# On Homeomorphically Irreducible Spanning Trees in Cubic Graphs 

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#### Abstract

A spanning tree without a vertex of degree two is called a Hist which is an abbreviation for homeomorphically irreducible spanning tree. We provide a necessary condition for the existence of a Hist in a cubic graph. As one consequence, we answer affirmatively an open question on Hists by Albertson, Berman, Hutchinson and Thomassen.


Keywords: Hist, cubic graph, cyclic edge-connectivity, bipartite, spanning tree, fullerene

## 1 Introduction

All graphs considered here are finite and simple. In a connected graph $G$, a spanning tree which does not have a vertex of degree two is called a homeomorphically irreducible spanning tree, or abbreviated a Hist. Several conditions which ensure the existence of a Hist in a graph are known, see for instance [1, 3, 9]. In this paper, we only consider Hists in cubic graphs. For an integer $k$, a connected cubic graph $G$ which contains two disjoint cycles is said to be cyclically $k$-edge-connected if deleting any set of at most $k-1$ edges from $G$ does not separate $G$ into two components both of which have a cycle. The following question was asked in [1, p. 253].

Question 1 Does there exist a cyclically $k$-edge-connected cubic graph without a Hist for each positive integer $k$ ?

Note that every Hist $T$ in a cubic graph has only vertices of degree one and three. Hence $E(G)$ has a partition into $E(T)$ and the edge set of a union of disjoint cycles.

Let us call a 2-regular subgraph $H$ of a connected graph $G$ non-separating if $G-E(H)$ is connected. For a set $S$ of edges in $G$, we denote by $\langle S\rangle$ the subgraph of $G$ induced by the edges in $S$. So, the vertex set of $\langle S\rangle$ is the set of end vertices of edges in $S$. We answer Question 1 by applying Corollary 3, a corollary of Theorem 2 which turns out to be useful for proving that certain cubic graphs do not have a Hist.

[^0]Theorem 2 Let $G$ be a cubic graph with a Hist $T$, and let $H=\langle E(G)-E(T)\rangle$. Then $H$ is a non-separating 2-regular subgraph of $G$ satisfying $|V(H)|=|V(G)| / 2+1$.

Proof. Let $G$ be a cubic graph with a Hist $T$ and let $H=\langle E(G)-E(T)\rangle$. Since $V(H)$ is the set of all leaves of $T$ and $G-E(H)=T, H$ is a non-separating 2-regular subgraph of $G$.

Let $t_{1}$ be the number of leaves in $T$, and let $t_{3}$ be the number of vertices of degree 3 in $T$. Since $T$ is a Hist, we have $t_{1}+t_{3}=|V(G)|$. On the other hand, it is easy to see that $t_{1}=t_{3}+2$. (This can be obtained straightforwardly by the Handshaking Lemma or by induction. For example, see [13, Exercise 2.1.23 on p. 70].) Therefore, $|V(G)|=2 t_{1}-2$. Since $V(H)$ is the set of all leaves of $T$, we have $|V(H)|=t_{1}$. By using the above equations the proof is completed.

Corollary 3 Let $G$ be a bipartite cubic graph. If $G$ has a Hist, then $|V(G)| \equiv 2(\bmod 4)$.
Proof. Let $G$ be a bipartite cubic graph with a Hist. By Theorem 2, $H=\langle E(G)-E(T)\rangle$ is a non-separating 2-regular subgraph of $G$ satisfying $|V(H)|=|V(G)| / 2+1$. Since $G$ is bipartite, $|V(H)|$ is even and hence $|V(G)| \equiv 2(\bmod 4)$.

Remark: Corollary 3 implies that no bipartite cubic graph $G$ with $|V(G)| \equiv 0(\bmod 4)$ has a Hist. However, if $G$ is a bipartite cubic graph with $|V(G)| \equiv 2(\bmod 4)$, then $G$ may or may not have a Hist. Both cases could happen, see Section 3,

Now we obtain a positive answer to Question 1 by applying Corollary 3 together with the following proposition.

Proposition 4 For every positive integer $k$, there exists a cyclically $k$-edge-connected bipartite cubic graph $G$ such that $|V(G)| \equiv 0(\bmod 4)$.

Proposition 4 can be directly proved by considering transitive graphs: it is known that for any positive integer $k$, there are infinitely many vertex-transitive bipartite cubic graphs $G$ of girth at least $k$ with $|V(G)| \equiv 0(\bmod 4)$, see for example [12]. Since the cyclic edge-connectivity of vertex-transitive graph is equal to its girth (see [11]), Proposition 4 holds. However, since this proof requires several algebraic tools, we prefer to present an elementary proof which also offers a new method to construct cubic bipartite graphs with high cyclic edge-connectivity, see Theorem 7 and Lemma 8 in Section 2,

In Section 3, we show other application of Theorem 2 to plane and toroidal cubic graphs.

