Subdivisions of oriented cycles in digraphs with large chromatic number*

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August 15, 2018

Abstract

An *oriented cycle* is an orientation of a undirected cycle. We first show that for any oriented cycle C, there are digraphs containing no subdivision of C (as a subdigraph) and arbitrarily large chromatic number. In contrast, we show that for any C a cycle with two blocks, every strongly connected digraph with sufficiently large chromatic number contains a subdivision of C. We prove a similar result for the antidirected cycle on four vertices (in which two vertices have out-degree 2 and two vertices have in-degree 2).

1 Introduction

What can we say about the subgraphs of a graph G with large chromatic number? Of course, one way for a graph to have large chromatic number is to contain a large complete subgraph. However, if we consider graphs with large chromatic number and small clique number, then we can ask what other subgraphs must occur. We can avoid any graph H that contains a cycle because, as proved by Erdős [8], there are graphs with arbitrarily high girth and chromatic number. Reciprocally, one can easily show that every n-chromatic graph contains every tree of order n as a subgraph.

The following more general question attracted lots of attention.

Problem 1. Which are the graph classes $\mathcal{G}$ such that every graph with sufficiently large chromatic number contains an element of $\mathcal{G}$?

*This work was supported by ANR under contract STINT ANR-13-BS02-0007.

If such a class is finite, then it must contain a tree, by the above-mentioned result of Erdős. If it is infinite however, it does not necessary contains a tree. For example, every graph with chromatic number at least 3 contains an odd cycle. This was strengthened by Erdős and Hajnal [9] who proved that every graph with chromatic number at least k contains an odd cycle of length at least k. A counterpart of this theorem for even length was settled by Mihók and Schiermeyer [17]: every graph with chromatic number at least k contains an even cycle of length at least k. Further results on graphs with prescribed lengths of cycles have been obtained [12, 17, 21, 16, 15].

In this paper, we consider the analogous problem for directed graphs, which is in fact a generalization of the undirected one. The *chromatic number* $\chi(D)$ of a digraph D is the chromatic number of its underlying graph. The *chromatic number* of a class of digraphs $\mathcal{D}$, denoted by $\chi(\mathcal{D})$, is the smallest k such that $\chi(D) \leqslant k$ for all $D \in \mathcal{D}$, or $+\infty$ if no such k exists. By convention, if $\mathcal{D} = \emptyset$, then $\chi(\mathcal{D}) = 0$. If $\chi(\mathcal{D}) \neq +\infty$, we say that *$\mathcal{D}$ has bounded chromatic number*.

We are interested in the following question : which are the digraph classes $\mathcal{D}$ such that every digraph with sufficiently large chromatic number contains an element of $\mathcal{D}$? Let us denote by $\text{Forb}(H)$ (resp. $\text{Forb}(\mathcal{H})$) the class of digraphs that do not contain H (resp. any element of $\mathcal{H}$) as a subdigraph. The above question can be restated as follows :

Problem 2. Which are the classes of digraphs $\mathcal{D}$ such that $\chi(\text{Forb}(\mathcal{D})) < +\infty$?

This is a generalization of Problem 1. Indeed, let us denote by $\text{Dig}(\mathcal{G})$ the set of digraphs whose underlying digraph is in $\mathcal{G}$; Clearly, $\chi(\mathcal{G}) = \chi(\text{Dig}(\mathcal{G}))$.

An *oriented graph* is an orientation of a (simple) graph; equivalently it is a digraph with no directed cycles of length 2. Similarly, an *oriented path* (resp. *oriented cycle*, *oriented tree*) is an orientation of a path (resp. cycle, tree). An oriented path (resp., an oriented cycle) is said *directed* if all nodes have in-degree and out-degree at most 1.

Observe that if D is an orientation of a graph G and $\text{Forb}(D)$ has bounded chromatic number, then $\text{Forb}(G)$ has also bounded chromatic number, so G must be a tree. Burr proved that every $(k-1)^2$-chromatic digraph contains every oriented tree of order k. This was slightly improved by Addario-Berry et al. [2] who proved the following.

Theorem 3 (Addario-Berry et al. [2])**.** *Every $(k^2/2 - k/2 + 1)$-chromatic oriented graph contains every oriented tree of order k. In other words, for every oriented tree T of order k, $\chi(\text{Forb}(T)) \leqslant k^2/2 - k/2$.*

Conjecture 4 (Burr [6])**.** Every $(2k-2)$-chromatic digraph D contains a copy of any oriented tree T of order k.

For special oriented trees T, better bounds on the chromatic number of $\text{Forb}(T)$ are known. The most famous one, known as Gallai-Roy Theorem, deals with directed paths (a *directed path* is an oriented path in which all arcs are in the same direction) and can be restated as follows, denoting by $P^+(k)$ the directed path of length k.

Theorem 5 (Gallai [11], Hasse [13], Roy [18], Vitaver [20])**.** *$\chi(\text{Forb}(P^+(k))) = k$.*

The chromatic number of the class of digraphs not containing a prescribed oriented path with two blocks (*blocks* are maximal directed subpaths) has been determined by Addario-Berry et al. [1].

Theorem 6 (Addario-Berry et al. [1]). *Let P be an oriented path with two blocks on n vertices.*

- *If $n = 3$, then $\chi(\mathrm{Forb}(P)) = 3$.*
- *If $n \geqslant 4$, then $\chi(\mathrm{Forb}(P)) = n - 1$.*

In this paper, we are interested in the chromatic number of $\mathrm{Forb}(\mathcal{H})$ when $\mathcal{H}$ is an infinite family of oriented cycles. Let us denote by $\text{S-Forb}(D)$ (resp. $\text{S-Forb}(\mathcal{D})$) the class of digraphs that contain no subdivision of D (resp. any element of $\mathcal{D}$) as a subdigraph. We are particularly interested in the chromatic number of $\text{S-Forb}(\mathcal{C})$, where $\mathcal{C}$ is a family of oriented cycles.

Let us denote by $\vec{C}_k$ the directed cycle of length k. For all k, $\chi(\text{S-Forb}(\vec{C}_k)) = +\infty$ because transitive tournaments have no directed cycle. Let us denote by $C(k, \ell)$ the oriented cycle with two blocks, one of length k and the other of length ℓ. Observe that the oriented cycles with two blocks are the subdivisions of $C(1,1)$. As pointed Gyárfás and Thomassen (see [1]), there are acyclic oriented graphs with arbitrarily large chromatic number and no oriented cycles with two blocks. Therefore $\chi(\text{S-Forb}(C(k, \ell))) = +\infty$. We first generalize these two results to every oriented cycle.

Theorem 7. *For any oriented cycle C,*

$$\chi(\text{S-Forb}(C)) = +\infty.$$

In fact, we show a stronger theorem (Theorem 20): for any positive integer b, there are digraphs of arbitrarily high chromatic number that contains no oriented cycles with less than b blocks. It directly implies the following generalization of the previous theorem.

Theorem 8. *For any finite family $\mathcal{C}$ of oriented cycles,*

$$\chi(\text{S-Forb}(\mathcal{C})) = +\infty.$$

In contrast, if $\mathcal{C}$ is an infinite family of oriented cycles, $\text{S-Forb}(\mathcal{C})$ may have bounded chromatic number. By the above argument, such a family must contain a cycle with at least b blocks for every positive integer b. A cycle C is *antidirected* if any vertex of C has either in-degree 2 or out-degree 2 in C. In other words, it is an oriented cycle in which all blocks have length 1. Let us denote by $\mathcal{A}_{\geqslant 2k}$ the family of antidirected cycles of length at least $2k$. In Theorem 13, we prove that $\chi(\mathrm{Forb}(\mathcal{A}_{\geqslant 2k})) \leqslant 8k - 8$. Hence we are left with the following problem.

Problem 9. What are the infinite families of oriented cycles $\mathcal{C}$ such that $\mathrm{Forb}(\mathcal{C}) < +\infty$? What are the infinite families of oriented cycles $\mathcal{C}$ such that $\text{S-Forb}(\mathcal{C}) < +\infty$?

On the other hand, considering strongly connected (strong for short) digraphs may lead to dramatically different result. An example is provided by the following celebrated result due to Bondy [4] : *every strong digraph of chromatic number at least k contains a directed cycle of length at least k.* Denoting the class of strong digraphs by $\mathcal{S}$, this result can be rephrased as follows.

Theorem 10 (Bondy [4])**.** $\chi(\text{S-Forb}(\vec{C}_k) \cap \mathcal{S}) = k-1$.

Inspired by this theorem, Addario-Berry et al. [1] posed the following problem.

Problem 11. Let k and ℓ be two positive integers. Does S-Forb$(C(k,\ell) \cap \mathcal{S})$ have bounded chromatic number?

In Subsection 5.2, we answer to this problem in the affirmative. In Theorem 23 we prove

$$\chi(\text{S-Forb}(C(k,\ell) \cap \mathcal{S}) \leqslant (k+\ell-2)(k+\ell-3)(2\ell+2)(k+\ell+1), \text{ for all } k \geqslant \ell \geqslant 2, k \geqslant 3.$$

Note that since $\chi(\text{S-Forb}(C(k',\ell') \cap \mathcal{S}) \leqslant \chi(\text{S-Forb}(C(k,\ell) \cap \mathcal{S})$ if $k' \leqslant k$ and $\ell' \leqslant \ell$, this gives also an upper bound when k or ℓ are small. However, in those cases, we prove better upper bounds. In Corollary 32, we prove

$$\chi(\text{S-Forb}(C(k,1)) \cap \mathcal{S}) \leqslant \max\{k+1, 2k-4\} \text{ for all } k.$$

We also give in Subsection 5.2 the exact value of S-Forb$(C(k,\ell) \cap \mathcal{S})$ for $(k,\ell) \in \{(1,2),(1,3),(2,3)\}$.

More generally, one may wonder what happens for other oriented cycles.

Problem 12. Let C be an oriented cycle with at least four blocks. Is $\chi(\text{S-Forb}(C) \cap \mathcal{S})$ bounded?

In Section 7, we show that $\chi(\text{S-Forb}(\hat{C}_4) \cap \mathcal{S}) \leqslant 24$ where $\hat{C}_4$ is the antidirected cycle of order 4.

2 Definitions

We follow [5] for basic notions and notations. Let D be a digraph. $V(D)$ denotes its vertex-set and $A(D)$ its arc-set.

If $uv \in A(D)$ is an arc, we sometimes write $u \to v$ or $v \leftarrow u$.

For any $v \in V(D)$, $d^+(v)$ (resp. $d^-(v)$) denotes the out-degree (resp. in-degree) of v. $\delta^+(D)$ (resp. $\delta^-(D)$) denotes the minimum out-degree (resp. in-degree) of D.

