Circuit covers of cubic signed graphs

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Abstract

A signed graph is a graph G associated with a mapping $\sigma : E(G) \to \{-1, +1\}$, denoted by (G, σ). A *cycle* of (G, σ) is a connected 2-regular subgraph. A cycle C is *positive* if it has an even number of negative edges, and negative otherwise. A *circuit* of of a signed graph (G, σ) is a positive cycle or a barbell consisting of two edge-disjoint negative cycles joined by a path. The definition of a circuit of signed graph comes from the signed-graphic matroid. A circuit cover of (G, σ) is a family of circuits covering all edges of (G, σ). A circuit cover with the smallest total length is called a shortest circuit cover of (G, σ) and its length is denoted by $\mathrm{scc}(G, \sigma)$. Bouchet proved that a signed graph with a circuit cover if and only if it is flow-admissible (i.e., has a nowhere-zero integer flow). Máčajová et. al. show that a 2-edge-connected signed graph (G, σ) has $\mathrm{scc}(G, \sigma) \leq 9|E(G)|$ if it is flow-admissible. This bound was improved recently by Cheng et. al. to $\mathrm{scc}(G, \sigma) \leq 11|E(G)|/3$ for 2-edge-connected signed graphs with even negativeness, and particularly, $\mathrm{scc}(G, \sigma) \leq 3|E(G)| + \epsilon(G, \sigma)/3$ for 2-edge-connected cubic signed graphs with even negativeness (where $\epsilon(G, \sigma)$ is the negativeness of (G, σ)). In this paper, we show that every 2-edge-connected cubic signed graph has $\mathrm{scc}(G, \sigma) \leq 26|E(G)|/9$ if it is flow-admissible, and $\mathrm{scc}(G, \sigma) \leq 23|E(G)|/9$ if it has even negativeness.

Keywords: Circuit Cover, Signed Graphs

1 Introduction

Let G be a graph. A cycle of G is a connected 2-regular subgraph. A graph G is *2-edge-connected* if G is connected and does not contain a *cutedge*, whose deletion disconnects the graph G. A *singed graph* (G, σ) is a graph associated with a mapping $\sigma : E(G) \to \{-1, +1\}$, which is called a *signature* of (G, σ). An edge e is positive if $\sigma(e) = 1$ and negative if $\sigma(e) = -1$. A graph is a special signed graph with only positive edges. Signed graphs are well-studied combinatorial structures due to their applications in combinatorics, geometry and matroid theory (cf. [23]).

A cycle C of a signed graph (G, σ) is *positive* if it contains an even number of negative edges, and *negative* otherwise. A *barbell* of a signed graph is a pair of edge-disjoint negative cycles joined by a path, which could have length zero. A *circuit* of a signed graph is a positive cycle or a barbell. The definition

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of circuit of signed graphs comes from the signed-graphic matroid (cf. [23]). For a graph G, a circuit of G is a cycle. A *circuit cover* $\mathscr{C}$ of a signed graph (G,σ) is a family of circuits which covers all edges of G. The length of a circuit cover $\mathscr{C}$ is defined as $\ell(\mathscr{C}) = \sum_{C\in\mathscr{C}} |E(C)|$. A *shortest circuit cover* $\mathscr{C}$ of (G,σ) is a circuit cover with the smallest length, i.e. $\ell(\mathscr{C})$ is minimum, over all circuit covers of (G,σ). The length of a shortest circuit cover of (G,σ) is denoted by $\mathrm{scc}(G,\sigma)$.

The shortest circuit cover problem has been well-studied for graphs (cf. [24]) and matroids (cf. [10, 21]). Thomassen [22] showed that for a given graph G, it is NP-complete to determine $\mathrm{scc}(G)$, which settled a problem proposed by Itai et. al. [18]. Bermond, Jackson and Jaeger [2], independently Alon and Tarsi [1] obtained the following result, which was further generalized by Fan [8] to 2-edge-connected graph with positive weights on edges.

Theorem 1.1 (Bermond, Jackson and Jaeger [2], Alon and Tarsi [1])**.** *Let G be a 2-edge-connected graph. Then* $\mathrm{scc}(G) \le 5|E(G)|/3$.

The bound in the above theorem was further improved to $44|E(G)|/27$ by Fan [7] for 2-edge-connected cubic graphs. For cubic graphs G with a nowhere-zero 5-flow, Jamshy, Raspaud and Tarsi [14] show that $\mathrm{scc}(G) \le 8|E(G)|/5$. With additional information on cycle or 2-factor structures, some upper bounds on shortest circuit cover of cubic graphs are obtained in [4, 12, 15]. In general, Alon and Tarsi made the following conjecture – the Shortest Circuit Cover Conjecture.

Conjecture 1.2 (Alon and Tarsi [1])**.** *Every 2-edge-connected cubic graph has a shortest circuit cover with length at most* $7|E(G)|/5$.

Jamshy and Tarsi [13] proved that Conjecture 1.2 implies the well-known Circuit Double Cover Conjecture, proposed independently by Szekeres [19] and Seymour [21].

Conjecture 1.3 (Szekeres [19] and Seymour [21])**.** *Every 2-edge-connected graph has a family of circuits which covers every edge twice.*

By the splitting lemma of Fleischner (Lemma III.26 in [9]), it suffices to show that Conjecture 1.3 holds for all 2-edge-connected cubic graphs. For 2-edge-connected cubic graphs, Conjecture 1.3 is equivalent to another long-standing problem, the Strong Embedding Conjecture due to Haggard [11], which says that every 2-connected graph has an embedding in a closed surface such that every face is an open disc and is bounded by a cycle, so-called a strong embedding. Based on the coloring-flow duality, the dual of a digraph embedded in an orientable surface is a graph (or balanced signed graph) but the dual of a digraph embedded in a non-orientable surface is a signed graph (cf. [6]). By the duality, the dual of a digraph strongly embedded in a non-orientable surface is a signed graph with an even number of negative edges. It is interesting to ask: for a given 2-connected signed graph (G,σ) with an even number of negative edges, is (G,σ) a dual of some digraph strongly embedded in a non-orientable surface? If so, then (G,σ) has a circuit double cover because every face boundary of (G,σ) is a positive cycle. A weaker question is whether a 2-connected signed graph with an even number of negative edges has a circuit double cover or not? The answer to this question is negative, even for 3-connected cubic signed graph. The signed graph in Figure 1 has no circuit double cover.