## 2 Proof of Proposition 4

In order to prove Proposition 4, we use the following fact which can be proved in several ways, for instance, by the probabilistic method (see [14, Theorems 2.5 and 2.10]) and by the constructive method (see [5]).

Fact 5 For every positive integer $d$, there exists a d-connected $4 d$-regular graph of girth at least $d$.

Then we apply the well known concept of an inflation (see for instance [6]):
Definition 6 Let $H$ be a graph and let $G$ be a cubic graph. Then $G$ is called an inflation of $H$ if $G$ contains a 2-factor $F$ consisting of chordless cycles such that the graph obtained from $G$ by contracting each cycle of $F$ to a vertex is isomorphic to $H$.

If the minimum degree of $H$ is at least 3 , then obviously an inflation of $H$ exists, since one obtains (informally speaking) an inflation of $H$ by expanding every vertex of $H$ to a cycle. The next theorem guarantees the high cyclic edge-connectivity for each inflation of graphs with high connectivity and girth.

Theorem 7 Let $k \geq 3$ and let $H$ be a $k$-connected graph with girth at least $k$. Then every inflation of $H$ is cyclically $k$-edge-connected.
Proof. Let $G$ be an inflation of $H$. For each vertex $x \in V(H)$, denote the unique cycle of $F$ (as in Definition (6) in $G$ corresponding to $x$ by $C_{x}$. We say that a cycle $C$ in $G$ is transverse if there are two distinct vertices $x_{1}$ and $x_{2}$ in $H$ with $V\left(C_{x}\right) \cap V(C) \neq \emptyset$ for each $x \in\left\{x_{1}, x_{2}\right\}$. Otherwise $C$ is said to be non-transverse, that is, $C=C_{x}$ for some vertex $x \in V(H)$.

Suppose by contradiction that $G$ is not cyclically $k$-edge-connected. Then $G$ has a set $S$ of edges with $|S| \leq k-1$ such that $G-S$ has precisely two components $D_{1}$ and $D_{2}$ both having a cycle. By taking such a set $S$ as small as possible, we may assume that $S$ is a matching.

For $i \in\{1,2\}$, let $D_{i}^{H}$ be the subgraph of $H$ induced by the vertex set

$$
\left\{x \in V(H): V\left(C_{x}\right) \cap V\left(D_{i}\right) \neq \emptyset\right\} .
$$

So, $D_{1}^{H}$ is obtained from $D_{1}$ in the following way: for each $x \in V(H)$ such that $C_{x} \cap D_{1}$ is not the null graph, where $C_{x} \cap D_{1}$ is the maximum common subgraph of $C_{x}$ and $D_{1}$, contract $C_{x} \cap D_{1}$ into one vertex and delete all resultant loops.

$$
\begin{aligned}
\text { Let } \quad S_{\mathrm{V}}^{H} & =\left\{x \in V(H): E\left(C_{x}\right) \cap S \neq \emptyset\right\} \\
\text { and } \quad & \quad S_{\mathrm{E}}^{H} \\
& =S \cap E(H) \\
& =\left\{e \in S: e \notin E\left(C_{x}\right) \text { for any } x \in V(H)\right\} .
\end{aligned}
$$

Note that $\left|S_{V}^{H}\right|+\left|S_{E}^{H}\right| \leq|S| \leq k-1$.
Suppose that $V\left(D_{i}^{H}\right)-S_{\mathrm{V}}^{H} \neq \emptyset$ for each $i \in\{1,2\}$. Then $H-S_{\mathrm{V}}^{H}-S_{\mathrm{E}}^{H}$ have two components $D_{1}^{H}$ and $D_{2}^{H}$. In this case, the number of vertex disjoint paths from a vertex of $D_{1}^{H}$ to a vertex of $D_{2}^{H}$ is at most $\left|S_{\mathrm{V}}^{H}\right|+\left|S_{\mathrm{E}}^{H}\right| \leq k-1$, which contradicts by Menger's Theorem that $H$ is $k$-connected.

