# Decomposing planar cubic graphs 

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#### Abstract

The 3-Decomposition Conjecture states that every connected cubic graph can be decomposed into a spanning tree, a 2-regular subgraph and a matching. We show that this conjecture holds for the class of connected plane cubic graphs.


Keywords: cubic graph, 3-regular graph, spanning tree, decomposition, separating cycle

## 1 Introduction

All graphs considered here are finite and without loops. A decomposition of a graph $G$ is a set of subgraphs whose edge sets partition the edge set of $G$. Any of these subgraphs may equal the empty graph - that is, a graph whose vertex set is empty - unless this is excluded by additional requirements (such as being a spanning tree). We regard matchings in decompositions as 1 -regular subgraphs.

The 3-Decomposition Conjecture (3DC) by the first author [2, 4] states that every connected cubic graph has a decomposition into a spanning tree, a 2 -regular subgraph and a matching. For an example, see the graph on the left

[^0]in Figure 1. The 2-regular subgraph in such a decomposition is necessarily nonempty whereas the matching can be empty.

The 3DC was proved for planar and projective-planar 3-edge-connected cubic graphs in [3]. It is also known that the conjecture holds for all hamiltonian cubic graphs. For a survey on the 3DC, see [5].

We call a cycle $C$ in a connected graph $G$ separating if $G-E(C)$ is disconnected. The 3DC was shown in [5] to be equivalent to the following conjecture, called the 2-Decomposition Conjecture (2DC). (See Proposition 14 at the end of this paper.)

Conjecture 1 (2DC) Let $G$ be a connected graph with vertices of degree two and three only such that every cycle of $G$ is separating. Then $G$ can be decomposed into a spanning tree and a nonempty matching.

For an example, see the graph on the right in Figure 1. The main result of this paper, Theorem 2, shows that the 2DC is true in the planar case. Call a graph subcubic if its maximum degree is at most 3 .


Figure 1: Decomposition of a cubic and a subcubic graph into a spanning tree (thick lines), a 2-regular subgraph (dotted lines), and a nonempty matching (thin lines).

Theorem 2 Every connected subcubic plane graph in which every cycle is separating has a decomposition into a spanning tree and a matching.

Note that the matching in Theorem 2 is empty if and only if the subcubic graph is a tree. It follows that the 2DC holds for the planar case. Finally, we will prove that Theorem 2 implies the planar case of the 3DC:

Corollary 3 Every connected cubic plane graph can be decomposed into a spanning tree, a nonempty 2 -regular subgraph and a matching.

## 2 Preliminary observations

Before we establish some facts needed for the proof of Theorem 2, we introduce some terminology and notation. We refer to [1, 6] for additional information.

A cycle is a connected 2-regular graph. Moreover, a 2-cycle is a cycle with precisely two edges. A $v w$-path is a path with endvertices $v$ and $w$. For $k \in\{2,3\}$, a $k$-vertex of a graph $G$ is a vertex of degree $k$. Similarly, for $k, \ell \in\{2,3\}$, a $(k, \ell)$-edge is one with endvertices of degrees $k$ and $\ell$. We let $V_{2}(G)$ and $V_{3}(G)$ denote the set of vertices of degree 2 and 3, respectively.

Definition 4 Let $\mathcal{G}_{2,3}$ be the class of all connected plane graphs with each vertex of degree 2 or 3 . Let $\mathcal{S}_{2,3}$ be the class of all graphs $G$ in $\mathcal{G}_{2,3}$, such that each cycle in $G$ is separating.

If a vertex $v$ of $G$ belongs to the boundary of a face $F$, we say that $v$ is incident with $F$ or simply that it is a vertex of $F$. If $A$ is a set of edges of $G$ and $e$ is an edge, we abbreviate $A \cup\{e\}$ to $A+e$ and $A \backslash\{e\}$ to $A-e$.

When contracting an edge, any resulting parallel edges are retained. The contraction of a parallel edge is not allowed. Suppressing a 2 -vertex (with two different neighbours) means contracting one of its incident edges. If $e \in E(G)$, then $G / e$ denotes the graph obtained from $G$ by contracting $e$.

The graph with two vertices and three edges joining them is denoted by $\Theta$.

Recall that an edge-cut $C$ in a connected graph $G$ is an inclusionwise minimal set of edges whose removal disconnects $G$. By the minimality, $G-C$ has exactly two components. The edge-cut $C$ is cyclic if both components of $G-C$ contain cycles. The graph $G$ is said to be cyclically $k$-edge-connected (where $k$ is a positive integer) if it contains no cyclic edge-cuts of size less than $k$. Note that cycles, trees and subdivisions of $\Theta$ or of $K_{4}$ are cyclically $k$-edge-connected for every $k$.

In this paper, the end of a proof is marked by $\square$, and the end of the proof of a claim (within a more complicated proof) is marked by $\triangle$.

The following lemma is a useful sufficient condition for a 2-edge-cut to be cyclic:

Lemma 5 Let $C$ be a 2-edge-cut in a 2-edge-connected graph $G \in \mathcal{G}_{2,3}$. If no component of $G-C$ is a path, then $C$ is a cyclic edge-cut.