An *oriented path* is any orientation of a *path*. The *length* of a path is the number of its arcs. Let $P = (v_1, \dots, v_n)$ be an oriented path. If $v_i v_{i+1} \in A(D)$, then $v_i v_{i+1}$ is a *forward arc*; otherwise, $v_{i+1} v_i$ is a *backward arc*. P is a *directed path* if all of its arcs are either forward or backward ones. For convenience, a directed path with forward arcs only is called a *dipath*. A *block* of P is a maximal directed subpath of P. A path is entirely determined by the sequence $(b_1, \dots, b_p)$ of the lengths of its blocks and the sign $+$ or $-$ indicating if the first arc is forward or backward respectively. Therefore we denote by $P^+(b_1, \dots, b_p)$ (resp. $P^-(b_1, \dots, b_p)$) an oriented path whose first arc is forward (resp. backward) with p blocks, such that the ith block along it has length b_i.

Let $P = (x_1, x_2, \dots, x_n)$ be an oriented path. We say that P is an (x_1, x_n)*-path*. For every $1 \leqslant i \leqslant j \leqslant n$, we note $P[x_i, x_j]$ (resp. $P]x_i, x_j[$, $P[x_i, x_j[$, $P]x_i, x_j]$) the oriented subpath $(x_i, \dots, x_j)$ (resp. $(x_{i+1}, \dots, x_{j-1})$, $(x_i, \dots, x_{j-1})$, $(x_{i+1}, \dots, x_j)$).

The vertex x_1 is the *initial vertex* of P and x_n its *terminal vertex*. Let P_1 be an (x_1,x_2)-dipath and P_2 an (x_2,x_3)-dipath which are disjoint except in x_2. Then $P_1 \odot P_2$ denotes the (x_1,x_3)-dipath obtained from the concatenation of these dipaths.

The above definitions and notations can also be used for oriented cycles. Since a cycle has no initial and terminal vertex, we have to choose one as well as a direction to run through the cycle.Therefore if $C = (x_1,x_2,\dots,x_n,x_1)$ is an oriented cycle, we always assume that x_1x_2 is an arc, and if C is not directed that x_1x_n is also an arc.

A path or a cycle (not necessarily directed) is *Hamiltonian* in a digraph if it goes through all vertices of D.

The digraph D is *connected* (resp. *k-connected*) if its underlying graph is connected (resp. k-connected). It is *strongly connected*, or *strong*, if for any two vertices u,v, there is a (u,v)-dipath in D. It is *k-strongly connected* or *k-strong*, if for any set S of $k-1$ vertices $D-S$ is strong. A *strong component* of a digraph is an inclusionwise maximal strong subdigraph. Similarly, a *k-connected component* of a digraph is an inclusionwise maximal k-connected subdigraph.

3 Antidirected cycles

The aim of this section is to prove the following theorem, that establish that $\chi(\text{Forb}(\mathcal{A}_{\geqslant 2k})) \leqslant 8k-8$.

Theorem 13. *Let D be an oriented graph and k an integer greater than* 1.
If $\chi(D) \geqslant 8k-7$, then D contains an antidirected cycle of length at least $2k$.

A graph G is *k-critical* if $\chi(G) = k$ and $\chi(H) < k$ for any proper subgraph H of G. Every graph with chromatic number k contains a k-critical graph. We denote by $\delta(G)$ the minimum degree of the graph G. The following easy result is well-known.

Proposition 14. *If G is a k-critical graph, then $\delta(G) \geqslant k-1$.*

Let (A,B) be a bipartition of the vertex set of a digraph D. We denote by $E(A,B)$ the set of arcs with tail in A and head in B and by $e(A,B)$ its cardinality.

Lemma 15 (Burr [7])**.** Every digraph D contains a partition (A,B) such that $e(A,B) \geqslant |E(D)|/4$.

Lemma 16 (Burr [7])**.** *Let G be a bipartite graph and p be an integer. If $|E(G)| \geqslant p|V(G)|$, then G has a subgraph with minimum degree at least $p+1$.*

Lemma 17. *Let $k \geqslant 1$ be an integer. Every bipartite graph with minimum degree k contains a cycle of order at least* $2k$.

Proof. Let G be a bipartite graph with bipartition (A,B). Consider a longest path P in G. Without loss of generality, we may assume that one of its ends a is in A. All neighbours of a are in P (otherwise P can be lengthened). Let b be the furthest neighbour of a in B along P. Then $C = P[a,b] \cup ab$ is a cycle containing at least k vertices in B, namely the neighbours of a. Hence C has length at least $2k$, since G is bipartite. □

Proof of Theorem 13. It suffices to prove that every $(8k-7)$-critical oriented graph contains an antidirected cycle of length at least $2k$.

Let D be a $(8k-7)$-critical oriented graph. By Proposition 14, it has minimum degree at least $8k-8$, so $|E(D)| \geqslant (4k-4)|V(D)|$. By Lemma 15, D contains a partition such that $e(A,B) \geqslant |E(D)|/4 \geqslant (k-1)|V(D)|$. Consequently, by Lemma 16, there are two sets $A' \subseteq A$ and $B' \subseteq B$ such that every vertex in A' (resp. B') has at least k out-neighbours in B' (resp. k in-neighbours in A'). Therefore, by Lemma 17, the bipartite oriented graph induced by $E(A',B')$ contains a cycle of length at least $2k$, which is necessarily antidirected. $\square$

Problem 18. Let ℓ be an even integer. What the minimum integer $a(\ell)$ such that every oriented graph with chromatic number at least $a(\ell)$ contains an antidirected cycle of length at least ℓ ?

4 Acyclic digraphs without cycles with few blocks

The aim of this section is to establish Theorems 7 and 8. To do so we will use a result on hypergraph colouring.

A *cycle* of *length* $k \geqslant 2$ in a hypergraph $\mathcal{H}$ is an alternating cyclic sequence $e_0, v_0, e_1, v_1, \dots e_{k-1}, v_{k-1}, e_0$ of distinct hyperedges and vertices in $\mathcal{H}$ such that $v_i \in e_i \cap e_{i+1}$ for all i modulo k. The *girth* of a hypergraph is the length of a shortest cycle.

A hypergraph $\mathcal{H}$ on a ground set X is said to be *weakly c-colourable* if there exists a colouring of the elements of X with c colours such that no hyperedge of $\mathcal{H}$ is monochromatic. The *weak chromatic number* of $\mathcal{H}$ is the least c such that $\mathcal{H}$ is weakly c-colourable. Erdős and Lovász [10] (and more recently Alon *et al.*[3]) proved the following result:

Theorem 19. *[10, Theorem 1'], [3] For $k,g,c \in \mathbb{N}$, there exists a k-uniform hypergraph with girth larger than g and weak chromatic number larger than c.*

Our construction relies on the hypergraphs whose existence is established by Theorem 19.

Theorem 20. *For any positive integers b,c, there exists an acyclic digraph D with $\chi(D) \geqslant c$ in which all oriented cycles have more than b blocks.*

Proof. We shall prove the result by induction on c, the result holding trivially for $c=2$ with D the directed path on two vertices. We thus assume our claim to hold for a graph D_c with $\chi(D_c)=c$, and show how extend it to $c+1$.

Let p be the number of proper c-colourings of D_c, and let those colourings be denoted by $col_c^1, \dots, col_c^p$. By Theorem 19 there exists a $c \times p$-uniform hypergraph $\mathcal{H}$ with weak chromatic number $> p$ and girth $> b/2$. Let $X = \{x_1, \dots, x_n\}$ be the ground set of $\mathcal{H}$.

We construct D_{c+1} from n disjoint copies $D_c^1, \dots, D_c^n$ of D_c as follows. For each hyperedge $S \in \mathcal{H}$, we do the following (see Figure 1) :

- We partition S into p sets $S_1, \dots, S_p$ of cardinality c.
- For each set $S_i = \{x_{k_1}, \dots, x_{k_c}\}$, we choose vertices $v_{k_1} \in D_c^{k_1}, \dots, v_{k_c} \in D_c^{k_c}$ such that $col_c^i(v_{k_1}) = 1, \dots, col_c^i(v_{k_c}) = c$, and add a new vertex $w_{S,i}$ with $v_{k_1}, \dots, v_{k_c}$ as in-neighbours.

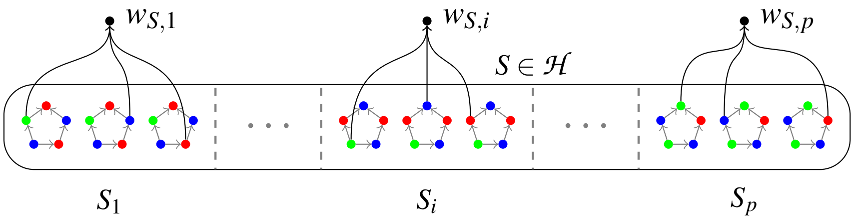

Figure 1: Construction of D_{c+1}

Let us denote by W the set of vertices of D_{c+1} that do not belong to any of the copies of D_c (i.e. the $w_{S,i}$). We now prove that the resulting digraph D_{c+1} is our desired digraph.

Firstly it is acyclic, as we only add sinks (the $w_{S,i}$) to disjoint copies of D_c, which are acyclic by the induction hypothesis.

Secondly, every oriented cycle C in D_{c+1} has more than b blocks. If C is in a copy of D_c, then we have the result by the induction hypothesis. Henceforth we may assume that S contains some vertices in W, say $w_1, \dots, w_{b'}$ in cyclic order around C. As the vertices of W are all sinks, the number of blocks of C is at least $2b'$. Let us denote by S_{w_i} the hyperedge of $\mathcal{H}$ which triggered the creation of w_i. Then two consecutive $S_{w_i}, S_{w_{i+1}}$ (indices are modulo b') have a vertex x_i of X in common (indeed, the vertices between w_i and w_{i+1} in C belong to some copy D_c^i). Therefore the sequence $x_{b'}, S_{w_1}, x_1, S_{w_2}, x_2, \dots, S_{w_{b'}}, x_{b'}$ contains a cycle in $\mathcal{H}$. Hence by our choice of $\mathcal{H}$, $b' > b/2$, so C has more than b blocks.

Finally, let us prove that $\chi(D_{c+1}) = c+1$. We added a stable set to the disjoint union of copies of D_c, so $\chi(D_{c+1}) \leqslant \chi(D_c) + 1 = c+1$.
Now suppose for a contradiction that D_{c+1} admits a proper c-colouring ϕ. It induces on $\mathcal{H}$ the p-colouring ψ where $\psi(x_k)$ is the index of the colouring of D_c on D_c^k, i.e. the restriction of ϕ on D_c^k is the colouring $col_c^{\psi(x_k)}$. Now since $\mathcal{H}$ is $(p+1)$-chromatic, there exists an hyperedge S of $\mathcal{H}$ which is monochromatic. Let i be the integer such that $\psi(x) = i$ for all $x \in S$. Consider $S_i = \{x_{k_1}, \dots, x_{k_c}\}$ and let $v_{k_1} \in D_c^{k_1}, \dots, v_{k_c} \in D_c^{k_c}$ be the in-neighbours of $w_{S,i}$. By construction, $col_c^i(v_{k_1}) = 1, \dots, col_c^i(v_{k_c}) = c$, so $\phi(v_{k_1}) = 1, \dots, \phi(v_{k_c}) = c$. Consequently $w_{S,i}$ has the same colour (by ϕ) as one of its in-neighbours. This contradicts the fact that ϕ is proper. Hence $\chi(D_{c+1}) \geqslant c+1$. $\square$

Theorems 7 and 8 directly follow from Theorem 20, since a cycle and its subdivision have the same number of blocks.