Therefore, we may assume without loss of generality that $V\left(D_{1}^{H}\right)-S_{\mathrm{V}}^{H}=\emptyset$.
Note that $D_{1}$ contains by assumption a cycle, say $C_{1}$, and $C_{1}$ must be transverse (otherwise, $C_{1}=C_{x}$ for some $x \in V\left(D_{1}^{H}\right)-S_{\mathrm{V}}^{H}$, but this contradicts that $\left.V\left(D_{1}^{H}\right)-S_{\mathrm{V}}^{H}=\emptyset\right)$. Thus, $C_{1}$ corresponds to a closed trail in $D_{1}^{H}$, say $C_{1}^{H}$. Since the girth of $H$ is at least $k$ and every closed trail contains a cycle, we have $\left|V\left(C_{1}^{H}\right)\right| \geq k$, which is a contradiction to the fact that $V\left(C_{1}^{H}\right) \subseteq V\left(D_{1}^{H}\right) \subseteq S_{\mathrm{V}}^{H}$ and $\left|S_{\mathrm{V}}^{H}\right| \leq k-1$.

Note that the statement of the above theorem does not hold if $H$ is only demanded to be $k$-edge-connected.


Figure 1: A graph $H$ with Eulerian orientation (the left side) and the bipartite graph $G$ obtained by an inflation of $H$ (the right side) for the case $k=2$. In $G$, the vertices with outdegree 3 are represented by white circles, while the vertices with indegree 3 are represented by black circles.

Lemma 8 Let $k \geq 2$ and let $H$ be a $2 k$-regular graph. Then there exists a bipartite cubic inflation of $H$ with $2 k|V(H)|$ vertices.

Proof. Since every inflation of $H$ has $2 k|V(H)|$ vertices, it suffices to show that $H$ has a bipartite inflation.

Since each component of $H$ is Eulerian, it has an Eulerian orientation, that is, the indegree equals the outdegree for every vertex of $H$. Then we can expand every vertex $x$ of $H$ to a cycle $C_{x}$ to obtain an inflation $G$ with the property that the oriented edges incident with the vertices of $C_{x}$ are alternately directed towards and outwards $C_{x}$. See Figure [1. Furthermore, since each cycle $C_{x}$ is of length exactly $2 k$, it is possible to extend this partial orientation to an orientation of $G$ (by orienting the edges of each cycle $C_{x}$ ) such that every vertex of $G$ has then either outdegree 3 or indegree 3. This shows a 2 -coloring of $G$, and hence $G$ is bipartite.

Proof of Proposition 4. Let $k$ be a positive integer. By Fact 5 there exists a $k$-connected $4 k$-regular graph $H$ of girth at least $k$. Since $H$ is $4 k$-regular, it follows from Lemma 8 that there exists a bipartite cubic inflation $G$ with $4 k|V(H)|$ vertices. Since $H$ is $k$-connected and has the girth at least $k$, it follows from Theorem 7 that $G$ is cyclically $k$-edge-connected, which completes the proof.

## 3 Hists in plane cubic graphs

Let us call a plane cubic graph with a Hist in short a pcH-graph. A pcH-graph is by its definition a generalization of a cubic Halin graph (defined in [8]) which is a pcH-graph with a Hist such that all the leaves of the Hist induce precisely one cycle. It is easy to see that any cubic Halin graph contains a triangle. In contrast to cubic Halin graphs, pcH-graphs can have girth 4 or even 5, see Figure 2, Note that it is NP-complete to determine whether a plane cubic graph has a Hist, see [4]. (To be exact, Douglas [4] proved that only for plane graphs of maximum degree at most 3 . However, replacing each vertex of degree at most 2 with a certain gadget, we can easily modify the proof to show the NP-completeness of the

Hist problem for plane cubic graphs.) Since any non-facial cycle of a cubic plane graph is separating, by restricting Theorem [2 to the planar case we obtain:

Corollary 9 Let $G$ be a plane cubic graph with a Hist. Then $G$ contains a non-separating 2-regular subgraph $H$ consisting of facial cycles such that $|V(H)|=|V(G)| / 2+1$.


Figure 2: A fullerene graph with a Hist.
Applying Corollary 9, we see for instance that the dodecahedron does not have a Hist, since it has 20 vertices and every facial cycle has length 5 . The dodecahedron belongs to the class of fullerene graphs, which are plane 3 -connected cubic graphs with facial cycles of length 5 and 6 only, see Figure 2 for an example. Using Corollary 9 one can prove straightforwardly that other plane cubic graphs, for instance the Buckminster fullerene graph [7, Figure 9.5. on p. 211] and the Grinberg graph [2, Fig.18.9. on p. 480], do not have a Hist. This should illustrate the usefulness of the above corollary. We asked in the first version of this paper whether there are finitely or infinitely many fullerene graphs with a Hist which is answered below.