It is natural to consider the shortest circuit cover problem for signed graphs. Let (G,σ) be a signed graph with a circuit cover. Does (G,σ) have a shortest circuit cover with length less than $2|E(G)|$, which follows directly if (G,σ) has a circuit double cover? However, the above examples show that some signed

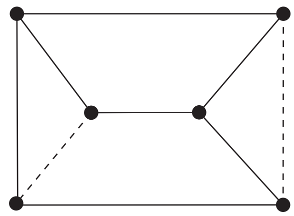

Figure 1: A 3-connected signed graph without a circuit double cover
(solid edges are positive and dashed edges are negative).

graphs do not have a circuit double cover. The shortest circuit cover problem for signed graph have been studies by Máčajová et. al. [17] and Cheng et. al. [5]. Before presenting their results, we need some terminologies.

The 2-edge-connectivity condition is sufficient for a graph to have a circuit cover, but it does not guarantee the existence of a circuit cover for a signed graph. A signed graph (G, σ) is *flow-admissible* if (G, σ) has a nowhere-zero flow. If (G, σ) has a circuit cover, then every edge of (G, σ) is contained by a circuit (a positive cycle or a barbell). Note that a positive cycle has a nowhere-zero 2-flow, but a barbell has a nowhere-zero 3-flow (cf. [3]). So a signed graph with a circuit cover is flow-admissible. Bouchet [3] proved that a signed graph (G, σ) has a circuit cover property if and only if it is flow-admissible. Máčajová et. al. [17] obtained the following result.

Theorem 1.4 (Máčajová, Raspaud, Rollová and Škoviera, [17]). *Let (G, σ) be a 2-edge-connected signed graph. If (G, σ) is flow-admissible, then* $\mathrm{scc}(G, \sigma) \leq 9|E(G)|$.

The above result was improved recently by Cheng et. al. [5] for 2-edge-connected signed graph (G, σ) with even negativeness (see Section 2 for definition of negtiveness) as follows.

Theorem 1.5 (Cheng, Lu, Luo and Zhang [5]). *Let (G, σ) be a 2-edge-connected signed graph with even negativeness. Then* $\mathrm{scc}(G, \sigma) \leq 11|E(G)|/3$.

For 2-connected cubic signed graphs (G, σ) with even negativeness, the above bound could be improved to $\mathrm{scc}(G, \sigma) \leq 3|E(G)| + \epsilon(G, \sigma)/3$ in terms of negativeness $\epsilon(G, \sigma)$ of (G, σ) (see [5]). In this paper, we consider the shortest circuit cover of cubic signed graphs and the following is our main result.

Theorem 1.6. *Let (G, σ) be a 2-edge-connected cubic signed graph. If (G, σ) is flow-admissible, then* $\mathrm{scc}(G, \sigma) \leq 26|E(G)|/9$.

For 2-connected cubic signed graphs (G, σ) with even negativeness, the bound in Theorem 1.6 can be improved to $\mathrm{scc}(G, \sigma) \leq 23|E(G)|/9$ as shown in Theorem 3.2.

2 Preliminaries

Let (G, σ) be a connected signed graph. If H is a subgraph of G, the *signed subgraph* of (G, σ) consisting of edges in H together with their signatures is denoted by (H, σ). An edge-cut S of (G, σ) is a minimal set of edges whose removal disconnects the signed graph. A *switch operation* ζ on S is a mapping $\zeta : E(G) \to \{-1, 1\}$ such that $\zeta(e) = -1$ if $e \in S$ and $\zeta(e) = 1$ otherwise. Two signatures σ and σ' are *equivalent* if there exists an edge cut S such that $\sigma(e) = \zeta(e) \cdot \sigma'(e)$ where ζ is the switch operation on S. For any edge-cut S, a cycle D of (G, σ) contains an even number of edges from S. So a circuit C of (G, σ) is also a circuit of (G, σ') for any equivalent signature σ' of σ. Therefore, we immediately have the following observation.

Observation 2.1. *Let (G, σ) be a flow-admissible signed graph and σ' be an equivalent signature of σ. Then* $\mathrm{scc}(G, \sigma) = \mathrm{scc}(G, \sigma')$.

The *negativeness* of a signed graph (G, σ) is the smallest number of negative edges over all equivalent signatures of σ, denoted by $\epsilon(G, \sigma)$. Máčajová and Škoviera [16] proved that a 2-edge-connected signed graph is flow-admissible if and only if $\epsilon(G, \sigma) \neq 1$. Combining it with Boucet's result [3] that a signed graph with a circuit cover if and only if it is flow-admissible, the following observation holds.

Observation 2.2. *Let (G, σ) be a 2-edge-connected signed graph. Then (G, σ) has a circuit cover if and only if $\epsilon(G, \sigma) \neq 1$.*

If (G, σ) has the smallest number of negative edges, an edge cut S has at most half number of negative edges. Otherwise, apply the switch operation on S and the number of negative edges of (G, σ) is reduced, contradicting that (G, σ) has the smallest number of negative edges.

Observation 2.3. *If a signed graph (G, σ) with $\epsilon(G, \sigma)$ negative edges, then every edge cut S contains at most $|S|/2$ negative edges.*

A connected graph H is called a *cycle-tree* if it has no vertices of degree-1 and all cycles of H are edge-disjoint. If H is a cycle-tree, then the graph obtained from H by contracting all edges in cycles is a tree. In other words, a cycle-tree can be obtained from a tree by blowing up all leaf vertices and some non-leaf vertices to edge disjoint cycles. A vertex v of H is a *cutvertex* if $H \backslash \{v\}$ has more components than H. A cutvertex v is said to *separate* a graph H into H_1 and H_2 if $H_1 \cup H_2 = H$ and $H_1 \cap H_2 = v$. Note that both H_1 and H_2 are connected since H is connected. A cycle D of H is a *leaf-cycle* if H has a vertex v separating D and $H \backslash (V(D) \backslash \{v\})$.

