

ORIENTATIONS OF GRAPHS WITH UNCOUNTABLE CHROMATIC NUMBER

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Dedicated to Professor András Hajnal.

ABSTRACT. Motivated by an old conjecture of P. Erdős and V. Neumann-Lara, our aim is to investigate digraphs with uncountable dichromatic number and orientations of undirected graphs with uncountable chromatic number. A graph has uncountable *chromatic number* if its vertices cannot be covered by countably many independent sets, and a digraph has uncountable *dichromatic number* if its vertices cannot be covered by countably many acyclic sets. We prove that consistently there are digraphs with uncountable dichromatic number and arbitrarily large digirth; this is in surprising contrast with the undirected case: any graph with uncountable chromatic number contains a 4-cycle. Next, we prove that several well known graphs (uncountable complete graphs, certain comparability graphs, and shift graphs) admit orientations with uncountable dichromatic number in ZFC. However, we show that the statement “every graph G of size and chromatic number ω_1 has an orientation D with uncountable dichromatic number” is independent of ZFC.

1. INTRODUCTION

The chromatic number of an undirected graph G , denoted by $\chi(G)$, is the minimal number of independent sets needed to cover the vertex set of G . A beautiful branch of graph theory deals with the problem of understanding the consequences of having large (finite or infinite) chromatic number. In particular, what subgraphs H must appear in graphs G with large, say uncountable chromatic number? Is it true that cycles, paths or certain highly connected sets must embed into every graph with large enough chromatic number? There are numerous deep results regarding these questions; the investigations started in the 1960s with a seminal paper of P. Erdős and A. Hajnal [8] and later on, significant contributions were made by P. Komjáth, S. Shelah, C. Thomassen, S. Todorcevic and several other people. In particular, it is now well understood exactly what cycles and finite graphs must embed into a graph G with $\chi(G) > \omega$. We shall review some of these results in later sections but the surveys [15, 16] offer great overview of this topic.

In the case of directed graphs, acyclic sets play the role of independent sets: the *dichromatic number of a directed graph* D , denoted again by $\chi(D)$, is defined to be the minimal number of *acyclic vertex sets* needed to cover the vertices of D [22]. The notion of the dichromatic number of digraphs is certainly well investigated (see [3, 12, 13, 19, 28] for various directions in research). Now, our paper is motivated by two fundamental questions: first, we aim to understand which classical results on chromatic number and obligatory subgraphs

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extend to the directed case. Second, we hope to shed more light on an old conjecture of Erdős and V. Neumann-Lara [4, 21]:

Conjecture 1.1. *There is a function $f : \mathbb{N} \rightarrow \mathbb{N}$ so that $\chi(G) \geq f(k)$ implies that $\chi(D) \geq k$ for some orientation D of G .*

Note that any graph G with $\chi(G) \geq 3$ must contain a cycle and hence there is an orientation D of G with a directed cycle i.e. $\chi(D) \geq 2$. In turn $f(2) = 3$ but no other value of the function f is currently known. Our aim will be to understand the possible values of $\chi(D)$ where D is an orientation of a graph G with $\chi(G) > \omega$. In [4], a related invariant is introduced and further investigated in [5]: let

$$\vec{\chi}(G) = \sup\{\chi(D) : D \text{ is an orientation of } G\}.$$

That is, $\vec{\chi}(G) \geq k$ means that there is an orientation of G so that whenever we colour the vertices of G with $< k$ colours then we can find a monochromatic directed cycle. As Erdős noted in [4], it is surprisingly hard to determine $\vec{\chi}(G)$ for rather simple graphs G ; we certainly can't refute this in the case of uncountable graphs either.

Before we summarize the results of our paper, let us introduce some notation: throughout the paper, G will denote an undirected graph and D a digraph. An orientation D of an undirected graph G is a digraph D with the same set of vertices as G and for every undirected edge ab in G either ab or ba (but not both) is an arc of D . We will use the well known arrow notation:

$$D \rightarrow (D_0)_r^1$$

means that for every r -colouring of the vertices of D one can find a monochromatic copy of D_0 . The negation is denoted by $D \not\rightarrow (D_0)_r^1$.

We let

$$G \xrightarrow{\text{ENL}} (D_0)_r^1$$

mean that there is an orientation D of G such that $D \rightarrow (D_0)_r^1$.

We will write

$$G \xRightarrow{\text{ENL}} (D_0)$$

to denote the fact that there is an orientation D of G such that D_0 is a subgraph of $D[W]$ whenever $\chi(G[W]) = \chi(G)$.

We start in Section 2 by proving an important lemma on amalgamating digraphs with large digirth; this will later be applied in multiple arguments. Next, in Section 3, we investigate what are those directed graphs that embed into any digraph D with $\chi(D) > \omega$. The two main results of this section are Theorem 3.5 and 3.7: we prove that consistently

- for each $k < \omega$ there is a digraph D with $\chi(D) > \omega$ so that D has no directed cycles of length $\leq k$;
- there is a digraph D with $\chi(D) > \omega$ so that $D \not\rightarrow (\vec{C}_k)_k^1$ for all $k < \omega$.

This is in surprising contrast with the undirected case: $\chi(G) > \omega$ implies that $G \rightarrow (C_{2k})_\omega^1$ for all $k < \omega$. We remark that a standard compactness argument combined with Theorem 3.5 shows the existence of finite digraphs D with arbitrary large digirth and dichromatic number, a result of D. Bokal et al [2].

Next, in Section 4, we construct various orientations of graphs G with uncountable chromatic number. First, we look at specific graphs: the complete graph on κ vertices, comparability graphs of Suslin trees and certain non-special trees and shift graphs. We show that these undirected graphs all admit orientations with large dichromatic number (in ZFC).

Now, we can see that any obligatory subgraph for digraphs D with uncountable dichromatic number must be bipartite and consistently acyclic.

Second, we show in Theorem 4.9 that $\chi(G) = \omega_1$ is equivalent to $\overline{\chi}(G) = \omega_1$ under \diamond^+ for any G of size ω_1 . Actually, we prove the much stronger relation

$$G \xrightarrow{\text{ENL}} (D)$$

where D is any orientation of the half graph $H_{\omega, \omega}$.

Finally, in Section 5, we show that consistently there is a graph G with $\chi(G) = |G| = \omega_1$ but $\overline{\chi}(G) \leq \omega$; that is, $\chi(D) \leq \omega$ for any orientation D of G . In particular, this provides some information on the Erdős-Neumann-Lara conjecture for uncountable graphs: the statement “ $\chi(G) = \omega_1$ implies $\overline{\chi}(G) = \omega_1$ for G of size ω_1 ” is independent of ZFC.

We end our paper with a healthy list of open problems which in our opinion worth the attention of the interested reader.

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2. PRELIMINARIES

In our paper, G always denotes an undirected graph i.e. a pair (V, E) so that $E \subseteq [V]^2$. A pair $D = (V, E)$ is a digraph if $E \subseteq V^2$, and we do not allow multiple arcs i.e. if $uv \in E$ then $vu \notin E$. We say that D is an orientation of G if D and G have the same set of vertices and D has an arc between two vertices u and v (in exactly one of the two directions) if and only if uv is an edge in G .

We will use $V(G)$ and $V(D)$ to denote the vertex set of G and D , and $E(G)$ and $E(D)$ to denote the edge/arc set of G and D , respectively. We let $N^+(v) = \{w \in V(D) : vw \in E(D)\}$ and $N^-(v) = \{w \in V(D) : wv \in E(D)\}$. For digraphs D_i , we use the convention that $D = \bigcup\{D_i : i < n\}$ is the pair $(\bigcup\{V(D_i) : i < n\}, \bigcup\{E(D_i) : i < n\})$ which may or may not be a digraph in our definition (since multi-edges could be introduced).

We write $G_0 \hookrightarrow G$ to denote the fact that G_0 embeds into G as a not necessarily induced subgraph; \hookrightarrow will also be used in the context of digraphs. We let $G[W]$ and $D[W]$ denote the induced subgraph of G and D on vertices W .

We say that the length of a path is the number of its edges. Let $\overrightarrow{P}_\omega$ denote the one way infinite directed path and let \overrightarrow{C}_n denote the directed cycle with n vertices. The *girth/digirth* of a graph/digraph is the length of its shortest cycle/directed cycle.

We will frequently use the following lemma on amalgamating digraphs with prescribed digirth.

Lemma 2.1. *Suppose that the digraphs D_i are on vertex sets V_i (finite or infinite) so that there is a single R such that $R = V_i \cap V_j$ and there is a digraph isomorphism $\psi_{i,j} : V_i \rightarrow V_j$ which is the identity on R for all $i < j < n$. Then $D = \bigcup\{D_i : i < n\}$ is a digraph.*

Fix $k \in \omega$ at least 3. If each D_i has digirth bigger than k then

- (1) *any path P from $\alpha \in V_i$ to $\alpha' = \psi_{i,j}(\alpha) \in V_j$ in D has length $> k$;*
- (2) *D has digirth bigger than k .*

Furthermore, suppose that $\alpha_i \in V_i \setminus R$ so that $\alpha_j = \psi_{i,j}(\alpha_i)$ for $i < j < n$.

(3) Let $n > k$ and define D^* by $V(D^*) = V(D)$ and $E(D^*) = E(D) \cup \{\alpha_{n-1}\alpha_0, \alpha_i\alpha_{i+1} : i < n-1\}$. Then D^* has digirth bigger than k .

Note that the analogue of Lemma 2.1 trivially fails for undirected graphs: it is easy to find G_0, G_1 both copies of the path of length 2 so that $G_0 \cup G_1$ is a copy of C_4 .

Proof. It is obvious that D is a digraph.

(1) Suppose that there is a path P on vertices $a_0 = \alpha, a_1, \dots, a_{\ell-1}, a_\ell = \alpha'$ from $\alpha \in V_{i^*}$ to $\alpha' = \psi_{i^*, j^*}(\alpha) \in V_{j^*}$ in D which has length $\ell \leq k$; we can suppose that ℓ is minimal. Let $\psi_{i,i}$ be the identity on V_i and let

$$\psi = \bigcup \{\psi_{i,j^*} : i < n\}.$$

Note that ψ is a digraph homomorphism from D to D_{j^*} . Furthermore, ψ is injective on $\{a_i : i < \ell\}$ by the minimality of ℓ . Hence $\psi(a_0) = \alpha', \dots, \psi(a_{\ell-1}), \psi(a_\ell) = \alpha'$ is a cycle in D_{j^*} of length $\ell \leq k$ which contradicts that D_{j^*} has digirth $> k$.

(2) Now, suppose that C on vertices $a_0, a_1, \dots, a_{\ell-1}, a_\ell = a_0$ is a cycle in D of length $\ell \leq k$. Let $j^* \in n$ so that $a_0 \in V_{j^*}$. If ψ is defined as above then ψ has to be injective on $\{a_i : i < \ell\}$ otherwise there is a path (a subgraph of C) contradicting (1). In particular, $\psi(a_0) = a_0, \dots, \psi(a_{\ell-1}), \psi(a_\ell) = a_0$ is a cycle in D_{j^*} so $\ell > k$; this is a contradiction.

(3) Suppose that C on vertices $a_0, \dots, a_{\ell-1}, a_\ell = a_0$ is a cycle in D^* of length $\ell \leq k$. (1) and (2) imply that C must contain at least 2 non adjacent edges from $D^* \setminus D$. Also, as $k < n$, there must be a vertex of C not in $A = \{\alpha_i : i < n\}$. Hence, for some $\ell_0 < \ell_1 < \ell$, $a_{\ell_0}, a_{\ell_1} \in A$ and $a_{\ell_0}, \dots, a_{\ell_1}$ is a directed path in D . However, this (and $\ell \leq k$) contradicts (1). \square

Finally, let us slightly extend the arrow notations: given a set of directed graphs \mathcal{D} we let

$$D \rightarrow (\bigwedge \mathcal{D})_r^1$$

mean that for every r -colouring of the vertices of D and every $D_0 \in \mathcal{D}$ there is a monochromatic copy of D_0 in D . Similarly,

$$D \rightarrow (\bigvee \mathcal{D})_r^1$$

means that for every r -colouring of the vertices of D there is a monochromatic copy of some digraph D_0 from \mathcal{D} in D .

