

# Hamiltonicity of planar graphs with a forbidden minor

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October 20, 2016

## Abstract

Tutte showed that 4-connected planar graphs are Hamiltonian, but it is well known that 3-connected planar graphs need not be Hamiltonian. We show that  $K_{2,5}$ -minor-free 3-connected planar graphs are Hamiltonian. This does not extend to  $K_{2,5}$ -minor-free 3-connected graphs in general, as shown by the Petersen graph, and does not extend to  $K_{2,6}$ -minor-free 3-connected planar graphs, as we show by an infinite family of examples.

## 1 Introduction

All graphs in this paper are finite and simple (no loops or multiple edges).

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\*Supported by National Security Agency grant H98230-13-1-0233 and Simons Foundation awards 245715 and 429625. The United States Government is authorized to reproduce and distribute reprints notwithstanding any copyright notation herein.

†Supported by National Security Agency grant H98230-13-1-0233 and Simons Foundation award 245715.

‡Supported in part by JSPS KAKENHI Grant Number 25871053 and by a Grant for Basic Science Research Projects from The Sumitomo Foundation.

Whitney [24] showed that every 4-connected plane triangulation is Hamiltonian, and Tutte [23] extended this to every 4-connected planar graph. Tutte’s result has been strengthened in various ways; see for example [4, 14, 18, 20, 21, 22].

If we relax the connectivity condition, it is not true that all 2- or 3-connected planar graphs are Hamiltonian. The smallest 2-connected planar graph that is not Hamiltonian is  $K_{2,3}$ . The smallest 3-connected planar graph that is not Hamiltonian is the so-called Herschel graph, with 11 vertices and 18 edges. It was known to Coxeter in 1948 [6, p. 8], but a proof that it is smallest relies on later work by Barnette and Jucovič [1] and Dillencourt [9]. If we restrict to triangulations, the smallest 2- or 3-connected planar triangulation that is not Hamiltonian is a triangulation obtained by adding 9 edges to the Herschel graph. It was known to C. N. Reynolds (in dual form) in 1931, as reported by Whitney [24, Fig. 9]. Again, the proof that this is smallest relies on [1] and [9]. This triangulation was also presented much later by Goldner and Harary [12], so it is sometimes called the Goldner-Harary graph.

It is therefore reasonable to ask what conditions can be imposed on a 2- or 3-connected planar graph to make it Hamiltonian. The main direction in which positive results have been obtained is to restrict the types of 2- or 3-cuts in the graph. Dillencourt [8, Theorem 4.1] showed that a near-triangulation (a 2-connected plane graph with all faces bounded by triangles except perhaps the outer face) with no separating triangles and certain restrictions on chords of the outer cycle is Hamiltonian. Sanders [19, Theorem 2] extended this to a larger class of graphs. Jackson and Yu [15, Theorem 4.2] showed that a plane triangulation is Hamiltonian if each ‘piece’ defined by decomposing along separating triangles connects to at most three other pieces. Our results explore a different kind of condition, based on excluding a complete bipartite minor.

Excluded complete bipartite minors have been used previously in more general settings to prove results involving concepts related to the existence of a Hamilton cycle, such as toughness, circumference, or the existence of spanning trees of bounded degree; see for example [2, 3, 17]. We are interested in graphs that have no  $K_{2,t}$  minor, for some  $t$ . Some general results are known for such graphs, including a rough structure theorem [10], upper bounds on the number of edges [5, 16], and a lower bound on circumference [3].

For 2-connected graphs, a  $K_{2,3}$ -minor-free graph is either outerplanar or  $K_4$ , and is therefore both planar and Hamiltonian. However, the authors [11] recently characterized all  $K_{2,4}$ -minor-free graphs, and there are many  $K_{2,4}$ -minor-free 2-connected planar graphs that are not Hamiltonian. For 3-connected graphs, the  $K_{2,4}$ -minor-free ones belong to a small number of small graphs, some of which are nonplanar, or a sparse infinite family of planar graphs; all are Hamiltonian. There are  $K_{2,5}$ -minor-free 3-connected nonplanar graphs that are not Hamiltonian, such as the Petersen graph, but in this paper we show that all  $K_{2,5}$ -minor-free 3-connected planar graphs are Hamiltonian. We also show that this cannot be extended to  $K_{2,6}$ -minor-free graphs, by constructing an infinite family of  $K_{2,6}$ -minor-free 3-connected planar graphs that are not Hamiltonian.

The number  $g(n)$  of nonisomorphic  $K_{2,5}$ -minor-free 3-connected planar graphs on  $n$  vertices grows at least exponentially (for  $n \geq 10$  with  $n$  even this is not hard to show using the family of graphs obtained by adding an optional diagonal chord across each quadrilateral face of a prism  $C_{n/2} \square K_2$ ). Some computed values of  $g(n)$  are as follows.

$n$	7	8	9	10	11	12
$g(n)$	31	194	918	3278	8346	18154

The exponential growth of  $g(n)$  contrasts with the growth of the number of nonisomorphic 3-connected  $K_{2,4}$ -minor-free graphs (planar or nonplanar), which is only linear [11]. Thus, our results apply to a sizable class of graphs.

In Section 2 we provide necessary definitions and preliminary results. The main result, Theorem 3.1, that  $K_{2,5}$ -minor-free 3-connected planar graphs are Hamiltonian, is proved in Section 3. In Section 4 we discuss  $K_{2,6}$ -minor-free 3-connected planar graphs.

## 2 Definitions and Preliminary Results

An edge, vertex, or set of  $k$  vertices whose deletion increases the number of components of a graph is a *cutedge*, *cutvertex*, or *k-cut*, respectively. The subgraph of  $G$  induced by  $S \subseteq V(G)$  is denoted by  $G[S]$ . If  $P$  is a path and  $x, y \in V(P)$  then  $P[x, y]$  represents the subpath of  $P$  between  $x$  and  $y$ .

### 2.1 Minors and models

A graph  $H$  is a *minor* of a graph  $G$  if  $H$  is isomorphic to a graph formed from  $G$  by contracting and deleting edges of  $G$  and deleting vertices of  $G$ . A graph is *H-minor-free* if it does not have  $H$  as a minor. Another way to think of a minor  $H$  is in terms of a function  $\beta$  mapping each  $u \in V(H)$  to  $\beta(u) \subseteq V(G)$ , the *branch set* of  $u$ , such that (a)  $\beta(u) \cap \beta(u') = \emptyset$  if  $u \neq u'$ ; (b)  $G[\beta(u)]$  is connected for each  $u$ ; and (c) if  $uu' \in E(H)$  then there is at least one edge between  $\beta(u)$  and  $\beta(u')$  in  $G$ . We call  $\beta$  a *model*, or more specifically an *edge-based model*, of  $H$  in  $G$ . More generally, we may replace condition (c) by the existence of a function  $\pi$  mapping each  $e = uu' \in E(H)$  to a path  $\pi(uu')$  in  $G$ , such that (c<sub>1</sub>)  $\pi(uu')$  starts in  $\beta(u)$  and ends in  $\beta(u')$ ; (c<sub>2</sub>) no internal vertex of  $\pi(uu')$  belongs to any  $\beta(u'')$  (even if  $u'' = u$  or  $u'$ ); and (c<sub>3</sub>)  $\pi(e)$  and  $\pi(e')$  are internally disjoint if  $e \neq e'$ . We call  $(\beta, \pi)$  a *path-based model* of  $H$  in  $G$ .