Proof. Let $K$ be a component of $G-C$ and let $u$ and $v$ be the endvertices of the edges of $C$ in $K$. Note that since $G$ is subcubic and 2-edge-connected, $C$ is a matching and thus $u \neq v$. Suppose that $K$ is acyclic. Since it is not
a path, it is a tree with at least 3 leaves, one of which is different from $u, v$ and so its degree in $G$ is 1 . Since $G \in \mathcal{G}_{2,3}$, this is impossible. Consequently, each component of $G-C$ contains a cycle and $C$ is cyclic.

Lemma 6 Every cyclically 3-edge-connected graph $G \in \mathcal{G}_{2,3}$ is bridgeless. Furthermore, $G$ contains no pair of parallel edges unless $G$ is a 2 -cycle or a subdivision of $\Theta$.

Proof. Suppose that $e$ is a bridge in $G$ and $K$ is a component of $G-e$. Since $G \in \mathcal{G}_{2,3}, K$ has at least two vertices. If $K$ contains no cycle, then $K$ is a tree and it has a leaf not incident with $e$. This contradicts the assumption that $G \in \mathcal{G}_{2,3}$. Thus, $\{e\}$ is a cyclic edge-cut, a contradiction.

Suppose that $x, y$ are two vertices in $G$ joined by a pair of parallel edges and that $G$ is neither a 2 -cycle nor a subdivision of $\Theta$. Since $G$ is bridgeless, both $x$ and $y$ are of degree 3 . Let $C$ consist of the two edges incident with just one of $x, y$. If the component of $G-C$ not containing $x$ were acyclic, it would be a tree with exactly two leaves, i.e., a path or a single vertex, and $G$ would be a subdivision of $\Theta$. Hence, $C$ is a cyclic 2-edge-cut of $G$ contradicting the assumption that $G$ is cyclically 3 -edge-connected.

Observation 7 Every cyclically 3-edge-connected graph in $\mathcal{G}_{2,3}$ is a cycle or a subdivision of a 3-edge-connected cubic graph.

Proof. Suppose that $G \in \mathcal{G}_{2,3}$ is cyclically 3-edge-connected and different from a cycle. Let the cubic graph $G^{\prime}$ be obtained by suppressing each vertex of degree two. (Since $G$ is bridgeless by Lemma 6, this does not involve contracting a parallel edge.) If $C$ is a 2-edge-cut in $G^{\prime}$, then each component of $G^{\prime}-C$ contains a 3 -vertex or is a 2 -cycle. Lemma 5 implies that $C$ corresponds to a cyclic 2 -edge-cut in $G$ which is a contradiction.

Lemma 8 Let $G \in \mathcal{G}_{2,3}$. If each face of $G$ is incident with a 2-vertex, then $G \in \mathcal{S}_{2,3}$. Moreover, if $G$ is cyclically 3-edge-connected, then $G \in \mathcal{S}_{2,3}$ if and only if each face of $G$ is incident with a 2 -vertex.

Proof. In a graph in $\mathcal{S}_{2,3}$, any cycle that is not a facial cycle is separating. Thus, if $G \in \mathcal{G}_{2,3}$ and each face is incident with a 2-vertex, then $G \in \mathcal{S}_{2,3}$. The second assertion is trivially true if $G$ is a cycle. Suppose thus, using Observation 7, that $G$ is a subdivision of a 3-edge-connected cubic graph. It is well known that in a 3 -edge-connected plane graph, facial cycles are exactly
the non-separating cycles. Thus, if $G \in \mathcal{S}_{2,3}$, then every face is incident with a 2 -vertex.

Graphs $G \in \mathcal{S}_{2,3}$ with cyclic 2-edge-cuts may have faces which are not incident with 2 -vertices. We will use in the next section the following subset of $\mathcal{S}_{2,3}$.

Definition 9 Let $\mathcal{S}_{2,3}^{f}$ be the class of all connected plane graphs $G \in \mathcal{S}_{2,3}$ such that each face of $G$ is incident with a 2 -vertex.

The next lemma will be used in the proof of Theorem 11.
Lemma 10 Let $G \in \mathcal{G}_{2,3}$ be cyclically 3-edge-connected and let $u$ be a 2vertex of the outer face, with distinct neighbours $x$ and $y$ of degree 3 (see Figure 22). Let the other neighbours of $y$ be denoted by $a, b$ and the other neighbours of $x$ by $c, d$, such that the clockwise order of the neighbours of $y(x)$ is uba (udc, respectively). Then all of the following conditions hold, unless $G$ is a subdivision of $\Theta$ or of $K_{4}$ :
(1) $\{a, b, c, d\} \cap\{x, y\}=\emptyset$,
(2) $\{a, d\} \cap\{b, c\}=\emptyset$, and
(3) $b \neq c$ or $a \neq d$.