Now, we write

$$G \xrightarrow{\text{ENL}} (\bigvee \mathcal{D})_r^1$$

to mean that there is an orientation D of G such that $D \rightarrow (\bigvee \mathcal{D})_r^1$. So the relation $\overline{\chi}(G) > \omega$ can be written as

$$G \xrightarrow{\text{ENL}} (\overrightarrow{C}_3 \vee \overrightarrow{C}_4 \vee \dots)_\omega^1$$

or $G \xrightarrow{\text{ENL}} (\bigvee_{3 \leq n < \omega} \overrightarrow{C}_n)_\omega^1$.

Let us omit the straightforward definitions of $G \xrightarrow{\text{ENL}} (\bigwedge \mathcal{D}_0)_r^1$, $G \xrightarrow{\text{ENL}} (\bigvee \mathcal{D}_0)$ and $G \xrightarrow{\text{ENL}} (\bigwedge \mathcal{D}_0)$.

2.1. Set theoretic preliminaries. In general, we use standard set theoretic notations and definitions but let us refer the reader to [17] for anything that is left undefined. However, we do include a short reminder of two key (and somewhat advanced) concepts that appear regularly: elementary submodels and forcing.

First, we say that a subset M of a model V is an elementary submodel if for any first order formula ϕ with parameters from M is true in (M, \in) (written as $M \models \phi$) if and only if it is true in (V, \in) . We write $M < V$ in this case. For technical reasons, one takes elementary submodels of $H(\theta)$, the collection of sets of hereditary cardinality $< \theta$, for a large θ , instead of the complete set theoretic universe V .

The idea is, that if G is an uncountable graph and M is a countable elementary submodel so that $G = (V, E) \in M$, then the countable graph $G \upharpoonright M := G[V \cap M]$ highly resembles the uncountable G . Now $(M_\xi)_{\xi < \zeta}$ is a continuous chain of models if $M_\nu \subseteq M_\xi$ for $\nu < \xi$ and $M_\xi = \bigcup_{\nu < \xi} M_\nu$ for any limit $\xi < \zeta$. If $(M_\xi)_{\xi < \zeta}$ covers a graph G then we can get a very useful decomposition of G by looking at $(G \upharpoonright M_{\xi+1} \setminus M_\xi)_{\xi < \zeta}$.

Probably the most useful thing to keep in mind is the following:

Fact 2.2. *Suppose that M is a countable elementary submodel of $H(\theta)$ and $A \in M$. If A is countable then $A \subseteq M$ or equivalently, if $A \setminus M$ is nonempty then A is uncountable.*

In particular, if $G \in M$ and a finite set of vertices $W \subseteq G \upharpoonright M$ has a single common neighbour outside $G \upharpoonright M$ then there must be uncountably many common neighbours to W in G (and infinitely many of these will be in $G \upharpoonright M$ too). Let us refer the reader to [27] for a complete introduction to elementary submodels and combinatorics.

Next, our main tool to prove the consistency of a statement is either invoking a combinatorial principle (like \diamond^+) or by forcing. With forcing, one looks at a (countable) model V of ZFC and a poset $\mathbb{P} \in V$ to form a larger model $V^{\mathbb{P}}$ by adding a filter $\mathcal{G} \subseteq \mathbb{P}$ which is generic with respect to V . For example, \mathbb{P} can be the set of all finite graphs (with a certain property) on say ω_1 ; when extending a graph $p \in \mathbb{P}$ to a larger graph $q \in \mathbb{P}$, we do not add new edges between vertices of p . Now any filter $\mathcal{G} \subseteq \mathbb{P}$ defines a graph $G = \bigcup \mathcal{G}$ which, in the case of a generic filter, is a quite random and useful object.

A key property of forcing is that any formula ϕ which is true in the extension (i.e. $V^{\mathbb{P}} \models \phi$) is *forced* by a condition p from the filter \mathcal{G} (written as $p \Vdash \phi$). Finally, in order to show that the forcing behaves nicely (i.e. no cardinals are collapsed) we will prove that our posets are ccc i.e. any set $Q \subset \mathbb{P}$ of uncountably many conditions contains $p \neq q \in Q$ with a common extension. The way to do this (in our case) is to find $p \neq q \in Q$ which are isomorphic and agree on their common vertices; this is done generally by applying the Δ -system lemma and Lemma 2.1.

Fact 2.3 (Δ -system lemma). *Suppose that \mathcal{S} is an uncountable set of finite sets. Then there is a single finite set r and uncountable $\mathcal{R} \subset \mathcal{S}$ so that $s \cap t = r$ for any $s \neq t \in \mathcal{R}$.*

Naturally, one can suppose that all elements of \mathcal{R} have the same size and, in case of finite graphs, each $s \in \mathcal{R}$ carries the same graph.

3. OBLIGATORY SUBGRAPHS OF DIGRAPHS WITH UNCOUNTABLE DICHROMATIC NUMBER

For directed graphs D , we can ask what implications does $\chi(D) > \omega$ have; in particular, what are those directed graphs that embed into any digraph D with $\chi(D) > \omega$? We will mention the undirected counterparts of our results as we proceed.

Proposition 3.1. *Suppose that $\chi(D) > \omega$. Then there is $D_0 \subseteq D$ so that $\chi(D_0) > \omega$ and each vertex in D_0 has infinite in and out degree.*

We thank one of our anonymous referees for simplifying the original proof of this result.

Proof. Suppose that D is a counterexample to the statement with minimal cardinality; in particular, any subgraph of D with size $< |D|$ has countable dichromatic number. Now, we can find an ordinal κ and vertices $\{v_\alpha : \alpha < \kappa\}$ so that $\chi(D[V \setminus \{v_\alpha : \alpha < \kappa\}]) \leq \omega$ and for all $\alpha \in \kappa$ either

$$(3.1) \quad |\{\beta \in \kappa \setminus \alpha : v_\alpha v_\beta \in E(D)\}| < \omega$$

or

$$(3.2) \quad |\{\beta \in \kappa \setminus \alpha : v_\beta v_\alpha \in E(D)\}| < \omega.$$

Indeed, given vertices v_α for $\alpha < \beta$ we look at $D[V \setminus \{v_\alpha : \alpha < \beta\}]$: if this digraph has countable dichromatic number then we stop. Otherwise, there must be a vertex $v_\beta \in V \setminus \{v_\alpha : \alpha < \beta\}$ so that v_β has only finitely many in or finitely many out neighbours in $D[V \setminus \{v_\alpha : \alpha < \beta\}]$.

We let V^+ and V^- denote the set of v_α so that (3.1) or (3.2) above holds, respectively.

Now, it suffices to show that $\chi(D[V^+]) \leq \omega$ and $\chi(D[V^-]) \leq \omega$ holds. This, together with $\chi(D[V \setminus \{v_\alpha : \alpha < \kappa\}]) \leq \omega$ implies that $\chi(D) \leq \omega$ which is a contradiction.

Consider V^+ and the set-mapping F^+ defined by $v_\alpha \mapsto \{v_\beta \in N^+(\alpha) : \alpha < \beta\}$. By Fodor's theorem [9], V^+ is the union of countably many F^+ -free sets $\{V_i^+ : i < \omega\}$ i.e. $u \notin F^+(v)$ if $u \neq v \in V_i^+$. In other words, each arc of $D[V_i^+]$ goes down with respect to the well order we defined and so $D[V_i^+]$ is acyclic. The argument for V^- is completely analogous. \square

Corollary 3.2. $\overrightarrow{P}_\omega$ embeds into D whenever $\chi(D) > \omega$. Moreover, if T is any orientation of the everywhere ω -branching rooted tree then T embeds into D whenever $\chi(D) > \omega$.

The undirected version of the above lemma and corollary appeared in [8] and we followed similar proofs.

Before proceeding further, we mention that the set of obligatory digraphs for graphs with $\chi(D) > \omega$ is closed under a simple operation: let $\text{rev}(D_0)$ denote the digraph on vertices $V(D_0)$ and edges $\{uv : vu \in E(D_0)\}$.

Observation 3.3. *If $D_0 \hookrightarrow D$ for every D such that $\chi(D) > \omega$ then $\text{rev}(D_0) \hookrightarrow D$ for every D such that $\chi(D) > \omega$ as well.*

Proof. Indeed, note that $\chi(\text{rev}(D)) = \chi(D)$ so $D_0 \hookrightarrow \text{rev}(D)$ as well which implies that $\text{rev}(D_0) \hookrightarrow \text{rev}(\text{rev}(D)) = D$. \square

One of the strongest results on obligatory subgraph was found by A. Hajnal and P. Komjáth: the half graph $H_{\omega,\omega}$ embeds into any graph G with $\chi(G) > \omega$ [10]. Recall that $H_{\omega,\omega}$ is the graph defined on vertices $\omega \times 2$ and $(k, i)(\ell, j)$ is an edge if and only if $k \leq \ell < \omega$ and $i = 0, j = 1$.

There are two simple orientations of $H_{\omega,\omega}$: $(k, 0)(\ell, 1)$ is an arc if and only if $k \leq \ell < \omega$ or $(\ell, 1)(k, 0)$ is an arc if and only if $k \leq \ell < \omega$. We will denote these graphs by $\overrightarrow{H}_{\omega,\omega}$ and $\overleftarrow{H}_{\omega,\omega}$, respectively.

Proposition 3.4. $\overrightarrow{H}_{\omega,\omega}$ and $\overleftarrow{H}_{\omega,\omega}$ both embed into D if $\chi(D) > \omega$.

Proof. It suffices to prove for $\overrightarrow{H_{\omega,\omega}}$ by Observation 3.3. Suppose that D is a digraph on vertex set V without a copy of $\overrightarrow{H_{\omega,\omega}}$ so that $\chi(D) > \omega$. Let us also suppose that D has minimal size among these graphs. Cover D by a continuous chain of elementary submodels $(M_\xi)_{\xi < \zeta}$ so that $|M_\xi| < |D|$ and $D \in M_\xi$.

Claim 3.4.1. *If $v \in V \cap M_{\xi+1} \setminus M_\xi$ then $N^-(v) \cap M_\xi$ is finite.*

Indeed, suppose that $x_0, x_1 \dots \in N^-(v) \cap M_\xi$. The set $N^+[\{x_i : i < n\}]$ must be uncountable otherwise $N^+[\{x_i : i < n\}] \subseteq M_\xi$ and so $v \in M_\xi$. Hence, we can find distinct $y_0, y_1 \dots$ so that $y_n \in N^+[\{x_i : i < n\}]$. Now $\overrightarrow{H_{\omega,\omega}} \hookrightarrow D[\{x_i, y_i : i < \omega\}]$. This contradicts our assumption that $\overrightarrow{H_{\omega,\omega}}$ does not embed into D .

By the minimal size of D , there are maps $f_\xi : V \cap M_{\xi+1} \setminus M_\xi \rightarrow \omega$ so that there are no monochromatic cycles with respect to f_ξ . Define $f = (f^1, f^2) : V \rightarrow \omega \times \omega$ so that $f^1 = \bigcup_{\xi < \zeta} f_\xi$ and $f^2(v) \neq f^2(w)$ if $v \in V \cap M_{\xi+1} \setminus M_\xi$ and $w \in N^-(v) \cap M_\xi$. This can be done as $N^-(v) \cap M_\xi$ is finite.

We claim that f witnesses that $\chi(D) \leq \omega$ which is a contradiction. Indeed, the definition of f^1 guarantees that if C is monochromatic with respect to f then C must have an arc of the form wv with $v \in V \cap M_{\xi+1} \setminus M_\xi$ and $w \in N^-(v) \cap M_\xi$. But in this case $f^2(v) \neq f^2(w)$ \square

At this point, we are uncertain of exactly what orientations of $H_{\omega,\omega}$ must embed into any D with $\chi(D) > \omega$.

3.1. Cycles and dichromatic number. Erdős proved in the groundbreaking [7] that there are graphs with arbitrary large finite chromatic number and arbitrary large girth. Rather surprisingly this fails for uncountable chromatic number: if $\chi(G) > \omega$ then G contains a 4-cycle. This was originally proved in [8] but also follows from the fact that $H_{\omega,\omega}$ embeds into G if $\chi(G) > \omega$.