Now we discuss  $K_{2,t}$  minors in particular. We assume that  $V(K_{2,t}) = \{a_1, a_2, b_1, b_2, \dots, b_t\}$  and  $E(K_{2,t}) = \{a_i b_j \mid 1 \leq i \leq 2, 1 \leq j \leq t\}$ .

Edge-based models are convenient for proving nonexistence of a minor. In fact, for  $K_{2,t}$  minors we can use an even more restrictive model. Consider an edge-based model  $\beta_0$  of  $K_{2,t}$ , where  $b_1$  and its incident edges correspond to a path  $v_1 v_2 \dots v_k$ ,  $k \geq 3$ , with  $v_1 \in \beta_0(a_1)$ ,  $v_k \in \beta_0(a_2)$ , and  $v_i \in \beta_0(b_1)$  for  $2 \leq i \leq k-1$ . Define  $\beta_1(b_1) = \{v_2\}$ ,  $\beta_1(a_2) = \beta_0(a_2) \cup \{v_3, \dots, v_{k-1}\}$ , and  $\beta_1(u) = \beta_0(u)$  for all other  $u$ . Then  $\beta_1$  is also an edge-based model of a  $K_{2,t}$  minor, and  $|\beta_1(b_1)| = 1$ . Applying the same procedure to  $b_2, b_3, \dots, b_t$  in turn, we obtain an edge-based model  $\beta = \beta_t$  with  $|\beta(b_j)| = 1$  for  $1 \leq j \leq t$ . Such a  $\beta$  is a *standard model* of  $K_{2,t}$ , and we denote it by either  $\Sigma(R_1, R_2 \mid s_1, s_2, \dots, s_t)$  or  $\Sigma(R_1, R_2 \mid S)$  where  $R_i = \beta(a_i)$  for  $1 \leq i \leq 2$ ,  $\{s_j\} = \beta(b_j)$  for  $1 \leq j \leq t$ , and  $S = \{s_1, s_2, \dots, s_t\}$ . Thus, if  $G$  has a  $K_{2,t}$  minor then it has a standard model of  $K_{2,t}$ .

On the other hand, path-based models are useful for proving existence of a minor. Moreover, in many situations where we find a  $K_{2,t}$  minor it would be tedious to give an exact description of the minor. So, we say that  $\Theta(R_1, R_2 \mid S_1, S_2, \dots, S_t)$  is an *approximate (path-based) model* of  $K_{2,t}$  if there exists a path-based model  $(\beta, \pi)$  such that  $R_i \subseteq \beta(a_i)$  for  $1 \leq i \leq 2$  and  $S_j \subseteq \beta(b_j)$  for  $1 \leq j \leq t$ . For convenience, we also allow each  $R_i$  to be a subgraph, not just a set of vertices, with  $V(R_i) \subseteq \beta(a_i)$ , and similarly for each  $S_j$ . Informally, we will specify enough of each branch set to make the existence of the minor clear; using subgraphs rather than vertex sets sometimes helps to clarify why a branch set induces a connected subgraph. We use this notation even when we actually have an exact description of a minor. For brevity, we just say that ‘ $\Theta(R_1, R_2 \mid S_1, S_2, \dots, S_t)$  is a  $K_{2,t}$  minor’. In our figures the sets or subgraphs  $R_i$  are enclosed by dotted curves, and the sets or subgraphs  $S_j$  (usually just single vertices) are indicated by triangles.

We also need one type of rooted minor. We say there is a  $K_{2,t}$  *minor rooted at  $R_1$  and  $R_2$*  if there is a path-based model  $(\beta, \pi)$  (or, equivalently, edge-based model  $\beta$ , or even standard model  $\beta$ ) with  $R_1 \subseteq \beta(a_1)$  and  $R_2 \subseteq \beta(a_2)$ . Again we extend this to allow  $R_1$  and  $R_2$  to be subgraphs, not just sets of vertices.

### 2.2 Path-outerplanar graphs

A graph is *outerplanar* if it has a plane embedding in which all vertices are on the outer face. We use a characterization that we proved elsewhere of graphs without rooted  $K_{2,2}$  minors in terms of special outerplanar graphs. (Along different lines, Demasi [7, Lemma 2.2.2] provided a description of graphs with no  $K_{2,2}$  minor rooting all four vertices, in terms of disjoint paths.) Our characterization uses the following definitions. Given  $x, y \in V(G)$ , an *xy-outerplane embedding* of a graph  $G$  is an embedding in a closed disk

$D$  such that a Hamilton  $xy$ -path  $P$  of  $G$  is contained in the boundary of  $D$ ;  $P$  is called the *outer path*. A graph is *xy-outerplanar*, or *path-outerplanar*, if it has an  $xy$ -outerplane embedding. A graph  $G$  is a *block* if it is connected and has no cutvertex; a block is either 2-connected,  $K_2$ , or  $K_1$ .

**Lemma 2.1** ([11]). *Suppose  $x, y \in V(G)$  where  $x \neq y$  and  $G' = G + xy$  is a block (which holds, in particular, if  $G$  has a Hamilton  $xy$ -path). Then  $G$  has no  $K_{2,2}$  minor rooted at  $x$  and  $y$  if and only if  $G$  is  $xy$ -outerplanar.*

The following results on Hamilton paths in outerplanar and  $xy$ -outerplanar graphs will be useful.

**Lemma 2.2.** *Let  $G$  be a 2-connected outerplanar graph. Let  $x \in V(G)$  and let  $xy$  be an edge on the outer cycle  $Z$  of  $G$ . Then for some vertex  $t$  with  $\deg_G(t) = 2$ , there exists a Hamilton path  $xy \dots t$  in  $G$ .*

*Proof.* Fix a forward direction on  $Z$  so that  $y$  follows  $x$ . Denote by  $v_1Zv_2$  the forward path from  $v_1$  to  $v_2$  on  $Z$ . Proceed by induction on  $|V(G)|$ . In the base case,  $G = K_3$  and the result is clear. Assume the lemma holds for all graphs with at most  $n - 1$  vertices and  $|V(G)| = n \geq 4$ . Let  $w \neq y$  be the other neighbor of  $x$  on  $Z$ . If  $\deg_G(w) = 2$ , then we take  $t = w$  and  $xZw$  is a desired Hamilton path in  $G$ . Otherwise let  $v$  be a neighbor of  $w$  such that  $vw \notin E(Z)$  (possibly  $v = y$ ). Let  $G'$  be the subgraph of  $G$  induced by  $vZw$ ;  $G'$  is a 2-connected  $vw$ -outerplanar graph with  $|V(G')| \leq n - 1$ . By the inductive hypothesis, there exists a Hamilton path  $Q = vw \dots t$  in  $G'$  where  $\deg_{G'}(t) = \deg_G(t) = 2$ . Then  $xZv \cup Q$  is the desired path in  $G$ .  $\square$

**Corollary 2.3.** *Let  $G$  be an  $xy$ -outerplanar graph with  $x \neq y$ . Then there exists a Hamilton path  $x \dots t$  in  $G - y$ , where  $t = x$  if  $|V(G)| = 2$ , and  $t$  is some vertex with  $\deg_G(t) = 2$  otherwise.*

*Proof.* If  $|V(G)| = 2$  this is clear, so suppose that  $|V(G)| \geq 3$ . Then  $G + xy$  is a 2-connected outerplanar graph, so by Lemma 2.2 it has a Hamilton path  $yx \dots t$  ending at a vertex  $t$  of degree 2. Now  $P - y$  is the required path.  $\square$

## 2.3 Connectivity and reducibility

The following observation will be useful.

