Extensions of a theorem of Erdős on nonhamiltonian graphs*

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March 29, 2017

Abstract

Let n, d be integers with $1 \leq d \leq \lfloor \frac{n-1}{2} \rfloor$, and set $h(n, d) := \binom{n-d}{2} + d^2$. Erdős proved that when $n \geq 6d$, each nonhamiltonian graph G on n vertices with minimum degree $\delta(G) \geq d$ has at most $h(n, d)$ edges. He also provides a sharpness example $H_{n,d}$ for all such pairs n, d. Previously, we showed a stability version of this result: for n large enough, every nonhamiltonian graph G on n vertices with $\delta(G) \geq d$ and more than $h(n, d+1)$ edges is a subgraph of $H_{n,d}$.

In this paper, we show that not only does the graph $H_{n,d}$ maximize the number of edges among nonhamiltonian graphs with n vertices and minimum degree at least d, but in fact it maximizes the number of copies of any fixed graph F when n is sufficiently large in comparison with d and $|F|$. We also show a stronger stability theorem, that is, we classify all nonhamiltonian n-graphs with $\delta(G) \geq d$ and more than $h(n, d+2)$ edges. We show this by proving a more general theorem: we describe all such graphs with more than $\binom{n-(d+2)}{k} + (d+2)\binom{d+2}{k-1}$ copies of K_k for any k. **Mathematics Subject Classification:** 05C35, 05C38.

Keywords: Subgraph density, hamiltonian cycles, extremal graph theory.

1 Introduction

Let $V(G)$ denote the vertex set of a graph G, $E(G)$ denote the edge set of G, and $e(G) = |E(G)|$. Also, if $v \in V(G)$, then $N(v)$ is the neighborhood of v and $d(v) = |N(v)|$. If $v \in V(G)$ and $D \subset V(G)$ then for shortness we will write $D + v$ to denote $D \cup \{v\}$. For $k, t \in \mathbb{N}$, $(k)_t$ denotes the falling factorial $k(k-1)\ldots(k-t+1) = \frac{k!}{(k-t)!}$.

The first Turán-type result for nonhamiltonian graphs was due to Ore [11]:

Theorem 1 (Ore [11]). *If G is a nonhamiltonian graph on n vertices, then $e(G) \leq \binom{n-1}{2} + 1$.*

This bound is achieved only for the n-vertex graph obtained from the complete graph K_{n-1} by adding a vertex of degree 1. Erdős [4] refined the bound in terms of the minimum degree of the graph:

*This paper started at SQUARES meeting of the American Institute of Mathematics (April 2016).

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Theorem 2 (Erdős [4])**.** *Let n, d be integers with $1 \le d \le \lfloor \frac{n-1}{2} \rfloor$, and set $h(n,d) := \binom{n-d}{2} + d^2$. If G is a nonhamiltonian graph on n vertices with minimum degree $\delta(G) \ge d$, then*

$$e(G) \le \max \left\{ h(n,d), h(n, \left\lfloor \frac{n-1}{2} \right\rfloor) \right\} =: e(n,d).$$

This bound is sharp for all $1 \le d \le \lfloor \frac{n-1}{2} \rfloor$.

To show the sharpness of the bound, for $n, d \in \mathbb{N}$ with $d \le \lfloor \frac{n-1}{2} \rfloor$, consider the graph $H_{n,d}$ obtained from a copy of K_{n-d}, say with vertex set A, by adding d vertices of degree d each of which is adjacent to the same d vertices in A. An example of $H_{11,3}$ is on the left of Fig 1.

Figure 1: Graphs $H_{11,3}$ (left) and $K'_{11,3}$ (right).

By construction, $H_{n,d}$ has minimum degree d, is nonhamiltonian, and $e(H_{n,d}) = \binom{n-d}{2} + d^2 = h(n,d)$. Elementary calculation shows that $h(n,d) > h(n, \lfloor \frac{n-1}{2} \rfloor)$ in the range $1 \le d \le \lfloor \frac{n-1}{2} \rfloor$ if and only if $d < (n+1)/6$ and n is odd or $d < (n+4)/6$ and n is even. Hence there exists a $d_0 := d_0(n)$ such that

$$e(n,1) > e(n,2) > \cdots > e(n,d_0) = e(n,d_0+1) = \cdots = e(n, \left\lfloor \frac{n-1}{2} \right\rfloor),$$

where $d_0(n) := \lceil \frac{n+1}{6} \rceil$ if n is odd, and $d_0(n) := \lceil \frac{n+4}{6} \rceil$ if n is even. Therefore $H_{n,d}$ is an extremal example of Theorem 2 when $d < d_0$ and $H_{n,\lfloor (n-1)/2 \rfloor}$ when $d \ge d_0$.

In [10] and independently in [6] a stability theorem for nonhamiltonian graphs with prescribed minimum degree was proved. Let $K'_{n,d}$ denote the edge-disjoint union of K_{n-d} and K_{d+1} sharing a single vertex. An example of $K'_{11,3}$ is on the right of Fig 1.

Theorem 3 ([10, 6])**.** *Let $n \ge 3$ and $d \le \lfloor \frac{n-1}{2} \rfloor$. Suppose that G is an n-vertex nonhamiltonian graph with minimum degree $\delta(G) \ge d$ such that*

$$e(G) > e(n, d+1) = \max \left\{ h(n, d+1), h(n, \left\lfloor \frac{n-1}{2} \right\rfloor) \right\}. \tag{1}$$

Then G is a subgraph of either $H_{n,d}$ or $K'_{n,d}$.

One of the main results of this paper shows that when n is large enough with respect to d and t, $H_{n,d}$ not only has the most edges among n-vertex nonhamiltonian graphs with minimum degree at least d, but also has the most copies of *any* t-vertex graph. This is an instance of a generalization of the Turán problem called *subgraph density problem*: for $n \in \mathbb{N}$ and graphs T and H, let $ex(n,T,H)$ denote the maximum possible number of (unlabeled) copies of T in an n-vertex H-free graph. When $T = K_2$, we have the usual extremal number $ex(n,T,H) = ex(n,H)$.

Some notable results on the function $ex(n,T,H)$ for various combinations of T and H were obtained in [5, 2, 1, 8, 9, 7]. In particular, Erdős [5] determined $ex(n,K_s,K_t)$, Bollobás and Győri [2] found the order of magnitude of $ex(n,C_3,C_5)$, Alon and Shikhelman [1] presented a series of bounds on $ex(n,T,H)$ for different classes of T and H.

In this paper, we study the maximum number of copies of T in nonhamiltonian n-vertex graphs, i.e. $ex(n,T,C_n)$. For two graphs G and T, let $N(G,T)$ denote the number of *labeled* copies of T that are subgraphs of G, i.e., the number of injections $\phi : V(T) \to V(G)$ such that for each $xy \in E(T)$, $\phi(x)\phi(y) \in E(G)$. Since for every T and H, $|Aut(T)|\, ex(n,T,H)$ is the maximum of $N(G,T)$ over the n-vertex graphs G not containing H, some of our results are in the language of labeled copies of T in G. For $k \in \mathbb{N}$, let $N_k(G)$ denote the number of unlabeled copies of K_k's in G. Since $|Aut(K_k)| = k!$, we have $N_k(G) = N(G,K_k)/k!$.

2 Results

As an extension of Theorem 2, we show that for each fixed graph F and any d, if n is large enough with respect to $|V(F)|$ and d, then among all n-vertex nonhamiltonian graphs with minimum degree at least d, $H_{n,d}$ contains the maximum number of copies of F.

Theorem 4. *For every graph F with $t := |V(F)| \geq 3$, any $d \in \mathbb{N}$, and any $n \geq n_0(d,t) := 4dt + 3d^2 + 5t$, if G is an n-vertex nonhamiltonian graph with minimum degree $\delta(G) \geq d$, then $N(G,H) \leq N(H_{n,d},F)$.*

On the other hand, if F is a star $K_{1,t-1}$ and $n \leq dt - d$, then $H_{n,d}$ does not maximize $N(G,F)$. At the end of Section 4 we show that in this case, $N(H_{n,\lfloor (n-1)/2\rfloor},F) > N(H_{n,d},F)$. So, the bound on $n_0(d,t)$ in Theorem 4 has the right order of magnitude when $d = O(t)$.