Proof. We prove (1). Consider the vertex $x$ and suppose that $x=a$. Then $c$ or $d$ is $y$ otherwise $x$ would have degree 4. Therefore $y=d$ since $y=c$ would imply that $x d$ is a bridge, contradicting Lemma 6. Then the set of edges $C=\{x c, y b\}$ is a 2-edge-cut. Lemma 5 implies that the component of $G-C$ not containing $x$ is a path. Hence, $G$ is a subdivision of $\Theta$ which is a contradiction. Thus, $x \neq a$. Essentially the same argument shows that $x \neq b$. Trivially, $c \neq x \neq d$, so $x \notin\{a, b, c, d\}$. By symmetry, we conclude that (1) holds.

To prove (2), note that $a \neq b$ by Lemma 6. If $a=c$, then $y b$ or $x d$ would be a bridge by a planarity argument, contradicting Lemma 6. Thus, $a \notin\{b, c\}$, and by symmetry, $d \notin\{b, c\}$.

Finally, we prove (3). Suppose that $b=c$ and $a=d$. If both $a$ and $b$ are 2-vertices, then $G$ is a subdivision of $\Theta$. Otherwise, they must both be 3 -vertices as $G$ would otherwise contain a bridge. If they are adjacent, then $G$ is a subdivision of $K_{4}$ contrary to the assumption. Thus, we may assume that there is a 2 -edge-cut $C$ such that one edge in $C$ is incident with $a$ and the other one with $b$, and none of these edges is incident with $x$ nor $y$. Since $G$ is cyclically 3 -edge-connected, the component of $G-C$ not containing $a$ is a path, so $G$ is a subdivision of $K_{4}$, which is a contradiction.


Figure 2: The situation in Lemma 10. The dotted line indicates part of the boundary of the outer face. A priori, some of the vertices $a, b, c, d$ may coincide and $b, c$ may be incident with the outer face.

## 3 Decomposition into a forest and a matching with prescribed edges

To find a decomposition of a connected graph into a spanning tree and a matching, it is clearly sufficient to decompose it into a forest and a matching. Thus, we define a 2-decomposition of a graph $G$ as a decomposition $E(G)=$ $E(F) \cup E(M)$ such that $F$ is a forest and $M$ is a matching (called the forest part and the matching part of the decomposition, respectively). If $B$ is a set of edges of $G$, then a $B$-2-decomposition (abbreviated $B-2 D$ ) of $G$ is a 2-decomposition whose forest part contains $B$. Obviously, if $B$ contains all edges of a cycle, then $G$ cannot have a $B-2 \mathrm{D}$. Note also that there are graphs in $\mathcal{S}_{2,3}$ without a $B$-2D where $B$ consists only of a few (2,3)-edges; for an example see Figure 3. Let us define $B(2,3)$ as the set of $(2,3)$-edges of $B$ and call a vertex sensitive if it is a 2 -vertex incident with an edge in $B(2,3)$.

The following theorem is the main statement needed to prove Theorem 2 . Examples in Figure 3 show some limitations to relaxing the conditions in Theorem 11 .


Figure 3: Two graphs $G \in \mathcal{S}_{2,3}^{f}$ and edge sets $B$ (bold) such that $G$ admits no $B-2 \mathrm{D}$. Left: example showing that condition (a) in Theorem 11 cannot be relaxed to allow $|B(2,3)|>1$. Right: example showing that condition (b3) cannot be dropped.

Theorem 11 Let $G \in \mathcal{S}_{2,3}^{f}$ be 2-edge-connected and not a cycle. Let $F_{0}$ be the outer face of $G$, and let $B$ be a set of edges contained in the boundary of $F_{0}$. Suppose that either
(a) $G$ is cyclically 3-edge-connected and $|B(2,3)| \leq 1$, or
(b) $G$ contains a cyclic 2-edge-cut and there are distinct vertices $v, w$ incident with $F_{0}$ such that $v$ is a 2 -vertex and all of the following hold:
(b1) $v, w$ are separated by every cyclic 2-edge-cut of $G$,
(b2) all edges in $B$ are contained in a vw-subpath of the boundary of $F_{0}$,
(b3) if $v$ is a sensitive vertex, then the inner face of $G$ incident with $v$ is incident with another 2-vertex, and
(b4) every sensitive vertex which is not $v$ is either $w$ or adjacent to $w$.
Then $G$ admits a $B-2 D$.
Note that if $G$ in Theorem 11 has a cyclic 2-edge-cut, then conditions (b2) and (b4) imply that $|B(2,3)| \leq 2$. Before we start with the proof, we explain how we use contraction in this section. Suppose we contract an edge $e=v w$ in a graph $H$ into the vertex $v$, then $w \notin V(H / e), v \in V(H / e)$ and each vertex of $H / e-v$ has the same vertex-label as the corresponding vertex in $H-v-w$. For the proof it will be essential that every edge of $H / e$ corresponds to an edge of $H-e$ and vice versa. We will use this edgecorrespondence between the graphs $H / e$ and $H$ for edges which are not $e$ and edge-sets that do not contain $e$, without referring to it. To avoid later confusion, note that an edge $v x \in E(H / e)$ can correspond to an edge in $H$ with other endvertices than in $H / e$, namely $w x$.

Proof. Suppose by contradiction that $G$ is a counterexample with $|V(G)|$ minimum. Moreover, let $B$ be a set of edges satisfying the assumptions of the theorem, such that $G$ has no $B-2 \mathrm{D}$ and $|B|$ is maximum.