**Lemma 2.4.** *Suppose  $G$  is a 2-connected plane graph and  $C$  is a cycle in  $G$ . Then the subgraph of  $G$  consisting of  $C$  and all edges and vertices inside  $C$  is 2-connected.*

*Proof.* Any cutvertex in the subgraph would also be a cutvertex of  $G$ .  $\square$

The following results will allow us to simplify the situations that we have to deal with in the proof of Theorem 3.1.

**Theorem 2.5** (Halin, [13, Theorem 7.2]). *Let  $G$  be a 3-connected graph with  $|V(G)| \geq 5$ . Then for every  $v \in V(G)$  with  $\deg(v) = 3$ , there is an edge  $e$  incident with  $v$  such that  $G/e$  is 3-connected.*

A  $k$ -separation in a graph  $G$  is a pair  $(H, K)$  of edge-disjoint subgraphs of  $G$  with  $G = H \cup K$ ,  $|V(H) \cap V(K)| = k$ ,  $V(H) - V(K) \neq \emptyset$ , and  $V(K) - V(H) \neq \emptyset$ .

**Lemma 2.6.** *Let  $G$  be a 3-connected graph and suppose  $(H, K)$  is a 3-separation in  $G$  with  $V(H) \cap V(K) = \{x, y, z\}$ . Suppose  $K' = K - V(H)$  is connected and  $H$  is 2-connected. Let  $G'$  be the graph formed from  $G$  by contracting  $K'$  to a single vertex. Then  $G'$  is 3-connected.*

*Proof.* Let  $v$  be the vertex in  $G'$  formed from contracting  $K'$ . Since  $G$  is 3-connected,  $xv, yv, zv \in E(G')$ . We claim that every pair of vertices in  $G'$  has three vertex-disjoint paths between them. By Menger's Theorem, it will follow that  $G'$  is 3-connected. We consider five different types of pairs of vertices.

First, suppose  $w_1, w_2 \in V(H) - \{x, y, z\}$ ; there are three internally disjoint paths from  $w_1$  to  $w_2$  in  $G$ :  $P_1$ ,  $P_2$ , and  $P_3$ . If  $V(P_i) \cap V(K') = \emptyset$  for  $i = 1, 2, 3$ , then  $P_1$ ,  $P_2$ , and  $P_3$  are the desired paths in  $G'$ .

If  $V(P_i) \cap V(K') \neq \emptyset$  for some  $i$ , then  $|V(P_i) \cap \{x, y, z\}| \geq 2$  since  $\{x, y, z\}$  separates  $K'$  from  $H$ . Thus  $V(P_i) \cap V(K') \neq \emptyset$  for at most one  $i$ . Suppose  $V(P_1) \cap V(K') \neq \emptyset$ . Then all vertices of  $V(P_1) \cap V(K')$  are in a single subpath of  $P_1$  which we replace by  $v$  to form a new path  $P'_1$ . The paths  $P'_1, P_2$ , and  $P_3$  are the desired paths in  $G'$ .

Second, consider  $w_1 \in V(H) - \{x, y, z\}$  and  $w_2 \in \{x, y, z\}$ , say  $w_2 = x$ . If there are not three internally disjoint paths between  $w_1$  and  $x$  in  $G'$ , then  $w_1$  and  $x$  are separated either by a 2-cut  $\{u_1, u_2\}$  (if  $w_1x \notin E(G)$ ) or by  $w_1x$  and some vertex  $u_1$  (if  $w_1x \in E(G)$ ). Since  $w_1$  and  $x$  are not separated by a 2-cut or by an edge and a vertex in  $G$ , we may assume that  $u_1 = v$ . But then  $u_2$  is a cutvertex in  $H$  or  $w_1x$  is a cutedge in  $H$ , which is a contradiction since  $H$  is 2-connected. Hence there are three internally disjoint paths between  $w_1$  and  $x$ .

Third, consider  $w_1, w_2 \in \{x, y, z\}$ , say  $w_1 = x$  and  $w_2 = y$ . Because  $H$  is 2-connected, there are two internally disjoint paths  $P_1$  and  $P_2$  from  $x$  to  $y$  in  $H$ . Take  $P_3 = xvy$ . Then  $P_1, P_2$ , and  $P_3$  are the desired paths in  $G'$ .

Fourth, consider  $w_1 \in V(H) - \{x, y, z\}$  and  $v$ . For any  $w_2 \in V(K')$ , there are three internally disjoint paths  $P_1, P_2$ , and  $P_3$  from  $w_2$  to  $w_1$  in  $G$ . Without loss of generality, say  $x \in V(P_1)$ ,  $y \in V(P_2)$ , and  $z \in V(P_3)$ . Form  $P'_1$  from  $P_1$  by replacing  $P_1[w_2, x]$  with  $vx$ , form  $P'_2$  from  $P_2$  by replacing  $P_2[w_2, y]$  with  $vy$ , and form  $P'_3$  from  $P_3$  by replacing  $P_3[w_2, z]$  with  $vz$ . The paths  $P'_1, P'_2$ , and  $P'_3$  are the desired paths in  $G'$ .

Finally, consider  $w_1 \in \{x, y, z\}$ , say  $w_1 = x$ , and  $v$ . By a consequence of Menger's Theorem, since  $H$  is 2-connected there are two internally disjoint paths from  $\{y, z\}$  to  $x$  in  $H$ , say  $P_1 = y \dots x$  and  $P_2 = z \dots x$ . Then  $P'_1 = vy \cup P_1$ ,  $P'_2 = vz \cup P_2$ , and  $P_3 = vx$  are the desired paths in  $G'$ .  $\square$

Lemma 2.6 is false without the hypothesis that  $H$  is 2-connected: then we could have  $V(H) = \{w, x, y, z\}$  and  $E(H) = \{wx, wy, wz\}$ , in which case  $G'$  would be isomorphic to  $K_{2,3}$ , which is not 3-connected.

Now we use the results above to set up a framework that will help to simplify the graph in our main proof. Suppose  $G$  is a 3-connected graph, and  $C$  is a cycle in  $G$ . We say that  $G$  is  $C$ -reducible to a graph  $G'$  provided (a)  $G'$  is obtained from  $G$  by contracting edges of  $G$  with at most one end on  $C$  and/or deleting edges in  $E(G) - E(C)$ , (b)  $G'$  is 3-connected, and (c) for every cycle  $Z'$  in  $G'$  there is a cycle  $Z$  in  $G$  with  $|V(Z)| \geq |V(Z')|$ . By (a),  $C$  is still a cycle in  $G'$ . From this, we see that  $C$ -reducibility is transitive. Also by (a),  $G'$  is a minor of  $G$ .