An immediate corollary of Theorem 4 is the following generalization of Theorem 1

Corollary 5. *For every graph F with $t := |V(F)| \geq 3$ and any $n \geq n_0(t) := 9t + 3$, if G is an n-vertex nonhamiltonian graph, then $N(G,H) \leq N(H_{n,1},F)$.*

We consider the case that F is a clique in more detail. For $n, k \in \mathbb{N}$, define on the interval $[1, \lfloor (n-1)/2 \rfloor]$ the function

$$h_k(n,x) := \binom{n-x}{k} + x\binom{x}{k-1}. \tag{2}$$

We use the convention that for $a \in \mathbb{R}$, $b \in \mathbb{N}$, $\binom{a}{b}$ is the polynomial $\frac{1}{b!}a \times (a-1) \times \ldots \times (a-b+1)$ if $a \geq b-1$ and 0 otherwise.

By considering the second derivative, one can check that for any fixed k and n, as a function of x, $h_k(n,x)$ is convex on $[1, \lfloor (n-1)/2 \rfloor]$, hence it attains its maximum at one of the endpoints, $x = 1$ or $x = \lfloor (n-1)/2 \rfloor$. When $k = 2$, $h_2(n,x) = h(n,x)$. We prove the following generalization of Theorem 2.

Theorem 6. *Let n, d, k be integers with $1 \leq d \leq \left\lfloor \frac{n-1}{2} \right\rfloor$ and $k \geq 2$. If G is a nonhamiltonian graph* *on n vertices with minimum degree $\delta(G) \geq d$, then the number $N_k(G)$ of k-cliques in G satisfies*

$$N_k(G) \leq \max\left\{h_k(n,d), h_k(n, \left\lfloor \frac{n-1}{2} \right\rfloor)\right\}$$

Again, graphs $H_{n,d}$ and $H_{n,\lfloor (n-1)/2 \rfloor}$ are sharpness examples for the theorem.

Finally, we present a stability version of Theorem 6. To state the result, we first define the family of extremal graphs.

Fix $d \leq \lfloor (n-1)/2 \rfloor$. In addition to graphs $H_{n,d}$ and $K'_{n,d}$ defined above, define $H'_{n,d}$: $V(H'_{n,d}) = A \cup B$, where A induces a complete graph on $n-d-1$ vertices, B is a set of $d+1$ vertices that induce exactly one edge, and there exists a set of vertices $\{a_1, \ldots, a_d\} \subseteq A$ such that for all $b \in B$, $N(b) - B = \{a_1, \ldots, a_d\}$. Note that contracting the edge in $H'_{n,d}[B]$ yields $H_{n-1,d}$. These graphs are illustrated in Fig. 2

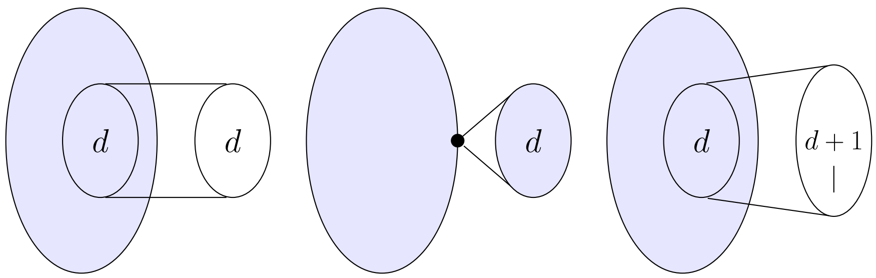

Figure 2: Graphs $H_{n,d}$ (left), $K'_{n,d}$ (center), and $H'_{n,d}$ (right), where shaded background indicates a complete graph.

We also have two more extremal graphs for the cases $d = 2$ or $d = 3$. Define the nonhamiltonian n-vertex graph $G'_{n,2}$ with minimum degree 2 as follows: $V(G'_{n,2}) = A \cup B$ where A induces a clique or order $n-3$, $B = \{b_1, b_2, b_3\}$ is an independent set of order 3, and there exists $\{a_1, a_2, a_3, x\} \subseteq A$ such that $N(b_i) = \{a_i, x\}$ for $i \in \{1, 2, 3\}$ (see the graph on the left in Fig. 3).

The nonhamiltonian n-vertex graph $F_{n,3}$ with minimum degree 3 has vertex set $A \cup B$, where A induces a clique of order $n-4$, B induces a perfect matching on 4 vertices, and each of the vertices in B is adjacent to the same two vertices in A (see the graph on the right in Fig. 3).

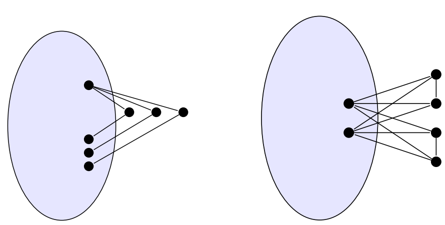

Figure 3: Graphs $G'_{n,2}$ (left) and $F_{n,3}$ (right).

Our stability result is the following:

Theorem 7. *Let $n \geq 3$ and $1 \leq d \leq \left\lfloor \frac{n-1}{2} \right\rfloor$. Suppose that G is an n-vertex nonhamiltonian graph* *with minimum degree $\delta(G) \geq d$ such that there exists $k \geq 2$ for which*

$$N_k(G) > \max\left\{h_k(n, d+2), h_k(n, \left\lfloor \frac{n-1}{2} \right\rfloor)\right\}. \tag{3}$$

Let $\mathcal{H}_{n,d} := \{H_{n,d}, H_{n,d+1}, K'_{n,d}, K'_{n,d+1}, H'_{n,d}\}$.
(i) If $d = 2$, then G is a subgraph of $G'_{n,2}$ or of a graph in $\mathcal{H}_{n,2}$;
(ii) if $d = 3$, then G is a subgraph of $F_{n,3}$ or of a graph in $\mathcal{H}_{n,3}$;
(iii) if $d = 1$ or $4 \leq d \leq \lfloor \frac{n-1}{2} \rfloor$, then G is a subgraph of a graph in $\mathcal{H}_{n,d}$.

The result is sharp because $H_{n,d+2}$ has $h_k(n, d+2)$ copies of K_k, minimum degree $d+2 > d$, is nonhamiltonian and is not contained in any graph in $\mathcal{H}_{n,d} \cup \{G'_{n,2}, F_{n,3}\}$.

The outline for the rest of the paper is as follows: in Section 3 we present some structural results for graphs that are edge-maximal nonhamiltonian to be used in the proofs of the main theorems, in Section 4 we prove Theorem 4, in Section 5 we prove Theorem 6 and give a cliques version of Theorem 3, and in Section 6 we prove Theorem 7.

3 Structural results for saturated graphs

We will use a classical theorem of Pósa (usually stated as its contrapositive).

Theorem 8 (Pósa [12])**.** *Let $n \geq 3$. If G is a nonhamiltonian n-vertex graph, then there exists $1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$ such that G has a set of k vertices with degree at most k.*

Call a graph G *saturated* if G is nonhamiltonian but for each $uv \notin E(G)$, $G + uv$ has a hamiltonian cycle. Ore's proof [11] of Dirac's Theorem [3] yields that

$$d(u) + d(v) \leq n - 1 \tag{4}$$

for every n-vertex saturated graph G and for each $uv \notin E(G)$.

We will also need two structural results for saturated graphs which are easy extensions of Lemmas 6 and 7 in [6].

Lemma 9. *Let G be a saturated n-vertex graph with $N_k(G) > h_k(n, \lfloor \frac{n-1}{2} \rfloor)$ for any $k \geq 2$. Then for some $1 \leq r \leq \lfloor \frac{n-1}{2} \rfloor$, $V(G)$ contains a subset D of r vertices of degree at most r such that $G - D$ is a complete graph.*

Proof. Since G is nonhamiltonian, by Theorem 8, there exists some $1 \leq r \leq \lfloor \frac{n-1}{2} \rfloor$ such that G has r vertices with degree at most r. Pick the maximum such r, and let D be the set of the vertices with degree at most r. Since $h_k(G) > h(n, \lfloor \frac{n-1}{2} \rfloor)$, $r < \lfloor \frac{n-1}{2} \rfloor$. So, by the maximality of r, $|D| = r$.

Suppose there exist $x, y \in V(G) - D$ such that $xy \notin E(G)$. Among all such pairs, choose x and y with the maximum $d(x)$. Since $y \notin D$, $d(y) > r$. Let $D' := V(G) - N(x) - \{x\}$ and $r' := |D'| = n - 1 - d(x)$. By (4),

$$d(z) \leq n - 1 - d(x) = r' \text{ for all } z \in D'. \tag{5}$$

So D' is a set of r' vertices of degree at most r'. Since $y \in D'$, $r' \geq d(y) > r$. Thus by the maximality of r, we get $r' = n-1-d(x) > \lfloor \frac{n-1}{2} \rfloor$. Equivalently, $d(x) < \lceil \frac{n-1}{2} \rceil$. For all $z \in D' + \{x\}$, either $z \in D$ where $d(z) \leq r \leq \lfloor \frac{n-1}{2} \rfloor$, or $z \in V(G) - D$, and so $d(z) \leq d(x) \leq \lfloor \frac{n-1}{2} \rfloor$.