Now, for finite directed graphs the analogue of Erdős' theorem was proved by Bokal et al [2]: there are digraphs with arbitrary large finite dichromatic number without short directed cycles. At this point, it is somewhat unexpected that this result extends to uncountably dichromatic directed graphs as well:

Theorem 3.5. *Consistently, for each natural number $n \geq 3$ there is a digraph D on vertex set ω_1 so that*

- (1) D has digirth bigger than n , and
- (2) $\overrightarrow{C}_{n+1} \hookrightarrow D[X]$ for every uncountable $X \subseteq \omega_1$.

In particular, $\chi(D) = \omega_1$.

Proof. We show that for any n there is a ccc poset of size ω_1 which introduces such a digraph D . We leave it to the reader to check that the finite support product or iteration of these countably many posets gives a model with the appropriate graphs for each n at the same time.

Fix $n \geq 3$ and simply let \mathbb{P} be the set of all finite digraphs on a subset of ω_1 which avoid \overrightarrow{C}_k for $3 \leq k \leq n$ i.e. each $p \in \mathbb{P}$ is a finite digraph $(V(p), E(p))$ with digirth $> n$. We write $p \leq q$ for $p, q \in \mathbb{P}$ if $V(p) \supseteq V(q)$ and $p[V(q)] = q$.

We say that $p, q \in \mathbb{P}$ are twins if $|V(p)| = |V(q)|$, $V(p) \cap V(q) < V(p) \setminus V(q) < V(q) \setminus V(p)$ (or vica versa $V(q) \setminus V(p) < V(p) \setminus V(q)$) and the unique order preserving map $\psi_{p,q}$ from $V(p)$ to $V(q)$ is a digraph isomorphism of p and q . Note that if p, q are twins then $p \cup q$ is a digraph as well.

Clearly, any generic filter $\mathcal{G} \subseteq \mathbb{P}$ gives a digraph \dot{D} on vertex set ω_1 with $E(\dot{D}) = \bigcup \{E(p) : p \in \mathcal{G}\}$.

Claim 3.5.1. \mathbb{P} is ccc.

Proof. Note that any uncountable set of conditions contains an uncountable subset of pairwise twins by the Δ -system lemma. Now, we claim that $p \cup q$ is a condition if p, q are twins. Indeed, apply Lemma 2.1 (2). □

The next claim finishes the proof of the theorem:

Claim 3.5.2. $V^{\mathbb{P}} \models \vec{C}_{n+1} \leftrightarrow \dot{D}[\dot{X}]$ for every uncountable $\dot{X} \subseteq \omega_1$.

Proof. Suppose that $p \Vdash |\dot{X}| = \omega_1$. There is an uncountable $Y \subseteq \omega_1$ and $p_\beta \in \mathbb{P}$ for $\beta \in Y$ so that $p_\beta \leq p$, $\beta \in V(p_\beta)$ and $p_\beta \Vdash \beta \in \dot{X}$. Apply the Δ -system lemma to find $\alpha_0 < \alpha_1 < \dots < \alpha_n \in Y$ so that p_{α_i} and p_{α_j} are twins whenever $i < j < n + 1$.

Define q by letting

$$(3.3) \quad V(q) = \bigcup_{i < n+1} V(p_{\alpha_i}) \text{ and } E(q) = \bigcup_{i < n+1} E(p_{\alpha_i}) \cup \{\alpha_n \alpha_0, \alpha_i \alpha_{i+1} : i < n\}.$$

Lemma 2.1 (3) implies that $q \in \mathbb{P}$ and of course $q \leq p_{\alpha_i}$. We clearly have $q \Vdash \dot{D}[\alpha_0 \dots \alpha_n] \leftrightarrow \dot{D}[\dot{X}]$ and that $q \Vdash \dot{D}[\alpha_0 \dots \alpha_n]$ is an induced copy of \vec{C}_{n+1} . □

□

Corollary 3.6. *Consistently, any digraph D_0 which embeds into all digraphs D with $\chi(D) > \omega$ must be acyclic.*

We don't know at this point how to construct digraphs with uncountable dichromatic number but with arbitrary large digirth in ZFC.

Now, the fact that C_4 appears in every graph G with $\chi(G) > \omega$ shows that the relation

$$G \rightarrow (C_4)_\omega^1$$

is equivalent to $\chi(G) > \omega$. The digraph version is (consistently) false by the above theorem, however at this point it seems possible that $\chi(D) > \omega$ implies $D \rightarrow (\vec{C}_k)_\omega^1$ for some $k < \omega$ for any D . We show now that this is not the case. Let us denote the set of nonzero, nondecreasing $f : \mathbb{N} \rightarrow \mathbb{N}$ so that $\lim_{k \rightarrow \infty} f(k) = \infty$ with \mathcal{F} for the next proof.

Theorem 3.7. *Consistently, for $f \in \mathcal{F}$ there is a digraph $D = D_f$ on vertex set ω_1 so that*

- (1) $\chi(D) = \omega_1$, and
- (2) $D \not\rightarrow (\vec{C}_k)_{f(k)}^1$ for all $k < \omega$.

Proof. Given $f \in \mathcal{F}$ we define the poset \mathbb{P}_f of conditions $p = (d^p, (g_k^p)_{3 \leq k \leq n^p})$ where

- (P1) $d^p = (V^p, E^p)$ is a finite digraph on ω_1 and $n^p = |V^p|$,
- (P2) $g_k^p : V(d^p) \rightarrow f(k)$, and
- (P3) $d^p[\{v : g_k^p(v) = i\}]$ has digirth $> k$ for all $i < f(k)$ and $3 \leq k \leq n^p$.

We let $p \leq q$ if

- (i) $V^p \supseteq V^q$ and $d^p[V^q] = d^q$,
- (ii) $g_k^q = g_k^p \upharpoonright V^q$ for all $3 \leq k \leq n^q$.

It is clear that a generic filter $\mathcal{G} \subseteq \mathbb{P}$ introduces a digraph $\dot{D} = \bigcup \{d^p : p \in \mathcal{G}\}$ and functions \dot{g}_k by $\dot{g}_k = \bigcup \{g_k^p : p \in \mathcal{G}, k \leq n^p\}$ for $k \in \omega$.

Claim 3.7.1. *The following holds for any generic filter $\mathcal{G} \subseteq \mathbb{P}$ and \dot{D}, \dot{g}_k defined as above:*

- (a) \dot{D} has vertex set ω_1 ,
- (b) $\text{dom}(\dot{g}_k) = \omega_1$, and
- (c) $\dot{D}[\{v \in \omega_1 : \dot{g}_k(v) = i\}]$ has digirth $> k$ for all $i < f(k)$ and $3 \leq k < \omega$.

Proof. (a) Indeed, it suffices to show that the set $\{p \in \mathbb{P} : v \in V^p\}$ is dense for every $v \in \omega_1$. Given any $q \in \mathbb{P}$ and $v \in \omega_1 \setminus V^q$ we let $V^p = V^q \cup \{v\}$ and $E^p = E^q$. Then simply define $g_k^p = g_k^q \cup \{(v, 0)\}$ where $3 \leq k \leq n^p = n^q + 1$ and $k^* = \min\{k, n^q\}$. It is easy to check that property (P3) is satisfied by p .

(b) follows from (a), and (c) follows from property (P3). □

We say that two conditions p, q are twins if

- (1) $n^p = n^q$ and $V(d^p) \cap V(d^q) < V(d^p) \setminus V(d^q) < V(d^q) \setminus V(d^p)$ (or vica versa $V(d^q) \setminus V(d^p) < V(d^p) \setminus V(d^q)$),
- (2) the unique order preserving map $\psi_{p,q}$ from $V(d^p)$ to $V(d^q)$ is an isomorphism of the digraphs d^p and d^q , and
- (3) $g_k^p(v) = g_k^q(\psi_{p,q}(v))$ for all $3 \leq k \leq n^p$ and $v \in V^p$.

Claim 3.8. \mathbb{P} is ccc.

Proof. By standard Δ -system arguments, it suffices to show that if $p, q \in \mathbb{P}$ are twins then they have a common extension $r \in \mathbb{P}$. We let $d^r = d^p \cup d^q$ and define $g_k^r = g_k^p \cup g_k^q$ where $3 \leq k \leq n^r$ and $k^* = \min\{k, n^p\}$. Note that $f(k^*) \leq f(k)$ for $k^* = \min\{k, n^p\}$ so $g_k^r : V^r \rightarrow f(k)$ i.e. property (P2) is satisfied.

We need to check that $d^r[\{v \in V^r : g_k^r(v) = i\}]$ has digirth $> k$ for all $i < f(k)$ and $3 \leq k \leq n^r$. Note that

$$d^r[\{v \in V^r : g_k^r(v) = i\}] = d^p[\{v \in V^p : g_{k^*}^p(v) = i\}] \cup d^q[\{v \in V^q : g_{k^*}^q(v) = i\}]$$

where $k^* = \min\{k, n^p\}$. Furthermore, the graphs $d^p[\{v \in V^p : g_{k^*}^p(v) = i\}]$ and $d^q[\{v \in V^q : g_{k^*}^q(v) = i\}]$ are isomorphic and have digirth $> k$. Hence Lemma 2.1 (2) implies that $d^r[\{v \in V^r : g_k^r(v) = i\}]$ still has digirth $> k$. In turn, r satisfies property (P3) and so $r \in \mathbb{P}$ is a common extension of p and q . □

Claim 3.8.1. $V^{\mathbb{P}} \models \chi(\dot{D}[\dot{W}]) = \omega_1$ for any uncountable $\dot{W} \subseteq \omega_1$.

Proof. Suppose that $p \Vdash \dot{W} \subseteq \omega_1$ is uncountable. Find $Y \in [\omega_1]^{\omega_1}, n \in \omega$ and $p_\alpha \leq p$ for $\alpha \in Y$ so that

- (i) $\{p_\alpha : \alpha \in Y\}$ are pairwise twins (with mappings $\psi_{\alpha, \alpha'}$ witnessing this) and $n^{p_\alpha} = n \geq 2$,
- (ii) $\alpha \in V^{p_\alpha}$ and $\psi_{\alpha, \alpha'}(\alpha) = \alpha'$ for $\alpha, \alpha' \in Y$, and
- (iii) $p_\alpha \Vdash \alpha \in \dot{W}$ for all $\alpha \in Y$.

This can be done by the Δ -system lemma. Let $N \in \mathbb{N}$ be minimal so that $f(N) > f(n)$; such a value exists as $f(k) \rightarrow \infty$ as $k \rightarrow \infty$ and $N > n$ as f is nondecreasing. Now, fix distinct $\alpha_j \in Y$ for $j < N$ and define q as follows:

$$(3.4) \quad V^q = \bigcup_{j < N} V^{p_{\alpha_j}} \quad \text{and} \quad E^q = \bigcup_{j < N} E^{p_{\alpha_j}} \cup \{\alpha_{N-1}\alpha_0, \alpha_j\alpha_{j+1} : j < N-1\}.$$

Now we define g_k^q for $3 \leq k \leq n^q$ as follows: let

$$g_k^q = \bigcup_{i < N} g_{k^*}^{p_{\alpha_j}}$$

if $3 \leq k \leq N - 1$ where $k^* = \min\{k, n\}$ and let

$$g_k^q = g_n^{p_{\alpha_0}} \upharpoonright (V^{p_{\alpha_0}} \setminus \{\alpha_0\}) \cup \{(\alpha_0, f(n))\} \cup \bigcup_{1 \leq j < N} g_n^{p_{\alpha_j}}$$

if $N \leq k \leq n^q$.

We need to check that $d^q[\{v \in V^q : g_k^q(v) = i\}]$ has digirth $> k$ for all $i < f(k)$ and $3 \leq k \leq n^q$. If $k \leq N - 1$ then

$$d^q[\{v \in V^q : g_k^q(v) = i\}] = \bigcup_{j < N} d^{p_{\alpha_j}}[\{v \in V^{p_{\alpha_j}} : g_k^{p_{\alpha_j}}(v) = i\}]$$

and we can apply either Lemma 2.1 (2) (if $g_k^{p_{\alpha_j}}(\alpha_j) \neq i$) or Lemma 2.1 (3) (if $g_k^{p_{\alpha_j}}(\alpha_j) = i$) to see that $d^q[\{v \in V^q : g_k^q(v) = i\}]$ has digirth $> k$.