Theorem 10 There are infinitely many fullerene graphs with a Hist.
Proof. Let $A, B$ and $H$ be the plane graphs shown in Figure 3 (every label of a vertex in the figure is shown left above the vertex). By identifying the cycle $C_{1}$ of $A$ and the cycle $C_{2}$ of $B$ such that $u$ and $v$ are identified, we obtain a fullerene graph with a Hist which is illustrated in bold edges. In order to construct infinitely many fullerene graphs with Hists, we use the graphs $H_{i}$ which are defined as follows (during the construction of $H_{i}$ we keep the bold edges of every copy of $H$ which will then define the edges of the Hist within $H_{i}$ in the fullerene graph). Firstly, let $H_{0}$ be a plane cycle of length 18 and let $H_{1} \simeq H$. Then define the graph $H_{i}(i \geq 2)$ recursively, by identifying the outer cycle $C_{1}^{\prime}$ in a copy of $H$ and
the cycle (18-gon) $C_{2}^{\prime}$ in $H_{i-1}$ so that $u^{\prime}$ in $C_{1}^{\prime}$ and $v^{\prime}$ in $C_{2}^{\prime}$ are identified. Now we construct for every nonnegative integer $k$ the fullerene graph $F_{k}$ with $36 k+46$ vertices. We identify the cycle $C_{1}$ in $A$ and the outer cycle $C_{1}^{\prime}$ in $H_{k}$ such that $u$ in $C_{1}$ and $u^{\prime}$ in $C_{1}^{\prime}$ are identified. Finally, we identify the cycle $C_{2}^{\prime}$ in $H_{k}$ and the outer cycle $C_{2}$ in $B$ such that $v^{\prime}$ in $C_{2}^{\prime}$ and $v$ in $C_{2}$ are identified. Note that the 12 shaded faces in Figure 3 are pentagons of $F_{k}$. It is not difficult to verify that the bold edges in $A, B$ and the bold edges of $H_{i}$ induce a Hist in $F_{k}$.


Figure 3: Plane graphs $A, B$ and $H$.
Remark: In the proof of Theorem 10, every facial cycle of the fullerene graph $F_{k}$ which is edge-disjoint with the defined Hist has length 6. In contrast to $F_{k}$, the fullerene graph in Figure 2 has facial cycles of length 5 which are edge-disjoint with the illustrated Hist. By computer search, T. Jatschka [10] showed that there are fullerene graphs with Hists with 38 vertices and that every fullerene graph with less than 38 vertices does not have a Hist.

A class of graphs similar to fullerene graphs are cubic hexangulations. Recall that a hexangulation of a surface is a 2-connected graph with an embedding on the surface such that every facial cycle has length 6 . For example, consider the dual of the triangulation in Figure 4. Using this type of construction, we see that there are infinitely many bipartite cubic hexangulations $G$ of the torus with $|V(G)| \equiv 0(\bmod 4)$. Corollary 3 directly shows that such hexangulations $G$ do not contain a Hist. We asked in the first version of this paper whether there are finitely or infinitely many hexangulations of the torus with a Hist. This question is answered in the next theorem.

Theorem 11 There are infinitely many cubic hexangulations of the torus with a Hist.
Proof. Let $G_{0}$ and $G_{1}$ be the hexangulations of the torus shown in the left and the center of Figure 5. (The top and the bottom, the left and the right are identified, respectively.) Let $T$ be the hexangulation of the annulus shown in the right of Figure 5. (The top and the bottom are identified.) We construct the cubic hexangulation $G_{k}(k \geq 2)$ of the torus recursively, by (i) cutting $G_{k-1}$ along the cycle $v_{1} v_{2} v_{3} v_{4} v_{5} v_{6}$, and (ii) inserting $T$ with appropriate identification. Then $G_{k}$ is a cubic hexangulation with $(12 k+10)$ vertices, and has a Hist, which is presented in Figure 5 by the bold edges.


Figure 4: The dual of bipartite cubic hexangulations $G$ of the torus satisfying $|V(G)| \equiv$ $0(\bmod 4)$. The top and the bottom, the left and the right are identified, respectively.

The length of a shortest non-contractible cycle of a graph embedded on a non-spherical surface is called the edge-width of the graph. Note that $G_{k}$ in the proof of Theorem 11 is bipartite for every $k \geq 0$, has girth 6 and edge-width exactly 6 for every $k \geq 1$.

After submitting the first version of this paper, the authors were informed that Zhai, Wei, He and Ye [15] also proved Theorem [11, together with the case of the Klein bottle. It was also announced that their constructed hexangulations can have arbitrary large edgewidth.

$G_{1}$


Figure 5: Hexangulations $G_{0}, G_{1}$ of the torus and $T$ of the annulus.

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