A *signed cycle-tree* (H, σ) is a signed graph such that H is a cycle-tree and **every cycle of** (H, σ) **contains at least one negative edge**. Let $\mathcal{F}$ be a family of circuits of (H, σ). A cycle D of (H, σ) is covered *t times* ($t \geq 1$) by $\mathcal{F}$ if every edge of D is covered by $\mathcal{F}$ and $\sum_{D' \in \mathcal{F}} |E(D) \cap E(D')| = t|E(D)|$, where t is a rational number.

Lemma 2.4. *Let (H, σ) be a signed cycle-tree with an even number of negative cycles. Then (H, σ) has a family of circuits $\mathcal{F}$ which covers all leaf-cycles once and all other cycles at most 3/2-times.*

Proof. Let (H, σ) be a counterexample with the smallest number of edges. First, we have the following claim:

Claim: *(H, σ) does not contain a cutvertex v which separates H into two subgraphs H_1 and H_2 such that both (H_1, σ) and (H_2, σ) contain an even number of negative cycles.*

Proof of Claim: suppose to the contrary that (H, σ) does have a such vertex v. Since both H_1 and H_2 are connected, both of them contains a cycle-tree with an even number of negative cycles. We may assume that H_i is a cycle-tree. (If H_i is not a cycle-tree, its maximum connected subgraph H_i' without vertices of degree 1 is a cycle-tree. Then use H_i' instead.) Furthermore, a cycle of H is contained in either H_1 or H_2.

Since (H, σ) is a counterexample to the lemma with minimum number of edges and $|E(H_i)| < |E(H)|$, both (H_1, σ) and (H_2, σ) have a family of circuits covering leaf-cycle exactly once and other cycles at most 3/2-times. Denote the two families of circuits by $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively. As H_1 and H_2 are separated by v, it follows that (H_1, σ) and (H_2, σ) have no cycle in common. Because a leaf-cycle of (H, σ) is either a leaf-cycle of (H_1, σ) or (H_2, σ), it follows that $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ is a family of circuits of (H, σ) which cover leaf-cycles once and other cycles at most 3/2-times, contradicting that (H, σ) is a counterexample. This completes the proof of Claim.

In the following, we may assume first that (H, σ) contains a positive cycle C. Since H is a cycle-tree, every component of $H \backslash E(C)$ has exactly one vertex on C, which is a cutvertex. By Claim, every component of $H \backslash E(C)$ contains an odd number of negative cycles. So the totally number of components of $H \backslash E(C)$ is even because (H, σ) contains an even number of negative cycles. Denote these components by $P_1, Q_1, P_2, Q_2, \cdots, P_k, Q_k$, which appear in clockwise order along the cycle C.

Let S_i be the segments of C joining P_i and Q_i for $i = 1, ..., k$, and R_i be the segments of C joining Q_i and P_{i+1} for $i = 1, ..., k$ (subscripts modulo k). Then $C = \bigcup_{i=1}^{k} (S_i \cup R_i)$. Without loss of generality, assume that

$$\sum_{i=1}^{k} |E(S_i)| \leq |E(C)|/2 \leq \sum_{i=1}^{k} |E(R_i)|. \tag{1}$$

Note that each component $(P_i \cup S_i \cup Q_i, \sigma)$ $(i = 1, ..., k)$ of $H \backslash E(\cup_i^k R_i)$ is a signed cycle-tree with an even number of negative cycles. Because $|E(P_i \cup S_i \cup Q_i)| < |E(H)|$ and (H, σ) is a counterexample with the smallest number of edges, the signed cycle-tree $(P_i \cup S_i \cup Q_i, \sigma)$ has a desired family of circuits $\mathcal{F}_i$. Let

$$\mathcal{F} := (\bigcup_{i=1}^{k} \mathcal{F}_i) \cup \{C\}.$$

By (1), C is covered by $\mathcal{F}$ at most 3/2-times. Note that every leaf-cycle of (H, σ) is also a leaf-cycle of $(P_i \cup S_i \cup Q_i, \sigma)$ for some unique $i \in \{1, ..., k\}$. Therefore, $\mathcal{F}$ is a desired family of circuits of (H, σ), contradicting that (H, σ) is a counterexample. So (H, σ) does not contain a positive cycle.

If (H, σ) contains exactly two negative cycles, then (H, σ) itself is a barbell, denoted by B. Then $\{B\}$ is a desired family of circuits. Hence assume that (H, σ) has at least four negative cycles. Choose a negative cycle D of (H, σ) such that the number of components of $H \backslash E(D)$ is maximum over all cycles of H. By Claim, every component has an odd number of negative cycles. Therefore, $H \backslash E(D)$ has an odd number of components which is at least three by the choice of D. By a similar argument as in the case when D is positive, we can label these components by $Q_0, P_1, Q_1, P_2, Q_2, ..., P_k, Q_k$ $(k \geq 1)$ in clockwise order along D such that

$$\sum_{i=1}^{k} |E(S_i)| \leq |E(D)|/2, \tag{2}$$

where S_i is a segment of D joining P_i and Q_i for $i = 1, ..., k$.

Let $H_0 = Q_0 \cup D$ and $H_i = P_i \cup S_i \cup Q_i$ for $i = 1, ..., k$. Then each (H_i, σ) $(i = 0, ..., k)$ is a signed cycle-tree with an even number of negative cycles. Since $k \geq 1$, $|V(H_i)| < |V(H)|$ for all $i \in \{0, ..., k\}$. As (H, σ) is a counterexample with smallest number of edges, each (H_i, σ) is not a counterexample and therefore has a family of circuits $\mathcal{F}_i$ which covers all leaf-cycle of H_i once and other cycle at most 3/2-times. Let

$$\mathcal{F} = \bigcup_{i=0}^{k} \mathcal{F}_i.$$

Since D is a leaf-cycle of H_0, it is covered by $\mathcal{F}_0$ once. By (2), all $\mathcal{F}_1, ..., \mathcal{F}_k$ together cover at most half number edges of D. Therefore, D is covered by $\mathcal{F}$ at most 3/2-times. Since any other cycle is covered by only one of $\mathcal{F}_i$'s, it follows that $\mathcal{F}$ is a desired family of circuits of (H, σ), a contradiction to that (H, σ) is a counterexample. This completes the proof. $\square$

Theorem 2.5. *Let (H, σ) be a signed cycle-tree with an even number of negative cycles. Then (H, σ) has a family of circuits $\mathcal{F}$ covering all cycles with length*

$$\ell(\mathcal{F}) \leq \frac{4}{3}|E(H)|.$$

Proof. Use induction on the number of edges of (H, σ). If (H, σ) has no edges, then the theorem holds trivially by taking $\mathcal{F} = \emptyset$. So in the following, assume that the theorem holds for all signed cycle-trees with at most $|E(H)| - 1$ edges.