Now, for k between N and n^q , Lemma 2.1 (3) might not apply directly as $k > N$. We distinguish 4 cases depending on value of $i < f(k)$. If $f(n) < i < f(k)$ then $d^q[\{v \in V^q : g_k^q(v) = i\}]$ is empty so we have nothing to prove. If $i = f(n)$ then $d^q[\{v \in V^q : g_k^q(v) = i\}] = \{\alpha_0\}$ so again we have nothing to prove. Now if $i < f(n)$ but $g_n^{p_{\alpha_j}}(\alpha_j) \neq i$ then again we can apply Lemma 2.1 (2) to see that $d^q[\{v \in V^q : g_k^q(v) = i\}]$ has digirth $> k$ as

$$d^q[\{v \in V^q : g_k^q(v) = i\}] = \bigcup_{j < N} d^{p_{\alpha_j}}[\{v \in V^{p_{\alpha_j}} : g_n^{p_{\alpha_j}}(v) = i\}]$$

as before.

Finally, let's look at the case when $g_n^{p_{\alpha_j}}(\alpha_j) = i$ (if this holds for one j then it holds for all $j < N$ as we are working with twin conditions). Suppose that C is a cycle in $d^q[\{v \in V^q : g_k^q(v) = i\}]$ of length $\leq k$. By Lemma 2.1 (2), C must contain a new edge of the form $\alpha_j \alpha_{j+1}$ where $1 \leq j < N - 1$. In particular, we can find $1 \leq j_0 < j_1 \leq N - 1$ so that C contains a directed path from α_{j_1} to α_{j_0} using only edges from $\bigcup_{j < N} d^{p_{\alpha_j}}[\{v \in V^{p_{\alpha_j}} : g_n^{p_{\alpha_j}}(v) = i\}]$. Let $\psi = \bigcup\{\psi_{\alpha_j, \alpha_{j_1}} : j < N\}$ where $\psi_{\alpha, \alpha}$ is the identity on V^{p_α} . Now ψ maps P into a walk from α_{j_1} back to $\alpha_{j_1} = \psi(\alpha_{j_0})$ in $d^{p_{\alpha_{j_1}}}[\{v \in V^{p_{\alpha_{j_1}}} : g_n^{p_{\alpha_{j_1}}}(v) = i\}]$. Also, this walk has length at most k so it must contain a cycle of length at most k as well. However, this contradicts that $d^{p_{\alpha_{j_1}}}[\{v \in V^{p_{\alpha_{j_1}}} : g_n^{p_{\alpha_{j_1}}}(v) = i\}]$ has digirth $> k$.

Hence, we showed that $(g_k^q)_{3 \leq k \leq n^q}$ satisfies property (P3) and so $q \in \mathbb{P}_f$. It is now clear that $q \Vdash \dot{D}[\{\alpha_j : j < N\}]$ is a copy of $\vec{\mathcal{C}}_N$ in \dot{W} . □

At this point, we showed that for any single $f \in \mathcal{F}$ there is a ccc extension of the ground model with the required digraph D_f .

Now, starting from a model of CH, we can define a finite support iteration $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta)_{\alpha \leq \omega_1, \beta < \omega_1}$ of length ω_1 where $V^{\mathbb{P}_\alpha} \models \dot{\mathbb{Q}}_\alpha = \mathbb{P}_j$ for some \mathbb{P}_α -name \dot{f} for a function in \mathcal{F} .

It follows from Claim 3.8 that each \mathbb{P}_α is ccc so we can arrange the iteration in such a way that any $f \in \mathcal{F}$ in the final model shows up at some intermediate stage i.e. $\dot{\mathbb{Q}}_\alpha = \mathbb{P}_j$ for some α and appropriate name \dot{f} for f . So it suffices to check that in the final model $V^{\mathbb{P}_{\omega_1}}$ we still have $\chi(D_f) = \omega_1$ for the graphs that we introduced by the intermediate forcings \mathbb{Q}_α . This can be done using determined conditions and the argument in Claim 3.8.1; more

precisely, we add edges to the finite approximation of D_f in coordinate α as in Claim 3.8.1 while not adding any new edges to graphs in other coordinates. We leave the details to the interested reader. \square

Let us also mention the following

Observation 3.9. *Suppose that D is a digraph.*

- (1) $D \rightarrow (\vec{C}_k)_\omega^1$ for all $k < \omega$ implies $\chi(D) \leq 2^\omega$.
- (2) The edges of D can always be partitioned into two acyclic sets.

Indeed, if f_k witnesses $D \rightarrow (\vec{C}_k)_\omega^1$ then $f : V(D) \rightarrow \omega^\omega$ defined by $v \mapsto (f_k(v))_{k \in \omega}$ witnesses $\chi(D) \leq 2^\omega$. To see (2), take an arbitrary well order on the vertices and consider the forward and backward edges.

Finally, we state without proof that the famous Erdős-de Bruijn compactness result also holds for directed graphs:

Theorem 3.10. *Suppose that any finite subgraph of the digraph D has finite dichromatic number at most k . Then $\chi(D) \leq k$ as well.*

In particular, we can deduce the result of Bokal et al [2] on finite digirth and dichromatic number from our forcing result in Theorem 3.5: given a model V of set theory and finite number k , we force to find an extension $V^{\mathbb{P}}$ with a graph D with digirth $> k$ and $\chi(D) > \omega$. By the above compactness result, there must be a finite subgraph D^* of D (in $V^{\mathbb{P}}$) which has dichromatic number $\geq k$. However, the models V and $V^{\mathbb{P}}$ have the same finite digraphs and hence $D^* \in V$ as well. Much like the probabilistic proof in [2] this forcing argument gives no information about these sparse digraphs with large dichromatic number. A simple, recursive construction of such graphs was actually given by M. Severino [24].

4. ORIENTATIONS OF UNDIRECTED GRAPHS WITH LARGE CHROMATIC NUMBER

There are two trivial orientations of any undirected graph G given a well order $<$ on the vertices: define the orientation \vec{G} of G by $uv \in E(\vec{G})$ if and only if $uv \in E(G)$ and $u < v$. Similarly, \overleftarrow{G} is defined by $uv \in E(\overleftarrow{G})$ if and only if $uv \in E(G)$ and $v < u$.

It is well known that if G has countable *colouring number* i.e. $\{u \in N(v) : u < v\}$ is finite for every vertex $v \in V(G)$ (for some well order $<$ of the vertices) then $\chi(G) \leq \omega$ (see [8]). This yields the following observations: let \vec{S} denote the countable star with all edges pointing out, and \overleftarrow{S} denote the countable star with all edges pointing in. Then the orientation \vec{G} of G witnesses $G \xrightarrow{\text{ENL}} (\vec{S})_\omega^1$ while \overleftarrow{G} witnesses $G \xrightarrow{\text{ENL}} (\overleftarrow{S})_\omega^1$.

As we saw in the previous section, $H_{\omega, \omega}$ embeds into any graph G with $\chi(G) > \omega$ [10]. In particular, the girth of G is at most 4 whenever $\chi(G) > \omega$; so it could be the case that

$$G \xrightarrow{\text{ENL}} (\vec{C}_4)_\omega^1 \text{ or even } G \xrightarrow{\text{ENL}} (\vec{C}_4)$$

whenever $\chi(G) > \omega$. Indeed, we are going to prove this, at least for some graphs.

First, let us look at complete graphs. Recall that

$$\kappa \rightarrow [\kappa; \kappa]_2^2$$

means that there is a function $f : [\kappa]^2 \rightarrow 2$ so that for all $A, B \in [\kappa]^\kappa$ and $i < 2$ there is $\alpha \in A$ and $\beta \in B$ so that $\alpha < \beta$ and $f(\alpha, \beta) = i$.

Theorem 4.1. *Suppose that κ is an infinite cardinal.*

- (1) If κ is regular and $\kappa \rightarrow [\kappa; \kappa]_2^2$ then $K_\kappa \xrightarrow{ENL} \left(\bigwedge_{3 \leq n \in \omega} \vec{C}_n \right)$.
- (2) If λ is uncountable then $K_{\lambda^+} \xrightarrow{ENL} \left(\bigwedge_{3 \leq n \in \omega} \vec{C}_n \right)_\lambda^1$.

Let us show a corollary first:

Corollary 4.2. $\vec{\chi}(K_\kappa) = \kappa$ for any infinite cardinal κ .

Proof. Recall that $\kappa^+ \rightarrow [\kappa^+; \kappa^+]_2^2$ holds whenever κ is a regular cardinal [23]. Hence Theorem 4.1 (1) and (2) implies that $\vec{\chi}(K_{\kappa^+}) = \kappa^+$ for any cardinal κ .

Now, given a limit cardinal κ let $(\kappa_i)_{i < \text{cf}(\kappa)}$ be a cofinal sequence of regular cardinals in κ . Let $V_i \in [\kappa]^{\kappa_i^+}$ pairwise disjoint for $i < \text{cf}(\kappa)$. Then K_κ restricted to V_i is just a copy of $K_{\kappa_i^+}$ so we can apply Theorem 4.1 (1) to find an orientation D_i witnessing $K_{\kappa_i^+} \xrightarrow{ENL} \left(\bigwedge_{3 \leq n \in \omega} \vec{C}_n \right)$ on V_i . Putting together these digraphs D_i (and orienting the edges outside arbitrarily) we defined an orientation D of K_κ that witnesses $K_\kappa \xrightarrow{ENL} \left(\bigwedge_{3 \leq n \in \omega} \vec{C}_n \right)_\mu^1$ for any $\mu < \kappa$. In particular, $\vec{\chi}(K_\kappa) = \kappa$. □

Our proof of Theorem 4.1 was motivated by the proof of Theorem 8 [11]; we will point out further connections to [11] later as well, in particular in Section 5.

Proof. (1) Let $f : [\kappa]^2 \rightarrow 2$ witness $\kappa \rightarrow [\kappa; \kappa]_2^2$. Now simply define $D = (\kappa, E)$ by $\alpha\beta \in E$ if $\alpha < \beta$ and $f(\alpha, \beta) = 0$, otherwise $\beta\alpha \in E$.

First, we show that any induced subgraph of size κ contains a copy of \vec{C}_3 . Let $W \in [\kappa]^\kappa$. Define $W^+ = \{v \in W : |N^+(v) \cap W| < \kappa\}$ and $W^- = \{v \in W : |N^-(v) \cap W| < \kappa\}$. If there is a $v \in W \setminus (W^+ \cup W^-)$ then by the choice of f we can find $\alpha \in N^+(v) \cap W$ and $\beta \in N^-(v) \cap W$ so that $v < \alpha < \beta$ and $f(\alpha, \beta) = 0$. Then $v\alpha, \alpha\beta, \beta v \in E$ so $\{v, \alpha, \beta\}$ is a copy of \vec{C}_3 .

Now, it suffices to show that $|W^+| = \kappa$ or $|W^-| = \kappa$ is not possible. If $|W^+| = \kappa$ then using the regularity of κ , one can find a $Y \in [W^+]^\kappa$ so that $\alpha < \beta \in Y$ implies that $\beta \notin N^+(\alpha)$. However, $f(\alpha, \beta) = 0$ for some $\alpha < \beta \in Y$ by the choice of f so $\beta \in N^+(\alpha)$; this is a contradiction. The proof that $|W^-| = \kappa$ is not possible is completely analogous.

Now, fix $n \in \omega$ at least 3 and $W \in [\kappa]^\kappa$; we will find a copy of \vec{C}_n in $D[W]$. Find pairwise disjoint paths $P_\xi = (\alpha_0^\xi \dots \alpha_{n-2}^\xi)$ in W of length $n - 2$ for $\xi < \kappa$. This can be done by applying Corollary 3.2; indeed, we already proved that $\chi(D[W \setminus \delta]) > \omega$ for any $\delta < \kappa$ so $\vec{P}_\omega \hookrightarrow D[W \setminus \delta]$.

Note that if there is a single ξ so that $N^-(\alpha_0^\xi) \cap W$ and $N^+(\alpha_{n-2}^\xi) \cap W$ both have size κ then we can extend P_ξ into a copy of \vec{C}_n in W . So suppose that this is not the case; then there is $I \in [\kappa]^\kappa$ so that either

- (i) $|N^-(\alpha_0^\xi) \cap W| < \kappa$ for all $\xi \in I$, or
- (ii) $|N^+(\alpha_{n-2}^\xi) \cap W| < \kappa$ for all $\xi \in I$.