5 Cycles with two blocks in strong digraphs

In this section we first prove that S-Forb$(C(k,\ell)) \cap \mathcal{S}$ has bounded chromatic number for every k, ℓ. We need some preliminaries.

5.1 Definitions and tools

5.1.1 Levelling

In a digraph D, the *distance* from a vertex x to another y, denoted by $\text{dist}_D(x,y)$ or simply $\text{dist}(x,y)$ when D is clear from the context, is the minimum length of an (x,y)-dipath or $+\infty$ if no such dipath exists. For a set $X \subseteq V(D)$ and vertex $y \in V(D)$, we define $\text{dist}(X,y) = \min\{\text{dist}(x,y) \mid x \in X\}$ and $\text{dist}(y,X) = \min\{\text{dist}(y,x) \mid x \in X\}$, and for two sets $X,Y \subseteq V(D)$, $\text{dist}(X,Y) = \min\{\text{dist}(x,y) \mid x \in X, y \in Y\}$.

An *out-generator* in a digraph D is a vertex u such that for any $x \in V(D)$, there is an (u,x)-dipath. Observe that in a strong digraph every vertex is an out-generator.

Let u be an out-generator of D. For every nonnegative integer i, the *ith level from u in D* is $L_i^u = \{v \mid \text{dist}_D(u,v) = i\}$. Because u is an out-generator, $\bigcup_i L_i^u = V(D)$. Let v be a vertex of D, we set $\text{lvl}^u(v) = \text{dist}_D(u,v)$, hence $v \in L^u_{\text{lvl}(v)}$. In the following, the vertex u is always clear from the context. Therefore, for sake of clarity, we drop the superscript u.

The definition immediately implies the following.

Proposition 21. *Let D be a digraph having an out-generator u. If x and y are two vertices of D with $\text{lvl}(y) > \text{lvl}(x)$, then every (x,y)-dipath has length at least $\text{lvl}(y) - \text{lvl}(x)$.*

Let D be a digraph and u be an out-generator of D. A *Breadth-First-Search Tree* or *BFS-tree* T with *root u*, is a sub-digraph of D spanning $V(D)$ such that T is an oriented tree and, for any $v \in V(D)$, $dist_T(u,v) = \text{dist}_D(u,v)$. It is well-known that if u is an out-generator of D, then there exist BFS-trees with root u.

Let T be a BFS-tree with root u. For any vertex x of D, there is an unique (u,x)-dipath in T. The *ancestors* of x are the vertices on this dipath. For an ancestor y of x, we note $y \geqslant_T x$. If y is an ancestor of x, we denote by $T[y,x]$ the unique (y,x)-dipath in T. For any two vertices v_1 and v_2, the *least common ancestor* of v_1 and v_2 is the common ancestor x of v_1 and v_2 for which $\text{lvl}(x)$ is maximal. (This is well-defined since u is an ancestor of all vertices.)

5.1.2 Decomposing a digraph

The *union* of two digraphs D_1 and D_2 is the digraph $D_1 \cup D_2$ with vertex set $V(D_1) \cup V(D_2)$ and arc set $A(D_1) \cup A(D_2)$. Note that $V(D_1)$ and $V(D_2)$ are not necessarily disjoint.

The following lemma is well-known.

Lemma 22. *Let D_1 and D_2 be two digraphs. $\chi(D_1 \cup D_2) \leqslant \chi(D_1) \times \chi(D_2)$.*

Proof. Let $D = D_1 \cup D_2$. For $i \in \{1,2\}$, let c_i be a proper colouring of D_i with $\{1,\dots,\chi(D_i)\}$. Extend c_i to $(V(D), A(D_i))$ by assigning the colour 1 to all vertices in V_{3-i}. Now the function c defined by $c(v) = (c_1(v), c_2(v))$ for all $v \in V(D)$ is a proper colouring of D with colour set $\{1,\dots,\chi(D_1)\} \times \{1,\dots,\chi(D_2)\}$. $\square$

5.2 General upper bound

Theorem 23. *Let k and ℓ be two positive integers such that $k \geqslant \max\{\ell, 3\}$, and let D be a digraph in* S-Forb$(C(k,\ell)) \cap \mathcal{S}$. *Then,* $\chi(D) \leqslant (k+\ell-2)(k+\ell-3)(2\ell+2)(k+\ell+1)$.

Proof. Since D is strongly connected, it has an out-generator u. Let T be a BFS-tree with root u. We define the following sets of arcs.

$$\begin{aligned}
A_0 &= \{xy \in A(D) \mid \mathrm{lvl}(x) = \mathrm{lvl}(y)\};\\
A_1 &= \{xy \in A(D) \mid 0 < |\mathrm{lvl}(x) - \mathrm{lvl}(y)| < k+\ell-3;\\
A' &= \{xy \in A(D) \mid \mathrm{lvl}(x) - \mathrm{lvl}(y) \geqslant k+\ell-3\}.
\end{aligned}$$

Since $k+\ell-3 > 0$ and there is no arc xy with $\mathrm{lvl}(y) > \mathrm{lvl}(x)+1$, (A_0, A_1, A') is a partition of $A(D)$. Observe moreover that $A(T) \subseteq A_1$. We further partition A' into two sets A_2 and A_3, where $A_2 = \{xy \in A' \mid y \text{ is an ancestor of } x \text{ in } T\}$ and $A_3 = A' \setminus A_2$. Then (A_0, A_1, A_2, A_3) is a partition of $A(D)$. Let $D_j = (V(D), A_j)$ for all $j \in \{0,1,2,3\}$.

Claim 23.1. $\chi(D_0) \leqslant k+\ell-2$.

Subproof. Observe that D_0 is the disjoint union of the $D[L_i]$ where $L_i = \{v \mid \mathrm{dist}_D(u,v) = i\}$. Therefore it suffices to prove that $\chi(D[L_i]) \leqslant k+\ell-2$ for all non-negative integer i.

$L_0 = \{u\}$ so the result holds trivially for $i = 0$.

Assume now $i \geqslant 1$. Suppose for a contradiction $\chi(D[L_i]) \geqslant k+\ell-1$. Since $k \geqslant 3$, by Theorem 6, $D[L_i]$ contains a copy Q of $P^+(k-1, \ell-1)$. Let v_1 and v_2 be the initial and terminal vertices of Q, and let x be the least common ancestor of v_1 and v_2. By definition, for $j \in \{1,2\}$, there exists a (x, v_j)-dipath P_j in T. By definition of least common ancestor, $V(P_1) \cap V(P_2) = \{x\}$, $V(P_j) \cap L_i = \{v_j\}$, $j = 1,2$, and both P_1 and P_2 have length at least 1. Consequently, $P_1 \cup P_2 \cup Q$ is a subdivision of $C(k,\ell)$, a contradiction. $\Diamond$

Claim 23.2. $\chi(D_1) \leqslant k+\ell-3$.

Subproof. Let ϕ_1 be the colouring of D_1 defined by $\phi_1(x) = \mathrm{lvl}(x) \pmod{k+\ell-3}$. By definition of D_1, this is clearly a proper colouring of D_1. $\Diamond$

Claim 23.3. $\chi(D_2) \leqslant 2\ell+2$.

Subproof. Suppose for a contradiction that $\chi(D_2) \geqslant 2\ell+3$. By Theorem 6, D_2 contains a copy Q of $P^-(\ell+1, \ell+1)$, which is the union of two disjoint dipaths which are disjoint except in there initial vertex y, say $Q_1 = (y_0, y_1, y_2, \dots, y_{\ell+1})$ and $Q_2 = (z_0, z_1, z_2, \dots, z_{\ell+1})$ with $y_0 = z_0 = y$. Since Q is in D_2, all vertices of Q belong to $T[u,y]$. Without loss of generality, we can assume $z_1 \geqslant_T y_1$.

If $z_{\ell+1} \geqslant_T y_{\ell+1}$, then let j be the smallest integer such that $z_j \geqslant_T y_{\ell+1}$. Then the union of $T[y_1,y] \odot Q_2[y,z_j] \odot T[z_j, y_{\ell+1}]$ and $Q_1[y_1, y_{\ell+1}]$ is a subdivision of $C(k,\ell)$, because $T[y_1,y]$ has length at least $k-2$ as $\mathrm{lvl}(y) \geqslant \mathrm{lvl}(y_1) + k + \ell - 3$. This is a contradiction.

Henceforth $y_{\ell+1} \geqslant_T z_{\ell+1}$. Observe that all the z_j, $1 \leqslant j \leqslant \ell+1$ are in $T[y_{\ell+1}, y_1]$. This, by the Pigeonhole principle, there exists $i, j \geqslant 1$ such that $y_{i+1} \geqslant_T z_{j+1} \geqslant_T z_j \geqslant_T y_i \geqslant_T z_{j-1}$.

If $\mathrm{lvl}(z_{j-1}) \geqslant \mathrm{lvl}(y_i) + \ell - 1$, then $T[y_i, z_{j-1}] \odot (z_{j-1}, z_j)$ has length at least ℓ. Hence its union with $(y_i, y_{i+1}) \odot T[y_{i+1}, z_j]$, which has length greater than k, is a subdivision of $C(k, \ell)$, a contradiction.

Thus $\mathrm{lvl}(z_{j-1}) < \mathrm{lvl}(y_i) + \ell - 1$ (in particular, in this case, $j > 1$ and $i > 2$). Therefore, by definition of A', $\mathrm{lvl}(y_i) \geqslant \mathrm{lvl}(z_j) + k - 1$ and $\mathrm{lvl}(y_{i-1}) \geqslant \mathrm{lvl}(z_{j-1}) + k - 1$. Hence both $T[z_{j-1}, y_{i-1}]$ and $T[z_j, y_i]$ have length at least $k-1$. So the union of $T[z_{j-1}, y_{i-1}] \odot (y_{i-1}, y_i)$ and $(z_{j-1}, z_j) \odot T[z_j, y_i]$ is a subdivision of $C(k,k)$ (and thus of $C(k,\ell)$), a contradiction. $\Diamond$

Claim 23.4. $\chi(D_3) \leqslant k + \ell + 1$.

Subproof. In this claim, it is important to note that $k + \ell - 3 \geqslant k - 1$ because $\ell \geqslant 2$. We use the fact that $\mathrm{lvl}(x) - \mathrm{lvl}(y) \geqslant k - 1$ if xy is an edge in A_3.

Suppose for a contradiction that $\chi(D_3) \geqslant k + \ell + 1$. By Theorem 6, D_3 contains a copy Q of $P^-(k, \ell)$ which is the union of two disjoint dipaths which are disjoint except in there initial vertex y, say $Q_1 = (y_0, y_1, y_2, \dots, y_k)$ and $Q_2 = (z_0, z_1, z_2, \dots, z_\ell)$ with $y_0 = z_0 = y$.