Now we count the number of k-cliques in G: Among $V(G) - D'$, there are at most $\binom{n-r'}{k}$ k-cliques. Also, each vertex in D' can be in at most $\binom{r'}{k-1}$ k-cliques. Therefore $N_k(G) \leq \binom{n-r'}{k} + r'\binom{r'}{k-1} \leq h_k(n, \lfloor \frac{n-1}{2} \rfloor)$, a contradiction. $\square$

Also, repeating the proof of Lemma 7 in [6] gives the following lemma.

Lemma 10 (Lemma 7 in [6])**.** *Under the conditions of Lemma 9, if $r = \delta(G)$, then $G = H_{n,\delta(G)}$ or $G = K'_{n,\delta(G)}$.*

4 Maximizing the number of copies of a given graph and a proof of Theorem 4

In order to prove Theorem 4, we first show that for any fixed graph F and any d, of the two extremal graphs of Lemma 10, if n is large then $H_{n,d}$ has at least as many copies of F as $K'_{n,d}$.

Lemma 11. *For any $d, t, n \in \mathbb{N}$ with $n \geq 2dt + d + t$ and any graph F with $t = |V(F)|$ we have $N(K'_{n,d}, F) \leq N(H_{n,d}, F)$.*

Proof. Fix F and $t = |V(F)|$. Let $K'_{n,d} = A \cup B$ where A and B are cliques of order $n-d$ and $d+1$ respectively and $A \cap B = \{v^*\}$, the cut vertex of $K'_{n,d}$. Also, let D denote the independent set of order d in $H_{n,d}$. We may assume $d \geq 2$, because $H_{n,1} = K'_{n,1}$. If x is an isolated vertex of F then for any n-vertex graph G we have $N(G,F) = (n-t+1)N(G, F-x)$. So it is enough to prove the case $\delta(F) \geq 1$, and we may also assume $t \geq 3$.

Because both $K'_{n,d}[A]$ and $H_{n,d} - D$ are cliques of order $n-d$, the number of embeddings of F into $K'_{n,d}[A]$ is the same as the number of embeddings of F into $H_{n,d} - D$. So it remains to compare only the number of embeddings in $\Phi := \{\varphi : V(F) \to V(K'_{n,d})$ such that $\varphi(F)$ intersects $B - v^*\}$ to the number of embeddings in $\Psi := \{\psi : V(F) \to V(H_{n,d})$ such that $\psi(F)$ intersects $D\}$.

Let $C \cup \overline{C}$ be a partition of the vertex set $V(F)$, $s := |C|$. Define the following classes of Φ and Ψ

— $\Phi(C) := \{\varphi : V(F) \to V(K'_{n,d})$ such that $\varphi(C)$ intersects $B - v^*$, $\varphi(C) \subseteq B$, and $\varphi(\overline{C}) \subseteq V - B\}$,

— $\Psi(C) := \{\psi : V(F) \to V(H_{n,d})$ such that $\psi(C)$ intersects D, $\psi(C) \subseteq (D \cup N(D))$, and $\psi(\overline{C}) \subseteq V - (D \cup N(D))\}$.

By these definitions, if $C \neq C'$ then $\Phi(C) \cap \Phi(C') = \emptyset$, and $\Psi(C) \cap \Psi(C') = \emptyset$. Also $\bigcup_{\emptyset \neq C \subseteq V(F)} \Phi(C) = \Phi$. We claim that for every $C \neq \emptyset$,

$$|\Phi(C)| \leq |\Psi(C)|. \tag{6}$$

Summing up the number of embeddings over all choices for C will prove the lemma. If $\Phi(C) = \emptyset$, then (6) obviously holds. So from now on, we consider the cases when $\Phi(C)$ is not empty, implying $1 \leq s \leq d+1$.

Case 1: There is an F-edge joining $\overline{C}$ and C. So there is a vertex $v \in C$ with $N_F(v) \cap \overline{C} \neq \emptyset$. Then for every mapping $\varphi \in \Phi(C)$, the vertex v must be mapped to v^* in $K'_{n,d}$, $\varphi(v) = v^*$. So this vertex v is uniquely determined by C. Also, $\varphi(C) \cap (B - v^*) \neq \emptyset$ implies $s \geq 2$. The rest of C can be mapped arbitrarily to $B - v^*$ and $\overline{C}$ can be mapped arbitrarily to $A - v^*$. We obtained that $|\Phi(C)| = (d)_{s-1}(n - d - 1)_{t-s}$.

We make a lower bound for $|\Psi(C)|$ as follows. We define a $\psi \in \Psi(C)$ by the following procedure. Let $\psi(v) = x \in N(D)$ (there are d possibilities), then map some vertex of $C - v$ to a vertex $y \in D$ (there are $(s-1)d$ possibilities). Since $N + y$ forms a clique of order $d+1$ we may embed the rest of C into $N - v$ in $(d-1)_{s-2}$ ways and finish embedding of F into $H_{n,d}$ by arbitrarily placing the vertices of $\overline{C}$ to $V - (D \cup N(D))$. We obtained that $|\Psi(C)| \geq d^2(s-1)(d-1)_{s-2}(n-2d)_{t-s} = d(s-1)(d)_{s-1}(n-2d)_{t-s}$.

Since $s \geq 2$ we have that

$$\begin{aligned}
\frac{|\Psi(C)|}{|\Phi(C)|} \geq \frac{d(s-1)(d)_{s-1}(n-2d)_{t-s}}{(d)_{s-1}(n-d-1)_{t-s}} &\geq d(2-1)\left(\frac{n-2d+1-t+s}{n-d-t+s}\right)^{t-s} \\
&= d\left(1 - \frac{d-1}{n-d-t+s}\right)^{t-s} \\
&\geq d\left(1 - \frac{(d-1)(t-s)}{n-d-t+s}\right) \\
&\geq d\left(1 - \frac{(d-1)t}{n-d-t}\right) \\
&> 1 \text{ when } n > dt + d + t.
\end{aligned}$$

Case 2: C and $\overline{C}$ are not connected in F. We may assume $s \geq 2$ since C is a union of components with $\delta(F) \geq 1$. In $K'_{n,d}$ there are at exactly $(d+1)_s(n-d-1)_{t-s}$ ways to embed F into B so that only C is mapped into B and $\overline{C}$ goes to $A - v^*$, i.e., $|\Phi(C)| = (d+1)_s(n-d-1)_{t-s}$.

We make a lower bound for $|\Psi(C)|$ as follows. We define a $\psi \in \Psi(C)$ by the following procedure. Select any vertex $v \in C$ and map it to some vertex in D (there are sd possibilities), then map $C - v$ into $N(D)$ (there are $(d)_{s-1}$ possibilities) and finish embedding of F into $H_{n,d}$ by arbitrarily placing the vertices of $\overline{C}$ to $V - (D \cup N(D))$. We obtained that $|\Psi(C)| \geq ds(d)_{s-1}(n-2d)_{t-s}$. We have

$$\begin{aligned}
\frac{|\Psi(C)|}{|\Phi(C)|} \geq \frac{ds(d)_{s-1}(n-2d)_{t-s}}{(d+1)_s(n-d-1)_{t-s}} &\geq \frac{ds}{d+1}\left(1 - \frac{(d-1)t}{n-d-t}\right) \\
&\geq \frac{2d}{d+1}\left(1 - \frac{(d-1)t}{n-d-t}\right) \text{ because } s \geq 2 \\
&> 1 \text{ when } n > 2dt + d + t.
\end{aligned}$$

□

We are now ready to prove Theorem 4.

Theorem 4. *For every graph F with $t := |V(F)| \geq 3$, any $d \in \mathbb{N}$, and any $n \geq n_0(d,t) := 4dt + 3d^2 + 5t$, if G is an n-vertex nonhamiltonian graph with minimum degree $\delta(G) \geq d$, then $N(G, H) \leq N(H_{n,d}, F)$.*

Proof. Let $d \geq 1$. Fix a graph F with $|V(F)| \geq 3$ (if $|V(F)| = 2$, then either $F = K_2$ or $F = \overline{K}_2$). The case where G has isolated vertices can be handled by induction on the number of isolated vertices, hence we may assume each vertex has degree at least 1. Set

$$n_0 = 4dt + 3d^2 + 5t. \tag{7}$$

Fix a nonhamiltonian graph G with $|V(G)| = n \geq n_0$ and $\delta(G) \geq d$ such that $N(G,F) > N(H_{n,d},F) \geq (n-d)_t$. We may assume that G is saturated, as the number of copies of F can only increase when we add edges to G.