First, assume that (H, σ) contains a positive leaf-cycle C. Let $H' \subset H$ be a cycle-tree containing all cycles of H except C. Then (H', σ) has an even number of negative cycles. Since $|E(H')| < |E(H)|$, by inductive hypothesis, (H', σ) has a family of circuits $\mathcal{F}'$ covering all cycles of (H', σ) with length $\ell(\mathcal{F}') \leq 4|E(H')|/3$. Then $\mathcal{F} = \mathcal{F}' \cup \{C\}$ is a family of circuits covering all cycles of (H, σ) because a cycle of H is either a cycle of H' or C. The length of $\mathcal{F}$ is

$$\ell(\mathcal{F}) = \ell(\mathcal{F}') + |E(C)| \leq \frac{4}{3}|E(H')| + |E(C)| \leq \frac{4}{3}(|E(H')| + |E(C)|) \leq \frac{4}{3}|E(H)|.$$

So (H, σ) has a family of circuits $\mathcal{F}$ covering all cycles with length at most $4|E(H)|/3$.

In the following, assume that all leaf-cycles of (H, σ) are negative. Let $D_1, D_2, ..., D_k$ be all leaf-cycles. Let l be the total length of non-leaf cycles of H. Since H is an outerplanar graph, H has an embedding in the plane such that all vertices of H appear on the boundary of the infinite face. Let W be the closed walk bounding the infinite face. Then all vertices of a leaf-cycle D_i appears as a consecutive segment in W. Without loss of generality, assume that the leaf-cycles of H appears in W in the order $D_1, D_2, ..., D_k,$. Let $S_{i,i+1}$ be the segment of W joining D_i and D_{i+1} (subscribes modulo k) such that all internal vertices of $S_{i,i+1}$ do not belong to any leaf-cycle of (H, σ). Then $S_{i,i+1}$ is a path because H does not have vertices of degree 1. Let

$$B_i = D_i \cup D_{i+1} \cup S_{i,i+1} \text{ for } i = 1, 2, ..., k \text{ (subscribes modulo } k\text{)}.$$

Then B_i is a barbell for $i = 1, ..., k$. Let $\mathcal{F}_1 = \{B_1, B_2, ..., B_k\}$, which covers all edges in non-leaf cycles exactly once and all other edges twice. So $\ell(\mathcal{F}_1) = 2|E(H)| - l$.

By Lemma 2.4, (H, σ) has a family of circuits $\mathcal{F}_2$ covering all cycles with length $\ell(\mathcal{F}_2) \leq |E(H)| + l/2$. Let $\mathcal{F}$ be the family of circuits with the smaller length between $\mathcal{F}_1$ and $\mathcal{F}_2$. Then

$$\ell(F) \leq \frac{1}{3}\big(\ell(\mathcal{F}_1) + \ell(\mathcal{F}_2) + \ell(\mathcal{F}_2)\big) = \frac{1}{3}\big((2|E(H)| - l) + 2(|E(H)| + l/2)\big) = \frac{4}{3}|E(H)|.$$

This completes the proof. $\square$

3 Shortest circuit covers

In this section, we consider the shortest circuit covers of cubic signed graphs. Let (G, σ) be a 2-edge-connected signed graph and let $E^-(G, \sigma) := \{e \mid \sigma(e) = -1\}$ and $E^+(G, \sigma) := \{e \mid \sigma(e) = 1\}$. By Observation 2.1, we may always assume (G, σ) has the smallest number of negative edges over all equivalent signatures of σ. In other words, $|E^-(G,\sigma)| = \epsilon(G,\sigma)$. Let G^+ be the subgraph of G induced by edges in $E^+(G,\sigma)$, i.e., $G^+ = G \backslash E^-(G,\sigma)$. By Observation 2.3, for any edge-cut S, the following inequalities hold

$$|E^-(G,\sigma) \cap S| \le |S|/2 \le |E^+(G,\sigma) \cap S|. \tag{3}$$

So G^+ is connected spanning subgraph of G.

Lemma 3.1. *Let (G,σ) be a 2-edge-connected signed graph with $|E^-(G,\sigma)| = \epsilon(G,\sigma)$. If (G,σ) has a family of circuits $\mathcal{F}$ such that every negative edge e is contained in a cycle of $\bigcup_{C\in\mathcal{F}} C$, then $\mathcal{F}$ covers all cutedges of $G^+ = G\backslash E^-(G,\sigma)$.*

Proof. Suppose to the contrary that G^+ has a cutedge e which is not covered by any circuit in $\mathcal{F}$. Let S be an edge-cut of G such that $S \cap E^+ = \{e\}$. Since $|E^-(G,\sigma)| = \epsilon(G,\sigma)$, then $|S|/2 \le |E^+(G,\sigma) \cap S| = 1$ by (3). The 2-edge-connectivity of (G,σ) implies that $|S| = 2$. Let e' be the other edge in S. Then $e' \in E^-(G,\sigma)$. Note that e' is contained by a cycle D of $\bigcup_{C\in\mathcal{F}} C$. Note that $|E(D) \cap S|$ is even. Therefore, the cycle D contains e too. So e is covered by $\mathcal{F}$, a contradiction. This completes the proof. $\square$

Let (G,σ) be a 2-edge-connected flow-admissible cubic signed graph. In order to show that (G,σ) has a small circuit cover, we need to find a family of circuits with a suitable length to cover all negative edges and all bridges of G^+, and another family of circuits to cover the rest of edges. By Theorem 1.1, there is a family of circuits of G^+ covering all edges of G^+ except these cutedges with length at most $5|E(G^+)|/3$. Hence, by Lemma 3.1, it suffices to find a family of circuits $\mathcal{F}$ with a suitable length such that every edge of $E^-(G,\sigma)$ is covered by a cycle of some circuit in $\mathcal{F}$.