If case (i) holds then, using that κ is regular, we can find $J \in [I]^\kappa$ so that $\xi < \zeta \in J$ implies that $\alpha_0^\xi < \alpha_0^\zeta$ and $\alpha_0^\zeta \notin N^-(\alpha_0^\xi)$. However, this clearly contradicts the choice of f as there is some $\xi < \zeta \in J$ such that $f(\alpha_0^\xi, \alpha_0^\zeta) = 1$.

Similarly, if case (ii) holds then we can find $J \in [I]^\kappa$ so that $\xi < \zeta \in J$ implies that $\alpha_0^\xi < \alpha_0^\zeta$ and $\alpha_0^\zeta \notin N^+(\alpha_0^\xi)$. This again contradicts the choice of f .

(2) Suppose that λ is uncountable. We fix a *club guessing sequence* $\{C_\alpha : \alpha \in E_\omega^{\lambda^+}\}$, that is: C_α is a cofinal sequence of type ω in α and whenever $E \subseteq \lambda^+$ is a club in λ^+ (i.e. a closed and unbounded subset) then $C_\alpha \subseteq E$ for stationary many $\alpha \in E_\omega^{\lambda^+}$. The existence of such guessing sequences was originally proved in Claim 2.3 [25] (for a detailed proof see [1]). We let $I(\alpha, 0) = C_\alpha(0)$ and $I(\alpha, n) = C_\alpha(n) \setminus C_\alpha(n-1)$ for $1 \leq n < \omega$ where $(C_\alpha(n))_{n \in \omega}$ is the increasing enumeration of C_α . Now, define the orientation D as follows: given $\alpha < \beta \in \lambda^+$ we let $\alpha\beta \in E(D)$ if and only if $n(\alpha, \beta)$ is even where $n(\alpha, \beta) = \min\{n \in \omega : \alpha \in I(\beta, n)\}$; otherwise $\beta\alpha \in E(D)$.

We will show that given a partition $\lambda^+ = \bigcup\{A_i : i < \lambda\}$ there is an $i < \lambda$ so that $D[A_i]$ contains a directed n -cycle for all $3 \leq n \in \omega$. Take a continuous, increasing sequence of elementary submodels $(M_\xi)_{\xi < \lambda^+}$ covering λ^+ so that $\{A_i, D : i < \lambda\} \subseteq M_\xi$ and $|M_\xi| = \lambda$ for all $\xi < \lambda^+$. Let $E = \{M_\xi \cap \lambda^+ : \xi < \lambda^+\}$. E is a club so there is an $i < \lambda$ and some stationary $S \subseteq A_i$ so that $C_\beta \subseteq E$ for all $\beta \in S$. Observe that $I(\beta, n) \cap A_i \neq \emptyset$ for every $\beta \in S$ and $n \in \omega$.

Claim 4.2.1. *For every $n \in \omega$ at least 3 and every $\delta < \lambda^+$ there is a path $P = (\alpha_0 \dots \alpha_{n-2})$ in $D[A_i \setminus \delta]$ so that $|N^+(\alpha_{n-2}) \cap A_i| = \lambda^+$.*

Proof. We prove by induction on $n \geq 3$. If $n = 3$ then let $\beta \in S \setminus \delta$ and pick $\alpha_0 \in I(\beta, 2k)$ where k is large enough so that $\delta < C_\beta(2k-1)$. We need that $|N^+(\alpha_0) \cap A_i| = \lambda^+$; if $|N^+(\alpha_0) \cap A_i| \leq \lambda$ and $C_\beta(2k) = M_\xi \cap \lambda^+$ then $N^+(\alpha_0) \cap A_i \subseteq M_\xi$ by elementarity as well. However, $\beta \in N^+(\alpha_0) \cap A_i \setminus M_\xi$.

Now suppose that $n > 3$, and again let $\beta \in S \setminus \delta$. Using the inductive hypothesis and the fact that $C_\beta(2k) = M_\xi \cap \lambda^+$ for some $\xi < \lambda^+$ find a path $P = (\alpha_0 \dots \alpha_{n-2})$ in $I(\beta, 2k)$ so that $|N^+(\alpha_{n-2}) \cap A_i| = \lambda^+$ where k is large enough so that $\delta < C_\beta(2k-1)$. By elementarity, we can find $\alpha_{n-1} \in I(\beta, 2k) \setminus \{\alpha_i : i < n-1\}$ so that $\alpha_{n-1} \in N^+(\alpha_{n-2})$. As before, it is easy to show that $|N^+(\alpha_{n-1}) \cap A_i| = \lambda^+$ and so $(\alpha_0 \dots \alpha_{n-1})$ is the desired path. \square

Now, fix $3 \leq n \in \omega$. Let $\beta \in S$ arbitrary and find a path $P = (\alpha_0 \dots \alpha_{n-2})$ in $A_i \cap I(\beta, 1)$ so that $|N^+(\alpha_{n-2}) \cap A_i| = \lambda^+$. This can be done by applying Claim 4.2.1 with $\delta = C_\beta(0)$ inside the appropriate elementary submodel M_ξ where $C_\beta(1) = M_\xi \cap \lambda^+$.

Note that $C_\beta(2) = M_{\xi'} \cap \lambda^+$ for some $\xi' < \lambda^+$ so we can find $\gamma \in N^+(\alpha_{n-2}) \cap A_i \cap I(\beta, 2)$. Now, $\beta\alpha_0 \dots \alpha_{n-2}\gamma$ is a copy of \vec{C}_n in A_i . \square

Recall that given a poset \mathbb{P} we define its comparability graph $G_{\mathbb{P}}$ on vertex set \mathbb{P} and let $st \in E(G_{\mathbb{P}})$ if and only if $s <_{\mathbb{P}} t$ or $t <_{\mathbb{P}} s$. A Suslin-tree is a poset \mathbb{S} so that each $p \in \mathbb{S}$ has a well ordered set of predecessors and each chain and antichain of \mathbb{S} is countable. Suslin-trees exist in some models of ZFC (e.g. if \diamond holds) and do not exist in others (e.g. if Martin's axiom holds without CH).

We can use the argument from Theorem 4.1 and a trick due to J. Steprans to get the following:

Proposition 4.3. *Suppose that \mathbb{S} is a Suslin-tree and $G_{\mathbb{S}}$ is its comparability graph. Then $G_{\mathbb{S}} \xrightarrow{ENL} \left(\bigwedge_{3 \leq n \in \omega} \vec{C}_n \right)$.*

Proof. We work in a model V of ZFC with a Suslin tree \mathbb{S} on ω_1 . As before, we pick $f : [\omega_1]^2 \rightarrow 2$ with the property that $f''\{\{a, b\} : a \in A, b \in B, a < b\} = 2$ whenever $A, B \in [\omega_1]^{\omega_1}$. Such functions were defined in [20] from a ladder system on ω_1 in a *robust*

way: if our model of set theory V is extended to another model W preserving ω_1 then $f(a, b)$ evaluated in V and W agree. Now let \vec{ab} if and only if $a <_{\mathbb{S}} b$ and $f(a, b) = 0$.

Given $T \in [\mathbb{S}]^{\omega_1}$ and $n \in \omega$ at least 3, we need to find a copy of \vec{C}_n in $G_{\mathbb{S}}[T]$. T as a subtree of \mathbb{S} is still Suslin, and hence forcing with T over our model V preserves cardinals (by ccc) and introduces an uncountable set $A \subseteq T$ so that $H = G_{\mathbb{S}}[A]$ is complete.

Now, working in the larger model V^T , the function f still witnesses $\omega_1 \rightarrow [\omega_1; \omega_1]_2^2$ and so by Theorem 4.1 we can find a directed n -cycle $v_0 \dots v_{n-1}$ in A . However, the fact that $v_0 \dots v_{n-1}$ forms an n -cycle in T is absolute (see the remark on f earlier) so this must be true in our original model V as well. □

Let us state (without presenting the proof) that another class of graphs defined from well behaved non-special trees admit similar orientations: suppose that $S \subseteq \omega_1$ is stationary and let $\sigma(S)$ denote the poset on $\{t \subseteq S : t \text{ is closed}\}$ where $s \leq t$ if and only if s is an initial segment of t .

Theorem 4.4. $G_{\sigma(S)} \xrightarrow{ENL} \left(\bigwedge_{3 \leq n \in \omega} \vec{C}_n \right)_{\omega}^1$ for any stationary $S \subseteq \omega_1$.

Indeed, one can combine the machinery of [26] and the \diamond^+ -argument from Theorem 4.9 to prove this result. Note that neither $G_{\sigma(S)}$ nor $G_{\mathbb{S}}$ for \mathbb{S} Suslin contains an uncountable complete subgraph.

Next, we prove that shift graphs defined on large enough sets have large dichromatic number. Let $Sh_n(\lambda)$ denote the graph on vertices $[\lambda]^n$ and edges $\{\xi_i : i < n\}\{\xi_j : 1 \leq j < n + 1\}$ where $\xi_0 < \dots < \xi_n \in \lambda$.

Theorem 4.5. $Sh_n(\exp_n(\kappa)) \xrightarrow{ENL} (\vec{C}_4)_{\kappa}^1$ for all $2 \leq n < \omega$. In particular,
 $\vec{\chi}(Sh_n(\exp_n(\kappa))) > \kappa$.

As $Sh_n(\lambda)$ has no odd cycles of length less than $2n$, we get:

Corollary 4.6. *There are digraphs with arbitrary large dichromatic number and large odd (undirected) girth.*

Corollary 4.7. *Any digraph F which embeds into all digraphs D with $\chi(D) > \omega$ must be bipartite.*

Lastly, we encourage the reader to keep the $n = 2$ and $\kappa = \omega$ case in mind when reading the following proof; otherwise the technical details might overshadow the actual ideas involved.

Proof of Theorem 4.5. Let $\lambda = \exp_n(\kappa)$. We construct an orientation D of $Sh_n(\lambda)$ so that whenever $G : [\lambda]^n \rightarrow \kappa$ then there is a monochromatic directed 4-cycle. In particular, we aim for a copy of \vec{C}_4 of the following form: the vertices will be $\{\alpha_0\} \cup R$, $R \cup \{\beta\}$, $\{\alpha_1\} \cup R$, $R \cup \{\alpha_3\}$ where $|R| = n - 1$ and $\alpha_0 < \alpha_1 < R < \alpha_3 < \beta$.

List all pairs (A, g) where $A \in [\lambda]^{\exp_{n-1}(\kappa)}$, $g : [A]^n \rightarrow \kappa$ as $\{(A_{\beta}, g_{\beta}) : \beta \in S_{\kappa^+}^{\lambda}\}$ so that $\sup A_{\beta} < \beta$.

By induction on β define the orientation of edges of the form $\{\alpha\} \cup R$, $R \cup \{\beta\}$ where $\alpha < R < \beta$ and $|R| = n - 1$. In short, the $n + 1$ -tuple $\{\alpha\} \cup R \cup \{\beta\}$ will be oriented either up (meaning $\{\alpha\} \cup R$, $R \cup \{\beta\} \in E(D)$) or down (meaning $R \cup \{\beta\}$, $\{\alpha\} \cup R \in E(D)$).

For notational simplicity we will use αR , $R\beta$, $\alpha R\beta$ for $\{\alpha\} \cup R$, $R \cup \{\beta\}$ and $\{\alpha\} \cup R \cup \{\beta\}$ respectively.

Now fix $\beta \in S_{\kappa^+}^\lambda$ and $R \in [\beta]^{n-1}$. We define by induction on $i < \kappa$ disjoint finite sets $a_{\beta,R,i} \in A_\beta \cap \min(R)$ and direct the $n+1$ -tuples of the form $\alpha R\beta$ where $\alpha \in a_{\beta,R,i}$.

Given $i < \kappa$ and the finite sets $a_{\beta,R,j}$ for $j < i$, we consider three cases:

Case 1. If there is $\alpha_0 < \alpha_1 \in (A_\beta \cap \min(R)) \setminus \bigcup \{a_{\beta,R,j} : j < i\}$ and $\alpha_3 \in A_\beta \setminus \max(R)$ so that

- (1) $\alpha_1 R\alpha_3$ and $\alpha_0 R\alpha_3$ are oriented differently (one up, other down), and
- (2) g_β is constant i on $\alpha_0 R, \alpha_1 R, R\alpha_3$.