We begin with a technical claim:
Claim 1 Let $r s$ be an edge of a graph $H \in \mathcal{G}_{2,3}$ where $d_{H}(r)=2$ and both neighbours of $r$ are distinct. Let $H^{\prime}$ be obtained from $H$ by contracting $r s$ into $r$ and let $B^{\prime} \subseteq E\left(H^{\prime}\right)$. If $H^{\prime}$ has a $B^{\prime}-2 D$, then $H$ admits a $\left(B^{\prime}+r s\right)-2 D$.

Let $\left(F^{\prime}, M^{\prime}\right)$ be a $B^{\prime}-2 \mathrm{D}$ of $H^{\prime}$. Let $F=F^{\prime}+r s$ and let $z$ denote the neighbour of $r$ in $H$ distinct from $s$. Then $r z \in F^{\prime}$ or $r z \notin F^{\prime}$. In each case, $F$ is a forest of $H$; in fact, $F$ is the forest part of a $\left(B^{\prime}+r s\right)-2 \mathrm{D}$ of $H$. The matching part of the desired 2 D is $E(H)-E(F)$.

We distinguish two main cases.

## Case I: $G$ satisfies condition (a) in the theorem.

We start with the following claim:
Claim $2 G$ contains no (2, 2)-edge.
For contradiction, suppose that $f$ is such an edge; contracting $f$, we obtain a 2-edge-connected graph in $\mathcal{S}_{2,3}^{f}$ satisfying condition (a) of the theorem. By the minimality of $G$, the resulting graph admits a $(B-f)-2 \mathrm{D}$. Then Claim 1 implies a $B-2 \mathrm{D}$ of $G$, a contradiction.

Using Claim 2 it is straightforward to verify that when $G$ is a subdivision of $\Theta$ or of $K_{4}$, then $G$ has a $B-2 \mathrm{D}$. Thus, we may assume that $G$ is not a subdivision of either of these graphs.

Note that we often refer to edges of $G$ only by their endvertices (for example, $x c$ ). This is sufficient, since by Lemma 6, $G$ contains no parallel edges.

Since $G \in \mathcal{S}_{2,3}^{f}$, the outer face is incident with a 2 -vertex, which is by Claim 2 incident with a $(2,3)$-edge. If $B(2,3)=\emptyset$, then we can add any (2,3)-edge into $B(2,3)$, preserving condition (a) in Theorem 11. Then by the maximality of $B$, we obtain a $B-2 \mathrm{D}$, a contradiction. Therefore, we may assume that $|B(2,3)|=1$.

Let $e=u x$ denote the unique edge in $B(2,3)$, with $u \in V_{2}(G)$, and let the neighbour of $u$ other than $x$ be denoted by $y$, see Figure 2, Note that $x, y \in V_{3}(G)$. Label the neighbours of $x, y$ distinct from $u$ by $a, b, c, d$ as in Lemma 10. Since $G$ is neither a subdivision of $\Theta$ nor of $K_{4}$, we may assume by Lemma 10 that the vertices $a, b, c, d, x, y, u$ are all distinct, except that possibly $a=d$ or $b=c$ (but not both).

Let $G^{\prime}$ be the graph obtained from $G$ by removing $u$ and contracting the edge $y b$ into $y$.

Claim $3 G^{\prime}$ is not cyclically 3-edge-connected.
For the sake of a contradiction, suppose that $G^{\prime}$ is cyclically 3-edge-connected. Let $B^{\prime} \subseteq E\left(G^{\prime}\right)$ with $B^{\prime}=B-u x+y a$. Assume first that $\left|B^{\prime}(2,3)\right| \leq 1$. Using the fact that $G \in \mathcal{S}_{2,3}^{f}$ and since $x$ is a 2-vertex of the outer face of $G^{\prime}$, it is not difficult to verify that $G^{\prime} \in \mathcal{S}_{2,3}^{f}$. It follows that $\left(G^{\prime}, B^{\prime}\right)$ satisfies the conditions of the theorem, so $G^{\prime}$ admits a $B^{\prime}-2 \mathrm{D}$ by the minimality of $G$. Adding the edges $y b$ and $u x$ to its forest part and the edge $u y$ to its matching part, we obtain a $B-2 \mathrm{D}$ of $G$, a contradiction.

Thus, $\left|B^{\prime}(2,3)\right| \geq 2$. Since $B(2,3)=\{u x\}, B^{\prime}(2,3)=\{y a, x d\}$ implying $x d \in B$, and since $|B(2,3)|=1$, we have $d_{G}(d)=d_{G^{\prime}}(d)=3$. Furthermore, since $y a \in B^{\prime}(2,3)$ either $d_{G}(a)=2$ and $d_{G}(b)=3$ or vice versa.