Assume that a vertex of $Q_1 - y$ is an ancestor of y. Let i be the smallest index such that y_i is an ancestor of y. If it exists, by definition of A_3, $i \geqslant 2$. Let x be the common ancestor of y_i and y_{i-1} in T. By definition of A_3, y_i is not an ancestor of y_{i-1}, so x is different from y_i and y_{i-1}. Moreover by definition of A', $\mathrm{lvl}(y) - k \geqslant \mathrm{lvl}(y_{i-1}) - k \geqslant \mathrm{lvl}(y_i) - 1 \geqslant \mathrm{lvl}(x)$. Hence $T[x, y_{i-1}]$ and $T[x, y]$ have length at least k. Moreover these two dipaths are disjoint except in x. Therefore, the union of $T[x, y_{i-1}]$ and $T[x, y] \odot Q_1[y, y_{i-1}]$ is a subdivision of $C(k,k)$ (and thus of $C(k, \ell)$), a contradiction.

Similarly, we get a contradiction if a vertex of $Q_2 - y$ is an ancestor of y. Henceforth, no vertex of $V(Q_1) \cup V(Q_2) \setminus \{y\}$ is an ancestor of y.

Let x_1 be the least common ancestor of y and y_1. Note that $|T[x_1, y]| \geqslant k$ so $|T[x_1, y_1]| < k$, for otherwise G would contain a subdivision of $C(k,k)$. Therefore $\mathrm{lvl}(y_1) - \mathrm{lvl}(x_1) < k$. We define inductively $x_2, \dots, x_k$ as follows: x_{i+1} is the least common ancestor of x_i and y_i. As above $|T[x_i, y_{i-1}]| \geqslant k$ so $\mathrm{lvl}(y_i) - \mathrm{lvl}(x_i) < k$. Symmetrically, let t_1 be the least common ancestor of y and z_1 and for $1 \leqslant i \leqslant \ell - 1$, let t_{i+1} be the least common ancestor of t_i and z_i. For $1 \leqslant i \leqslant \ell$, we have $\mathrm{lvl}(z_i) - \mathrm{lvl}(t_i) < k$. Moreover, by definition all x_i and t_j are ancestors of y, so they all are on $T[u, y]$.

Let P_y (resp. P_z) be a shortest dipath in D from y_k (resp. z_ℓ) to $T[u, y] \cup Q_1[y_1, y_{k-1}] \cup Q_2[z_1, z_{\ell-1}]$. Note that P_y and P_z exist since D is strongly connected. Let y' (resp. z') be the terminal vertex of P_y (resp. P_z). Let w_y be the last vertex of $T[x_k, y_k]$ in P_y (possibly, $w_y = y_k$.) Similarly, let w_z be the last vertex of $T[t_\ell, z_\ell]$ in P_z (possibly, $w_z = z_\ell$.) Note that $P_y[w_y, y']$ is a shortest dipath from w_y to y' and $P_z[w_z, z']$ is a shortest dipath from w_z to z'.

If $y' = y_j$ for $0 \leqslant j \leqslant k-1$, consider $R = T[x_k, w_y] \odot P_y[w_y, y_j]$ is an (x_k, y_j)-dipath. By Proposition 21, R has length at least k because $\mathrm{lvl}(y_j) - \mathrm{lvl}(x_k) \geqslant \mathrm{lvl}(y_j) - \mathrm{lvl}(y_k) + 1 \geqslant k$. Therefore the union of R and $T[x_k, y] \cup Q_1[y, y_j]$ is a subdivision of $C(k,k)$, a contradiction.

Similarly, we get a contradiction if z' is in $\{z_1, \dots, z_{\ell-1}\}$. Consequently, P_y is disjoint from $Q_1[y, y_{k-1}]$ and P_z is disjoint from $Q_2[y, z_{\ell-1}]$.

If P_y and P_z intersect in a vertex s. By the above statement, $s \notin V(Q) \setminus \{y_k, z_\ell\}$. Therefore the union of $Q_1 \odot P_y[y_k, s]$ and $Q_2 \odot P_z[z_\ell, s]$ is a subdivision of $C(k, \ell)$, a contradiction. Henceforth P_y and P_z are disjoint.

Assume both y' and z' are in $T[u, y]$. If $y' \geqslant_T z'$, then the union of $Q_1 \odot P_y \odot T[y', z']$ and $Q_2 \odot P_z$ form a subdivision of $C(k, \ell)$; and if $z' \geqslant_T y'$, then the union of $Q_1 \odot P_y$ and $Q_2 \odot P_z \odot T[z', y']$ form a subdivision of $C(k, \ell)$. This is a contradiction.

Henceforth a vertex among y' and z' is not in $T[u, y]$. Let us assume that y' is not in $T[u, y]$ (the case $z' \notin T[u, y]$ is similar), and so $y' = z_i$ for some $1 \leqslant i \leqslant \ell - 1$. If $\mathrm{lvl}(y') \geqslant \mathrm{lvl}(x_k) + k$, then both $T[x_k, w_y] \odot P_y[w_y, y']$ and $T[x_k, y] \odot Q_2[y, z_i]$ have length at least k by Proposition 21, so their union is a subdivision of $C(k, k)$, a contradiction. Hence $\mathrm{lvl}(x_k) \geqslant \mathrm{lvl}(z_i) - k + 1 \geqslant \mathrm{lvl}(z_\ell) \geqslant \mathrm{lvl}(t_\ell)$.

If $z' = y_j$ for some j, then necessarily $\mathrm{lvl}(z') \geqslant \mathrm{lvl}(x_k) + k \geqslant \mathrm{lvl}(t_\ell) + k$ and both $T[t_\ell, w_z] \odot P_z[w_z, z']$ and $T[t_\ell, y] \odot Q_1[y, y_j]$ have length at least k, so their union is a subdivision of $C(k, k)$, a contradiction.

Therefore $z' \in T[u, y]$. The union of $T[t_\ell, z']$ and $T[t_\ell, w_z] \odot P_z[w_z, z']$ is not a subdivision of $C(k, k)$ so by Proposition 21, $\mathrm{lvl}(z') \leqslant \mathrm{lvl}(t_\ell) + k - 1 \leqslant \mathrm{lvl}(z_\ell) + k - 1 \leqslant \mathrm{lvl}(z_{\ell-1})$.

If $\mathrm{lvl}(z') \leqslant \mathrm{lvl}(x_k)$, then the union of Q_1 and $Q_2 \odot P_z \odot T[z', y_k]$ is a subdivision of $C(k, \ell)$, a contradiction. Hence $\mathrm{lvl}(z') > \mathrm{lvl}(x_k)$. Therefore $\mathrm{lvl}(y') = \mathrm{lvl}(z_i) \leqslant \mathrm{lvl}(x_k) + k - 1 \leqslant \mathrm{lvl}(z') + k - 2 \leqslant \mathrm{lvl}(z_\ell) + 2k - 3$, which implies that $i = \ell - 1$ that is $y' = z_i = z_{\ell-1}$. Now the union of $[T[x_1, y_1]] \odot Q_1[y_1, y_k] \odot P_y$ and $T[x_1, y] \odot Q_2[y, z_{\ell-1}]$ is a subdivision of $C(k, \ell)$, a contradiction. $\Diamond$

Claims 23.1, 23.2, 23.3, and 23.4, together with Lemma 22 yield the result. $\square$

5.3 Better bound for Hamiltonian digraphs

We now improve on the bound of Theorem 23 in case of digraphs having a Hamiltonian directed cycle. Therefore we define

$$\phi(k, \ell) = \max\{\chi(D) \mid D \in \text{S-Forb}(C(k, \ell)) \text{ and } D \text{ has a Hamiltonian directed cycle}\}.$$

This section aims at proving that $\phi(k, k) \leqslant 6k - 6$.

Let D be a digraph and let $C = (v_1, \ldots, v_n, v_1)$ be a Hamiltonian cycle in D (C may be directed or not).

For any $i, j \leqslant n$, let $d_C(v_i, v_j)$ be the distance between v_i and v_j in the undirected cycle C. That is, $d_C(v_i, v_j) = \min\{j - i, n - j + i\}$ if $j > i$ and $d_C(v_i, v_j) = \min\{i - j, n - i + j\}$ otherwise.

A *chord* is an arc of $A(D) \setminus A(C)$. The *span* $\mathrm{span}_C(a)$ of a chord $a = v_i v_j \in F$ is $d_C(i, j)$. We denote by $\mathrm{span}_C(D)$ be the maximum span of a chord in D.

Lemma 24. *If D is a digraph with a Hamiltonian cycle C and at least one chord, then* $\chi(D) < 2 \cdot \mathrm{span}_C(D)$.

Proof. Set $C = (v_1, \ldots, v_n, v_1)$ and set $\ell = \mathrm{span}_C(D)$. If $n < 2\ell$, then the result trivially holds. Let us assume that $n = k\ell + r$ with $k \geqslant 2$ and $r < \ell$. Consider the following colouring. For any

$1 \leqslant i \leqslant k\ell$, let us colour v_i with colour $i - \lfloor i/\ell \rfloor \ell$. For any $1 < t \leqslant r$, let us colour $v_{k\ell+t}$ with $\ell + t - 1$. This colouring uses the $\ell + r$ colours of $\{0, \ldots, \ell + r - 1\}$.

Moreover, for any $1 \leqslant i \leqslant n$, all neighbours (in-neighbours and out-neighbours) of v_i belong to $\{v_{i-\ell}, \ldots, v_{i-1}\} \cup \{v_{i+1}, \ldots, v_{i+\ell}\}$ (all indices must be taken moduo n), for otherwise there would be a chord with span strictly larger than ℓ. Hence, the colouring is proper. $\square$

Let $A \subseteq V(D)$, let $N(A) \subseteq V(D) \setminus A$ be the set of vertices not in A that are adjacent to some vertex in A.

Lemma 25. *Let D be a digraph and let (A, B) be a partition of $V(D)$. Then*

$$\chi(D) = \max\{\chi(D[A]) + |N(A)|, \chi(D[B])\}.$$

Proof. Let us consider a proper colouring of $D[B]$ with colour set $\{1, \ldots, \chi(D[B])\}$. W.l.o.g., vertices in $N(A)$ have received colours in $\{1, \ldots, |N(A)|\}$. Let us colour $D[A]$ using colours in $\{|N(A)| + 1, \ldots, |N(A)| + \chi(D[A])\}$. We obtain a proper colouring of D using $\max\{\chi(D[A]) + |N(A)|, \chi(D[B])\}$ colours. $\square$

Lemma 26. *Let D be a digraph containing no subdivision of $C(k,k)$ and having a Hamiltonian directed cycle $C = (v_1, \ldots, v_n, v_1)$. Assume that D contains a chord v_iv_j with span at least $2k-2$ and let $A = \{v_{i+1}, \ldots, v_{j-1}\}$ and $B = \{v_{j+1}, \ldots, v_{i-1}\}$ (indices are taken modulo n). Then $|N(A)| \leqslant 2k+1$ and $|N(B)| \leqslant 2k+1$.*

Proof. W.l.o.g., assume that D has a chord v_1v_j with $2k-1 \leqslant j \leqslant n-2k+3$.