Let T be a spanning tree of G^+. Then T is also a spanning tree of G because G^+ is a spanning subgraph of G. For any $e \in E^-(G,\sigma) \subseteq E(G)\backslash E(T)$, let D_e be the elementary cycle of $T \cup \{e\}$. Since G is cubic, the symmetric difference of all cycles D_e, denoted by $\mathscr{D}$, consists of disjoint cycles. Let Q consists of all cycles of $\mathscr{D}$ with negative edges. Because a negative edge e is contained by only D_e, Q contains all negative edges of (G,σ), i.e., $E^-(G,\sigma) \subseteq E(Q)$. Let H be a minimal connected subgraph of G such that $Q \subseteq H \subseteq Q \cup T$. By the minimality of H, H has no vertices of degree 1 and any edge e of $E(H)\backslash E(Q)$ is a cutedge. (Otherwise, $H\backslash\{e\}$ is still connected and satisfies $Q \subseteq H \cup \{e\} \subseteq Q \cup T$, a contradiction to the minimality of H.) So $H/E(Q)$ is a tree and hence H is a cycle-tree. So (H,σ) is a signed cycle-tree of (G,σ) such that $E^-(G,\sigma) \subseteq E(H,\sigma)$.

Before proceed to prove our main result — Theorem 1.6, we show a better bound for 2-edge-connected cubic signed graphs with even negativeness. By Obeservation 2.2, a 2-edge-connected signed graph with even negativeness always has a circuit cover.

Theorem 3.2. *Let (G,σ) be a 2-edge-connected cubic signed graph with even negativeness. Then*

$$\mathrm{scc}(G,\sigma) < \frac{23}{9}|E(G)|.$$

Proof. If $\epsilon(G,\sigma) = 0$, then (G,σ) is a graph and hence $\mathrm{scc}(G,\sigma) \le 5|E(G)|/3$ by Theorem 1.1. The theorem holds immediately. So in the following, assume that $\epsilon(G,\sigma) \ge 2$ and $|E^-(G,\sigma)| = \epsilon(G,\sigma)$ by Observation 2.1.

Recall that (G,σ) has a signed cycle-tree (H,σ) such that $E^-(G,\sigma) \subseteq E(H,\sigma)$. Since $\epsilon(G,\sigma)$ is even, it follows that (G,σ) has an even number of negative cycles. By Theorem 2.5, (H,σ) has a family of circuits

$\mathcal{F}_1$ which covers all cycles of (H,σ) and hence covers all negative edges of (G,σ) with length

$$\ell(\mathcal{F}_1) \leq \frac{4}{3}|E(H)| \leq \frac{4}{3}(|V(G)| - 1 + |E^-(G,\sigma)|) = \frac{8}{9}|E(G)| + \frac{4}{3}|E^-(G,\sigma)| - \frac{4}{3}. \tag{4}$$

By Lemma 3.1, $\mathcal{F}_1$ covers all cutedges of G^+. Deleting all cutedges from G^+, every component of the resulting graph is 2-edge-connected. By Theorem 1.1, all shortest circuit covers of these components together form a family of circuits $\mathcal{F}_2$ of G^+, which covers all edges of G^+ except cutedges with length

$$\ell(\mathcal{F}_2) \leq \frac{5}{3}|E(G^+)| = \frac{5}{3}(|E(G)| - |E^-(G,\sigma)|).$$

So $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ is a circuit cover of (G,σ) with length

$$\ell(\mathcal{F}) = \ell(\mathcal{F}_1)+\ell(\mathcal{F}_2) \leq (\frac{8}{9}|E(G)|+\frac{4}{3}|E^-(G,\sigma)|-\frac{4}{3})+\frac{5}{3}(|E(G)|-|E^-(G,\sigma)|) \leq \frac{23}{9}|E(G)|-\frac{1}{3}|E^-(G,\sigma)|-\frac{4}{3}.$$

It follows that $\mathrm{scc}(G,\sigma) < 23|E(G)|/9$. So the theorem holds. □

In the following, we consider signed cubic graphs (G,σ) with odd negativeness, i.e., $\epsilon(G,\sigma)$ is odd. The *signed-girth* of a signed graph (G,σ) is length of a shortest circuit containing negative edges, denoted by $g_s(G,\sigma)$. Before proceed to prove our main result, we need some technical lemmas.

Lemma 3.3. *Let (G,σ) be a signed cubic graph, and (N,σ) be a signed cycle-tree of (G,σ). If $g_s(G,\sigma) \geq |E(G)|/3 + 2$, then:*

(1) (G,σ) does not contain two disjoint circuits both containing negative edges;

(2) (N,σ) has at most three leaf-cycles and at most one non-leaf cycle. Furthermore, if it has a non-leaf cycle, then all leaf-cycles are negative.

Proof. If (G,σ) has only one circuit, the lemma holds trivially. So assume that (G,σ) has at least two distinct circuits. Let C_1 and C_2 be two distinct circuits. If $V(C_1) \cap V(C_2) = \emptyset$, then $|V(C_1)| + |V(C_2)| \leq |V(G)|$. Note that $E(C_i) \leq |V(C_i)| + 1$ $(i = 1, 2)$ and equality holds if and only if C_i is a barbell. Since G is cubic, $|V(G)| = 2|E(G)|/3$. It follows that

$$|E(C_1)| + |E(C_2)| \leq |V(C_1)| + |V(C_2)| + 2 \leq |V(G)| + 2 = \frac{2}{3}|E(G)| + 2.$$

Without loss of generality, assume that $|E(C_1)| \leq |E(C_2)|$. Hence $|E(C_1)| \leq |E(G)|/3 + 1$, contradicting $g_s(G,\sigma) \geq |E(G)|/3 + 2$. This completes the proof of (1).