Because $n \geq 4dt + t$ by (7),

$$\begin{aligned}\frac{(n-d)_t}{(n)_t} &\geq \left(\frac{n-d-t}{n-t}\right)^t = \left(1 - \frac{d}{n-t}\right)^t \\ &\geq 1 - \frac{dt}{n-t} \geq 1 - \frac{1}{4} = \frac{3}{4}.\end{aligned}$$

So, $(n-d)_t \geq \frac{3}{4}(n)_t$.

After mapping edge xy of F to an edge of G (in two labeled ways), we obtain the loose upper bound,

$$2e(G)(n-2)_{t-2} \geq N(G,F) \geq (n-d)_t \geq \frac{3}{4}(n)_t,$$

therefore

$$e(G) \geq \frac{3}{4}\binom{n}{2} > h_2(n, \lfloor (n-1)/2 \rfloor). \tag{8}$$

By Pósa's theorem (Theorem 8), there exists some $d \leq r \leq \lfloor (n-1)/2 \rfloor$ such that G contains a set R or r vertices with degree at most r. Furthermore by (8), $r < d_0$. So by integrality, $r \leq d_0 - 1 \leq (n+3)/6$. If $r = d$, then by Lemma 10, either $G = H_{n,d}$ or $G = K'_{n,d}$. By Lemma 11 and (7), $G = H_{n,d}$, a contradiction. So we have $r \geq d+1$.

Let $\mathcal{I}$ denote the family of all nonempty independent sets in F. For $I \in \mathcal{I}$, let $i = i(I) := |I|$ and $j = j(I) := |N_F(I)|$. Since F has no isolated vertices, $j(I) \geq 1$ and so $i \leq t-1$ for each $I \in \mathcal{I}$. Let $\Phi(I)$ denote the set of embeddings $\varphi : V(F) \to V(G)$ such that $\phi(I) \subseteq R$ and I is a maximum independent subset of $\phi^{-1}(R \cap \varphi(F))$. Note that $\varphi(I)$ is not necessarily independent in G. We show that

$$|\Phi(I)| \leq (r)_i r (n-r)_{t-i-1}. \tag{9}$$

Indeed, there are $(r)_i$ ways to choose $\phi(I) \subseteq R$. After that, since each vertex in R has at most r neighbors in G, there are at most r^j ways to embed $N_F(I)$ into G. By the maximality of I, all vertices of $F - I - N_F(I)$ should be mapped to $V(G) - R$. There are at most $(n-r)_{t-i-j}$ to do it. Hence $|\Phi(I)| \leq (r)_i r^j (n-r)_{t-i-j}$. Since $2r + t \leq 2(d_0 - 1) + t < n$, this implies (9).

Since each $\varphi : V(F) \to V(G)$ with $\varphi(V(F)) \cap R \neq \emptyset$ belongs to $\Phi(I)$ for some nonempty $I \in \mathcal{I}$, (9) implies

$$N(G,F) \leq (n-r)_t + \sum_{\emptyset \neq I \in \mathcal{I}} |\Phi(I)| \leq (n-r)_t + \sum_{i=1}^{t-1} \binom{t}{i} (r)_i r (n-r)_{t-i-1}. \tag{10}$$

Hence

$$\begin{aligned}
\frac{N(G,F)}{N(H_{n,d},F)} &\leq \frac{(n-r)_t + \sum_{i=1}^{t-1}\binom{t}{i}(r)_i r(n-r)_{t-i-1}}{(n-d)_t} \\
&\leq \frac{(n-r)_t}{(n-d)_t} + \frac{1}{(n-d)_t} \times \frac{r}{n-r-t+2}\sum_{i=1}^{t-1}\binom{t}{i}(r)_i(n-r)_{t-i} \\
&= \frac{(n-r)_t}{(n-d)_t} + \frac{(n)_t-(n-r)_t-(r)_t}{(n-d)_t} \times \frac{r}{n-r-t+2} \\
&\leq \frac{(n-r)_t}{(n-d)_t} \times \frac{n-t+2-2r}{n-t+2-r} + \frac{(n)_t}{(n-d)_t} \times \frac{r}{n-t+2-r} := f(r).
\end{aligned}$$

Given fixed n, d, t, we claim that the real function $f(r)$ is convex for $0 < r < (n-t+2)/2$.

Indeed, the first term $g(r) := \frac{(n-r)_t}{(n-d)_t} \times \frac{n-t+2-2r}{n-t+2-r}$ is a product of t linear terms in each of which r has a negative coefficient (note that the $n-t+2-r$ term cancels out with a factor of $n-r-t+2$ in $(n-r)_t$). Applying product rule, the first derivative g' is a sum of t products, each with $t-1$ linear terms. For $r < (n-t+2)/2$, each of these products is negative, thus $g'(r) < 0$. Finally, applying product rule again, g'' is the sum of $t(t-1)$ products. For $r < (n-t+2)/2$ each of the products is positive, thus $g''(r) > 0$.

Similarly, the second factor of the second term (as a real function of r, of the form $r/(c-r)$) is convex for $r < n-t+2$.

We conclude that in the interval $[d+1, (n+3)/6]$ the function $f(r)$ takes its maximum either at one of the endpoints $r = d+1$ or $r = (n+3)/6$. We claim that $f(r) < 1$ at both end points.

In case of $r = d+1$ the first factor of the first term equals $(n-d-t)/(n-d)$. To get an upper bound for the first factor of the second term one can use the inequality $\prod(1+x_i) < 1 + 2\sum x_i$ which holds for any number of non-negative x_i's if $0 < \sum x_i \leq 1$. Because $dt/(n-d-t+1) \leq 1$ by (7), we obtain that

$$\begin{aligned}
f(d+1) &< \frac{n-d-t}{n-d} \times \frac{n-t-2d}{n-t-d+1} + \left(1 + \frac{2dt}{n-d-t+1}\right) \times \frac{d+1}{n-t-d+1} \\
&= \left(1 - \frac{t}{n-d}\right) \times \left(1 - \frac{d+1}{n-t-d+1}\right) + \left(\frac{d+1}{n-t-d+1}\right) + \left(\frac{2dt(d+1)}{(n-t-d+1)^2}\right) \\
&= 1 - \frac{t}{n-d} + \frac{t}{n-d} \times \frac{d+1}{n-t-d+1} + \frac{t}{n-d} \times \frac{2d(d+1)}{n-t-d+1} \times \frac{n-d}{n-t-d+1} \\
&= 1 - \frac{t}{n-d} \times \left(1 - \frac{d+1}{n-t-d+1} - \frac{2d(d+1)}{n-t-d+1} \times \left(1 + \frac{t-1}{n-t-d+1}\right)\right) \\
&< 1 - \frac{t}{n-d} \times (1 - \frac{1}{4t} - \frac{2}{3}(1 + \frac{1}{4d})) \\
&\leq 1 - \frac{t}{n-d} \times (1 - 1/12 - 2/3 \times 5/4) \\
&< 1.
\end{aligned}$$

Here we used that $n \geq 3d^2 + 2d + t$ and $n \geq 4dt + 5t + d$ by (7), $t \geq 3$, and $d \geq 1$.

To bound $f(r)$ for other values of r, let us use $1 + x \le e^x$ (true for all x). We get

$$f(r) < \exp\left\{-\frac{(r-d)t}{n-d-t+1}\right\} + \frac{r}{n-r-t+2} \times \exp\left\{\frac{dt}{n-d-t+1}\right\}.$$

When $r = (n+3)/6$, $t \ge 3$, and $n \ge 24d$ by (7), the first term is at most $e^{-18/46} = 0.676....$ Moreover, for $n \ge 9t$ (7) (therefore $n \ge 27$) we get that $\frac{r}{n-r-t+2}$ is maximized when t is maximized, i.e., when $t = n/9$. The whole term is at most $(3n+9)/(13n+27) \times e^{1/4} \le 5/21 \times e^{1/4} = 0.305...$, so in this range, $f((n+3)/6) < 1$.

By the convexity of $f(r)$, we have $N(G,F) < N(H_{n,d},F)$. □

When F is a star, then it is easy to determine $\max N(G,F)$ for all n.