Assume first that v_av_b is an arc from A to B.

(1) we cannot have $a \leqslant j-k$ and $b \leqslant n-k+1$, for otherwise the two dipaths $C[v_a, v_j]$ and $(v_a, v_b) \odot C[v_b, v_1] \odot (v_1, v_j)$ have length at least k and so their union is a subdivision of $C(k,k)$, a contradiction.

(2) we cannot have $a \geqslant k$ and $b \geqslant j+k-1$, for otherwise the two dipaths $C[v_1, v_a] \odot (v_a, v_b)$ and $(v_1, v_j) \odot C[v_j, v_b]$ have length at least k and so their union is a subdivision of $C(k,k)$, a contradiction.

Since $j \geqslant 2k-1$, either $a \leqslant j-k$ or $a \geqslant k$, so $v_b \in \{v_{j+1}, \ldots, v_{j+k-2}\} \cup \{v_{n-k+2}, \ldots, v_n\}$. Similarly, since $j \leqslant n-2k+3$, either $b \leqslant n-k+1$ or $b \geqslant j+k-1$, so $v_a \in \{v_2, \ldots, v_{k-1}\} \cup \{v_{j-k+1}, \ldots, v_{j-1}\}$.

Analogously, if v_bv_a is an arc from B to A, we obtain that $v_a \in \{v_2, \ldots, v_k\} \cup \{v_{j-k+2}, \ldots, v_{j-1}\}$ and $v_b \in \{v_{j+1}, \ldots, v_{j+k-2}\} \cup \{v_{n-k+3}, \ldots, v_n\}$.

Therefore $N(A) \subseteq \{v_1, \ldots, v_k\} \cup \{v_{j-k+1}, \ldots, v_j\}$, and $N(B) \subseteq \{v_j, \ldots, v_{j+k-2}\} \cup \{v_{n-k+2}, \ldots, v_n, v_1\}$. Hence $|N(A)| \leqslant 2k+1$ and $|N(B)| \leqslant 2k+1$. $\square$

Theorem 27. *Let D be a digraph and let $k \geqslant 1$ be an integer. If D has a Hamiltonian directed cycle and $\chi(D) > 6k-6$, then D contains a subdivision of a $C(k,k)$. In other words, $\phi(k,k) \leqslant 6k-6$.*

Proof. If $k = 2$, then we have the result by Theorem 37. Henceforth, we assume $k \geqslant 3$.

For sake of contradiction, let us consider a counterexample (i.e a digraph D with a Hamiltonian directed cycle, $\chi(D) > 6k-6$ and no subdivision of $C(k,k)$) with the minimum number of vertices.

Let $C = (v_1, \ldots, v_n, v_1)$ be a Hamiltonian directed cycle of D. By Lemma 24 and because $\chi(D) \geqslant 4k-4$, D contains a chord of span at least $2k-2$. Let s be the minimum span of a chord of span at least $2k-2$ and consider a chord of span s. W.l.o.g., this chord is $v_1 v_{s+1}$. Let $D_1 = D[v_1, \ldots, v_{s+1}]$ and let $D_2 = D[v_{s+1}, \ldots, v_n, v_1]$. By minimality of the span of $v_1 v_{s+1}$, either D_1 or D_2 contains no chord of span at least $2k-2$. There are two cases to be considered.

- Assume first that D_1 contains no chord of span at least $2k-2$. By Lemma 24, $\chi(D_1) \leqslant 4k-7$. Let $A = \{v_2, \ldots, v_s\}$. We have $\chi(D[A]) \leqslant \chi(D_1) \leqslant 4k-7$. Moreover, by Lemma 26, $|N(A)| \leqslant 2k+1$.

  Now D_2 has a Hamiltonian directed cycle and contains no subdivision of $C(k,k)$. Therefore, $\chi(D_2) \leqslant 6k-6$ since D has been chosen minimum. Finally, by Lemma 25, since $\chi(D[A]) + |N(A)| \leqslant 6k-6$ and $\chi(D_2) \leqslant 6k-6$, we get that $\chi(D) \leqslant 6k-6$, a contradiction.

- Assume now that D_2 contains no chord of span at least $2k-2$. Set $B = \{v_{s+1}, \ldots, v_n\}$. Similarly as in the previous case, we have $\chi(D[B]) \leqslant \chi(D_2) \leqslant 4k-7$ and $|N(B)| \leqslant 2k+1$.

  Let D'_1 be the digraph obtained from D_1 by reversing the arc $v_1 v_s$. Clearly D'_1 is Hamiltonian. Moreover, D'_1 contains no subdivision of a $C(k,k)$; indeed if it had such a subdivision S, replacing the arc $v_s v_1$ by $C[v_s, v_1]$ if it is in S, we obtain a subdivision of $C(k,k)$ in D, a contradiction. Therefore $\chi(D_1) = \chi(D'_1) \leqslant 6k-6$, by minimality of D.

  Hence by Lemma 25, since $\chi(D[B]) + |N(B)| \leqslant 6k-6$ and $\chi(D_1) \leqslant 6k-6$, we get that $\chi(D) \leqslant 6k-6$, a contradiction.

□

5.4 Better bound when $\ell = 1$

We now improve on the bound of Theorem 23 when $\ell = 1$. To do so, reduce the problem to digraphs having a Hamiltonian directed cycle. Recall that

$$\phi(k,\ell) = \max\{\chi(D) \mid D \in \text{S-Forb}(C(k,\ell)) \text{ and } D \text{ has a Hamiltonian directed cycle}\}.$$

Theorem 28. *Let k be an integer greater than 1.* $\chi(\text{S-Forb}(C(k,1)) \cap \mathcal{S}) \leqslant \max\{2k-4, \phi(k,1)\}$.

To prove this theorem, we shall use the following lemma.

Lemma 29. *Let D be a digraph containing a directed cycle C of length at least $2k-3$. If there is a vertex y in $V(D-C)$ and two distinct vertices $x_1, x_2 \in V(C)$ such that for $i = 1,2$, there is a (x_i, y)-dipath P_i in D with no internal vertices in C, then D contains a subdivision of $C(k,1)$.*

Proof. Since C has length at least $2k-3$, then one of $C[x_1,x_2]$ and $C[x_2,x_1]$ has length at least $k-1$. Without loss of generality, assume that $C[x_1,x_2]$ has length at least $k-1$. Let z be the first vertex along P_2 which is also in P_1. Then the union of $C[x_1,x_2] \odot P_2[x_2,z]$ and $P_1[x_1,z]$ is a subdivision of $C(k,1)$. $\square$

Proof of Theorem 28. Suppose for a contradiction that there is a strong digraph D with chromatic number greater than $\max\{2k-4, \phi(k,1)\}$ that contains no subdivision of $C(k,1)$. Let us consider the smallest such counterexample.

All 2-connected components of D are strong, and one of them has chromatic number $\chi(D)$. Hence, by minimality, D is 2-connected. Let C be a longest directed cycle in D. By Bondy's theorem (Theorem 10), C has length at least $2k-3$, and by definition of $\phi(k,1)$, C is not Hamiltonian.

Because D is strong, there is a vertex $v \in C$ with an out-neighbour $w \notin C$. Since D is 2-connected, $D-v$ is connected, so there is a (not necessarily directed) oriented path in $D-v$ between $C-v$ and w. Let $Q=(a_1,\dots,a_q)$ be such a path so that all its vertices except the initial one are in $V(D)\setminus V(C)$. By definition $a_q=w$ and $a_1 \in V(C)\setminus\{v\}$.

- Let us first assume that $a_1a_2 \in A(D)$. Let t be the largest integer such that there is a dipath from $C-v$ to a_t in $D-v$. Note that $t>1$ by the hypothesis. If $t=q$, then by Lemma 29, C contains a subdivision of $C(k,1)$, a contradiction. Henceforth we may assume that $t<q$. By definition of t, $a_{t+1}a_t$ is an arc. Let P be a shortest (v,a_{t+1})-dipath in D. Such a dipath exists because D is strong. By maximality of t, P has no internal vertex in $(C-v)\cup Q[a_1,a_t]$. Hence, $a_t \in D-C$ and there are an (a_1,a_t)-dipath and a (v,a_t)-dipath with no internal vertices in C. Hence, by Lemma 29, D contains a subdivision of $C(k,1)$, a contradiction.

- Now, we may assume that any oriented path $Q=(a_1,\dots,a_q)$ from $C-v$ to w starts with a backward arc, i.e., $a_2a_1 \in A(D)$. Let W be the set of vertices x such that there exists a (not necessarily directed) oriented path from w to x in $D-C$. In particular, $w \in W$.

  By the assumption, all arcs between $C-v$ and W are from W to $C-v$. Since D is strong, this implies that, for any $x \in W$, there exists a directed (w,x)-dipath in W. In other words, w is an out-generator of W. Let T_w be a BFS-tree of W rooted in w (see definitions in Section 5.1.1).

  Because D is strong and 2-connected, there must be a vertex $y \in C-v$ such that there is an arc ay from a vertex $a \in W$ to y.

  For purpose of contradiction, let us assume that there exists $z \in C-y$ such that there is an arc bz from a vertex $b \in W$ to z. Let r be the least common ancestor of a and b in T_w. If $|C[y,z]| \geqslant k$, then $T_w[r,a]\odot(a,y)\odot C[y,z]$ and $T_w[r,b]\odot(b,z)$ is a subdivision of $C(k,1)$. If $|C[z,y]| \geqslant k$, then $T_w[r,a]\odot(a,y)$ and $T_w[r,b]\odot(b,z)\odot C[z,y]$ is a subdivision of $C(k,1)$. In both cases, we get a contradiction.

  From previous paragraph and the definition of W, we get that all arcs from W to $D\setminus W$ are from W to $y \neq v$, and there is a single arc from $D\setminus W$ to W (this is the arc vw). Note that, since D is strong, this implies that $D-W$ is strong.

Let D_1 be the digraph obtained from $D-W$ by adding the arc vy (if it does not already exist). D_1 contains no subdivision of $C(k,1)$, for otherwise D would contain one (replacing the arc vy by the dipath $(v,w)\odot T_w[w,a]\odot(a,y)$). Since D_1 is strong (because $D-W$ is strong), by minimality of D, $\chi(D_1)\leqslant \max\{2k-4,\phi(k,1)\}$.

Let D_2 be the digraph obtained from $D[W\cup\{v,y\}]$ by adding the arc yv. D_2 contains no subdivision of $C(k,1)$, for otherwise D would contain one (replacing the arc yv by the dipath $C[y,v]$). Moreover, D_2 is strong, so by minimality of D, $\chi(D_2)\leqslant \max\{2k-4,\phi(k,1)\}$.

Consider now D^* the digraph $D_1\cup D_2$. It is obtained from D by adding the two arcs vy and yv (if they did not already exist). Since $\{v,y\}$ is a clique-cutset in D^*, we get $\chi(D^*)\leqslant \max\{\chi(D_1),\chi(D_2)\}\leqslant \max\{2k-4,\phi(k,1)\}$. But $\chi(D)\leqslant\chi(D^*)$, a contradiction.