Then we let $a_{\beta,R,i} = \{\alpha_0, \alpha_1\}$ and define the orientation of $\alpha_0 R\beta$ and $\alpha_1 R\beta$ so that $\alpha_0 R, R\beta, \alpha_1 R, R\alpha_3$ is a copy of \vec{C}_4 .

Case 2. Suppose that Case 1 fails but there is $\alpha_0 < \alpha_1 \in A_\beta \cap \min(R) \setminus \bigcup \{a_{\beta,R,j} : j < i\}$ so that g_β is constant i on $\alpha_0 R, \alpha_1 R$. Then we let $a_{\beta,R,i} = \{\alpha_0, \alpha_1\}$ and define the orientation of $\alpha_1 R\beta$ and $\alpha_0 R\beta$ differently.

Case 3. If both Case 1 and Case 2 fails then we let $a_{\beta,R,i} = \emptyset$.

This finishes the induction on $i < \kappa$ and in turn completes the definition of D .

Now, suppose that $G : [\lambda]^n \rightarrow \kappa$ and our aim is to find a monochromatic 4-cycle. Take a κ -closed elementary submodel M of size $\exp_{n-1}(\kappa)$ so that $D, G \in M$ and $\exp_{n-1}(\kappa), \mathcal{P}^k(\kappa) \in M$ for $k \leq n-1$. Here \mathcal{P} is the power set operator, $\mathcal{P}^0(\kappa) = \kappa$ and $\mathcal{P}^{k+1}(\kappa) = \mathcal{P}(\mathcal{P}^k(\kappa))$.

Find $\beta \in S_{\kappa^+}^\lambda$ so that

$$(A_\beta, g_\beta) = (M \cap \lambda, G \upharpoonright [M \cap \lambda]^2).$$

Now we define a sequence of maps $G_0, G_1 \dots G_{n-1}$ so that

$$G_{n-k} : [\lambda]^k \rightarrow \mathcal{P}^{n-k}(\kappa)$$

as follows. We define $G_0 : [\lambda]^n \rightarrow \kappa$ simply by $G_0 = G$. Next, we define $G_1 : [\lambda]^{n-1} \rightarrow \mathcal{P}(\kappa)$ by

$$(4.1) \quad G_1(x_1, \dots, x_{n-1}) = \{i \in \kappa : |\{\xi < x_1 : G_0(\xi, x_1, \dots, x_{n-1}) = i\}| \geq \exp_{n-1}(\kappa)\} \in \mathcal{P}(\kappa).$$

In general, given G_{n-k-1} , we let

$$(4.2) \quad G_{n-k}(x_{n-k}, \dots, x_{n-1}) = \{i \in \mathcal{P}^{n-k-1}(\kappa) : |\{\xi < x_{n-k} : G_{n-k-1}(\xi, x_{n-k}, \dots, x_{n-1}) = i\}| \geq \exp_k(\kappa)\}.$$

Finally, for $k = 1$, we let

$$(4.3) \quad G_{n-1}(x_{n-1}) = \{i \in \mathcal{P}^{n-2}(\kappa) : |\{\xi < x_{n-1} : G_{n-2}(\xi, x_{n-1}) = i\}| \geq \exp_1(\kappa)\} \in \mathcal{P}^{n-1}(\kappa).$$

Note that $G_{n-1}(\beta) \in M$ by the assumptions on M .

Claim 4.8. *There is a decreasing sequence of ordinals $\xi_{n-1}, \xi_{n-2} \dots \xi_0$ and \in -decreasing $i_{n-1}, i_{n-2} \dots i_0$ with the following properties:*

- (1) $\xi_{n-1} = \beta$ and $i_{n-1} = G_{n-1}(\beta)$,
- (2) $\xi_{n-2} \in A_\beta \cap \xi_{n-1} = A_\beta$ so that
 - (a) $G_{n-1}(\xi_{n-2}) = i_{n-1} = G_{n-1}(\xi_{n-1})$, and
 - (b) $i_{n-2} = G_{n-2}(\xi_{n-2}, \xi_{n-1}) \in i_{n-1}$;
- (3) in general, $\xi_{n-k-1} \in A_\beta \cap \xi_{n-k}$ so that
 - (a) $G_{n-k}(\xi_{n-k-1} \dots \xi_{n-2}) = i_{n-k} = G_{n-k}(\xi_{n-k} \dots \xi_{n-1})$, and

(b) $i_{n-k-1} = G_{n-k-1}(\xi_{n-k-1}, \xi_{n-k} \dots \xi_{n-1}) \in i_{n-k}$
 where $k = 0 \dots n-1$.

Proof. Given $\xi_{n-1} = \beta$ and $i_{n-1} = G_{n-1}(\beta)$, observe that $G \in M, \text{ran}(G_{n-1}) \subseteq M$ implies that

$$\Lambda = \{\xi \in S_{\kappa^+}^\lambda : G_{n-1}(\xi) = G_{n-1}(\beta)\} \in M$$

as well. Hence $|\Lambda| \geq (\exp_{n-1}(\kappa))^+$ (as $\beta \in \Lambda \setminus M$) and so $|\Lambda \cap M| = \exp_{n-1}(\kappa)$. In turn, $cf(\exp_{n-1}(\kappa)) > \exp_{n-2}(\kappa)$ implies that there is an $i_{n-2} \in \mathcal{P}^{n-2}(\kappa)$ so that

$$(4.4) \quad |\{\xi \in \Lambda \cap M : G_{n-2}(\xi, \xi_{n-1}) = i_{n-2}\}| \geq \exp_{n-2}(\kappa).$$

In particular, $i_{n-2} \in i_1 = G_{n-1}(\xi_{n-1})$. Now pick any $\xi_{n-2} \in \Lambda \cap M$ with $G_{n-2}(\xi_{n-2}, \xi_{n-1}) = i_{n-2}$.

Suppose we found ξ_{n-k} and i_{n-k} as required. By assumption, $i_{n-k} \in i_{n-k+1} = G_{n-k+1}(\xi_{n-k} \dots \xi_{n-2})$. Hence

$$|\{\xi < \xi_{n-k} : G_{n-k}(\xi, \xi_{n-k} \dots \xi_{n-2}) = i_{n-k}\}| \geq \exp_k(\kappa)$$

and so

$$|\{\xi \in M \cap \xi_{n-k} : G_{n-k}(\xi, \xi_{n-k} \dots \xi_{n-2}) = i_{n-k}\}| \geq \exp_{n-k}(\kappa)$$

holds as well. By cofinality considerations, there is $i_{n-k-1} \in \mathcal{P}^{n-k-1}(\kappa)$ so that

$$|\{\xi \in M \cap \xi_{n-k} : G_{n-k}(\xi, \xi_{n-k} \dots \xi_{n-2}) = i_{n-k}, G_{n-k-1}(\xi, \xi_{n-k} \dots \xi_{n-1}) = i_{n-k-1}\}| \geq \exp_{n-k}(\kappa).$$

Note that this implies that $i_{n-k-1} \in i_{n-k} = G_{n-k}(\xi_{n-k} \dots \xi_{n-1})$ and we can pick any $\xi_{n-k-1} \in M \cap \xi_{n-k}$ so that $G_{n-k}(\xi_{n-k-1}, \xi_{n-k} \dots \xi_{n-2}) = i_{n-k}, G_{n-k-1}(\xi_{n-k-1}, \xi_{n-k} \dots \xi_{n-1}) = i_{n-k-1}$. Hence conditions (a) and (b) above are satisfied. \square

At last we get $\xi_0 \in A_\beta \cap \xi_1$ and $i_0 \in \kappa$ so that

- (a) $G_1(\xi_0 \dots \xi_{n-2}) = i_1 = G_1(\xi_1 \dots \xi_{n-1})$, and
- (b) $i_0 = G_0(\xi_0 \dots \xi_{n-1}) \in i_1$.

We let $R = \{\xi_0 \dots \xi_{n-2}\} \in [A_\beta]^{n-1}$ and look at the construction of the orientation D when we considered β and R . In particular, we consider the step when the colour $i_0 \in \kappa$ came up.

If the assumption of Case 1 was satisfied then we constructed a copy of \vec{C}_4 on vertices $\alpha_0 R, R\beta, \alpha_1 R, R\alpha_3$ and g_β is constant i_0 on $\alpha_0 R, \alpha_1 R, R\alpha_3$; g_β and G agree on A_β and $G(R\beta) = i_0$ so this is the desired monochromatic \vec{C}_4 .

Now, we suppose that Case 1 fails and reach a contradiction. We claim that the assumption of Case 2 is satisfied. Indeed, $i_0 \in i_1 = G_1(\xi_0 \dots \xi_{n-2})$ implies that there are κ many $\xi < \xi_0$ with the property that $G(\xi, \xi_0 \dots \xi_{n-2}) = i_0$. Also, M is κ -closed so $\bigcup\{a_{\beta, R, j} : j < i\} \in M$. Hence, using elementarity, we can select $\alpha_0 < \alpha_1 \in M \cap \xi_0 \setminus \bigcup\{a_{\beta, R, j} : j < i_0\}$ so that

$$G(\alpha_0, \xi_0 \dots \xi_{n-2}) = G(\alpha_1, \xi_0 \dots \xi_{n-2}) = i_0.$$

Hence, following the instructions in Case 2, we oriented $\alpha_0 R\beta$ and $\alpha_1 R\beta$ differently. In turn, the set

$$B = \{\xi \in \lambda \setminus \max(R) : \text{the orientation of } \alpha_0 R\xi, \alpha_1 R\xi \text{ are different and } G(R\xi) = i_0\}$$

is not empty. However, $B \in M$ so we can choose $\alpha_3 \in B \cap M$ and hence $\alpha_1, \alpha_2, \alpha_3$ witnesses that Case 1 holds. This contradicts our assumption and ends the proof. \square

Now, we prove that consistently *any graph* with size and chromatic number ω_1 has uncountable dichromatic number as well in a very strong sense.

Recall that \diamond^+ asserts the existence of sets $\mathcal{S}_\beta = \{S_{\beta,n} : n \in \omega\}$ where $\beta \in \omega_1$ so that for every $X \subseteq \omega_1$ there is a club $C \subseteq \omega_1$ such that $X \cap \beta = S_{\beta,n}$ for some $n \in \omega$ whenever $\beta \in C$.

Theorem 4.9. *Assume that \diamond^+ holds and the graph G has size and chromatic number ω_1 . Then*

$$G \xrightarrow{ENL} \left(\bigwedge \{D : D \text{ is an orientation of } H_{\omega,\omega}\} \right).$$

In other words, there is an orientation D^* of G so that whenever $G[A]$ is uncountably chromatic and D is an orientation of $H_{\omega,\omega}$ then D embeds into $D^*[A]$.

Proof. Let $\mathcal{S}_\beta = \{S_{\beta,n} : n \in \omega\}$ be the \diamond^+ sequence and let $\{D_\beta : \beta < \omega_1\}$ list all orientations of $H_{\omega,\omega}$. We suppose that each D_β has vertices $2 \times \omega$.

By induction on β we orient the edges of G from β to $\beta \cap N(\beta)$ to define an orientation $D_{\beta+1}^*$ of $G[\beta+1]$.

Let

$$\{(\delta_j, n_j, \xi_j) : j < \omega\}$$

list $\Gamma_\beta = \{(\delta, n, \xi) : \delta, \xi \leq \beta, n < \omega, |N(\beta) \cap S_{\delta,n}| = \omega\}$.

Suppose that the orientation D_β^* of $G[\beta]$ is defined already. By induction on $j < \omega$, we will select $a_j \in [N(\beta) \cap S_{\delta_j, n_j}]^{<\omega}$ such that $a_j \cap a_{j'} = \emptyset$ for $j < j' < \omega$ and orient the edges between β and a_j as follows.

At step j , we look at the set Φ_j of partial digraph embeddings φ of D_{ξ_j} into $D_\beta^*[S_{\delta_j, n_j}]$ such that $\text{dom}(\varphi) = 2 \times k$ for some $k \leq \omega$ and

$$\varphi[\{0\} \times k] \subseteq S_{\delta_j, n_j} \cap N(\beta) \setminus \bigcup_{j' < j} a_{j'}.$$

Note that Φ_j might only contain \emptyset . In any case, take a $\varphi_j \in \Phi_j$ which is maximal with respect to inclusion.