Claim 12. *Suppose $F = K_{1,t-1}$ with $t := |V(F)| \ge 3$, and $t \le n$ and d are integers with $1 \le d \le \lfloor (n-1)/2 \rfloor$. If G is an n-vertex nonhamiltonian graph with minimum degree $\delta(G) \ge d$, then*

$$N(G,F) \le \max\left\{H_{n,d}, H_{n,\lfloor (n-1)/2 \rfloor}\right\}, \tag{11}$$

and equality holds if and only if $G \in \left\{H_{n,d}, H_{n,\lfloor (n-1)/2 \rfloor}\right\}$.

Proof. The number of copies of stars in a graph G depends only on the degree sequence of the graph: if a vertex v of a graph G has degree $d(v)$, then there are $(d(v))_{t-1}$ labeled copies of F in G where v is the center vertex. We have

$$N(G,F) = \sum_{v \in V(G)} \binom{d(v)}{t-1}. \tag{12}$$

Since G is nonhamiltonian, Pósa's theorem yields an $r \le \lfloor (n-1)/2 \rfloor$, and an r-set $R \subset V(G)$ such that $d_G(v) \le r$ for all $v \in R$. Take the minimum such r, then there exists a vertex $v \in R$ with $\deg(v) = r$. We may also suppose that G is edge-maximal nonhamiltonian, so Ore's condition (4) holds. It implies that $\deg(w) \le n - r - 1$ for all $w \notin N(v)$. Altogether we obtain that G has r vertices of degree at most r, at least $n - 2r$ vertices (those in $V(G) - R - N(v)$) of degree at most $(n - r - 1)$. This implies that the right hand side of (12) is at most

$$r \times (r)_{t-1} + (n-2r) \times (n-r-1)_{t-1} + r \times (n-1)_{t-1} = N(H_{n,r}, F).$$

(Here equality holds only if $G = H_{n,r}$). Note that $r \in [d, \lfloor \frac{1}{2}(n-1) \rfloor]$. Since for given n and t the function $N(H_{n,r}, F)$ is strictly convex in r, it takes its maximum at one of the endpoints of the interval. □

Remark 13. *As it was mentioned in Section 2, $O(dt)$ is the right order for $n_0(d,t)$ when $d = O(t)$.*

To see this, fix $d \in \mathbb{N}$ and let F be the star on $t \ge 3$ vertices. If $d < \lfloor (n-1)/2 \rfloor$, $t \le n$ and $n \le dt - d$, then $H_{n,\lfloor (n-1)/2 \rfloor}$ contains more copies of F than $H_{n,d}$ does, the maximum in (11) is reached for $r = \lfloor (n-1)/2 \rfloor$. We present the calculation below only for $2d + 7 \le n \le dt - d$, the case $2d + 3 \le n \le 2d + 6$ can be checked by hand by plugging n into the first line of the formula below. We can proceed as follows.

$$
\begin{aligned}
N(H_{n,\lfloor (n-1)/2\rfloor},F)-N(H_{n,d},F) &= \Big(\lfloor (n-1)/2\rfloor (n-1)_{t-1}+\lceil (n+1)/2\rceil (\lfloor (n-1)/2\rfloor)_{t-1}\Big)\\
&\quad -\Big(d(n-1)_{t-1}+(n-2d)(n-d-1)_{t-1}+d(d)_{t-1}\Big)\\
&= \Big(\lfloor (n-1)/2\rfloor - d\Big)(n-1)_{t-1}-(n-2d)(n-d-1)_{t-1}\\
&\quad +\lceil (n+1)/2\rceil (\lfloor (n-1)/2\rfloor)_{t-1}-d(d)_{t-1}\\
&> \Big(\lfloor (n-1)/2\rfloor - d\Big)(n-1)_{t-1}-\Big((n-2d)(1-d/n)^{t-1}\Big)(n-1)_{t-1}\\
&> (n-1)_{t-1}\Big(\lfloor (n-1)/2\rfloor - d-(n-2d)e^{-(dt-d)/n}\Big)\\
&\geq (n-1)_{t-1}\left(\lfloor (n-1)/2\rfloor - d-(n-2d)/e\right)\\
&\geq 0.
\end{aligned}
$$

5 Theorem 6 and a stability version of it

In general, it is difficult to calculate the exact value of $N(H_{n,d},F)$ for a fixed graph F. However, when $F=K_k$, we have $N(H_{n,d},K_k)=h_k(n,d)k!$. Recall Theorem 6:

Let n,d,k be integers with $1\leq d\leq \left\lfloor \frac{n-1}{2}\right\rfloor$ and $k\geq 2$. If G is a nonhamiltonian graph on n vertices with minimum degree $\delta(G)\geq d$, then

$$N_k(G)\leq \max\left\{h_k(n,d),h_k(n,\left\lfloor \frac{n-1}{2}\right\rfloor)\right\}.$$

Proof of Theorem 6. By Theorem 8, because G is nonhamiltonian, there exists an $r\geq d$ such that G has r vertices of degree at most r. Denote this set of vertices by D. Then $N_k(G-D)\leq \binom{n-r}{k}$, and every vertex in D is contained in at most $\binom{r}{k-1}$ copies of K_k. Hence $N_k(G)\leq h_k(n,r)$. The theorem follows from the convexity of $h_k(n,x)$. □

Our older stability theorem (Theorem 3) also translates into the the language of cliques, giving a stability theorem for Theorem 6:

Theorem 14. *Let $n\geq 3$, and $d\leq \left\lfloor \frac{n-1}{2}\right\rfloor$. Suppose that G is an n-vertex nonhamiltonian graph with minimum degree $\delta(G)\geq d$ and there exists a $k\geq 2$ such that*

$$N_k(G)>\max\left\{h_k(n,d+1),h_k(n,\left\lfloor \frac{n-1}{2}\right\rfloor)\right\}. \tag{13}$$

Then G is a subgraph of either $H_{n,d}$ or $K'_{n,d}$.

Proof. Take an edge-maximum counterexample G (so we may assume G is saturated). By Lemma 9, G has a set D of $r\leq\lfloor (n-1)/2\rfloor$ vertices such that $G-D$ is a complete graph. If $r\geq d+1$, then $N_k(G)\leq \max\left\{h_k(n,d+1),h_k(n,\left\lfloor \frac{n-1}{2}\right\rfloor)\right\}$. Thus $r=d$, and we may apply Lemma 10. □

6 Discussion and proof of Theorem 7

One can try to refine Theorem 3 in the following direction: What happens when we consider n-vertex nonhamiltonian graphs with minimum degree at least d and less than $e(n, d+1)$ but more than $e(n, d+2)$ edges?

Note that for $d < d_0(n) - 2$,

$$e(n, d) - e(n, d+2) = 2n - 6d - 7,$$

which is greater than n. Theorem 7 answers the question above in a more general form—in terms of s-cliques instead of edges. In other words, we classify all n-vertex nonhamiltonian graphs with more than $\max\left\{h_s(n, d+2), h_s(n, \lfloor \frac{n-1}{2} \rfloor)\right\}$ copies of K_s.

As in Lemma 14, such G can be a subgraph of $H_{n,d}$ or $K'_{n,d}$. Also, G can be a subgraph of $H_{n,d+1}$ or $K'_{n,d+1}$. Recall the graphs $H_{n,d}, K'_{n,d}, H'_{n,d}, G'_{n,2}$, and $F_{n,3}$ defined in the first two sections of this paper and the statement of Theorem 3:

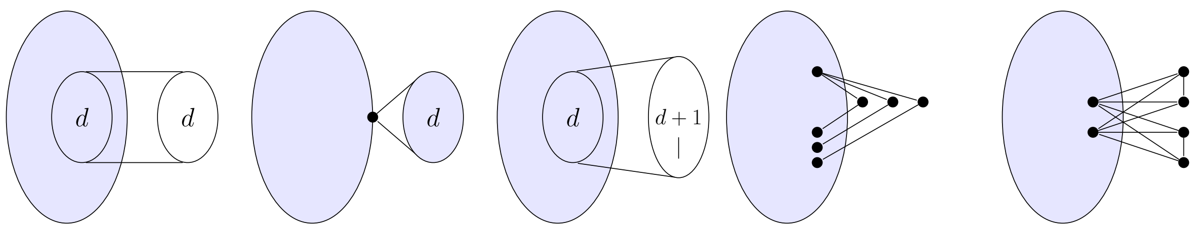

Figure 4: Graphs $H_{n,d}, K'_{n,d}, H'_{n,d}, G'_{n,2}$, and $F_{n,3}$.