We distinguish two cases according to $d_{G}(c)$. If $d_{G}(c)=3$, we let $G^{\prime \prime}$ be the graph obtained from $G^{\prime}$ by contracting $x c$ into $x$, and let $B^{\prime \prime}=B^{\prime}$. Then $\left|B^{\prime \prime}(2,3)\right|=1, G^{\prime \prime} \in \mathcal{S}_{2,3}^{f}$ and $G^{\prime \prime}$ is cyclically 3-edge-connected. By the minimality of $G, G^{\prime \prime}$ admits a $B^{\prime \prime}-2 \mathrm{D}$. To obtain a $B-2 \mathrm{D}$ of $G$, it suffices to add $u x, x c$ and $y b$ to the forest part, and $u y$ to the matching part, respectively, of the $B^{\prime \prime}-2 \mathrm{D}$. This contradicts the choice of $G$.

It remains to discuss the case $d_{G}(c)=2$. In this case, we let $G^{\prime \prime}=G^{\prime}$ and $B^{\prime \prime}=B^{\prime}-x d$ implying $\left|B^{\prime \prime}(2,3)\right|=1$. By the minimality of $G$, there is a $B^{\prime \prime}-2 \mathrm{D}$ of $G^{\prime \prime}$, say $\left(F^{\prime \prime}, M^{\prime \prime}\right)$, where $F^{\prime \prime}$ is a forest and $M^{\prime \prime}$ is a matching. Consider the 2-decomposition $(F, M)$ of $G$, where $F=F^{\prime \prime}+y b+u x$ and $M=M^{\prime \prime}+u y$. We must have $x d \notin F$, for otherwise this would be a $B-2 \mathrm{D}$. In fact, $F+x d$ must contain a cycle $Z$. Since $u y \notin F$ and since $d_{G}(c)=2$, $Z$ contains both edges incident with $c$. It follows that $F+x d-x c$ is acyclic and that $(F+x d-x c, M+x c-x d)$ is a $B-2 \mathrm{D}$ of $G$, a contradiction.

Let $B^{\prime} \subseteq E\left(G^{\prime}\right)$ and let $B^{\prime}=B-u x+y a$. We will show that $\left(G^{\prime}, B^{\prime}\right)$ satisfies condition (b). Then, by the minimality of $G, G^{\prime}$ will have a $B^{\prime}-2 \mathrm{D}$ implying a $B-2 \mathrm{D}$ of $G$, which will finish Case I. Firstly, $G^{\prime}$ contains a cyclic 2-edge-cut by Claim 3. Comparing faces of $G^{\prime}$ to those of $G$, we conclude that every face of $G^{\prime}$ is incident with a 2 -vertex. Thus, $G^{\prime} \in \mathcal{S}_{2,3}^{f}$. Let $v=x$ and $w=y$. We check conditions (b1)-(b4), starting with (b1). Any cyclic 2-edge-cut of $G^{\prime}$ not separating $x$ from $y$ would be a cyclic 2-edge-cut in $G$, contrary to the assumption that $G$ is cyclically 3 -edge-connected. Condition (b2) follows from the fact all edges of $B$ are edges of the boundary of the outer face of $G$, and all of this boundary (except for the edges $u x$ and $u y$ ) is covered by an $x y$-path in the boundary of the outer face of $G^{\prime}$. As for condition (b3), $x$ is indeed a 2-vertex of $G^{\prime}$, and since $d_{G}(x)=3$ and $G \in \mathcal{S}_{2,3}^{f}$, the inner face of $G^{\prime}$ incident with $x$ is also incident with some other 2-vertex. Finally, we consider condition (b4). Since for every vertex $z$ of $G^{\prime}, d_{G^{\prime}}(z)=d_{G}(z)$ except if $z \in\{x, y\}$, and since $B(2,3)=\{u x\}$ and all edges in $B$ are contained in the boundary of the outer face of $G$, we have $B^{\prime}(2,3) \subseteq\{x d, y a\}$. Then condition (b4) follows.

Hence, $G^{\prime}$ satisfies condition (b) and thus admits a $B^{\prime}-2 \mathrm{D}$, say $\left(F^{\prime}, M^{\prime}\right)$. Then $\left(F^{\prime}+u x+y b, M^{\prime}+u y\right)$ is a $B-2 \mathrm{D}$ of $G$, a contradiction to the choice of $G$ which finishes the discussion of Case I.

Case II: $G$ satisfies condition (b) in the theorem.
Let $C=\left\{e_{1}, e_{2}\right\}$ be a cyclic 2-edge-cut of $G$ such that the component $K_{1}$ of $G-C$ containing $v$ is inclusionwise minimal, i.e., there is no other cyclic

2-edge-cut $C^{\prime}$ such that the component of $G-C^{\prime}$ containing $v$ is contained in $K_{1}$. We refer to this property of $C$ as the minimality.

Let $K_{2}$ be the other component of $G-C$; note that $w \in V\left(K_{2}\right)$. For $i=1,2$, let $G_{i}$ denote the graph obtained from $G$ by contracting all edges of $K_{3-i}$. The vertex of $G_{i}$ incident with $e_{1}$ and $e_{2}$ is denoted by $u_{i}$. Thus, $G_{1}$ contains $v$ and $u_{1}$, while $G_{2}$ contains $w$ and $u_{2}$.