Case 1: If φ_j is a complete embedding then let $a_j = \emptyset$ and move to step $j+1$ in the induction.

Case 2: if Case 1 fails then there is a $k < \omega$ such that $\text{dom}(\varphi_j) = 2 \times k$ i.e. φ_j is a digraph embedding of $D_{\xi_j}[2 \times k]$. Let $a_j = \varphi_j[\{0\} \times k] \cup \{\alpha\}$ for some

$$\alpha \in S_{\delta_j, n_j} \cap N(\beta) \setminus \left(\bigcup_{j' < j} a_{j'} \cup \varphi_j[\{0\} \times k] \right).$$

Lets define the orientation between β and a_j so that

$$\varphi_j^* = \varphi_j \cup \{((0, k), \alpha), ((1, k), \beta)\}$$

is a digraph embedding of $D_{\xi_j}[2 \times (k+1)]$.

Edges from β to $(N(\beta) \cap \beta) \setminus \bigcup \{a_j : j < \omega\}$ are oriented arbitrarily. This finishes the definition of the orientation $D^* = \bigcup_{\beta < \omega_1} D_\beta^*$ of G .

Now take any A such that $G[A]$ is uncountably chromatic and an orientation D of $H_{\omega,\omega}$. There is a club of elementary submodels $\{M_\alpha : \alpha < \omega_1\}$ of $H(\aleph_2)$ so that $D, A \in M_\alpha$ and whenever $\delta = M_\alpha \cap \omega_1$ for some $\alpha < \omega_1$ then $A \cap \delta = S_{\delta,n}$ for some $n \in \omega$.

As $G[A]$ is uncountably chromatic, there is $\delta = M_\alpha \cap \omega_1$ for some $\alpha < \omega_1$ and $\beta \in A \setminus \delta$ such that $N(\beta) \cap A \cap \delta$ is infinite. We can suppose that there is a $\xi \leq \beta$ so that $D = D_\xi$.

Let $n < \omega$ such that $A \cap \delta = S_{\delta,n}$; so $N(\beta) \cap S_{\delta,n}$ is infinite. We claim that there is a copy of D in $D^*[S_{\delta,n}] \subseteq D^*[A]$.

Let us look at how the orientation was defined between β and $N(\beta) \cap \beta$. There is a $j < \omega$ such that $(\delta_j, n_j, \xi_j) = (\delta, n, \xi)$.

At step j , we selected a maximal partial embedding φ_j of D into $D_\beta^*[S_{\delta,n}]$. If Case 1 held for φ_j then φ_j is a complete embedding witnessing the existence of a copy of D in $D^*[S_{\delta,n}]$.

Otherwise, we are in Case 2: φ_j is an embedding of $D[2 \times k]$ for some $k < \omega$. This case, we extended φ_j into a strictly larger embedding φ_j^* with

$$\text{ran}(\varphi_j^*) \subseteq S_{\delta_j, n_j} \cup \{\beta\} \subseteq A.$$

Note that φ_j and A are both in M_α and

$$H(\aleph_2) \models \varphi_j \text{ can be extended to an embedding of } D[2 \times (k+1)] \text{ into } A \setminus \bigcup_{j' < j} a_{j'}.$$

Hence this sentence must be true in M_α as well, that is, there is an embedding $\varphi \in M_\alpha$ of $D[2 \times (k+1)]$ into $A \setminus \bigcup_{j' < j} a_{j'}$ extending φ_j . Of course the range of φ now has to be in $A \cap M_\alpha = S_{\delta,n}$ which contradicts the maximality of φ_j . \square

Corollary 4.10. *If \diamond^+ holds and G has size ω_1 then $\vec{\chi}(G) = \omega_1$ if and only if $\chi(G) = \omega_1$.*

5. ON THE LACK OF ORIENTATIONS WITH LARGE CHROMATIC NUMBER

In this final section, we show that consistently there is a graph G with uncountable chromatic number without an orientation with uncountable dichromatic number. In other words, $\chi(G) > \omega$ does not imply $\vec{\chi}(D) > \omega$.

In [11], Hajnal and Komjáth study an intriguing problem which can be roughly stated as follows: given a graph G with uncountable chromatic number, can we colour the edges of G with 2 (alternatively, ω or ω_1) colours so that each colour appears on each *large enough* subgraph.

Let us observe some straightforward connections to our investigations:

Observation 5.1. (1) *If $G \xrightarrow{ENL} (\bigvee \{\vec{C}_n : 3 \leq n < \omega\})$ then we can define a 2-edge colouring of G with the property that every colour appears on every uncountably chromatic induced subgraph.*

(2) *If $\vec{\chi}(G) > \omega$ then there is a 2-colouring of the edges of G such that whenever the vertices $V(G)$ are partitioned into countably many pieces $\{V_i : i < \omega\}$ then both colours appear on one of the spanned subgraphs $G[V_i]$*

Proof. Let us prove (1) and leave the completely analogous proof of (2) to the reader. Given the orientation D of $G = (\lambda, E)$ witnessing $G \xrightarrow{ENL} (\bigvee \{\vec{C}_n : 3 \leq n < \omega\})$ define $f(ab) = 0$ if $a < b \in \lambda$ and $\vec{ab} \in E(D)$, otherwise $f(ab) = 1$. Now, if $\chi(G[W]) > \omega$ then there is a directed cycle C in $D[W]$. Let $b = \max C$ (where C is considered as a set of ordinals in λ). If a and a' are the neighbours of b in C then we must have $f(ab) \neq f(a'b)$. \square

It was shown in [11, Theorem 5] that the consequence stated in Observation 5.1 (1) can consistently fail for a graph of size and chromatic number ω_1 . Hence, we get the following:

Corollary 5.2. *Consistently, there is a graph G with size and chromatic number ω_1 such that $G \xrightarrow{ENL} (\bigvee \{\vec{C}_n : 3 \leq n < \omega\})$ fails.*

However, it is unknown if there is a graph G of chromatic number ω_1 which fails the consequence stated in Observation 5.1 (2) (even consistently). Hence it is rather interesting that $\vec{\chi}(G) \leq \omega$ is possible for a graph G with uncountable chromatic number:

Theorem 5.3. *Consistently, there is a graph G on vertex set ω_1 such that*

- (1) $\vec{\chi}(G) \leq \omega$, however
- (2) $C_3 \leftrightarrow G[X]$ for every uncountable $X \subseteq \omega_1$, and so $\chi(G) = \omega_1$.

At this point, we don't know if the implication in Observation 5.1 (2) can or cannot be reversed.

Proof. We start from a model V of CH.

Let $\mathbb{P}_0 = \{(s, g) : s \in [\omega_1]^{<\omega}, g \subseteq [s]^2\}$ with the usual ordering.

Given a generic filter $\mathcal{G} \subseteq \mathbb{P}_0$ we get a graph $\dot{G} = \{g^p : p \in \mathcal{G}\}$ in the extension $V^{\mathbb{P}_0}$.

We will not use this particular fact, but let us mention that \dot{G} has uncountable dichromatic number in $V^{\mathbb{P}_0}$:

Claim 5.4. $V^{\mathbb{P}_0} \models \dot{G} \xrightarrow{ENL} (\vec{C}_3)$.

Proof. Let $f_\beta : \omega \rightarrow \beta$ be a bijection for $\beta \in \omega_1$. We define an orientation \dot{D} as follows: if $\alpha < \beta < \omega_1$ and $\alpha\beta \in \dot{G}$ then $\alpha\beta \in E(\dot{D})$ if and only if β and $\beta + n$ are connected in \dot{G} for $\alpha = f_\beta(n)$; otherwise, $\beta\alpha \in E(\dot{D})$.

Suppose that $p_0 \Vdash \dot{W}$ is uncountable and let $p_0, \dot{W}, \mathbb{P}_0, (f_\alpha)_{\alpha < \omega_1} \in M_0 < M_1$ where M_0, M_1 are countable elementary submodels of $H(\aleph_2)$. Find $\beta \in \omega_1 \setminus M_1$ and $p \leq p_0$ so that $p \Vdash \beta \in \dot{W}$. Let $I = \{n : \beta + n \in s^p\}$ and find a $p' \in M_1$ so that p and p' are compatible and there is $\beta' \in s^{p'} \setminus \text{ran}(f_\beta \upharpoonright I)$ with $p' \Vdash \beta' \in \dot{W}$.

Let $J = \{n : \beta' + n \in s^{p'}\}$ and find $p'' \in M_0$ compatible with both p and p' such that there is

$$\beta'' \in s^{p''} \setminus (\text{ran}(f_\beta \upharpoonright I) \cup \text{ran}(f_{\beta'} \upharpoonright J))$$

with $p'' \Vdash \beta'' \in \dot{W}$.

It is easy to see that we can find a common extension q of p, p' and p'' which forces that $\{\beta, \beta', \beta''\}$ is a copy of \vec{C}_3 . \square

Let us remark that the same argument shows $V^{\mathbb{P}_0} \models \dot{G} \xrightarrow{ENL} (D_0)$ for any finite digraph D_0 .

Now define a finite support iteration $(\mathbb{P}_\alpha, \dot{Q}_\alpha)_{\alpha \leq \omega_2, \beta < \omega_2}$ as follows: in $V^{\mathbb{P}_\alpha}$, we consider an orientation \dot{D}_α of \dot{G} and let

$$\dot{Q}_\alpha = \{q \in Fn(\omega_1, \omega) : \dot{D}_\alpha[q^{-1}(n)] \text{ is acyclic for all } n \in \omega\}.$$

Recall that $Fn(\lambda, \kappa)$ denotes the set of all finite partial functions from λ to κ .

Clearly, the forcing \dot{Q}_α introduces an ω -partition $\{\dot{W}_n : n < \omega\}$ of ω_1 such that $\dot{D}_\alpha[\dot{W}_n]$ has no directed cycles. In turn, $V^{\mathbb{P}_\alpha * \dot{Q}_\alpha} \models \chi(\dot{D}_\alpha) \leq \omega$.

Our goal is to prove that this iteration is ccc and in the final model $V^{\mathbb{P}_{\omega_2}}$ our graph \dot{G} is uncountably chromatic.

Claim 5.4.1. *The set of $p \in \mathbb{P}_\alpha$ which satisfy*

- (1) $p(\xi) \in V$ and $\text{dom}(p(\xi)) \subseteq s^{p(0)}$, and
- (2) $p \upharpoonright \xi$ decides the orientation of \dot{D}_ξ on $\text{dom}(p(\xi))$

for all $\xi \in \text{supp}(p)$ is dense in \mathbb{P}_α .

We will call these conditions *determined* and we only work with determined conditions from now on if not mentioned otherwise.

Proof. Easy induction on α . □

We say that $p, p' \in \mathbb{P}_0$ are *twins* if $|s^p| = |s^{p'}|$, $s^p \cap s^{p'} < s^p \setminus s^{p'} < s^{p'} \setminus s^p$ (or vica versa $s^{p'} \setminus s^p < s^p \setminus s^{p'}$) and the unique monotone bijection $\psi_{p,p'} : s^p \rightarrow s^{p'}$ (which fixes $s^p \cap s^{p'}$) gives a graph isomorphism between the graphs p and p' . We say two conditions from the iteration $q, q' \in \mathbb{P}_\alpha$ are *twins* if

- (1) $p(0)$ and $p'(0)$ are twins,

and for all $\xi \in \text{supp}(p) \cap \text{supp}(p')$

- (2) $\text{dom}(p'(\xi)) = \psi_{p,p'}[\text{dom}(p(\xi))]$,

- (3) $p(\xi)(\delta) = p'(\xi)(\psi_{p,p'}(\delta))$ for all $\delta \in \text{dom}(p(\xi))$, and

- (4) $p \upharpoonright \xi \Vdash a_0 a_1 \in \dot{D}_\xi$ if and only if $p' \upharpoonright \xi \Vdash a'_0 a'_1 \in \dot{D}_\xi$ for all $\xi \in \text{supp}(p) \cap \text{supp}(p')$ where $a'_i = \psi_{p,p'}(a_i)$.

