(\frac{3}{4}, \frac{3}{4}, \frac{1}{2}, \frac{3}{4}, \frac{3}{4})$ to $f(v)$, and $\mu^*(v) \geq 4 - 4 \times \frac{3}{4} - \frac{1}{2} - 2 \times \frac{1}{6} = \frac{1}{6} > 0$. Then let v_2 be a rich 4-vertex or a 5^+-vertex. By (R2.2) and (R2.3), v gives a charge sequence $(\frac{5}{6}, \frac{3}{4}, \frac{1}{2}, \frac{3}{4}, 1)$ to $f(v)$, and $\mu^*(v) \geq 4 - 1 - \frac{5}{6} - 2 \cdot \frac{3}{4} - \frac{1}{2} - \frac{1}{6} = 0$.

Case 3. $N(v)$ has no 3-vertex.

If $M(v)$ has at most one 3-vertex, then $f(v)$ has at least four $(5, 4^+, 4^+, 4^+)$-faces, by (R2.2) and (R2.3), v gives the charge sequence $(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ to $f(v)$, thus $\mu^*(v) \geq 4 - 1 - 4 \times \frac{1}{2} - 5 \times \frac{1}{6} > 0$. Hence by Lemma 2.10 (1), we assume that $M(v)$ has exactly two 3-vertices. By symmetry, we assume that $d(u_2) = d(u_3) = 3$ or $d(u_2) = d(u_4) = 3$. In the former case, by Lemma 2.9(4) v_3 is not a poor 4-vertex, which implies that $v_3 \notin Q_4(v)$, thus f_2, f_3 are light $(5, 4, 3, 4^+)$-faces or $(5, 5^+, 3, 4^+)$-faces. By (R2.2) and (R2.3), v gives a charge sequence $(\frac{1}{2}, \frac{5}{6}, \frac{5}{6}, \frac{1}{2}, \frac{1}{2})$ to $f(v)$, thus, $\mu^*(v) \geq 4 - 2 \times \frac{5}{6} - 3 \times \frac{1}{2} - 4 \times \frac{1}{6} = \frac{1}{6} > 0$. In the latter case, by Lemma 2.10 (2), $d(v_1) \geq 5$ or $d(v_3) \geq 5$ as $d(u_4) = 3$, and $d(v_1) \geq 5$ or $d(v_4) \geq 5$ as $d(u_2) = 3$, so $|Q_4(v)| \leq 4$. Note that $|Q_4(v)| \neq 4$, for otherwise, $d(v_1) \geq 5$, $d(v_j) = 4$ for $j \in [5] \setminus \{1\}$, and f_2 is a $(5, 4, 3, 4)$-face with $v_2, v_3 \in Q_4(v)$, a contradiction to Lemma 2.10(4). Therefore, by (R2.2) and (R2.3), v gives a charge sequence $(\frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2})$ to $f(v)$, and we have $\mu^*(v) \geq 4 - 2 \times 1 - 3 \times \frac{1}{2} - 3 \times \frac{1}{6} = 0$. □

Lemma 3.6. *For each $v \in int(C_0)$, $\mu^*(v) \geq 0$.*

Proof. By Lemmas 3.2 and 3.5, we may assume that $d(v) \geq 6$. We may further assume that $d(v) = 6$, as when $d(v) \geq 7$, $\mu^*(v) \geq \frac{7}{8} \times 7 - 6 = \frac{1}{8} > 0$ by (3).

If $t_3 = 0$, then $t_4 + t_p \leq 6$, so by (2), $\mu^*(v) \geq 6 - (t_4 + t_p) + \frac{1}{2}t_p \geq 0$. It $t_3 = 1$, then $t_4 \leq 3$, so $\mu^*(v) \geq 6 - \frac{9}{4} - t_4 - \frac{1}{2}t_p > 0$. If $t_3 = 2$, then by Proposition 2.1 $t_4 \leq 1$, so $\mu^*(v) \geq 6 - 2 \times \frac{9}{4} - 1 > 0$. Thus we assumer that $t_3 = 3$.