Theorem 7. *Let $n \geq 3$ and $1 \leq d \leq \lfloor \frac{n-1}{2} \rfloor$. Suppose that G is an n-vertex nonhamiltonian graph with minimum degree $\delta(G) \geq d$ such that exists a $k \geq 2$ for which*

$$N_k(G) > \max\left\{h_k(n, d+2), h_k(n, \left\lfloor \frac{n-1}{2} \right\rfloor)\right\}.$$

Let $\mathcal{H}_{n,d} := \{H_{n,d}, H_{n,d+1}, K'_{n,d}, K'_{n,d+1}, H'_{n,d}\}$.

(i) If $d = 2$, then G is a subgraph of $G'_{n,2}$ or of a graph in $\mathcal{H}_{n,2}$;
(ii) if $d = 3$, then G is a subgraph of $F_{n,3}$ or of a graph in $\mathcal{H}_{n,3}$;
(iii) if $d = 1$ or $4 \leq d \leq \lfloor \frac{n-1}{2} \rfloor$, then G is a subgraph of a graph in $\mathcal{H}_{n,d}$.

Proof. Suppose G is a counterexample to Theorem 7 with the most edges. Then G is saturated. In particular, degree condition (4) holds for G. So by Lemma 9, there exists an $d \leq r \leq \lfloor (n-1)/2 \rfloor$ such that $V(G)$ contains a subset D of r vertices of degree at most r and $G - D$ is a complete graph.

If $r \geq d+2$, then because $h_k(n, x)$ is convex, $N_k(G) \leq h_k(n, r) \leq \max\left\{h_k(n, d+2), h_k(n, \lfloor \frac{n-1}{2} \rfloor)\right\}$. Therefore either $r = d$ or $r = d+1$. In the case that $r = d$ (and so $r = \delta(G)$), Lemma 10 implies that $G \subseteq H_{n,d}$. So we may assume that $r = d+1$.

If $\delta(G) \geq d+1$, then we simply apply Theorem 3 with $d+1$ in place of d and get $G \subseteq H_{n,d+1}$ or

$G \subseteq K'_{n,d+1}$. So, from now on we may assume

$$\delta(G) = d. \tag{14}$$

Now (14) implies that our theorem holds for $d = 1$, since each graph with minimum degree exactly 1 is a subgraph of $H_{n,1}$. So, below $2 \le d \le \lfloor \frac{n-1}{2} \rfloor$.

Let $N := N(D) - D \subseteq V(G) - D$. The next claim will be used many times throughout the proof.

Lemma 15. *(a) If there exists a vertex $v \in D$ such that $d(v) = d+1$, then $N(v) - D = N$.*
(b) If there exists a vertex $u \in N$ such that u has at least 2 neighbors in D, then u is adjacent to all vertices in D.

Proof. If $v \in D$, $d(v) = d+1$ and some $u \in N$ is not adjacent to v, then $d(v)+d(u) \ge d+1+(n-d-2)+1 = n$. A contradiction to (4) proves (a).

Similarly, if $u \in N$ has at least 2 neighbors in D but is not adjacent to some $v \in D$, then $d(v) + d(u) \ge d + (n-d-2) + 2 = n$, again contradicting (4). □

Define $S := \{u \in V(G) - D : u \in N(v) \text{ for all } v \in D\}$, $s := s$, and $S' := V(G) - D - S$. By Lemma 15 (b), each vertex in S' has at most one neighbor in D. So, for each $v \in D$, call the neighbors of v in S' *the private neighbors of* v.

We claim that

$$D \textit{ is not independent.} \tag{15}$$

Indeed, assume D is independent. If there exists a vertex $v \in D$ with $d(v) = d+1$, then by Lemma 15 (b), $N(v) - D = N$. So, because D is independent, $G \subseteq H_{n,d+1}$. Assume now that every vertex $v \in D$ has degree d, and let $D = \{v_1, \ldots, v_{d+1}\}$.

If $s \ge d$, then because each $v_i \in D$ has degree d, $s = d$ and $N = S$. Then $G \subseteq H_{n,d+1}$. If $s \le d-2$, then each vertex $v_i \in D$ has at least two private neighbors in S'; call these private neighbors x_{v_i} and y_{v_i}. The path $x_{v_1}v_1y_{v_1}x_{v_2}v_2y_{v_2}\ldots x_{v_{d+1}}v_{d+1}y_{v_{d+1}}$ contains all vertices in D and can be extended to a hamiltonian cycle of G, a contradiction.

Finally, suppose $s = d-1$. Then every vertex $v_i \in D$ has exactly one private neighbor. Therefore $G = G'_{n,d}$ where $G'_{n,d}$ is composed of a clique A of order $n-d-1$ and an independent set $D = \{v_1, \ldots, v_{d+1}\}$, and there exists a set $S \subset A$ of size $d-1$ and distinct vertices $z_1, \ldots, z_{d+1}$ such that for $1 \le i \le d+1$, $N(v_i) = S \cup z_i$. Graph $G'_{n,d}$ is illustrated in Fig. 6.

For $d = 2$, we conclude that $G \subseteq G'_{n,2}$, as claimed, and for $d \ge 3$, we get a contradiction since $G'_{n,d}$ is hamiltonian. This proves (15).

Call a vertex $v \in D$ *open* if it has at least two private neighbors, *half-open* if it has exactly one private neighbor, and *closed* if it has no private neighbors.

We say that *paths $P_1, \ldots, P_q$ partition* D, if these paths are vertex-disjoint and $V(P_1) \cup \ldots \cup V(P_q) = D$. The idea of the proof is as follows: because $G - D$ is a complete graph, each path with endpoints in $G - D$ that covers all vertices of D can be extended to a hamiltonian cycle of G. So such a path does not exist, which implies that too few paths cannot partition D:

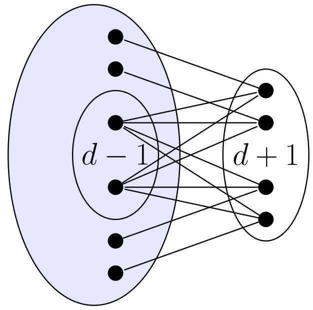

Figure 5: $G'_{n,d}$.

Lemma 16. *If $s \geq 2$ then the minimum number of paths in $G[D]$ partitioning D is at least s.*

Proof. Suppose D can be partitioned into $\ell \leq s-1$ paths $P_1, \dots, P_\ell$ in $G[D]$. Let $S = \{z_1, \dots, z_s\}$. Then $P = z_1 P_1 z_2 \dots z_\ell P_\ell z_{\ell+1}$ is a path with endpoints in $V(G)-D$ that covers D. Because $V(G)-D$ forms a clique, we can find a $z_1, z_{\ell+1}$ - path P' in $G-D$ that covers $V(G)-D-\{z_2, \dots, z_\ell\}$. Then $P \cup P'$ is a hamiltonian cycle of G, a contradiction. □

Sometimes, to get a contradiction with Lemma 16 we will use our information on vertex degrees in $G[D]$:

Lemma 17. *Let H be a graph on r vertices such that for every nonedge xy of H, $d(x)+d(y) \geq r-t$ for some t. Then $V(H)$ can be partitioned into a set of at most t paths. In other words, there exist t disjoint paths $P_1, \dots, P_t$ with $V(H) = \bigcup_{i=1}^{t} V(P_i)$.*

Proof. Construct the graph H' by adding a clique T of size t to H so that every vertex of T is adjacent to each vertex in $V(H)$. For each nonedge $x, y \in H'$,

$$d_{H'}(x) + d_{H'}(y) \geq (r-t) + t + t = r + t = |V(H')|.$$

By Ore's theorem, H' has a hamiltonian cycle C'. Then $C' - T$ is a set of at most t paths in H that cover all vertices of H. □

The next simple fact will be quite useful.

Lemma 18. *If $G[D]$ contains an open vertex, then all other vertices are closed.*

Proof. Suppose $G[D]$ has an open vertex v and another open or half-open vertex u. Let v', v'' be some private neighbors of v in S' and u' be a neighbor of u in S'. By the maximality of G, graph $G + vu'$ has a hamiltonian cycle. In other words, G has a hamiltonian path $v_1 v_2 \dots v_n$, where $v_1 = v$ and $v_n = u'$. Let $V' = \{v_i : vv_{i+1} \in E(G)\}$. Since G has no hamiltonian cycle, $V' \cap N(u') = \emptyset$.