**Lemma 2.7.** *Suppose  $C$  is a cycle in a 3-connected graph  $G$ . If  $B$  is a component of  $G - V(C)$  with exactly three neighbors on  $C$  then  $G$  is  $C$ -reducible to  $G/E(B)$ , in which  $B$  becomes a degree 3 vertex.*

*Proof.* Let  $G_0 = G - V(B)$ . If  $G_0$  is not 2-connected, then there is a cutvertex  $u$ . Now  $u \notin V(C)$  and  $V(C)$  must be entirely in one component of  $G_0 - u$ . Since the neighbors of  $B$  are all on  $C$ , vertices of  $B$  are only adjacent to vertices on one side of the cut. Hence  $u$  is also a cutvertex in  $G$ , which is a contradiction. Thus,  $G_0$  is 2-connected. Consider  $G' = G/E(B)$ . Clearly (a) holds, and (b) follows from Lemma 2.6.

Let  $a_1, a_2, a_3$  be the neighbors of  $B$  on  $C$ , and let  $b$  be the vertex of  $G'$  corresponding to  $B$ . Let  $Z'$  be a cycle in  $G'$ . If  $b \notin V(Z')$ , then  $Z = Z'$  is also a cycle in  $G$ . If  $b \in V(Z')$  then  $Z'$  uses a path  $a_i b a_j$ . Form a cycle  $Z$  in  $G$  from  $Z'$  by replacing  $a_i b a_j$  by a path from  $a_i$  to  $a_j$  through  $B$ . Clearly  $|V(Z)| \geq |V(Z')|$ , so (c) holds.  $\square$

**Lemma 2.8.** *Suppose  $C$  is a cycle in a 3-connected graph  $G$ . If  $b \in V(G) - V(C)$  has degree 3 then there is an edge  $bc$  so that  $G$  is  $C$ -reducible to  $G/bc$ .*

*Proof.* By Theorem 2.5 there is an edge  $bc$  such that  $G' = G/bc$  is 3-connected. Clearly (a) and (b) hold for  $G'$ ; we must show (c). Let  $a_1, a_2$  and  $c$  be the neighbors of  $b$  in  $G$ . Call the vertex that results from the contraction  $z$ . Suppose  $Z'$  is a cycle in  $G'$ . If  $a_1z, a_2z \notin E(Z')$ , then take  $Z = Z'$ . If  $|\{a_1z, a_2z\} \cap E(Z')| = 1$ , say  $a_1z \in E(Z')$ , form  $Z$  from  $Z'$  by replacing  $a_1z$  with the path  $a_1bc$ . If  $a_1z, a_2z \in E(Z')$ , form  $Z$  from

$Z'$  by replacing the subpath  $a_1za_2$  with  $a_1ba_2$ . In all cases,  $Z$  is a cycle in  $G$  with  $|V(Z)| \geq |V(Z')|$ , so (c) holds.  $\square$

**Lemma 2.9.** *Suppose  $C$  is a cycle in a 3-connected graph  $G$ . Suppose that  $a_1a_2 \in E(G) - E(C)$ , and there are three internally disjoint  $a_1a_2$ -paths in  $G - a_1a_2$ . Then  $G$  is  $C$ -reducible to  $G - a_1a_2$ .*

*In particular,  $G$  is  $C$ -reducible to  $G - a_1a_2$  if  $a_1$  and  $a_2$  are neighbors on  $C$  of a component of  $G - V(C)$  and  $a_1a_2 \in E(G) - E(C)$ .*

*Proof.* Clearly (a) and (c) hold for  $G' = G - a_1a_2$ ; we must show (b). Since  $G$  is 3-connected,  $G'$  is 2-connected, and if  $G'$  has a 2-cut then  $a_1$  and  $a_2$  must be in different components, which cannot happen because of the three internally disjoint  $a_1a_2$ -paths.

If  $a_1$  and  $a_2$  are neighbors of a component  $B$  of  $G - V(C)$  then there are three internally disjoint  $a_1a_2$ -paths in  $G - a_1a_2$ , namely the two paths between  $a_1$  and  $a_2$  in  $C$ , and a path from  $a_1$  to  $a_2$  through  $B$ .  $\square$

### 3 Main Result

We are now ready to prove the main result.

**Theorem 3.1.** *Let  $G$  be a 3-connected planar  $K_{2,5}$ -minor-free graph. Then  $G$  is Hamiltonian.*

Theorem 3.1 is proved by assuming  $G$  is not Hamiltonian, taking a longest cycle  $C$  in  $G$  and finding a contradiction with either a longer cycle or a  $K_{2,5}$  minor.

*Proof.* Assume that  $G$  is not Hamiltonian and assume  $G$  is represented as a plane graph. Let  $H$  and  $J$  be two subgraphs of  $G$ . Let  $R_0$  be the outside face of  $J$  (an open set),  $R_1$  the boundary of  $R_0$ , and  $R_2 = \mathbb{R}^2 - R_0 - R_1$ . We say  $H$  is *outside*  $J$  if as subsets of the plane we have  $H \subseteq R_0 \cup R_1$ , and *inside*  $J$  if  $H \subseteq R_1 \cup R_2$ .

Let  $C$  be a longest non-Hamilton cycle in  $G$ . A *longer cycle* means a cycle longer than  $C$ . Fix a forward direction on  $C$ , which we assume is clockwise. Denote by  $x^+$  the vertex directly after the vertex  $x$  on  $C$  and by  $x^-$  the vertex directly before  $x$ . Define  $C[x, y]$  to be the forward subpath of  $C$  from  $x$  to  $y$  which includes  $x$  and  $y$ . If  $x = y$  then  $C[x, y] = \{x\}$ . Define  $C(x, y) = C[x, y] - \{x, y\}$ ,  $C^-(x, y) = C[x, y] - x$ , and  $C^+(x, y) = C[x, y] - y$ . Define  $[x, y]$  to be  $V(C[x, y])$  and  $G[x, y]$  to be the induced subgraph  $G[C[x, y]]$ ; also define  $(x, y)$ ,  $G(x, y)$ , etc. similarly. We say a vertex  $z$  is *between*  $x$  and  $y$  if  $z \in (x, y)$ .

Let  $D$  be a component of  $G - V(C)$  with the most neighbors on  $C$ . We fix  $D$  in our arguments, and assume that  $D$  is inside  $C$ . Let  $u_0, u_1, \dots, u_{k-1}$  be the neighbors of  $D$  along  $C$  in forward order. Because  $G$  is 3-connected,  $k \geq 3$ . For any distinct  $u_i$  and  $u_j$  there is at least one path from  $u_i$  to  $u_j$  through  $D$ ; we use  $u_iDu_j$  to denote such a path. The sets  $U_i = (u_i, u_{i+1})$  (subscripts interpreted modulo  $k$ ) are called *sectors*. If  $U_i = \emptyset$  for some  $i$ , then there is a longer cycle: replace  $C[u_i, u_{i+1}]$  with  $u_iDu_{i+1}$ . Thus,  $U_i \neq \emptyset$  for all  $i$ .

A *jump*  $x - y$  is an  $xy$ -path where  $x \neq y$ ,  $x, y \in V(C)$ , and no edge or internal vertex of the path belongs to  $C$  or  $D$ . If  $S, T \subseteq V(C)$  then a *jump from  $S$  to  $T$*  or  $S - T$  *jump* is a jump  $x - y$  with  $x \in S$ ,  $y \in T$ ; if  $S = T$  we say this is a *jump on  $S$* . If  $S$  is a set of consecutive vertices on  $C$  then a *jump out of  $S$*  is a jump  $x - y$  where  $x \in S$ ,  $y \notin S$ , and  $y$  is not adjacent in  $C$  to a vertex of  $S$ . Whenever  $v, w \in V(C)$  are not equal and not consecutive on  $C$  and  $(v, w)$  contains no neighbor of  $D$  there is at least one jump out of  $(v, w) = [v^+, w^-]$ , because  $\{v, w\}$  is not a 2-cut.