Since (G,σ) does not contain two disjoint circuits, every signed cycle-tree (N,σ) of (G,σ) does not contain two disjoint circuits neither. Hence (N,σ) has at most three leaf-cycles.

If (N,σ) has two non-leaf cycles D_1 and D_2, then there is a leaf cycle D'_i is connected to D_i by a path P_i for $i = 1$ and 2 such that $P_1 \cap P_2 = \emptyset$. Then $D'_i \cup P_i \cup D_i$ contains a circuit for both $i = 1$ and 2, contradicting (N,σ) does not have two disjoint circuits. So (N,σ) has at most one non-leaf cycle.

If (N,σ) has exactly one non-leaf cycle D_1, then D_1 is connected to at least two leaf-cycles. If one of the leaf-cycles C is positive, the other leaf-cycles together with D_1 contains a circuit disjoint from C, a contradiction. This completes the proof. □

Lemma 3.4. *Let (G,σ) be a 2-edge-connected cubic signed graph with $\epsilon(G,\sigma) \geq 3$ negative edges, and G^+ be the subgraph induced by positive edges. If $g_s(G,\sigma) \geq |E(G)|/3 + 2$, then (G,σ) has a family of circuits $\mathcal{F}$ covering all negative edges of (G,σ) and all cutedges of G^+ such that*

$$\ell(\mathcal{F}) < \frac{11}{9}|E(G)| + \frac{5}{3}\epsilon(G,\sigma).$$

Proof. Let (G, σ) be a cubic signed graph with $g_s(G, \sigma) \geq |E(G)|/3 + 2$. If $\epsilon(G, \sigma)$ is even, the lemma follows from (4). So assume that $\epsilon(G, \sigma)$ is odd. Suppose to the contrary that (G, σ) is a counterexample.

Since $\epsilon(G, \sigma) \geq 3$, (G, σ) has a circuit cover by Observation 2.2. So (G, σ) has a circuit C containing negative edges. The circuit C has a negative edge in a cycle.

Claim 1. *For any negative edge e contained in a cycle of some circuit, the signed graph (G, σ) has a signed cycle-tree (H, σ) which has all negative edges in cycles and $E^-(H, \sigma) = E^-(G, \sigma) \backslash \{e\}$.*

Proof of Claim 1. Let T be a spanning tree of G^+. For any $e' \in E^-(G, \sigma) \backslash \{e\}$, let $D_{e'}$ be the elementary cycle of $T \cup \{e'\}$. The symmetric difference $\mathscr{D}_e = \oplus_{e' \in E^-(G,\sigma) \backslash \{e\}} D_{e'}$ consists of disjoint cycles because G is cubic. Let Q_e consist of all cycles of $\mathscr{D}_e$ containing at least one negative edge. Let H be a minimal connected subgraph satisfying $Q_e \subseteq H \subseteq Q_e \cup T$. By the minimality of H, we can conclude that (H, σ) is a signed cycle-tree of (G, σ) such that every edge in $E^-(G, \sigma) \backslash \{e\}$ is contained by a cycle of (H, σ). Note that $e \notin E(H, \sigma)$. So $E^-(H, \sigma) = E^-(G, \sigma) \backslash \{e\}$. This completes the proof of Claim 1.

For any negative edge e contained by a cycle of some circuit, among all such signed cycle-trees with property in Claim 1, choose a signed cycle-tree (H_e, σ) with the smallest number of cycles. Since $\epsilon(G, \sigma) - 1$ is even, it follows that (H_e, σ) has an even number of negative cycles.

Claim 2. *The signed cycle-tree (H_e, σ) is a circuit.*

Proof of Claim 2. Suppose on the contrary that (H_e, σ) is not a circuit. Then it has a non-leaf cycle D_0. Let $D_1, ..., D_k$ be all leaf-cycles of (H_e, σ). Since $g_s(G, \sigma) \geq |E(G)|/3 + 2$, by (ii) of Lemma 3.3, D_0 is the only non-leaf cycle of (H_e, σ), and $D_1, ..., D_k$ are negative cycles where $2 \leq k \leq 3$. Further, (H_e, σ) has $k + 1$ cycles.

Since G is 2-edge-connected and cubic, there are two disjoint paths P_1 and P_2 from D_1 to D_0. Since (G, σ) does not contain two disjoint circuits, for both $i = 1$ and 2, we have $P_i \cap D_t = \emptyset$ where $t = 2$ or k. Let v_1 and v_2 be two endvertices of P_1, and u_1 and u_2 be two endvertices of P_2 such that $v_1, v_2 \in V(D_1)$ and $u_1, u_2 \in V(D_0)$. The two vertices u_1 and u_2 separate D_0 into two internally disjoint segements S_1 and S_2. Without loss of generality, assume $|E(S_1)| \leq |E(S_2)|$. Then $D_1 \cup P_1 \cup P_2 \cup S_1$ has a positive cycle, denoted by C_1. If C_1 does not contain a negative edge, then both S_1 and $C_1 \cap D_1$ do not contain a negative edges. So deleting all internal vertices of S_1 and $C_1 \cap D_1$ from $H_e \cup (P_1 \cup P_2)$ results in a signed cycle tree with k cycles, contradicting that (H_e, σ) has the smallest number of cycles. Hence C_1 is a positive cycle with negative edges, and $|E(C_1)| \leq (|E(D_1)| - 1) + |E(P_1)| + |E(P_2)| + |E(S_1)|$.