By property (b2), $B$ is contained in a $v w$-path in the boundary of the outer face of $G$; since $C$ separates $v$ from $w$, we may henceforth assume that $e_{1} \notin B$. For $i=1,2$, let $B_{i}=B \cap E\left(G_{i}\right)$. Let $G_{1}^{*}$ be the graph obtained from $G_{1}$ by contracting $e_{1}$.

The following claim will sometimes be used without explicit reference:
Claim 4 The following hold:
(i) the graphs $G_{1}, G_{2}$ and $G_{1}^{*}$ are 2-edge-connected,
(ii) the endvertices of $e_{1}$ and $e_{2}$ in $G_{1}$ other than $u_{1}$ have degree 3, and
(iii) the graphs $G_{1}, G_{1}^{*}$ are cyclically 3-edge-connected and $G_{1} \in \mathcal{S}_{2,3}^{f}$.

Part (i) follows from the fact that edge contraction preserves the property of being 2 -edge-connected. Part (ii) is a consequence of the minimality of $C$. Part (iii): suppose by contradiction that $G_{1}$ has a cyclic 2-edge-cut $C_{1}$. Then $C_{1}$ does not separate $v$ from $u_{1}$ by the minimality of $C$. Hence one component of $G_{1}-C_{1}$ contains $v$ and $u_{1}$. Thus, $C_{1}$ in $G$ does not separate $v$ from $w$, which contradicts (b1). Finally, $G_{1} \in \mathcal{S}_{2,3}^{f}$ follows from the fact that $G \in \mathcal{S}_{2,3}^{f}$.

Note that $G_{1}^{*} \notin \mathcal{S}_{2,3}^{f}$ if the inner facial cycle of $G_{1}$ containing $u_{1}$ has no other 2-vertex. Then by Lemma 8, even $G_{1}^{*} \notin \mathcal{S}_{2,3}$ holds.

Claim 5 The following hold:
(i) The graph $G_{1}$ admits a $\left(B_{1}+e_{2}\right)-2 D$.
(ii) If $G_{1}^{*} \in \mathcal{S}_{2,3}$, then $G_{1}$ admits a $\left(B_{1}+e_{1}+e_{2}\right)-2 D$.
(i) If $v$ is not sensitive, then the desired decomposition is easy to obtain by noting that the pair $\left(G_{1}, B_{1}+e_{2}\right)$ satisfies condition (a) in the theorem. Suppose thus that $v$ is sensitive, and let $v^{\prime}$ be the unique neighbour of $v$ in $G_{1}$ such that $v v^{\prime} \in B_{1}(2,3)$. Let $G_{1}^{\prime}$ be obtained from $G_{1}$ by contracting $v v^{\prime}$ into $v$. Then $G_{1}^{\prime} \in \mathcal{S}_{2,3}$ thanks to property (b3) of $G, G_{1}^{\prime}$ is cyclically 3 -edgeconnected and $B_{1}+e_{2}$ contains at most one (2,3)-edge, so condition (a) is
satisfied for $\left(G_{1}^{\prime}, B_{1}+e_{2}-v v^{\prime}\right)$. Consequently, there is a $\left(B_{1}+e_{2}-v v^{\prime}\right)-2 \mathrm{D}$ of $G_{1}^{\prime}$. By Claim 1, $G_{1}$ admits a $\left(B_{1}+e_{2}\right)$-2D.
(ii) Suppose that $G_{1}^{*} \in \mathcal{S}_{2,3}$ and consider the set of edges $B_{1}^{*}=B_{1}+e_{2}$ in $G_{1}^{*}$. (Note that $e_{2}$ is an edge of $G_{1}^{*}$ while $e_{1}$ has been contracted in its construction.) By Claim 4 (iii) and Lemma 8, $G_{1}^{*} \in \mathcal{S}_{2,3}^{f}$. By property (b4) and the fact that $e_{2}$ is a (3,3)-edge in $G_{1}^{*}$, any (2,3)-edge in $B_{1}^{*}$ is incident with $v$. By property (b2), there is at most one such edge. Thus, the pair $\left(G_{1}^{*}, B_{1}^{*}\right)$ satisfies condition (a), and consequently $G_{1}^{*}$ admits a $B_{1}^{*}-2 \mathrm{D}$ by the minimality of $G$. By Claim 1, $G_{1}$ admits a $\left(B_{1}+e_{1}+e_{2}\right)-2 \mathrm{D}$.

Claim 6 The following hold:
(i) The graph $G_{2}$ admits a $\left(B_{2}-e_{2}\right)-2 D$.
(ii) If $G_{1}^{*} \notin \mathcal{S}_{2,3}$, then $G_{2}$ admits a $\left(B_{2}+e_{2}\right)-2 D$.
(i) Suppose first that $G_{2}$ contains at least one cyclic 2-edge-cut. Since $G_{2}$ arises by contracting all edges of $K_{1}$ 'into' the vertex $u_{2}$, it is straightforward to check that the pair $\left(G_{2}, B_{2}-e_{2}\right)$ satisfies condition (b) in the theorem with $u_{2}$ playing the role of $v$. (In relation to property (b3), note that $u_{2}$ is not sensitive with respect to $B_{2}-e_{2}$.) Thus, a ( $B_{2}-e_{2}$ )-2D of $G_{2}$ exists by the minimality of $G$.