A jump out of a sector  $U_i$  is a *sector jump*; since every  $U_i$  is nonempty, there is a sector jump out of every sector. A jump is an *inside* or *outside jump* if it is respectively inside or outside  $C$ . An inside jump must have both ends in  $[u_i, u_{i+1}]$  for some  $i$ . Thus, all sector jumps are outside jumps.

If there is a jump  $u_i^+ - u_j^+$ , then  $C[u_j^+, u_i] \cup u_iDu_j \cup C[u_i^+, u_j] \cup u_i^+ - u_j^+$  is a longer cycle. Denote such a longer cycle as  $L(u_i^+ - u_j^+)$ . If there is a jump  $u_i^- - u_j^-$ , then there is a symmetric longer cycle denoted  $L(u_i^- - u_j^-)$ . Call such cycles *standard longer cycles*. Figure 1 shows  $L(u_1^-, u_2^-)$  when  $k = 4$ .

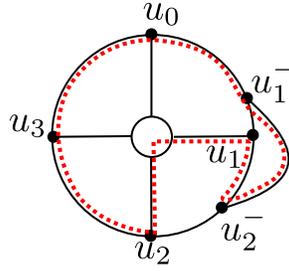


Figure 1

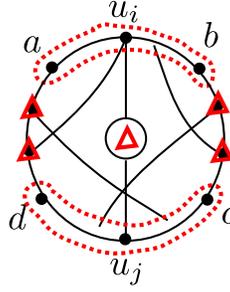


Figure 2

If  $x, y \in V(C)$ ,  $x \neq y$ ,  $W \subseteq G - V(C) - V(D)$ , and  $G[[x, y] \cup W]$  contains a  $K_{2,2}$  minor rooted at  $x$  and  $y$ , then we say there is a  $K_{2,2}$  minor along  $[x, y]$ . If there is no such minor then for any  $[x', y'] \subseteq [x, y]$  with  $x' \neq y'$  there is no  $K_{2,2}$  minor rooted at  $x'$  and  $y'$  in  $G[x', y']$ . Thus,  $G[x', y']$  is  $x'y'$ -outerplanar by Lemma 2.1 and we may apply Corollary 2.3 to  $G[x', y']$ .

Suppose  $a, b, c, d$  with  $c \neq b$ ,  $a \neq d$  appear in that order along  $C$ . Let  $W_1, W_2 \subseteq G - V(C) - V(D)$  with  $W_1 \cap W_2 = \emptyset$ . If there is a  $K_{2,2}$  minor in  $G[[a, d] \cup W_1]$  rooted at  $[a, b]$  and  $[c, d]$ , represented as  $\Sigma(R_1, R_2 | s_1, s_2)$ , and a  $K_{2,2}$  minor in  $G[[c, b] \cup W_2]$  rooted at  $[a, b]$  and  $[c, d]$ , represented as  $\Sigma(R'_1, R'_2 | s'_1, s'_2)$ , and there exist  $u_i \in [a, b]$  and  $u_j \in [c, d]$ , then there is a  $K_{2,5}$  minor  $\Theta(R_1 \cup R'_1, R_2 \cup R'_2 | s_1, s_2, s'_1, s'_2, D)$  in  $G$ . Denote such a minor by  $M([a, b], [c, d])$ . An example is shown in Figure 2.

For  $x \in V(C)$ , define  $\sigma(x) \in \{0, \frac{1}{2}, 1, 1\frac{1}{2}, \dots, k - \frac{1}{2}\}$  by  $\sigma(u_i) = i$ , and  $\sigma(x) = i + \frac{1}{2}$  if  $x \in U_i$ . Define the length of a jump  $x - y$  as  $\min\{|\sigma(x) - \sigma(y)|, k - |\sigma(x) - \sigma(y)|\}$ . A sector jump has length at least 1.

**Claim 1.** For every jump  $x - y$  of length greater than 1, there is a sector jump  $x_1 - y_1$  of length 1 with  $x_1, y_1 \in [x, y]$  and another sector jump  $x_2 - y_2$  of length 1 with  $x_2, y_2 \in [y, x]$ .

For any jump  $u - v$ , define the linear length as  $|\sigma(u) - \sigma(v)|$ . We claim that for any jump (not necessarily a sector jump)  $x' - y'$  of linear length  $\ell' > 1$  with  $\sigma(x') < \sigma(y')$ , there is a sector jump  $x'' - y''$  of linear length less than  $\ell'$  with  $x'', y'' \in [x', y']$ . The jump  $x' - y'$  must be outside  $C$ , and there is a sector  $U_j \subset (x', y')$ . Let  $x'' - y''$  be any jump out of  $U_j$ ; then  $\sigma(x') < \sigma(x'') < \sigma(y')$ . If  $x'' - y''$  does not contain an interior vertex of  $x' - y'$ , then by planarity  $x'' - y''$  has linear length less than  $\ell'$ . If  $x'' - y''$  contains an interior vertex of  $x' - y'$ , then we have jumps  $x'' - x'$  and  $x'' - y'$  with linear length less than  $\ell'$ , at least one of which is a sector jump. We may repeat this process until we reach a sector jump  $x^* - y^*$  with  $x^*, y^* \in [x', y']$  of linear length 1, and hence also length 1.

If we relabel  $u_0, u_1, \dots, u_{k-1}$  keeping the same cyclic order so that  $x \in \{u_0\} \cup U_0$  and repeatedly apply the previous paragraph beginning with the jump  $x - y$ , we obtain the required jump  $x_1 - y_1$ . Similarly, relabeling so that  $y \in \{u_0\} \cup U_0$  yields the jump  $x_2 - y_2$ . This completes the proof of Claim 1.

**Claim 2.**  $k = 3$ .

Assume that  $k \geq 4$ . Suppose there is a component  $D'$  of  $G - V(C)$  with neighbors in three consecutive sectors, say  $z_1 \in U_0$ ,  $z_2 \in U_1$ , and  $z_3 \in U_2$  ( $D'$  may also have neighbors in other sectors). Then since  $k \geq 4$ ,  $z_1 - z_3$  is a jump of length greater than 1. Therefore by Claim 1, there is a sector jump  $x - y$  of length 1 with  $u_i \in [x, y] \subseteq [z_3, z_1]$ . At most one of  $x \in U_2$ ,  $y \in U_0$  is true; we may assume that  $y \notin U_0$ . Then there is a  $K_{2,5}$  minor  $\Theta(D \cup \{u_1\}, D' \cup [z_3, x] \cup x - y | u_0, z_1, z_2, u_2, u_i)$  as shown in Figure 3. This minor applies even if  $x - y$  intersects  $D'$ .