Since $d(v) + d(u') = n-1$, we have $V(G) = V' \cup N(u') + u'$. Suppose that $v' = v_i$ and $v'' = v_j$. Then $v_{i-1}, v_{j-1} \in V'$, and $v_{i-1}, v_{j-1} \notin N(u')$. But among the neighbors of v_i and v_j, only v is not adjacent to u', a contradiction. □

Now we show that S is non-empty and not too large.

Lemma 19. *$s \geq 1$.*

Proof. Suppose $S = \emptyset$. If D has an open vertex v, then by Lemma 18, all other vertices are closed. In this case, v is the only vertex of D with neighbors outside of D, and hence $G \subseteq K'_{n,d}$, in which v is the cut vertex. Also if D has at most one half-open vertex v, then similarly $G \subseteq K'_{n,d}$.

So suppose that D contains no open vertices but has two half-open vertices u and v with private neighbors z_u and z_v respectively. Then $\delta(G[D]) \geq d - 1$. By Pósa's Theorem, if $d \geq 4$, then $G[D]$ has a hamiltonian v, u-path. This path together with any hamiltonian z_u, z_v-path in the complete graph $G - D$ and the edges uz_u and vz_v forms a hamiltonian cycle in G, a contradiction.

If $d = 3$, then by Dirac's Theorem, $G[D]$ has a hamiltonian cycle, i.e. a 4-cycle, say C. If we can choose our half-open v and u consecutive on C, then $C - uv$ is a hamiltonian v, u-path in $G[D]$, and we finish as in the previous paragraph. Otherwise, we may assume that $C = vxuy$, where x and y are closed. In this case, $d_{G[D]}(x) = d_{G[D]}(y) = 3$, thus $xy \in E(G)$. So we again have a hamiltonian v, u-path, namely $vxyu$, in $G[D]$. Finally, if $d = 2$, then $|D| = 3$, and $G[D]$ is either a 3-vertex path whose endpoints are half-open or a 3-cycle. In both cases, $G[D]$ again has a hamiltonian path whose ends are half-open. □

Lemma 20. *$s \leq d - 3$.*

Proof. Since by (14), $\delta(G) = d$, we have $s \leq d$. Suppose $s \in \{d-2, d-1, d\}$.

Case 1: All vertices of D have degree d.

Case 1.1: $s = d$. Then $G \subseteq H_{n,d+1}$.

Case 1.2: $s = d - 1$. In this case, each vertex in graph $G[D]$ has degree 0 or 1. By (15), $G[D]$ induces a non-empty matching, possibly with some isolated vertices. Let m denote the number of edges in $G[D]$.

If $m \geq 3$, then the number of components in $G[D]$ is less than s, contradicting Lemma 16. Suppose now $m = 2$, and the edges in the matching are x_1y_1 and x_2y_2. Then $d \geq 3$. If $d = 3$, then $D = \{x_1, x_2, y_1, y_2\}$ and $G = F_{n,3}$ (see Fig 3 (right)). If $d \geq 4$, then $G[D]$ has an isolated vertex, say x_3. This x_3 has a private neighbor $w \in S'$. Then $|S + w| = d$ which is more than the number of components of $G[D]$ and we can construct a path from w to S visiting all components of $G[D]$.

Finally, suppose $G[D]$ has exactly one edge, say x_1y_1. Recall that $d \geq 2$. Graph $G[D]$ has $d - 1$ isolated vertices, say $x_2, \dots, x_d$. Each of x_i for $2 \leq i \leq d$ has a private neighbor u_i in S'. Let $S = \{z_1, \dots, z_{d-1}\}$. If $d = 2$, then $S = \{z_1\}$, $N(D) = \{z_1, u_2\}$ and hence $G \subset H'_{n,2}$. So in this case the theorem holds for G. If $d \geq 3$, then G contains a path $u_d x_d z_{d-1} x_{d-1} z_{d-2} x_{d-2} \dots z_2 x_1 y_1 z_1 x_2 u_2$ from u_d to u_2 that covers D.

Case 1.3: $s = d - 2$. Since $s \geq 1$, $d \geq 3$. Every vertex in $G[D]$ has degree at most 2, i.e., $G[D]$ is a union of paths, isolated vertices, and cycles. Each isolated vertex has at least 2 private neighbors in S'. Each endpoint of a path in $G[D]$ has one private neighbor in S'. Thus we can find disjoint paths from S' to S' that cover all isolated vertices and paths in $G[D]$ and all are disjoint from S. Hence if the number c of cycles in $G[D]$ is less than $d - 2$, then we have a set of disjoint paths from $V(G) - D$ to $V(G) - D$ that cover D (and this set can be extended to a hamiltonian cycle in G). Since each cycle has at least 3 vertices and $|D| = d + 1$, if $c \geq d - 2$, then $(d + 1)/3 \geq d - 2$, which is possible only when $d < 4$, i.e. $d = 3$. Moreover, then $G[D] = C_3 \cup K_1$ and $S = N$ is a single vertex. But then $G = K'_{n,3}$.

Case 2: There exists a vertex $v^* \in D$ with $d(v^*) = d+1$. By Lemma 15 (b), $N = N(v^*) - D$, and so G has at most one open or half-open vertex. Furthermore,

$$\text{if } G \text{ has an open or half-open vertex, then it is } v^*, \text{ and by Lemma 15, there are no other vertices of degree } d+1. \tag{16}$$

Case 2.1: $s = d$. If v^* is not closed, then it has a private neighbor $x \in S'$, and the neighborhood of each other vertex of D is exactly S. In this case, there exists a path from x to S that covers D. If v^* is closed (i.e., $N = S$), then $G[D]$ has maximum degree 1. Therefore $G[D]$ is a matching with at least one edge (coming from v^*) plus some isolated vertices. If this matching has at least 2 edges, then the number of components in $G[D]$ is less than s, contradicting Lemma 16. If $G[D]$ has exactly one edge, then $G \subseteq H'_{n,d}$.

Case 2.2: $s = d-1$. If v^* is open, then $d_{G[D]}(v^*) = 0$ and by (16), each other vertex in D has exactly one neighbor in D. In particular, d is even. Therefore $G[D - v^*]$ has $d/2$ components. When $d \geq 3$ and d is even, $d/2 \leq s-1$ and we can find a path from S to S that covers $D - v^*$, and extend this path using two neighbors of v^* in S' to a path from $V(G) - D$ to $V(G) - D$ covering D. Suppose $d = 2$, $D = \{v^*, x, y\}$ and $S = \{z\}$. Then z is a cut vertex separating $\{x, y\}$ from the rest of G, and hence $G \subseteq K'_{n,2}$.

If v^* is half-open, then by (16), each other vertex in D is closed and hence has exactly one neighbor in D. Let $x \in S'$ be the private neighbor of v^*. Then $G[D]$ is 1-regular and therefore has exactly $(d+1)/2$ components, in particular, d is odd. If $d \geq 2$ and is odd, then $(d+1)/2 \leq d-1 = s$, and so we can find a path from x to S that covers D.

Finally, if v^* is closed, then by (16), every vertex of $G[D]$ is closed and has degree 1 or 2, and v^* has degree 2 in $G[D]$. Then $G[D]$ has at most $\lfloor d/2 \rfloor$ components, which is less than s when $d \geq 3$. If $d = 2$, then $s = 1$ and the unique vertex z in S is a cut vertex separating D from the rest of G. This means $G \subseteq K'_{n,3}$.

Case 2.3: $s = d-2$. Since $s \geq 1$, $d \geq 3$. If v^* is open, then $d_{G[D]}(v^*) = 1$ and by (16), each other vertex in D is closed and has exactly two neighbors in D. But this is not possible, since the degree sum of the vertices in $G[D]$ must be even. If v^* is half-open with a neighbor $x \in S'$, then $G[D]$ is 2-regular. Thus $G[D]$ is a union of cycles and has at most $\lfloor (d+1)/3 \rfloor$ components. When $d \geq 4$, this is less than s, contradicting Lemma 16. If $d = 3$, then $s = 1$ and the unique vertex z in S is a cut vertex separating D from the rest of G. This means $G \subseteq K'_{n,4}$.

If v^* is closed, then $d_{G[D]}(v^*) = 3$ and $\delta(G[D]) \geq 2$. So, for any vertices x, y in $G[D]$,

$$d_{G[D]}(x) + d_{G[D]}(y) \geq 4 \geq (d+1) - (d-2-1) = |V(G[D])| - (s-1).$$

By Lemma 17, if $s \geq 2$, then we can partition $G[D]$ into $s-1$ paths $P_1, ..., P_{s-1}$. This would contradict Lemma 16. So suppose $s = 1$ and $d = 3$. Then as in the previous paragraph, $G \subseteq K'_{n,4}$.
□

Next we will show that we cannot have $2 \leq s \leq d-3$.