Similary, there are two disjoint paths P'_1 and P'_2 from D_2 to D_0. Let S'_1 be the segment with smaller length of D_0 separated by two endvertices of P'_1 and P'_2. And $D_2 \cup P'_1 \cup P'_2 \cup S'_1$ contains a positive cycle C_2 wih negative edges. Since (G, σ) does not contain two disjoint circuit, it follows that for both $i = 1$ and 2, $P'_i \cap (P_1 \cup P_2 \cup D_1) = \emptyset$ and $P'_i \cap D_3 = \emptyset$ if $k = 3$. So

$$\begin{aligned}
|E(C_1)| + |E(C_2)| &\leq (|E(D_1)| - 1) + \sum_{i=1}^{2} |E(P_i)| + |E(S_1)| + (|E(D_2)| - 1) + \sum_{i=1}^{2} |E(P'_i)| + |E(S'_1)| \\
&\leq |E(D_1)| + |E(D_2)| + |E(D_0)| + |E(P_1 \cup P_2)| + |E(P'_1 \cup P'_2)| - 2 \\
&= |V(D_1 \cup D_2 \cup D_0 \cup P_1 \cup P_2 \cup P'_1 \cup P'_2)| + 4 - 2 \\
&\leq |V(G)| + 2 \\
&\leq \frac{2}{3}|E(G)| + 2.
\end{aligned}$$

Without loss of generality, assume $|E(C_1)| \leq |E(C_2)|$. Hence $|E(C_1)| \leq |E(G)|/3 + 1$, contradicting $g_s(G, \sigma) \geq |E(G)|/3 + 2$. This completes the proof of Claim 2.

By Claim 2, in the following, for any negative edge e contained in a cycle of some circuit,t (H_e, σ) is a circuit. In other words, (H_e, σ) is a positive cycle or a barbell.

Claim 3. *Let C be a positive cycle with negative edges or the union of two disjoint negative cycles. Then*

$$|E(C)| \geq \frac{4}{9}|E(G)| + 6.$$

Proof of Claim 3. If C is the union of two negative cycles D_1 and D_2, then there are two disjoint paths P and P' joining D_1 and D_2 since G is 2-edge-connected and cubic. For the case that C is a positive cycle, let $P = P' = \emptyset$.

Let e be a negative edge in a cycle of the circuit C. Note that (H_e, σ) is a circuit. Then both $\mathcal{F}_1 = \{C \cup P\} \cup \{H_e\}$ and $\mathcal{F}_2 = \{C \cup P'\} \cup \{H_e\}$ are two families of circuits covering all edges in $E^-(G, \sigma)$. Since every negative edge is contained either in a cycle of C or a cycle of (H_e, σ), by Lemma 3.1, both $\mathcal{F}_1$ and $\mathcal{F}_2$ cover all cutedges of G^+. Since (G, σ) is a countexample, both $\mathcal{F}_1$ and $\mathcal{F}_2$ have length at least $11|E(G)|/9 + 5\epsilon(G, \sigma)/3$. So

$$\ell(\mathcal{F}_1) + \ell(\mathcal{F}_2) = 2|E(C)| + |E(P)| + |E(P')| + 2|E(C')| \geq \frac{22}{9}|E(G)| + \frac{10}{3}|\epsilon(G, \sigma)|.$$

Since $C \cup P \cup P'$ is a connected subgraph of G with at most four vertices of degree 3, it follows that $|E(C)| + |E(P)| + |E(P')| \leq |V(G)| + 2$. Note that the circuit H_e has at most two cycles and hence has at most two vertices of degree 3. Therefore, $|E(H_e)| \leq |V(G)| + 1$. It follows that

$$\begin{aligned} |E(C)| &\geq \frac{22}{9}|E(G)| + \frac{10}{3}|\epsilon(G, \sigma)| - (|V(G)| + 2) - 2(|V(G)| + 1) \\ &= \frac{22}{9}|E(G)| + \frac{10}{3}|\epsilon(G, \sigma)| - 2|E(G)| - 4 \\ &\geq \frac{4}{9}|E(G)| + 6. \end{aligned}$$

This completes the proof of Claim 3.

Since (G, σ) contains at least three negative edges, let e_i $(i = 1, 2, 3)$ be negative edges of (G, σ) and D_{e_i} be the elementary cycle $T \cup \{e_i\}$. Let $C_{ij} \subseteq D_{e_i} \oplus D_{e_j}$ be either a positive cycle or the union of two disjoint negative cycles, which contains both e_i and e_j.

By Claim 3, we have

$$|E(C_{12})| + |E(C_{13})| + |E(C_{23})| \geq \frac{4}{3}|E(G)| + 18. \tag{5}$$

On the other hand, since $C_{23} \subseteq D_{e_2} \oplus D_{e_3} = (D_{e_1} \oplus D_{e_3}) \oplus (D_{e_1} \oplus D_{e_2}) = C_{12} \oplus C_{23}$, it follows that $\{C_{12}, C_{13}, C_{23}\}$ covers each edge of $T \cup \{e_1, e_2, e_3\}$ at most twice. Therefore,

$$|E(C_{12})| + |E(C_{13})| + |E(C_{23})| \leq 2|E(T \cup \{e_1, e_2, e_3\})| = 2(|V(G)| + 2) = \frac{4}{3}|E(G)| + 4,$$

a contradiction to (5). This completes the proof of the lemma. □

Now we are going to prove the main result. Recall our main result here.

Theorem 1.6. *Let (G, σ) be a 2-connected cubic signed graph. If (G, σ) is flow-admissible, then*

$$\mathrm{scc}(G, \sigma) < \frac{26}{9}|E(G)|.$$

Proof. Let (G,σ) be a 2-edge-connected flow-admissible cubic signed graph. If $\epsilon(G,\sigma)$ is even, the theorem follows from Theorem 3.2. So in the following, we always assume that $\epsilon(G,\sigma)$ is odd. By Observations 2.1 and 2.2, we further assume that $|E^-(G,\sigma)| = \epsilon(G,\sigma) \geq 3$.

Let $G^+ = G \backslash E^-(G,\sigma)$. By Theorem 1.1, G^+ has a family of circuits $\mathcal{F}_1$ covering all edges of G^+ except cutedges with length

$$\ell(\mathcal{F}_2) \leq \frac{5}{3}|E(G^+)| = \frac{5}{3}(|E(G)| - |E^-(G,\sigma)| = \frac{5}{3}(|E(G)| - \epsilon(G,\sigma)).$$

If the signed-girth of (G,σ) satisfies $g_s(G,\sigma) \geq |E(G)|/3+2$, then by Lemma 3.4, (G,σ) has a family of circuits $\mathcal{F}_2$ covering edges in $E^-(G,\sigma)$ and all cutedges of $G^+ = G\backslash E^-(G,\sigma)$ with length

$$\ell(\mathcal{F}_2) < \frac{11}{9}|E(G)| + \frac{5}{3}\epsilon(G,\sigma).$$

So $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ is a circuit cover of (G,σ) with length

$$\ell(\mathcal{F}) = \ell(\mathcal{F}_1) + \ell(\mathcal{F}_2) < \frac{5}{3}(|E(G)| - \epsilon(G,\sigma)) + \frac{11}{9}|E(G)| + \frac{5}{3}\epsilon(G,\sigma) = \frac{26}{9}|E(G)|.$$

So the theorem holds for all signed graphs with $g_s(G,\sigma) \geq |E(G)|/3+2$.