Now suppose there is a component  $D'$  of  $G - V(C)$  with neighbors in three sectors that are not consecutive (this requires  $k \geq 5$ ; again  $D'$  may also have neighbors in other sectors). We may assume that these are  $z_1 \in U_h$ ,  $z_2 \in U_i$ ,  $z_3 \in U_j$  in order along  $C$ , where  $U_h, U_i$  may be consecutive but  $U_i, U_j$  and  $U_j, U_h$

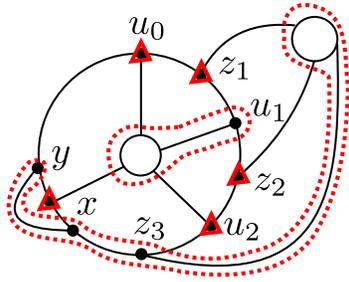


Figure 3

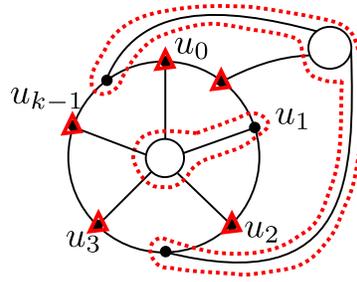


Figure 4

are not. Then there is a  $K_{2,5}$ -minor  $\Theta(D \cup \{u_{i+1}\}, D' \cup \{z_1, z_3\} | u_h, u_i, z_2, u_j, u_{j+1})$ . An example with  $(h, i, j) = (k - 1, 0, 2)$  is shown in Figure 4.

Hence, every component of  $G - V(C)$  other than  $D$  has neighbors in at most two sectors. Therefore, a sector jump of length 1, from  $U_{i-1}$  to  $U_i$ , cannot intersect any sector jump with an end in  $U_j$ ,  $j \notin \{i - 1, i\}$ , which includes all sector jumps of length at least 2.

From Claim 1 it follows that there are at least two distinct pairs of sectors with jumps of length 1 between them. Suppose there are three distinct pairs of sectors with jumps of length 1 between them, say  $x_1 - y_1$ ,  $x_2 - y_2$  and  $x_3 - y_3$  in order along  $C$ , where  $u_g \in (x_1, y_1)$ ,  $u_h \in (x_2, y_2)$  and  $u_i \in (x_3, y_3)$ . Since  $k \geq 4$ , we may assume there is some  $u_j \in (y_3, x_1)$ . Then there is a  $K_{2,5}$  minor  $\Theta(D \cup \{u_j\}, [y_1, x_2] \cup x_2 - y_2 \cup [y_2, x_3] | x_1, u_g, u_h, u_i, y_3)$ . An example with  $(g, h, i) = (0, 1, 2)$  is shown in Figure 5.

Therefore, we may assume that there are exactly two distinct pairs of sectors with jumps of length 1 between them, say  $x_1 - y_1$  and  $y_2 - x_2$  in order along  $C$ , where  $u_g \in (x_1, y_1)$  and  $u_h \in (y_2, x_2)$ . Suppose some sector has no jump of length 1 out of it. Without loss of generality we may assume this sector is  $U_0 \subseteq (x_2, x_1)$ . There is some sector jump  $x - y$  out of  $U_0$ . Then  $y \in [y_1, y_2]$ , otherwise Claim 1 would give a jump of length 1 between a third pair of sectors. Therefore there is a  $K_{2,5}$  minor  $\Theta(D \cup \{u_0, u_1\}, [y_1, y_2] | x, x_1, u_g, u_h, x_2)$  as shown in Figure 6.

Therefore, every sector has a jump of length 1 out of it, which means that  $k = 4$ , and we may assume that there are jumps  $U_3 - U_0$  and  $U_1 - U_2$ , but no jumps  $U_0 - U_1$  or  $U_2 - U_3$ . Let  $z_3 - z_0$  be the sector jump  $U_3 - U_0$  such that  $z_3$  is closest to  $u_3$  and  $z_0$  is closest to  $u_1$ . Similarly, let  $z_1 - z_2$  be the sector jump  $U_1 - U_2$  such that  $z_1$  is closest to  $u_1$  and  $z_2$  is closest to  $u_3$ . Each  $U_j$  is divided into two parts by  $z_j$ : let  $A_0 = (u_0, z_0)$ ,  $B_0 = (z_0, u_1)$ ,  $B_1 = (u_1, z_1)$ ,  $A_1 = (z_1, u_2)$ ,  $A_2 = (u_2, z_2)$ ,  $B_2 = (z_2, u_3)$ ,  $B_3 = (u_3, z_3)$  and  $A_3 = (z_3, u_0)$ .

We may assume that  $z_3 - z_0$  and  $z_1 - z_2$  are embedded in the plane so that  $D$  is outside both cycles  $Z_0 = C[z_3, z_0] \cup z_3 - z_0$  and  $Z_2 = C[z_1, z_2] \cup z_1 - z_2$ . Let  $H_0$  be the subgraph of  $G$  consisting of  $Z_0$  and all vertices and edges inside  $Z_0$ , and define  $H_2$  similarly; these are 2-connected by Lemma 2.4.

For any  $j$ , define  $N_j$  to be the set of vertices of  $V(G) - V(C) - V(D)$  inside a cycle  $C[u_j, u_{j+1}] \cup u_{j+1} D u_j$  (the exact path through  $D$  does not matter). Loosely, these are the vertices inside  $C$  associated with the sector  $U_j$ . We now claim that there is a  $K_{2,2}$  minor along  $[u_3, u_1]$  using only vertices in  $[u_3, u_1] \cup V(H_0) \cup N_3 \cup N_0$ .

If  $N_3 \neq \emptyset$ , then there is a component  $D'$  of  $G - V(C)$  with  $V(D') \subseteq N_3$ . Now  $D'$  has (at least) three neighbors in  $[u_3, u_0]$ , say  $w_1, w_2, w_3$  in order along  $C$ . So  $\Theta([u_3, w_1], [w_3, u_1] | w_2, D')$  is the required  $K_{2,2}$  minor. Thus, we may assume that  $N_3 = \emptyset$ , and symmetrically that  $N_0 = \emptyset$ .

Let  $H'_0 = H_0 \cup G[z_3, z_0]$ . Then  $V(H'_0) = V(H_0)$ , so  $H'_0$  is also 2-connected, but possibly  $E(H'_0) \neq E(H_0)$  because  $H'_0$  contains any edges inside  $C$  joining two vertices of  $[z_3, u_0]$  or two vertices of  $[u_0, z_0]$ . If  $H'_0$  has a  $K_{2,2}$  minor rooted at  $z_3$  and  $z_0$ , such as a minor  $\Theta(z_3, z_0 | u_0, q)$  when  $z_3 - z_0$  has an internal vertex  $q$ , then we can extend this minor using  $[u_3, z_3]$  and  $[z_0, u_1]$  to get the required  $K_{2,2}$  minor. If there is an inside jump out of any of  $B_3, A_3, A_0, B_0$ , then this jump together with  $z_3 - z_0$  forms the required  $K_{2,2}$  minor.