$\square$

From Theorem 28, one easily derives an upper bound on $\chi(\text{S-Forb}(C(k,1))\cap\mathcal{S})$.

Corollary 30. $\chi(\text{S-Forb}(C(k,1))\cap\mathcal{S})\leqslant 2k-1$.

Proof. By Theorem 28, it suffices to prove $\phi(k,1)\leqslant 2k-1$.

Let $D\in\text{S-Forb}(C(k,1))$ with a Hamiltonian directed cycle $C=(v_1,\dots,v_n,v_1)$. Observe that if v_iv_j is an arc, then $j\in C[v_{i+1},v_{i+k-1}]$ for otherwise the union of $C[v_i,v_j]$ and (v_i,v_j) would be a subdivision of $C(k,1)$. In particular, every vertex had both its in-degree and out-degree at most $k-1$, and so degree at most $2k-2$. As $\chi(D)\leqslant\Delta(D)+1$, the result follows. $\square$

The bound $2k-1$ is tight for $k=2$, because of the directed odd cycles. However, for larger values of k, we can get a better bound on $\phi(k,1)$, from which one derives a slightly better one for $\chi(\text{S-Forb}(C(k,1))\cap\mathcal{S})$.

Theorem 31. $\phi(k,1)\leqslant\max\{k+1,\frac{3k-3}{2}\}$.

Proof. For $k=2$, the result holds because $\phi(2,1)\leqslant\phi(2,2)\leqslant 3$ by Corollary 38.

Let us now assume $k\geqslant 3$. We prove by induction on n, that every digraph $D\in\text{S-Forb}(C(k,1))$ with a Hamiltonian directed cycle $C=(v_1,\dots,v_n,v_1)$ has chromatic number at most $\max\{k+1,\frac{3k-3}{2}\}$, the result holding trivially when $n\leqslant\max\{k+1,\frac{3k-3}{2}\}$.

Assume now that $n\geqslant\max\{k+1,\frac{3k-3}{2}\}+1$ All the indices are modulo n. Observe that if v_iv_j is an arc, then $j\in C[v_{i+1},v_{i+k-1}]$ for otherwise the union of $C[v_i,v_j]$ and (v_i,v_j) would be a subdivision of $C(k,1)$. In particular, every vertex had both its in-degree and out-degree at most $k-1$.

Assume that D contains a vertex v_i with in-degree 1 or out-degree 1. Then $d(v_i)\leqslant k$. Consider D_i the digraph obtained from $D-v_i$ by adding the arc $v_{i-1}v_{i+1}$. Clearly, D_i has a Hamiltonian directed cycle. Moreover is has no subdivision of $C(k,1)$ for otherwise, replacing the arc $v_{i-1}v_{i+1}$ by (v_{i-1},v_i,v_{i+1}) if necessary, yields a subdivision of $C(k,1)$ in D. By the induction hypothesis, D_i a $\max\{k+1,\frac{3k-3}{2}\}$-colouring which can be extended to v_i because $d(v_i)\leqslant k$.

Henceforth, we may assume that $\delta^-(D),\delta^+(D)\geqslant 2$.

Claim 31.1. *$d^+(v_i)+d^-(v_{i+1})\leqslant 3k-n-3$ for all i.*

Subproof. Let v_{i^+} be the first out-neighbour of v_i along $C[v_{i+2}, v_{i-1}]$ and let v_{i^-} be the last in-neighbour of v_{i+1} along $C[v_{i+3}, v_i]$. There are $d^+(v_i) - 1$ out-neigbours of v_i in $C[v_{i^+}, v_{i-1}]$ which all must be in $C[v_{i^+}, v_{i+k-1}]$ by the above observation. Therefore $i^+ \leqslant i + k - d^+(v_i)$. Similarly, $i^- \geqslant i - k + d^-(v_{i+1})$.

- if $v_i \in C[v_{i^-}, v_{i^+}]$, $C[v_{i^-}, v_{i^+}]$ has length $i^+ - i^- \leqslant 2k - d^+(v_i) - d^-(v_{i+1})$. Hence $C[v_{i^+}, v_{i^-}]$ has length at least $n - 2k + d^+(v_i) + d^-(v_{i+1})$. But the union of $(v_i, v_{i^+}) \odot C[v_{i^+}, v_{i^-}] \odot (v_{i^-}, v_{i+1})$ and (v_i, v_{i+1}) is not a subdivision of $C(k,1)$, so $C[v_{i^+}, v_{i^-}]$ has length at most $k-3$. Hence, $k - 3 \geqslant n - 2k + d^+(v_i) + d^-(v_{i+1})$, so $d^+(v_i) + d^-(v_{i+1}) \leqslant 3k - n - 3$.

- otherwise, $v_{i^+} \in C[v_{i^-}, v_{i+1}]$ and $v_{i^-} \in C[v_i, v_{i^+}]$. Both $C[v_{i^-}, v_{i+1}]$ and $C[v_i, v_{i^+}]$ have length less than k as $v_{i^-}v_{i+1}$ and $v_{i^-}v_{i+1}$ are arcs. Moreover, the union of these two dipaths is C and their intersection contains the three distinct vertices v_i, v_{i+1}, v_{i^-}. Consequently, $n = |C| \leqslant |C[v_{i^-}, v_{i+1}]| + |C[v_i, v_{i^+}]| - 3 \leqslant 2k - 3$. Let v_{i_0} be the last out-neighbour of v_i along $C[v_{i+2}, v_{i-1}]$. All the out-neighbours of v_i and all the in-neighbours of v_{i+1} are in $C[v_i, v_{i_0}]$ which has length less than k because $v_i v_{i_0}$ is an arc. Hence $d^+(v_i) + d^-(v_{i+1}) \leqslant k$, so $d^+(v_i) + d^-(v_{i+1}) \leqslant 3k - n - 3$ because $n \geqslant 2k - 3$. $\Diamond$

But $n \geqslant \frac{3k-1}{2}$, so by the above claim, $d^+(v_i) + d^-(v_{i+1}) \leqslant \frac{3k-5}{2}$ for all i.

Summing these inequalities over all i, we get $\sum_{i=1}^{n}(d^+(v_i) + d^-(v_{i+1}) \leqslant \frac{3k-5}{2} \cdot n$. Thus $\sum_{i=1}^{n} d(v_i) = \sum_{i=1}^{n}(d^+(v_i) + d^-(v_i)) \leqslant \frac{3k-5}{2} \cdot n$. Therefore there exists an index i such that v_i has degree at most $\frac{3k-5}{2}$. Consider the digraph D_i defined above. It is Hamiltonian and contains no subdivision of $C(k,1)$. By the induction hypothesis, D_i has a $\max\{k+1, \frac{3k-3}{2}\}$-colouring which can be extended to v because $d(v_i) \leqslant \frac{3k-5}{2}$. $\square$

Corollary 32. *Let k be an integer greater than 1. Then $\chi(\text{S-Forb}(C(k,1)) \cap \mathcal{S}) \leqslant \max\{k+1, 2k-4\}$.*

Proof. By Theorems 28 and 31, $\chi(\text{S-Forb}(C(k,1)) \cap \mathcal{S}) \leqslant \max\{2k-4, k+1, \frac{3k-3}{2}\} = \max\{k+1, 2k-4\}$. $\square$

6 Small cycles with two blocks in strong digraphs

6.1 Handle decomposition

Let D be a strongly connected digraph. A *handle* h of D is a directed path $(s, v_1, \dots, v_\ell, t)$ from s to t (where s and t may be identical) such that:

- $d^-(v_i) = d^+(v_i) = 1$, for every i, and

- removing the internal vertices and arcs of h leaves D strongly connected.

The vertices s and t are the *endvertices* of h while the vertices v_i are its *internal vertices*. The vertex s is the *initial vertex* of h and t its *terminal vertex*. The *length* of a handle is the number of its arcs, here $\ell + 1$. A handle of length 1 is said to be *trivial*.

Given a strongly connected digraph D, a *handle decomposition* of D starting at $v \in V(D)$ is a triple $(v, (h_i)_{1 \leqslant i \leqslant p}, (D_i)_{0 \leqslant i \leqslant p})$, where $(D_i)_{0 \leqslant i \leqslant p}$ is a sequence of strongly connected digraphs and $(h_i)_{1 \leqslant i \leqslant p}$ is a sequence of handles such that:

- $V(D_0) = \{v\}$,
- for $1 \leqslant i \leqslant p$, h_i is a handle of D_i and D_i is the (arc-disjoint) union of D_{i-1} and h_i, and
- $D = D_p$.

A handle decomposition is uniquely determined by v and either $(h_i)_{1 \leqslant i \leqslant p}$, or $(D_i)_{0 \leqslant i \leqslant p}$. The number of handles p in any handle decomposition of D is exactly $|A(D)| - |V(D)| + 1$. The value p is also called the *cyclomatic number* of D. Observe that $p = 0$ when D is a singleton and $p = 1$ when D is a directed cycle.

A handle decomposition $(v, (h_i)_{1 \leqslant i \leqslant p}, (D_i)_{0 \leqslant i \leqslant p})$ is *nice* if all handles except the first one h_1 have distinct endvertices (i.e., for any $1 < i \leqslant p$, the initial and terminal vertices of h_i are distinct).

A digraph is *robust* if it is 2-connected and strongly connected. The following proposition is well-known (see [5] Theorem 5.13).

Proposition 33. *Every robust digraph admits a nice handle decomposition.*

Lemma 34. *Every strong digraph D with $\chi(D) \geqslant 3$ has a robust subdigraph D' with $\chi(D') = \chi(D)$ and which is an oriented graph.*

Proof. Let D be a strong digraph D with $\chi(D) \geqslant 3$. Let D' be a 2-connected components of D with the largest chromatic number. Each 2-connected component of a strong digraph is strong, so D' is strong. Moreover, $\chi(D') = \chi(D)$ because the chromatic number of a graph is the maximum of the chromatic numbers of its 2-connected components. Now by Bondy's Theorem (Theorem 10), D' contains a cycle C of length at least $\chi(D') \geqslant 3$. This can be extended into a handle decomposition $(v, (h_i)_{1 \leqslant i \leqslant p}, (D_i)_{0 \leqslant i \leqslant p})$ of D such that $D_1 = C$. Let D'' be the digraph obtained from D' by removing the arcs (u, v) which are trivial handles h_i and such that (v, u) is in $A(D_{i-1})$, we obtain an oriented graph D'' which is robust and with $\chi(D'') = \chi(D') = \chi(D)$. □

6.2 $C(1,2)$

Proposition 35. *A robust digraph containing no subdivision of $C(1,2)$ is a directed cycle.*

Proof. Let D be a robust digraph containing no subdivision of $C(1,2)$. Assume for a contradiction that a robust digraph of D is not a directed cycle. By Proposition 33, it contains a directed cycle C and a nice handle h_2 from u to v. Now the union of h_2 and $C[u, v]$ is a subdivision of $C(1,2)$. □

Corollary 36. $\chi(\text{S-Forb}(C(1,2)) \cap \mathcal{S}) = 3$.