Lemma 21. $s = 1$.

Proof. Suppose $s = d - k$ where $3 \le k \le d - 2$.

Case 1: $G[D]$ has an open vertex v. By Lemma 18, every other vertex in D is closed. Let $G' = G[D] - v$. Then $\delta(G') \ge k - 1$ and $|V(G')| = d$. In particular, for any $x, y \in D - v$,

$$d_{G'}(x) + d_{G'}(y) \ge 2k - 2 \ge k + 1 = d - (d - k - 1) = |V(G')| - (s - 1).$$

By Lemma 17, we can find a path from S to S in G containing all of $V(G')$. Because v is open, this path can be extended to a path from $V(G) - D$ to $V(G) - D$ including v, and then extended to a hamiltonian cycle of G.

Case 2: D has no open vertices and $4 \le k \le d - 2$. Then $\delta(G[D]) \ge k - 1$ and again for any $x, y \in D$, $d_{G[D]}(x) + d_{G[D]}(y) \ge 2k - 2$. For $k \ge 4$, $2k - 2 \ge k + 2 = (d+1) - (d-k-1) = |D| - (s-1)$. Since $k \le d - 2$, by Lemma 17, $G[D]$ can be partitioned into $s - 1$ paths, contradicting Lemma 16.

Case 3: D has no open vertices and $s = d - 3 \ge 2$. If there is at most one half-open vertex, then for any nonadjacent vertices $x, y \in D$, $d_{G[D]}(x) + d_{G[D]}(y) \ge 2 + 3 = 5 \ge (d + 1) - (d - 3 - 1)$, and we are done as in Case 2.

So we may assume G has at least 2 half-open vertices. Let D' be the set of half-open vertices in D. If $D' \ne D$, let $v^* \in D - D'$. Define a subset D^- as follows: If $|D'| \ge 3$, then let $D^- = D'$, otherwise, let $D^- = D' + v^*$. Let G' be the graph obtained from $G[D]$ by adding a new vertex w adjacent to all vertices in D^-. Then $|V(G')| = d + 2$ and $\delta(G') \ge 3$. In particular, for any $x, y \in V(G'), d_{G'}(x) + d_{G'}(y) \ge 6 \ge (d + 2) - (d - 3 - 1) = |V(G')| - (s - 1)$. By Lemma 17, $V(G')$ can be partitioned into $s - 1$ disjoint paths $P_1, \ldots, P_{s-1}$. We may assume that $w \in P_1$. If w is an endpoint of P_1, then D can also be partitioned into $s - 1$ disjoint paths $P_1 - w, P_2, \ldots, P_{s-1}$ in $G[D]$, a contradiction to Lemma 16.

Otherwise, let $P_1 = x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_k$ where $x_i = w$. Since every vertex in $(D^-) - v^*$ is half-open and $N_{G'}(w) = D^-$, we may assume that x_{i-1} is half-open and thus has a neighbor $y \in S'$. Let $S = \{z_1, \ldots, z_{d-3}\}$. Then

$$y x_{i-1} x_{i-2} \ldots x_1 z_1 x_{i+1} \ldots x_k z_2 P_2 z_3 \ldots z_{d-4} P_{d-4} z_{d-3}$$

is a path in G with endpoints in $V(G) - D$ that covers D. □

Now we may finish the proof of Theorem 7. By Lemmas 19–21, $s = 1$, say, $S = \{z_1\}$. Furthermore, by Lemma 20,

$$d \ge 3 + s = 4. \tag{17}$$

Case 1: D has an open vertex v. Then by Lemma 18, every other vertex of D is closed. Since $s = 1$, each $u \in D - v$ has degree $d - 1$ in $G[D]$. If v has no neighbors in D, then $G[D] - v$ is a clique of order d, and $G \subseteq K'_{n,d}$. Otherwise, since $d \ge 4$, by Dirac's Theorem, $G[D] - v$ has a hamiltonian cycle, say C. Using C and an edge from v to C, we obtain a hamiltonian path P in $G[D]$ starting with v. Let $v' \in S'$ be a neighbor of v. Then $v' P z_1$ is a path from S' to S that covers D, a contradiction.

Case 2: D has a half-open vertex but no open vertices. It is enough to prove that

$$G[D] \text{ has a hamiltonian path } P \text{ starting with a half-open vertex } v, \tag{18}$$

since such a P can be extended to a hamiltonian cycle in G through z_1 and the private neighbor of v. If $d \geq 5$, then for any $x, y \in D$,

$$d_{G[D]}(x) + d_{G[D]}(y) \geq d - 2 + d - 2 = 2d - 4 \geq d + 1 = |V(G[D])|.$$

Hence by Ore's Theorem, $G[D]$ has a hamiltonian cycle, and hence (18) holds.

If $d < 5$ then by (17), $d = 4$. So $G[D]$ has 5 vertices and minimum degree at least 2. By Lemma 17, we can find a hamiltonian path P of $G[D]$, say $v_1v_2v_3v_4v_5$. If at least one of v_1, v_5 is half-open or $v_1v_5 \in E(G)$, then (18) holds. Otherwise, each of v_1, v_5 has 3 neighbors in D, which means $N(v_1) \cap D = N(v_5) \cap D = \{v_2, v_3, v_4\}$. But then $G[D]$ has hamiltonian cycle $v_1v_2v_5v_4v_3v_1$, and again (18) holds.

Case 3: All vertices in D are closed. Then $G \subseteq K'_{n,d+1}$, a contradiction. This proves the theorem. $\square$

7 A comment and a question

- It was shown in Section 4 that the right order of magnitude of $n_0(d,t)$ in Theorem 4 when $d = O(t)$ is dt. We can also show this when $d = O(t^{3/2})$. It could be that dt is the right order of magnitude of $n_0(d,t)$ for all d and t.
- Is there a graph F and positive integers d, n with $n < n_0(d,t)$ and $d \leq \lfloor (n-1)/2 \rfloor$ such that for some n-vertex nonhamiltonian graph G with minimum degree at least d,

$$N(G,F) > \max\{N(H_{n,d}), F), N(K'_{n,d}, F), N(H_{n,\lfloor (n-1)/2 \rfloor}, F)\}?$$

References

[1] N. Alon and C. Shikhelman, Many T copies in H-free graphs, J. of Combin. Theory Ser. B. **121** (2016), 146–172.

[2] B. Bollobás, E. Győri, Pentagons vs. triangles, Discrete Math. 308 (2008) 4332–4336.

[3] G. Dirac, Some theorems on abstract graphs, Proc. London Math. Soc. (3) **2** (1952), 69–81.

[4] P. Erdős, Remarks on a paper of Pósa, Magyar Tud. Akad. Mat. Kutató Int. Közl. **7** (1962), 227–229.

[5] P. Erdős, On the number of complete subgraphs contained in certain graphs, Magy. Tud. Akad. Mat. Kut. Intéz. Kőzl. 7 (1962) 459–474.

[6] Z. Füredi, A. Kostochka, and R. Luo, A stability version for a theorem of Erdős on nonhamiltonian graphs, to appear in Discrete Math.
Also see: `arXiv:1608.05741`, posted on August 19, 2016, 4 pp.

[7] Z. Füredi and L. Özkahya, On 3-uniform hypergraphs without a cycle of a given length, *Discrete Applied Mathematics* **216** (2017), 582–588.

[8] A. Grzesik, On the maximum number of five-cycles in a triangle-free graph, J. Combin. Theory Ser. B 102 (2012) 1061–1066.

[9] H. Hatami, J. Hladký, D. Král', S. Norine, and A. Razborov, On the number of pentagons in triangle-free graphs, J. Combin. Theory Ser. A. **120** (2013), 722–732.

[10] B. Li and B. Ning, Spectral analougues of Erdős' and Moon-Moser's theorems on Hamilton cycles, Linear Multilinear Algebra, **64** (2016), no.11, 1152–1169.

[11] O. Ore, Arc coverings of graphs, Acta Math. Acad. Sci. Hung. **10** (1959), 337–356.

[12] L. Pósa, A theorem concerning Hamilton lines, Magyar Tud. Akad. Mat. Kutató Int. Közl. **7** (1962), 225–226.

[13] L. Pósa, On the circuits of finite graphs, Magyar. Tud. Akad. Mat. Kutató Int. Kőzl. **8** (1963/1964), 355–361.