If $G_{2}$ is cyclically 3-edge-connected, then by properties (b2) and (b4) of $(G, B), B_{2}-e_{2}$ contains at most one (2,3)-edge (incident with $w$ if such an edge exists). Therefore, $\left(G_{2}, B_{2}-e_{2}\right)$ satisfies condition (a) in the theorem. The minimality of $G$ implies that $G_{2}$ has a $\left(B_{2}-e_{2}\right)-2 \mathrm{D}$.
(ii) Let us consider possible reasons why $G_{1}^{*} \notin \mathcal{S}_{2,3}$. Since $\mathcal{S}_{2,3}^{f} \subseteq \mathcal{S}_{2,3}$, there is a face of $G_{1}^{*}$ not incident with a 2 -vertex. Since $G_{1} \in \mathcal{S}_{2,3}^{f}$ (Claim 4 (iii)) and since the 2-vertex $v$ is contained in the outer face of $G_{1}^{*}$, there is only one such face, namely the inner face whose boundary contains $e_{2}$. Let $Q$ be the inner face of $G$ whose boundary contains the edge-cut $C$. Since $G \in \mathcal{S}_{2,3}^{f}, Q$ is incident with a 2 -vertex $z$. Since $G_{1}^{*} \notin \mathcal{S}_{2,3}^{f}, z$ and $u_{2}$ are both incident with the same inner face in $G_{2}$.

Suppose first that $G_{2}$ contains a cyclic 2-edge-cut. The existence of the vertex $z$ proves property (b3) for the pair $\left(G_{2}, B_{2}+e_{2}\right)$ with $u_{2}$ playing the role of $v$ (note that $u_{2}$ is sensitive). The other parts of condition (b) are straightforward to check. By the minimality of $G$, the desired $\left(B_{2}+e_{2}\right)-2 \mathrm{D}$ of $G_{2}$ exists.

It remains to consider that $G_{2}$ is cyclically 3 -edge-connected. If $e_{2}$ is the unique ( 2,3 )-edge in $B_{2}+e_{2}$, then ( $G_{2}, B_{2}+e_{2}$ ) satisfies condition (a) in the theorem, and hence the minimality of $G$ implies that $G_{2}$ admits a $\left(B_{2}+e_{2}\right)$ 2 D . Therefore, we may assume that there is another (2,3)-edge in $B_{2}+e_{2}$,
and in particular, there is a sensitive vertex $z^{\prime}$ incident with the outer face of $G_{2}$ with $z^{\prime} \neq u_{2}$. By condition (b4), $z^{\prime}$ has to be either $w$ or a vertex adjacent to $w$. Let $G_{2}^{*}$ be obtained from $G_{2}$ by contracting $e_{2}$ into $u_{2}$. Since $G \in \mathcal{S}_{2,3}^{f}$ and since $z$ and $z^{\prime}$ are 2-vertices, $G_{2}^{*} \in \mathcal{S}_{2,3}^{f}$. Hence the pair $\left(G_{2}^{*}, B_{2}-e_{2}\right)$ satisfies condition (a) of the theorem. By the minimality of $G$, there is a $\left(B_{2}-e_{2}\right)-2 \mathrm{D}$ of $G_{2}^{*}$. Claim 1 implies a $\left(B_{2}+e_{2}\right)-2 \mathrm{D}$ of $G_{2}$.

By the above claims we obtain the sought contradiction. Suppose first that $G_{1}^{*} \in \mathcal{S}_{2,3}$. By Claims 5 (ii) and $6(\mathrm{i})$, there is a $\left(B_{1}+e_{1}+e_{2}\right)-2 \mathrm{D}\left(F_{1}, M_{1}\right)$ of $G_{1}$ and a $\left(B_{2}-e_{2}\right)-2 \mathrm{D}\left(F_{2}, M_{2}\right)$ of $G_{2}$. Since $e_{1}, e_{2} \in E\left(F_{1}\right), F_{1} \cup F_{2}$ is acyclic, regardless of whether $e_{1}, e_{2} \in E\left(F_{2}\right)$. Clearly, $M_{1} \cup\left(M_{2}-e_{1}-e_{2}\right)$ is a matching in $G$, so we obtain a $B-2 \mathrm{D}$ of $G$, contradicting the choice of $G$.

Thus, $G_{1}^{*} \notin \mathcal{S}_{2,3}$. By Claims $5(\mathrm{i})$ and 6 (ii), there exists a $\left(B_{1}+e_{2}\right)-2 \mathrm{D}$ $\left(F_{1}^{\prime}, M_{1}^{\prime}\right)$ of $G_{1}$ and a $\left(B_{2}+e_{2}\right)-2 \mathrm{D}\left(F_{2}^{\prime}, M_{2}^{\prime}\right)$ of $G_{2}$. Since $e_{2}$ is contained in both $F_{1}^{\prime}$ and $F_{2}^{\prime}$, the 2-decompositions combined produce a $B-2 \mathrm{D}\left(F_{1}^{\prime} \cup F_{2}^{\prime}, M_{1}^{\prime} \cup M_{2}^{\prime}\right)$ if $e_{1} \notin E\left(F_{1} \cup F_{2}\right)$, or $\left(F_{1}^{\prime} \cup F_{2}^{\prime}, M_{1}^{\prime} \cup M_{2}^{\prime}-e_{1}\right)$ if $e_{1} \in E\left(F_{1} \cup F_{2}\right)$, a contradiction.