Proof. Lemma 34, Proposition 35, and the fact that every directed cycles is 3-colourable imply $\chi(\text{S-Forb}(C(1,2)) \cap \mathcal{S}) \leqslant 3$.

The directed cycles of odd length have chromatic number 3 and contain no subdivision of $C(1,2)$. Therefore, $\chi(\text{S-Forb}(C(1,2)) \cap \mathcal{S}) = 3$. □

6.3 $C(2,2)$

Theorem 37. *Let D be a strong digraph. If $\chi(D) \geqslant 4$, then D contains a subdivision of $C(2,2)$.*

Proof. By Lemma 34, we may assume that D is robust.

By Proposition 33, D has a nice handle decomposition. Consider a nice decomposition $(v,(h_i)_{1\leqslant i\leqslant p},(D_i)_{0\leqslant i\leqslant p})$ that maximizes the sequence $(\ell_1,\dots,\ell_p)$ of the length of the handles with respect to the lexicographic order.

Let q be the largest index such that h_q is not trivial.

Assume first that $q \neq 1$. Let s and t be the initial and terminal vertex of h_q respectively. There is an (s,t)-path P in D_{q-1}. If $P=(s,t)$, let r be the index of the handle containing the arc (s,t). Obviously, $r<q$. Now replacing h_r by the handle h'_r obtained from it by replacing the arc (s,t) by h_q and replacing h_q by (s,t), we obtain a nice handle decomposition contradicting the minimality of $(v,(h_i)_{1\leqslant i\leqslant p},(D_i)_{0\leqslant i\leqslant p})$. Therefore P has length at least 2. So $P\cup h_q$ is a subdivision of $C(2,2)$.

Assume that $q=1$, that is D has a hamiltonian directed cycle C. Let us call *chords* the arcs of $A(D)\setminus A(C)$. Suppose that two chords (u_1,v_1) and (u_2,v_2) *cross*, that is $u_2\in C]u_1,v_1[$ and $v_2\in C]v_1,u_1[$. Then the union of $C[u_1,u_2]\odot(u_2,v_2)$ and $(u_1,v_1)\odot C[v_1,v_2]$ forms a subdivision of $C(2,2)$.

If no two chords cross, then one can draw C in the plane and all chords inside it without any crossing. Therefore the graph underlying D is outerplanar and has chromatic number at most 3. $\square$

Since the directed odd cycles are in S-Forb$(C(2,2))$ and have chromatic number 3, Theorem 37 directly implies the following.

Corollary 38. $\chi(\text{S-Forb}(C(2,2))\cap\mathcal{S})=3$.

6.4 $C(1,3)$

Theorem 39. *Let D be a strong digraph. If $\chi(D) \geqslant 4$, then D contains a subdivision of $C(1,3)$.*

Proof. By Lemma 34, we may assume that D is robust. Thus, by Proposition 33, D has a nice handle decomposition. Consider a nice decomposition $(v,(h_i)_{1\leqslant i\leqslant p},(D_i)_{0\leqslant i\leqslant p})$ that maximizes the sequence $(\ell_1,\dots,\ell_p)$ of the length of the handles with respect to the lexicographic order.

Let q be the largest index such that h_q is not trivial.

Case 1: Assume first that $q\neq 1$. Let s and t be the initial and terminal vertex of h_q respectively. Since D_{q-1} is strong, there is an (s,t)-dipath P in D_{q-1}. If $P=(s,t)$, let r be the index of the handle containing the arc (s,t). Obviously, $r<q$. Now replacing h_r by the handle h'_r obtained from it by replacing the arc (s,t) by h_q and replacing h_q by (s,t), we obtain a nice handle decomposition contradicting the minimality of $(v,(h_i)_{1\leqslant i\leqslant p},(D_i)_{0\leqslant i\leqslant p})$. Therefore P has length at least 2. If either P or h_q has length at least 3, then $P\cup h$ is a subdivision of $C(1,3)$. Henceforth, we may assume that both P and h_q have length 2. Set $P=(s,u,t)$ and $h=(s,x,t)$. Observe that $V(D)=V(D_{q-1})\cup\{x\}$.

Assume that x has a neighbour t' distinct from s and t. By directional duality (i.e., up to reversing all arcs), we may assume that $x \to t'$. Considering the handle decomposition in which h_q is replaced by (s,x,t') and (x,t') by (x,t), we obtain that there is a dipath (s,u',t') in D_{q-1}. Now, if $u' = t$, then the union of (s,x,t') and (s,u,t,t') is a subdivision of $C(1,3)$. Henceforth, we may assume that $t \notin \{s,u,u',t'\}$. Since D_{q-1} is strong, there is a dipath Q from t to $\{s,u,u',t'\}$, which has length at least one by the preceding assumption. Note that $x \notin Q$ since Q is a dipath in D_{q-1}. Whatever vertex of $\{s,u,u',t'\}$ is the terminal vertex z of Q, we find a subdivision of $C(1,3)$:

- If $z = s$, then the union of (x,t') and $(x,t) \odot Q \odot (s,u',t')$ is a subdivision of $C(1,3)$;
- If $z = u$, then the union of (s,u) and $h_q \odot Q$ is a subdivision of $C(1,3)$;
- If $z = u'$, then the union of (s,u') and $h_q \odot Q$ is a subdivision of $C(1,3)$;
- If $z = t'$, then the union of (s,x,t') and $(s,u,t) \odot Q$ is a subdivision of $C(1,3)$.

Case 2: Assume that $q = 1$, that is D has a hamiltonian directed cycle C. Assume that two chords (u_1,v_1) and (u_2,v_2) cross. Without loss of generality, we may assume that the vertices u_1, u_2, v_1 and v_2 appear in this order along C. Then the union of $C[u_2,v_1]$ and $(u_2,v_2) \odot C[v_2,u_1] \odot (u_1,v_1)$ forms a subdivision of $C(1,3)$.

If no two chords cross, then one can draw C in the plane and all chords inside it without any crossing. Therefore the graph underlying D is outerplanar and has chromatic number at most 3. □

Since the directed odd cycles are in S-Forb$(C(1,3))$ and have chromatic number 3, Theorem 39 directly implies the following.

Corollary 40. $\chi(\text{S-Forb}(C(1,3)) \cap \mathcal{S}) = 3$.

6.5 $C(2,3)$

Theorem 41. *Let D be a strong directed graph. If $\chi(D) \geqslant 5$, then D contains a subdivision of $C(2,3)$.*

Proof. By Lemma 34, we may assume that D is a robust oriented graph. Thus, by Proposition 33, D has a nice handle decomposition. Let $\text{HD} = ((h_i)_{1 \leqslant i \leqslant p}, (D_i)_{1 \leqslant i \leqslant p})$ be a nice decomposition that maximizes the sequence $(\ell_1, \dots, \ell_p)$ of the length of the handles with respect to the lexicographic order. Recall that D_i is strongly connected for any $1 \leqslant i \leqslant p$. In particular, h_1 is a longest directed cycle in D. Let q be the largest index such that h_q is not trivial. Observe that for all $i > q$, h_i is a trivial handle by definition of q and, for $i \leqslant q$, all handles h_i have length at least 2.

Claim 41.1. *For any $1 < i \leqslant q$, h_i has length exactly 2.*

Subproof. For sake of contradiction, let us assume that there exists $2 \leqslant r \leqslant q$ such that $h_r = (x_1, \dots, x_t)$ with $t \geqslant 4$. Since D_{r-1} is strong, there is a (x_1, x_t)-dipath P in D_{r-1}. Note that P does not meet $\{x_2, \dots, x_{t-1}\}$. If P has length at least 2, then $P \cup h_r$ is a subdivision of $C(2,3)$. If $P = (x_1, x_t)$, let r' be the handle containing the arc $h_{r'}$. Now the handle decomposition obtained from HD by replacing $h_{r'}$by the handle derived from it by replacing the arc (x_1, x_t) by h_r, and replacing h_r by (x_1, x_t), contradicts the maximality of HD. $\Diamond$

For $1 < i \leqslant q$, set $h_i = (a_i, b_i, c_i)$. Since h_1 is a longest directed cycle in D and $\chi(D) \geqslant 5$, by Bondy's Theorem, h_1 has length at least 5. Set $h_1 = (u_1, \dots, u_m, u_1)$.

A *clone* of u_i is a vertex whose unique out-neighbour in D_q is u_{i+1} and whose unique in-neighbour in D_q is u_{i-1} (indices are taken modulo m).

Claim 41.2. *Let $v \in V(D) \setminus V(D_1)$. Let $1 < i \leqslant q$ such that $v = b_i$, the internal vertex of h_i. There is an index j such that b_i is a clone of u_j, that is $a_i = u_{j-1}$ and $c_i = u_{j+1}$.*

Subproof. We prove the result by induction on i.

By the induction hypothesis (or trivially if $i = 2$), there exists i^- and i^+ such that a_i is u_{i^-} or a clone of u_{i^-} and c_i is u_{i^+} or a clone of u_{i^+}. If $i^+ \notin \{i^- + 1, i^- + 2\}$, then the union of h_i and $(a_i, u_{i^-+1}, \dots, u_{i^+-1}, c_i)$ is a subdivision of $C(2,3)$, a contradiction If $i^+ = i^- - 1$, then $(a_i, b_i, c_i, h_1[u_{i^++1}, \dots, u_{i^--1}], a_i)$ is a cycle longer than h_1, a contradiction. Henceforth $i^+ = i^- + 2$. If c_i is not u_{i^+}, then it is a clone of u_{i^+}. Thus the union of $(a_i, b_i, c_i, u_{i^++1})$ and $(a_i, u_{i^-+1}, u_{i^+}, u_{i^++1})$ is a subdivision of $C(2,3)$, a contradiction Similarly, we obtain a contradiction if $a_i \neq u_{i^-}$. Therefore, $a_i = u_{i^--1}$ and $c_i = u_{i^-+1}$, that is b_i is a clone of u_{i^-+1}. Moreover all $b_{i'}$ for $i' < i$ are not adjacent to b_i and thus are still clones of some u_j. $\Diamond$

For $1 \leqslant i \leqslant m$, let S_i be the set of clones of u_i.

Claim 41.3.

(i) If $S_i \neq \emptyset$, then $S_{i-1} = S_{i+1} = \emptyset$.

(ii) If $x \in S_i$, then $N^+_D(x) = \{u_{i+1}\}$ and $N^-_D(x) = \{u_{i-1}\}$.

Subproof. (i) Assume for a contradiction, that both S_i and S_{i+1} are non-empty, say $x_i \in S_i$ and $x_{i+1} \in S_{i+1}$. Then the union of $(u_{i-1}, u_i, x_{i+1}, u_{i+2})$ and $(u_{i-1}, x_i, u_{i+1}, u_{i+2})$ is a subdivision of $C(2,3)$, a contradiction.