So we may assume that  $H'_0$  has no  $K_{2,2}$  minor rooted at  $z_3$  and  $z_0$ . Thus,  $z_3 - z_0$  has no internal vertex and so  $z_3 z_0$  is an outer edge of  $H'_0$ . Also, by Lemma 2.1,  $H'_0$  is  $z_3 z_0$ -outerplanar. If there is an edge of

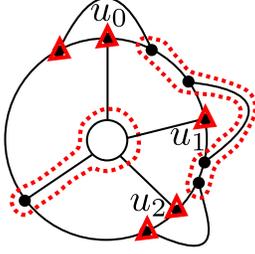


Figure 5

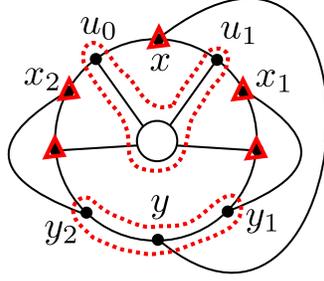


Figure 6

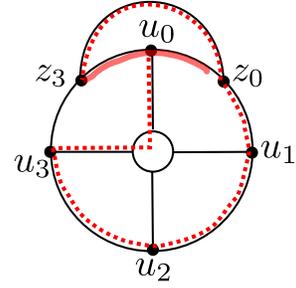


Figure 7

$G$  leaving  $H'_0$  at a vertex of  $A_3$  or  $A_0$  then, since  $N_3 = N_0 = \emptyset$ , that edge is an inside jump, creating the required  $K_{2,2}$  minor. Hence, any edges of  $G$  leaving  $H'_0$  leave at  $z_3$ ,  $u_0$  or  $z_0$ . Since  $G$  is 3-connected these are the only vertices that can have degree 2 in  $H'_0$ .

Suppose that  $B_3 = \emptyset$ . By Lemma 2.2 there is a Hamilton path  $P = z_0 z_3 \dots t$  in  $H'_0$  where  $t$  has degree 2 in  $H'_0$ ; then we must have  $t = u_0$ . Thus,  $P \cup C[z_0, u_3] \cup u_3 D u_0$  is a longer cycle, a contradiction. This cycle is shown in Figure 7, where we use the convention that paths found using Lemma 2.2 or Corollary 2.3 are shown by heavily shading the part of the graph covered by the paths; the rest of the cycle is shown using dotted curves. Thus,  $B_3$  is nonempty, and by a symmetric argument  $B_0$  is also nonempty.

Suppose  $r_0 - t$  is an outside jump out of  $B_0$ . This jump cannot contain an internal vertex of  $z_3 - z_0$ , and  $t \notin (u_3, z_3]$ , by choice of  $z_3 - z_0$ . The jump cannot contain an internal vertex of  $z_1 - z_2$ , and  $t \notin (u_1, z_1]$ , because there are no  $U_0 - U_1$  jumps. Thus,  $t \in [z_2, u_3]$ . Similarly, an outside jump  $r_3 - t'$  out of  $B_3$  must have  $t' \in [u_1, z_1]$ . Hence we cannot have outside jumps out of both  $B_0$  and  $B_3$  because the jumps  $r_0 - [z_2, u_3]$  and  $r_3 - [u_1, z_1]$  would intersect by planarity, giving a jump  $r_3 - r_0$  that contradicts the choice of  $z_3 - z_0$ . Therefore, there is an inside jump out of one of  $B_0$  or  $B_3$ , giving the required  $K_{2,2}$  minor along  $[u_3, u_1]$ .

By a symmetric argument there is also a  $K_{2,2}$  minor along  $[u_1, u_3]$  using only vertices in  $[u_1, u_3] \cup V(H_2) \cup N_1 \cup N_2$ . The two minors intersect only at  $u_1$  and  $u_3$ , so together they give a  $K_{2,5}$  minor  $M(u_3, u_1)$ . This concludes the proof of Claim 2.

Henceforth we assume  $k = 3$ . The next claim simplifies the structure of the graph we are looking at and makes further analysis easier.

**Claim 3.** *Without loss of generality, we may assume that  $D$  consists of a single degree 3 vertex  $d$  and that  $V(G) = V(C) \cup \{d\}$ . Thus, every jump is a single edge. We may also assume that there are no edges  $xy \in E(G) - E(C)$  where  $G$  has three internally disjoint  $xy$ -paths of length 2 or more; in particular  $u_i u_j \notin E(G)$  for all  $i, j \in \{0, 1, 2, \dots, k-1\}$ .*

Since  $k = 3$  and  $G$  is 3-connected, every component of  $G - V(C)$  has exactly three neighbors on  $C$ . Applying Lemma 2.7 to each of these components in turn, including  $D$ , we find that  $G$  is  $C$ -reducible to  $G_1$  for which every component of  $G_1 - V(C)$  is a single degree 3 vertex of  $G_1$ . Let  $d$  be the degree 3 vertex corresponding to  $D$ . Applying Lemma 2.8 to each vertex of  $V(G_1) - V(C) - \{d\}$  in turn, we find that  $G_1$  is  $C$ -reducible to  $G_2$  for which  $V(G_2) = V(C) \cup \{d\}$ . Starting from  $G_2$  and applying Lemma 2.9 repeatedly to any edge  $xy$  not on  $C$  where there are three internally disjoint  $xy$ -paths of length 2 or more, we find that  $G_2$  is  $C$ -reducible to  $G_3$  in which there are no such edges  $xy$ . Since  $u_i u_j \notin E(C)$  for all  $i$  and  $j$ ,  $G_3$  has no edges  $u_i u_j$  by the second part of Lemma 2.9.

Since  $C$ -reducibility is transitive,  $G$  is  $C$ -reducible to  $G_3$ .  $G_3$  is 3-connected and has all the properties stated in the claim. Since  $G_3$  is a minor of  $G$ ,  $G_3$  is planar, and showing that  $G_3$  has a  $K_{2,5}$  minor also shows that  $G$  has a  $K_{2,5}$  minor. By (c) of the definition of  $C$ -reducibility, showing that  $G_3$  has a cycle longer than

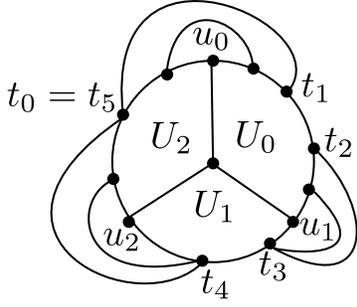


Figure 8

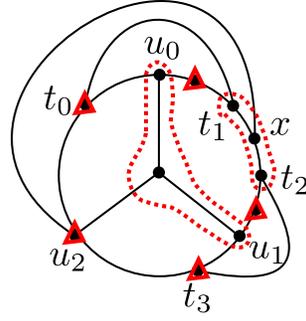


Figure 9

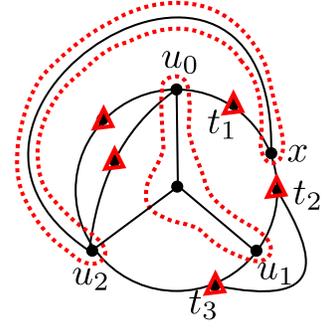


Figure 10

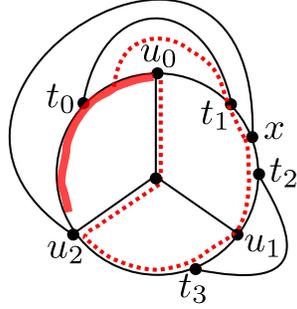


Figure 11

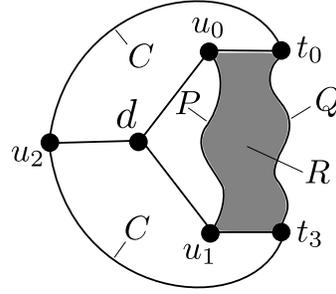


Figure 12

$C$  also shows that  $G$  has a cycle longer than  $C$ . Therefore, we may replace  $G$  by  $G_3$  in our arguments. This concludes the proof of Claim 3.