By (R2.1), v gives at most $\frac{9}{4}$ to a $(6, 4^-, 3)$-face, $\frac{7}{4}$ to a $(6, 5^+, 3)$-face, and $\frac{3}{2}$ to other incident 3-faces, thus $\mu^*(v) \geq 6 - \frac{9}{4}k_1 - 2k_2 - \frac{7}{4}k_3 - \frac{3}{2}k_4$, where k_1, k_2, k_3, k_4 are the numbers of 3-faces that receive $\frac{9}{4}, 2, \frac{7}{4}$, at most $\frac{3}{2}$ from v, respectively. Note that $k_1 + k_2 + k_3 + k_4 = 3$, and by Lemma 5 (5), v is incident with at most two $(6, 4^-, 3)$-faces, thus $k_1 + k_2 \leq 2$. Clearly, $\mu^*(v) \geq 9 - \frac{9}{4} \cdot 2 - \frac{7}{4} = -\frac{1}{4}$, and $\mu^*(v) < 0$ only if $k_1 = 2$ and $k_3 = 1$, in which case, v is weak, so by (R2c), v should give $\frac{5}{4}$ instead of $\frac{7}{4}$ to the $(6, 5^+, 3)$-face, a contradiction. □

Lemma 3.7. *For each $v \in C_0$, $\mu^*(v) \geq 0$.*

Proof. Let $d(v) = k$. By Proposition 2.1, $k \geq 2$.

If $k = 2$, then by (R4), $\mu^*(v) = 2 \times 2 - 6 + 2 = 0$. If $k = 3$, then v cannot be incident with faces in $F_3' \cup F_4'$. In this case, v may be incident with a face in $F_3'' \cup F_4''$. By (R4) and (R5), $\mu^*(v) \geq \frac{3}{2} - \frac{3}{2} = 0$. Let $k = 4$. If v is incident with a 3-face in F_3', then it is not incident with other 3- or 4-faces, thus by (R4) and (R5), $\mu^*(v) \geq 2 - 3 + 1 = 0$; if v is incident with faces from $F_3'' \cup F_4''$, then by (R4) and (R5), $\mu^*(v) \geq 2 - \frac{3}{2} \cdot 2 + 1 = 0$.

Let $k \geq 5$. The vertex v is incident with at most $\lfloor \frac{k-2}{2} \rfloor$ faces in F'. By (R3), (R4) and (R5),

$$\mu^*(v) \geq (2k - 6) - 3 \cdot \lfloor \frac{k-2}{2} \rfloor - \frac{3}{2} \cdot (k - 2 - 2 \cdot \lfloor \frac{k-2}{2} \rfloor) = \frac{k}{2} - 3.$$

Thus, $\mu^*(v) \geq 0$ if $k \geq 6$. When $k = 5$, v gains $\frac{1}{2}$ from C_0, so $\mu^*(v) \geq 0$ as well. □

Finally, we consider $\mu^*(C_0)$. For $i \in \{2, 3, 4, 5\}$, let s_i be the number of i-vertices on C_0. Then $|C_0| \geq s_2 + s_3 + s_4 + s_5$. By (R5),

$$\begin{aligned} \mu^*(C_0) &\geq |C_0| + 6 - 2s_2 - \frac{3}{2}s_3 - s_4 - \frac{1}{2}s_5 \geq |C_0| + 6 - \frac{3}{2}(s_2 + s_3 + s_4 + s_5) - \frac{1}{2}s_2 \\ &\geq |C_0| + 6 - \frac{3}{2}|C_0| - \frac{1}{2}s_2 = 6 - \frac{1}{2}(|C_0| + s_2) \end{aligned}$$

Note that $|C_0| = 3$ or 7. If $|C_0| = 3$ or $s_2 \leq 5$, then $\mu^*(C_0) \geq 0$. Hence we may assume that $|C_0| = 7$ and $(s_2, s_3, s_4, s_5) \in \{(6, 1, 0, 0), (7, 0, 0, 0)\}$. If $s_2 = 7$, then $G = C_0$ and it is trivially superextendable. If $s_2 = 6$ and $s_3 = 1$, then by (R5), C_0 gains 1 from the adjacent face which has degree more than 7. Thus, $\mu^*(C_0) \geq \frac{1}{2} > 0$.

We have shown that all vertices and faces have non-negative final charges. Furthermore, the outer-face has positive charges, except when $|C_0| = 7$ and $s_2 = 5$ and $s_3 = 2$ (the two 3-vertices must be adjacent and has a common neighbor not on C_0), in which there must be a face other than C_0 having degree more than 7. Thus the face has positive final charge. Therefore, $\sum_{x \in V(G) \cup F(G)} \mu^*(x) > 0$, a contradiction.

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