Claim 5.4.2. *If $p, p' \in \mathbb{P}_\alpha$ are determined and twins then they have a minimal common extension $p \vee p'$.*

Proof. We define $p \vee p'$ by

$$(p \vee p')(\alpha) = \begin{cases} (s^p \cup s^{p'}, g^p \cup g^{p'}), & \text{for } \alpha = 0, \text{ and} \\ p(\xi) \cup p'(\xi), & \text{for } \xi \in \alpha \setminus \{0\}. \end{cases}$$

It is clear that $(p \vee p')(0) \in \mathbb{P}_0$ so let us show $(p \vee p') \upharpoonright \xi$ forces that there are no monochromatic cycles with respect to $p(\xi) \cup p'(\xi)$. We do this by induction on $\xi < \alpha$.

Suppose that $p \vee p' \upharpoonright \xi$ is a condition. If $\xi \in \text{supp}(p) \setminus \text{supp}(p')$ or $\xi \in \text{supp}(p') \setminus \text{supp}(p)$ then $p \vee p' \upharpoonright \xi + 1 \in \mathbb{P}_{\xi+1}$ as well.

Now let $\xi \in \text{supp}(p) \cap \text{supp}(p')$. We need to show that $p \vee p' \upharpoonright \xi$ forces that there are no monochromatic directed cycles in \dot{D}_ξ with respect to $(p \vee p')(\xi)$. This will easily follow from Lemma 2.1: fix $k < \omega$ and consider $D = \dot{D}_\xi[\{v \in s^p : p(\xi)(v) = k\}]$ and $D' = \dot{D}_\xi[\{v \in s^{p'} : p'(\xi)(v) = k\}]$. $p \vee p' \upharpoonright \xi$ forces that D and D' are isomorphic, acyclic digraphs satisfying the assumptions of Lemma 2.1 as p and p' are twins. Hence, by Lemma 2.1 (2), $D \cup D' = \dot{D}_\xi[\{v \in s^{p \vee p'} : (p \vee p')(\xi)(v) = k\}]$ is acyclic as well. □

Claim 5.4.3. \mathbb{P}_α is ccc for all $\alpha \leq \omega_2$.

Proof. Indeed, the Δ -system lemma implies that any uncountable set of conditions must contain an uncountable set of pairwise twins, and Claim 5.4.2 implies that any two twin conditions are comparable. □

Now, by a standard bookkeeping argument, we can choose the names \dot{D}_α of \dot{G} so that

$$V^{\mathbb{P}^{\omega_2}} \models \overline{\chi}(G) \leq \omega.$$

Before we show that the chromatic number of G is still uncountable after the iteration, let us emphasize a simple fact which again follows Lemma 2.1 (1):

Claim 5.4.4. *Suppose that $p, p' \in \mathbb{P}_{\omega_2}$ are determined and twins and $\delta \in s^p \setminus s^{p'}$. Then $p \vee p' \upharpoonright \xi$ forces that there is no directed path from δ to $\delta' = \psi_{p,p'}(\delta)$ in \dot{D}_ξ which is monochromatic with respect to $p \vee p'(\xi)$.*

Now, we can prove the following:

Claim 5.4.5. *Suppose that $p, p' \in \mathbb{P}_{\omega_2}$ are determined and twins and $\delta \in s^p \setminus s^{p'}$. Then there is a minimal extension $p \vee_{\delta} p'$ of $p \vee p'$ which forces that δ is connected to $\delta' = \psi_{p,p'}(\delta)$ in \dot{G} .*

Proof. We define $q = p \vee_{\delta} p'$ by

$$q(\alpha) = \begin{cases} (s^{p \vee p'}, g^{p \vee p'} \cup \{\delta, \delta'\}), & \text{for } \alpha = 0, \text{ and} \\ (p \vee p')(\alpha), & \text{for } \alpha \in \omega_2 \setminus \{0\}. \end{cases}$$

That is, we essentially take $p \vee p'$ and add the single edge $\{\delta, \delta'\}$. If we prove that q is a condition then we are done.

We prove that $q \upharpoonright \xi$ is a condition by induction on $\xi < \omega_2$. Suppose we proved that $q \upharpoonright \xi$ is a condition but there is some $r \leq q \upharpoonright \xi$ and C so that r forces that C is a monochromatic cycle with respect to $q(\xi)$ in \dot{D}_{ξ} . C must contain the edge $\{\delta, \delta'\}$ and so r forces that δ and δ' are connected by a directed monochromatic path P (not containing the edge $\{\delta, \delta'\}$). However, orientation and colouring on P is decided by $p \vee p' \upharpoonright \xi$ already, which contradicts Claim 5.4.4. □

Note that $p \vee_{\delta} p'$ is not necessarily determined. In any case, to finish our proof, it suffices to show the following:

Claim 5.4.6. $V^{\mathbb{P}_{\omega_2}} \models C_3 \leftrightarrow \dot{G}[\dot{X}]$ for every uncountable $\dot{X} \subseteq \omega_1$.

Proof. Suppose $p \Vdash \dot{X}$ is uncountable for some $p \in \mathbb{P}_{\omega_2}$; we will find $q \leq p$ and $\beta, \beta', \beta'' \in \omega_1$ such that $q \Vdash \{\beta, \beta', \beta''\}$ is a triangle in $\dot{G}[\dot{X}]$.

As \mathbb{P}_{ω_2} is ccc, there is an uncountable set $Y \subseteq \omega_1$ in V and determined conditions $p_{\beta} \leq p$ so that $\beta \in s^{p_{\beta}}$ and $p_{\beta} \Vdash \beta \in \dot{X}$ whenever $\beta \in Y$.

Take a countable elementary submodel M of some large enough $H(\theta)$ containing everything relevant and let $\beta \in Y \setminus M$. We let $s = s^{p_{\beta}} \cap M$. Using elementarity, find $\beta' \in Y \cap M$ so that

- (1) p_{β} and $p_{\beta'}$ are twins,
- (2) $s = s^{p_{\beta}} \cap s^{p_{\beta'}}$, and
- (3) $\beta = \psi_{p',p}(\beta')$.

Let r be a determined condition extending $p_{\beta'} \vee_{\beta'} p_{\beta}$.

Now, let $\tilde{s} = s^r \cap M$ and, using elementarity again, find a condition $r' \in M$ so that

- (4) r and r' are twins,
- (5) $\tilde{s} = s^r \cap s^{r'}$, and
- (6) $\beta = \psi_{r',r}(\beta'')$.

Finally, let $q = r' \vee_{\beta''} r$. Clearly, $q \Vdash \{\beta, \beta', \beta''\}$ is a triangle in $\dot{G}[\dot{X}]$. □

□

In particular, if one is looking for the value of $f(\aleph_1)$ from Conjecture 1.1, it needs to be larger than \aleph_1 , at least consistently.

6. OPEN PROBLEMS

It seems that it is non trivial to find in ZFC a single digraph D with uncountable dichromatic number. Indeed, even to show that K_{ω_1} has an orientation with uncountable dichromatic number required the application of $\omega_1 \rightarrow [\omega_1; \omega_1]_2^2$ which is a deep result of J. Moore [20]. Hence, we have the following meta-problem:

Problem 6.1. *Provide a simple/elementary proof of the fact that K_{ω_1} has an orientation with uncountable dichromatic number.*

An obvious question which comes to mind regarding the definition of $\overline{\chi}(G)$ is whether the sup is actually a max:

Question 6.2. *Suppose that an undirected graph G has orientations D_ξ for $\xi < \text{cf}(\kappa)$ so that $\sup\{\chi(D_\xi) : \xi < \text{cf}(\kappa)\} = \kappa$. Is there a single orientation D of G so that $\chi(D) = \kappa$?*

We conjecture that the answer is yes if $\text{cf}(\kappa) = \omega$ but no if $\text{cf}(\kappa) > \omega$.

Regarding obligatory subgraphs and Proposition 3.4, we ask:

Problem 6.3. *Characterize those orientations D^* of $H_{\omega, \omega}$ so that D^* embeds into any digraph D with uncountable dichromatic number.*

At this point, we don't even have a list of those digraphs say on 4 vertices which embed into any digraph D with uncountable dichromatic number.

6.1. More on cycles. It is known that $\chi(G) > \omega$ implies that G has cycles of all but finitely many lengths [8, 29].

Question 6.4. *Are there digraphs D with $\chi(D) > \omega$ so that \overrightarrow{C}_k does not embed into D for infinitely many k ?*

More generally, describe what sets of cycles can be omitted by D with $\chi(D) > \omega$. In particular, answer the following:

Question 6.5. *Does $D \rightarrow (\overrightarrow{C}_3)_\omega^1$ imply that $\overrightarrow{C}_4 \hookrightarrow D$?*

The answer is yes for the undirected version: $G \rightarrow (C_3)_\omega^1$ obviously implies $\chi(G) > \omega$ and so $C_4 \hookrightarrow G$ and even $G \rightarrow (C_{2k})_\omega^1$ holds for all $2 \leq k < \omega$.

It was shown by Erdős and R. Rado [6] that triangle-free graphs G with size and chromatic number κ can be constructed without additional set theoretic assumptions.

Question 6.6. *Is there in ZFC a digraph D of size and dichromatic number ω_1 such that D has no directed triangles?*

If we omit the requirement on size then the shift graphs provide an example by Theorem 4.5.

Finally, it would be very interesting to construct digraphs as in Theorem 3.5 in ZFC:

Question 6.7. *Is there in ZFC, for every $k < \omega$, a digraph D of uncountable dichromatic number such that D has digirth at least k ?*

6.2. Connected subgraph. It is clear that every graph with uncountable chromatic number has a connected component with uncountable chromatic number. Similarly:

Observation 6.8. *Suppose that $\chi(D) > \omega$. Then there is $D_0 \subseteq D$ so that $\chi(D_0) > \omega$ and D_0 is strongly connected.*

Komjáth [14] showed that every graph G with uncountable chromatic number contains a k -connected subgraph with uncountable chromatic number where $k \in \omega$.

Question 6.9. *Suppose that $\chi(D) > \omega$ and $k \in \omega$. Is there a $D_0 \subseteq D$ so that D_0 is strongly k -connected?*

Even the case $k = 2$ is open. In the undirected case, the balanced complete bipartite graph on $2k$ vertices is a k -connected subgraph of any graph G with uncountable chromatic number. However, any strongly connected digraph D_0 contains directed cycles and hence, by Theorem 3.5, there is a digraph D with uncountable dichromatic number so that D_0 does not embed into D . Hence no single strongly connected graph will be a universal witness providing a positive answer to Question 6.9.

If the answer is yes to Question 6.9, a more ambitious goal would be to find a $D_0 \subseteq D$ so that $\chi(D_0) > \omega$ while D_0 is strongly k -connected.

Regarding Theorem 3.7, the most burning question is the following:

Question 6.10. *Is there (even consistently) a digraph D with $\chi(D) > \omega$ so that $D \xrightarrow{ENL} (\vec{C}_n)_2^1$ fails for every $n \in \omega$?*

Naturally, any ZFC example would be very warmly welcome:

Question 6.11. *Is there in ZFC a digraph D with $\chi(D) > \omega$ so that $D \xrightarrow{ENL} (\vec{C}_n)_\omega^1$ fails for every $n \in \omega$?*

6.3. Various questions. Regarding the Erdős-Neumann-Lara problem, we ask:

Problem 6.12. *Does $\chi(G) > 2^\omega$ imply $\vec{\chi}(G) > \omega$?*

It would be rather natural to look into the following with regards to Theorem 4.9:

Problem 6.13. *Does $\chi(G) = \omega_1$ imply $\vec{\chi}(G) = \omega_1$ consistently (without restricting the size of G)?*

The following might be easier to answer:

Problem 6.14. *Does*

$$G \xrightarrow{ENL} (\vec{P}_\omega)_\omega^1$$

hold in ZFC for G with chromatic number ω_1 where \vec{P}_ω is the one-way infinite directed path.

Finally, we close with a fascinating open problem of Neumann-Lara from 1985:

Problem 6.15. *Does every planar digraph have dichromatic number at most 2?*

The answer is yes if the digirth is at most four [13, 18]. We mention that this is a problem on finite digraphs: if there is an infinite counterexample then it must have a finite subdigraph of dichromatic number greater than 2 as well by Theorem 3.10.

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