Corollary 12 If $G \in \mathcal{S}_{2,3}$ is 2-edge-connected and $e \in E(G)$ is a (2,3)-edge, then $G$ admits an $\{e\}-2 D$.

Proof. We proceed by induction on the order of $G$. By choosing a suitable embedding of $G$, we may assume that $e$ is contained in the boundary of the outer face. If $G$ is cyclically 3 -edge-connected, then $G \in \mathcal{S}_{2,3}^{f}$ by Lemma 8 , and the existence of a 2-decomposition follows from Theorem 11 (with $B=\{e\}$ ). Hence, we assume that $G$ contains a cyclic 2 -edge-cut $C=\left\{e_{1}, e_{2}\right\}$. Let $K_{1}$ and $K_{2}$ be the components of $G-C$. Just as in Case II of the proof of Theorem 11, we contract all edges in $K_{1}$ or $K_{2}$ to obtain the smaller graphs $G_{1}$ and $G_{2}$ with new vertices $u_{1}$ and $u_{2}$, respectively. Note that $G_{i} \in \mathcal{S}_{2,3}$, $i=1,2$. We may assume that $e$ is contained in $G_{1}$.

By induction, there is an $\{e\}-2 \mathrm{D}\left(F_{1}, M_{1}\right)$ of $G_{1}$. First, suppose that $e \notin\left\{e_{1}, e_{2}\right\}$. Since $M_{1}$ is a matching, we may assume that $e_{1} \in E\left(F_{1}\right)$. Again by induction, there is an $\left\{e_{1}\right\}-2 \mathrm{D}\left(F_{2}, M_{2}\right)$ of $G_{2}$. Since each of $F_{1}$ and $F_{2}$ contains $e_{1}, G$ has an $\{e\}$-2D $\left(F_{1} \cup F_{2}, M_{1} \cup M_{2}\right)$ if $e_{2} \notin E\left(F_{1} \cup F_{2}\right)$ and an $\{e\}-2 \mathrm{D}\left(F_{1} \cup F_{2}, M_{1} \cup M_{2}-e_{2}\right)$ if $e_{2} \in E\left(F_{1} \cup F_{2}\right)$.

In the remaining case that $e \in\left\{e_{1}, e_{2}\right\}$, we assume without loss of generality that $e=e_{1}$ and proceed as above.

Recall that a 2-decomposition of a connected graph implies a decomposition into a spanning tree and a matching.

Theorem 2 now follows by induction: since Theorem 2 holds for cycles and Corollary 12 implies the 2-edge-connected case, it remains to show that
every graph $G$ satisfying the conditions of the theorem, with a bridge $e$, has a 2 -decomposition. By combining 2-decompositions of the components of $G-e$ (found by induction), we obtain an $\{e\}-2 \mathrm{D}$ of $G$, which completes the proof of Theorem 2.

Corollary 13 Every connected subcubic plane graph can be decomposed into a spanning tree, a 2-regular subgraph and a matching.

Proof. Let $G$ be a connected subcubic plane graph and let $\left\{C_{1}, \ldots, C_{k}\right\}$ be a maximal collection of disjoint cycles such that $G^{\prime}:=G-\bigcup_{i=1}^{k} E\left(C_{i}\right)$ is connected. Thus, $G^{\prime}$ is a connected subcubic plane graph in which every cycle is separating, so $G^{\prime}$ is decomposed into a spanning tree and a matching by Theorem 2. Adding the union of $C_{1}, \ldots, C_{k}$, we obtain the desired decomposition of $G$.

Finally, for the sake of completeness, we prove the following statement.
Proposition 14 The 3DC and the 2DC are equivalent conjectures.
Proof. The proof of Corollary 13, which applies for an arbitrary (not necessarily plane) connected subcubic graph, effectively shows that the 2DC implies the 3DC. Therefore, it suffices to prove the converse direction.

Let $H$ be a connected graph such that every cycle of $H$ is separating and each vertex of $H$ has degree 2 or 3 . Let $X$ denote the graph resulting from the graph $\Theta$ by subdividing one edge of $\Theta$ precisely once, i.e. $\left|V_{2}(X)\right|=1$. We construct from $H$ a cubic graph $G$ by adding $\left|V_{2}(H)\right|$ many copies of $X$ to $H$ and by connecting each 2-vertex of $H$ by an edge with a 2 -vertex of a copy of $X$. By the 3DC, there is a 3-decomposition of $G$. The edges connecting $H$ to copies of $X$ are obviously bridges of $G$ and are thus contained in the tree part, say $T$, of the 3 -decomposition. Since every cycle of $H$ is separating, every cycle of $G$ which is not separating is contained in some copy of $X$. Hence, we obtain a 2-decomposition of $H$ in which $T \cap H$ is the tree part and the matching part consists of the remaining edges of $H$.

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