In the following, assume that (G,σ) has a circuit C with length at most $|E(G)|/3+1$. Let e be a negative edge contained in a cycle of C, and let (H_e,σ) be a signed cycle-tree of (G,σ) containing all negative edges in $E^-(G,\sigma)\backslash\{e\}$ in cycles of (H_e,σ). (Note that, such signed cycle-trees exists as shown in Claim 1 in Lemma 3.4). By Theorem 2.5, (H_e,σ) has a family of circuits $\mathcal{F}_2$ covering all cycles of (H_e,σ) with length

$$\ell(\mathcal{F}_2) \leq \frac{4}{3}|E(H_e)| \leq \frac{4}{3}(|V(G)| - 1 + |E^-(G,\sigma)\backslash\{e\}|) = \frac{8}{9}|E(G)| + \frac{4}{3}\epsilon(G,\sigma) - \frac{8}{3}.$$

So $\mathcal{F}_2 \cup \{C\}$ covers all negative edges of (G,σ) and every negative edge is contained by a cycle of some circuit of $\mathcal{F}_2 \cup \{C\}$. Hence $\mathcal{F}_2 \cup \{C\}$ covers all negative edges of (G,σ) and all cutedges of G^+ by Lemma 3.1.

Note that G^+ has a family of circuits $\mathcal{F}_1$ covering all edges of G^+ except cutedges. So $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \{C\}$ is a circuit cover of (G,σ) with length

$$\begin{aligned}
\ell(\mathcal{F}) &= \ell(\mathcal{F}_1) + \ell(\mathcal{F}_2) + |E(C)| \\
&\leq \frac{5}{3}(|E(G)| - \epsilon(G,\sigma)) + \frac{8}{9}|E(G)| + \frac{4}{3}\epsilon(G,\sigma) - \frac{8}{3} + \frac{1}{3}|E(G)| + 1 \\
&\leq \frac{26}{9}|E(G)| - \frac{1}{3}\epsilon(G,\sigma) - \frac{5}{3} \\
&< \frac{26}{9}|E(G)|.
\end{aligned}$$

This completes the proof of Theorem 1.6. $\square$

4 Concluding remarks

A 2-edge-connected signed graph (G,σ) with a circuit cover may not have a circuit double cover. In the following, we construct infinitly many 2-edge-connected signed graphs (G,σ) with even negativeness but without circuit double cover properties.

Proposition 4.1. *Let (G, σ) be a cubic signed graph with a circuit double cover $\mathcal{F}$. If v is a vertex of degree-3 in a barbell $B \in \mathcal{F}$, then v is a vertex of degree-3 in another barbell $B' \in \mathcal{F}$.*

Proof. Since (G, σ) is cubic, there are exactly three edges e_1, e_2, e_3 incident with v. Since v is a vertex of degree-3 in B, e_1, e_2 and e_3 are covered once by B. So e_1, e_2 and e_3 are covered once by $\mathcal{F} \backslash \{B\}$. Hence, e_1, e_2 and e_3 belong to exactly one circuit in $\mathcal{F}$, which must be a barbell B'. So v is a vertex of degree-3 in B'. □

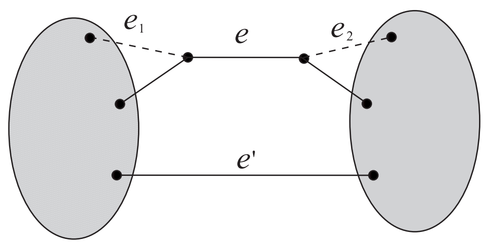

Figure 2: Infinitly many 2-connected signed graphs without a circuit double cover (solid edges are positive and dashed edges are negative).

Let G be a 2-connected cubic graph and $S = \{e, e'\}$ be a two edge-cut of G. Assume $e = uv$ and let e_1 incident with u and e_2 incdient with v. The signed graph (G, σ) is obatined from G by assigning -1 to both e_1 and e_2, and assigning 1 to all other edges. Suppose on the contrary that (G, σ) have a circuit cover $\mathcal{F}$. If $\mathcal{F}$ has a barbell B, then $B \cap S \neq \emptyset$ since e_1 and e_2 belong two different cycles of B. We may assume that $e \in B$ (a similar argument works for $e' \in B$). Then e is the path of B joining the two cycles of B. Hence both u and v are vertices of degree 3 in B. Then v is a vertex of degree 3 in another barbell B' in $\mathcal{F}$ by the above proposition. It follows that e' can not be covered by any circuit of (G, σ). So $\mathcal{F}$ does not have any barbell. Hence e_1 and e_2 are contained by two positive cycles C_1 and C_2 of $\mathcal{F}$. Then both C_1 and C_2 contain S. It follows that the third edge incident with u or v different from e_1, e_2 and e can not be covered by circuits in $\mathcal{F}$. Hence (G, σ) is a counterexample. This construction works for all cubic graphs with 2-edge-cut. Hence there are infinitly many 2-connected cubic signed graphs with a circuit cover but having no circuit double covers.

The example in Figure 1 shows that a 3-connected cubic signed graph with even negativeness may not have a circuit double cover. By above proposition, any circuit double cover of the signed graph does not have a barbell. Because a circuit containing the two negative edges of the signed graph in Figure 1 has length either 5 or 6, a counting of lengths of circuits shows that the signed graph has no circuit double covers.

As many 2-edge-connected signed graphs have no circuit double covers, it is interesting to ask, is there an integer k such that every 2-connected flow-admissible signed graph (G, σ) has a circuit k-cover?

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