(ii) Let $x \in S_i$. Assume for a contradiction that x has an out-neighbour y distinct from u_{i+1}. By (i), $y \notin S_{i-1}$, and $y \neq u_{i-1}$ because D is an oriented graph. If $y \in S_i \cup \{u_i\}$, then $(x, y, h_1[u_{i+1}, u_{i-1}], x)$ is a directed cycle longer than h. If $y \in S_j \cup \{u_j\}$ for $j \notin \{i-2\}$, then the union of (u_{i-1}, x, y, u_{j+1}) and $h_1[u_{i-1}, u_{j+1}]$ is a subdivision of $C(2,3)$, a contradiction. If $y \in S_{i-2}$, then the union of (x, y, u_{i-1}) and $(x, h_1[u_{i+1}, u_{i-1}]$ is a subdivision of $C(2,3)$, a contradiction. If $y = u_j$ for $j \notin \{i-1, i, i+1\}$, then the union of (u_{i-1}, x, y) and $h_1[u_{i-1}, y]$ is a subdivision of $C(2,3)$, a contradiction. $\Diamond$

This implies that $q = 1$. Indeed, if $q \geqslant 2$, then there is $i \leqslant m$ such that $b_2 \in S_i$. But $D - b_q = D_{q-1}$ is strong, and $\chi(D - b_q) \geqslant 5$, because $\chi(D) \geqslant 5$ and b_q has only two neighbours in D by

Claim 41.3-(ii). But then by minimality of D, $D - b_q$ contains a subdivision of $C(2,3)$, which is also in D, a contradiction.

Hence $m = |V(D)|$. Because $\chi(D) \geqslant 5$, D is not outerplanar, so there must be $i < j < k < \ell < i+m$ such that $(u_i, u_k) \in A(D)$ and $(u_j, u_\ell) \in A(D)$. We must have $j = i+1$ and $\ell = k+1$ since otherwise $(u_i, \dots, u_j, u_\ell)$ and $(u_i, u_k, \dots, u_\ell)$ form a subdivision of $C(2,3)$. In addition, $k = j+1$ since otherwise, $(u_j, u_\ell, \dots, u_i, u_k)$ and $(u_j, \dots, u_k)$ form a subdivision of $C(2,3)$. Therefore, any two "crossing" arcs must have their ends being consecutive in D_1. This implies that $N^+(u_j) = \{u_{j+1}, u_{j+2}\}$, $N^-(u_j) = \{u_{j-1}\}$, $N^+(u_k) = \{u_{k+1}\}$ and $N^-(u_k) = \{u_{k-1}, u_{k-2}\}$.

Now let D' be the digraph obtained from $D - \{u_j, u_k\}$ by adding the arc (u_i, u_ℓ). Because u_j and u_k have only three neighbours in D, $\chi(D') \geqslant 5$. By minimality of D, D' contains a subdivision of $C(2,3)$, which can be transformed into a subdivision of $C(2,3)$ in D by replacing the arc (u_i, u_ℓ) by the directed path $(u_i, u_j, u_{k,l})$. $\square$

Since every semi-complete digraph of order 4 does not contain $C(2,3)$ (which has order 5), we have the following.

Corollary 42. $\chi(\text{S-Forb}(C(2,3)) \cap \mathcal{S}) = 4$.

7 Cycles with four blocks in strong digraphs

Theorem 43. *Let D be a digraph in* S-Forb$(\hat{C}_4)$. *If D admits an out-generator, then $\chi(D) \leqslant 24$.*

Proof. The general idea is the same as in the proof of Theorem 23.

Suppose that D admits an out-generator u and let T be an BFS-tree with root u (See Subsubsection 5.1.1.). We partition $A(D)$ into three sets according to the levels of u.

$$\begin{aligned} A_0 &= \{(x,y) \in A(D) \mid \text{lvl}(x) = \text{lvl}(y)\}; \\ A_1 &= \{(x,y) \in A(D) \mid |\text{lvl}(x) - \text{lvl}(y)| = 1\}; \\ A_2 &= \{(x,y) \in A(D) \mid \text{lvl}(y) \leqslant \text{lvl}(x) - 2\}. \end{aligned}$$

For $i = 0,1,2$, let $D_i = (V(D), A_i)$.

Claim 43.1. $\chi(D_0) \leqslant 3$.

Subproof. Suppose for a contradiction that $\chi(D) \geqslant 4$. By Theorem 6, it contains a $P^-(1,1)$ (y_1, y, y_2), that is y, y_1 and y, y_2 are in $A(D_0)$. Let x be the least common ancestor of y_1 and y_2 in T. The union of $T[x, y_1]$, (y, y_1), (y, y_2), and $T[x, y_2]$ is a subdivision of $\hat{C}_4$, a contradiction. $\Diamond$

Claim 43.2. $\chi(D_1) \leqslant 2$.

Subproof. Since the arc are between consecutive levels, then the colouring ϕ_1 defined by $\phi_1(x) = lvl(x) \mod 2$ is a proper 2-colouring of D_1. $\Diamond$

Let $y \in V_i$ we denote by $N'(y)$ the out-degree of y in $\bigcup_{0 \leqslant j \leqslant i-1} V_j$. Let $D' = (V, A')$ with $A' = \cup_{x \in V} \{(x,y), y \in N'(x)\}$ and $D_x = (V, A_x)$ where A_x is the set of arc inside the level and from V_i to V_{i+1} for all i. Note that $A = A' \cup A_x$ and

Claim 43.3. $\chi(D_2) \leqslant 4$.

Subproof. Let x be a vertex of $V(D)$. If y and z are distinct out-neighbours of x in D_2, then their least common ancestor w is either y or z, for otherwise the union of $T[w,y]$, (x,y), (x,z), and $T[w,z]$ is a subdivision of $\hat{C}_4$. Consequently, there is an ordering $y_1,\dots,y_p$ of $N^+_{D_2}(x)$ such that the y_i appear in this order on $T[u,x]$.

Let us prove that $N^+(y_i) = \emptyset$ for $2 \leqslant i \leqslant p-1$. Suppose for a contradiction that y_i has an out-neighbour z in D_2. Let t be the least common ancestor of y_1 and z. If $t = z$, then the union of $(y_i,z) \odot T[z,y_1]$, (x,y_1), (x,y_p), and $T[y_i,y_p]$ is a subdivision of $\hat{C}_4$; if $t = y \neq z$, then the union of (y_i,z), $(x,y_1) \odot T[y_1,z]$, (x,y_p), and $T[y_i,y_p]$ is a subdivision of $\hat{C}_4$. Otherwise, if $t \notin \{y,z\}$, $T[t,y_1]$, $T[t,z]$, $(x,y_i) \odot (y_i,z)$ and (x,y_1) is a subdivision of $\hat{C}_4$.

Henceforth, in D_2, every vertex has at most two out-neighbours that are not sinks. Let V_0 be the set of sinks in D_2. It is a stable set in D_2. Furthermore $\Delta^+(D_2 - V_0) \leqslant 2$, so $D_2 - V_0$ is 3-colourable, because D_2 (and so $D_2 - V_0$) is acyclic. Therefore $\chi(D_2) \leqslant 4$. ◊

Claims 43.1, 43.2, 43.3, and Lemma 22 implies $\chi(D) \leqslant 24$. □

8 Further research

The upper bound of Theorem 23 can be lowered when considering 2-strong digraphs.

Theorem 44. *Let k and ℓ be two integers such that, $k \geqslant \ell$, $k+\ell \geqslant 4$ and $(k,\ell) \neq (2,2)$. Let D be a 2-strong digraph. If $\chi(D) \geqslant (k+\ell-2)(k-1)+2$, then D contains a subdivision of $C(k,\ell)$.*

Proof. Let D be a 2-strong digraph with chromatic number at least $(k+\ell-2)(k-1)+2$. Let u be a vertex of D. For every positive integer i, let $L_i = \{v \mid \mathrm{dist}_D(u,v) = i\}$.

Assume first that $L_k \neq \emptyset$. Take $v \in L_k$. In D, there are two internally disjoint (u,v)-dipaths P_1 and P_2. Those two dipaths have length at least k (and ℓ as well) since $\mathrm{dist}_D(u,v) \geqslant k$. Hence $P_1 \cup P_2$ is a subdivision of $C(k,\ell)$.

Therefore we may assume that L_k is empty, and so $V(D) = \{u\} \cup L_1 \cup \cdots \cup L_{k-1}$. Consequently, there is i such that $\chi(D[L_i]) \geqslant k+\ell-1$. Since $k+l-1 \geqslant 3$ and $(k-1,\ell-1) \neq (1,1)$, by Theorem 6, $D[L_i]$ contains a copy Q of $P^+(k-1,\ell-1)$. Let v_1 and v_2 be the initial and terminal vertices of Q. By definition, for $j \in \{1,2\}$, there is a (u,v_j)-dipath P_j in D such that $V(P_j) \cap L_i = \{v_j\}$. Let w be the last vertex along P_1 that is in $V(P_1) \cap V(P_2)$. Clearly, $P_1[w,v_1] \cup P_2[w,v_2] \cup Q$ is a subdivision of $C(k,\ell)$. □

To go further, it is natural to ask what happens if we consider digraphs which are not only strongly connected but k-strongly connected (k-strong for short).

Proposition 45. *Let C be an oriented cycle of order n. Every $(n-1)$-strong digraph contains a subdivision of C.*

Proof. Set $C = (v_1,v_2,\dots,v_n,v_1)$. Without loss of generality, we may assume that $(v_1,v_n) \in A(C)$. Let D be an $(n-1)$-strong digraph. Choose a vertex x_1 in $V(D)$. Then for $i = 2$ to n, choose a vertex x_i in $V(D) \setminus \{x_1,\dots,x_{i-1}\}$ such that $x_{i-1}x_i$ is an arc in D if $v_{i-1}v_i$ is an arc in

C and $x_i x_{i-1}$ is an arc in D if $v_i v_{i-1}$ is an arc in C. This is possible since every vertex has in- and out-degree at least $n-1$. Now, since D is $(n-1)$-strong, $D - \{x_2, \ldots, x_{n-1}\}$ is strong, so there exists a (x_1, x_n)-dipath P in $D - \{x_2, \ldots, x_{n-1}\}$. The union of P and $(x_1, x_2, \ldots, x_n)$ is a subdivision of C. □

Let $\mathcal{S}_p$ be the class of p-strong digraphs. Proposition 45 implies directly that $\text{S-Forb}(C) \cap \mathcal{S}_p = \emptyset$ and so $\chi(\text{S-Forb}(C) \cap \mathcal{S}_p) = 0$ for any oriented cycle C of length $p+1$. This yields the following problems.

Problem 46. Let C be an oriented cycle and p a positive integer. What is $\chi(\text{S-Forb}(C) \cap \mathcal{S}_p)$?

Note that $\chi(\text{S-Forb}(C) \cap \mathcal{S}_{p+1}) \leqslant \chi(\text{S-Forb}(C) \cap \mathcal{S}_p)$ for all p, because $\mathcal{S}_{p+1} \subseteq \mathcal{S}_p$.

Problem 47. Let C be an oriented cycle.

1) What is the minimum integer p_C such that $\chi(\text{S-Forb}(C) \cap \mathcal{S}_{p_C}) < +\infty$?

2) What is the minimum integer p_C^0 such that $\chi(\text{S-Forb}(C) \cap \mathcal{S}_{p_C^0}) = 0$?

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