We are now in the general situation where there are three sectors labeled  $U_0$ ,  $U_1$ , and  $U_2$ . Let  $t_0 - t_1$  be the outermost  $U_2 - U_0$  jump (if any  $U_2 - U_0$  jump exists), meaning that  $t_0 \in U_2$  is closest to  $u_2$  and  $t_1 \in U_0$  is closest to  $u_1$ . Similarly let  $t_2 - t_3$  be the outermost  $U_0 - U_1$  jump, and  $t_4 - t_5$  the outermost  $U_1 - U_2$  jump, when such jumps exist. Because every sector must have a jump out of it and by Claim 1, there are at least two sector jumps of length 1; without loss of generality, assume there are jumps  $t_0 - t_1$  and  $t_2 - t_3$ . Define  $X_0 = (t_0, u_0)$ ,  $X_1 = (u_0, t_1)$ ,  $X_2 = (t_2, u_1)$ ,  $X_3 = (u_1, t_3)$ ,  $X_4 = (t_4, u_2)$ , and  $X_5 = (u_2, t_5)$ , whenever the necessary  $t_i$  vertices exist. An example of the overall situation is shown in Figure 8.

**Claim 4.** *There are no sector jumps  $x - u_2$  where  $x \in (t_1, t_2)$ .*

Let  $x - u_2$  be a sector jump with  $x \in (t_1, t_2)$ . If there exist  $q_1 \in X_1$  and  $q_2 \in X_2$  then there is a  $K_{2,5}$  minor  $\Theta(\{d, u_0, u_1\}, [t_1, t_2] | q_1, q_2, t_3, u_2, t_0)$  as shown in Figure 9. So at least one of  $X_1$  and  $X_2$  is empty; without loss of generality, assume  $X_1 = \emptyset$ . Since  $X_1 = \emptyset$  and by choice of  $t_0 - t_1$ , all jumps out of  $U_2$  must go to  $t_1$ . If there is a  $K_{2,2}$  minor  $\Theta(u_2, u_0 | s_1, s_2)$  along  $[u_2, u_0]$ , then there is a  $K_{2,5}$  minor  $\Theta(\{d, u_0, u_1\}, x - u_2 | t_1, t_2, t_3, s_1, s_2)$  as shown in Figure 10. So we may assume there is no such minor, and apply Corollary 2.3 to  $G[u_2, u_0]$  to find a path  $P = u_0 \dots t$  such that  $V(P) = (u_2, u_0)$  and  $t$  is a degree 2 vertex in  $G[u_2, u_0]$ ; then we must have  $tt_1 \in E(G)$ . Thus,  $P \cup tt_1 \cup C[t_1, u_2] \cup u_2du_0$  is a longer cycle, as shown in Figure 11. This completes the proof of Claim 4.

**Claim 5.** *Either  $t_0 \neq u_0^-$  or  $t_3 \neq u_1^+$  ( $X_0$  and  $X_3$  cannot both be empty).*

Assume that  $t_0 = u_0^-$  and  $t_3 = u_1^+$ . See Figure 12. Let  $R = G[t_0, t_3]$ ; we may assume that  $d$  is outside  $R$ . There are three internally disjoint  $t_0t_3$ -paths of length 2 or more, namely  $t_0 - t_1 \cup C[t_1, t_2] \cup t_2 - t_3$ ,  $t_0u_0du_1t_3$  and  $C[t_3, t_0]$ , so by Claim 3,  $t_0t_3 \notin E(G)$ . Also by Claim 3,  $u_0u_1 \notin E(G)$ .

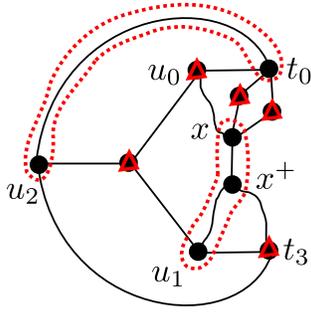


Figure 13

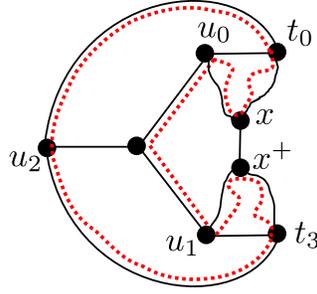


Figure 14

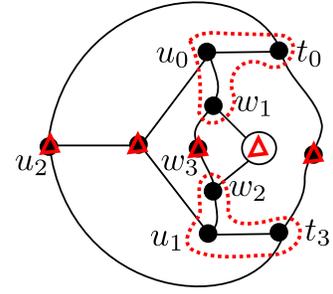


Figure 15

Let  $P$  be the walk from  $u_0$  to  $u_1$  counterclockwise along the outer face of  $R$  and  $Q$  be the walk from  $t_0$  to  $t_3$  clockwise along the outer face of  $R$ . The outer face of  $R$  is bounded by  $P \cup Q \cup \{u_0t_0, u_1t_3\}$ . If  $P = (u_0 = p_0)p_1p_2 \dots p_{r-1}(p_r = u_1)$  then each  $p_i$ ,  $1 \leq i \leq r$ , is closer to  $t_3$  along  $C[t_0, t_3]$  than  $p_{i-1}$ , so  $P$  has no repeated vertices and is a path; similarly,  $Q$  is a path. Additionally,  $|V(P)| \geq 3$  because  $u_0u_1 \notin E(G)$  and  $|V(Q)| \geq 3$  because  $t_0t_1, t_2t_3 \in E(Q)$  (possibly  $t_1 = t_2$ ).

The paths  $P$  and  $Q$  may intersect but only in limited ways. Any intersection vertex must belong to  $[t_1, t_2]$ . If  $P$  and  $Q$  intersect at two non-consecutive vertices on  $C$ , then using Claim 4 these two vertices would form a 2-cut in  $G$ . Hence there are three possibilities for  $P$  and  $Q$ :  $V(P) \cap V(Q) = \{x, x^+\}$ ,  $V(P) \cap V(Q) = \{x\}$ , or  $V(P) \cap V(Q) = \emptyset$ .

(1) First assume  $V(P) \cap V(Q) = \{x, x^+\} \subseteq [t_1, t_2]$ . We will show that there is a longer cycle. Let  $R_1 = G[t_0, x]$  and  $R_2 = G[x^+, t_3]$ . Then  $t_0t_1 \in E(R_1)$  and  $t_2t_3 \in E(R_2)$ . Let  $P_1 = P \cap R_1$  and  $Q_1 = Q \cap R_1$ ; then  $u_0 \in V(P_1)$ ,  $t_0t_1 \in E(Q_1)$ , and  $V(P_1) \cap V(Q_1) = \{x\}$ . First we construct a new  $u_0x$ -path  $P'_1$  and a new  $t_0x$ -path  $Q'_1$  such that  $V(P'_1 \cup Q'_1) = V(R_1)$  and  $V(P'_1) \cap V(Q'_1) = \{x\}$ . If  $Q_1$  is just the edge  $t_0t_1$  (so  $t_1 = x$ ) we may take  $P'_1 = C[u_0, x]$  and  $Q'_1 = Q_1$ . So we may assume that  $|V(Q_1)| \geq 3$ .

Let  $P'_1$  be a  $u_0x$ -path in  $R_1$  and  $Q'_1$  a  $t_0x$ -path in  $R_1$  so that  $V(P_1) \subseteq V(P'_1)$ ,  $V(Q_1) \subseteq V(Q'_1)$  and  $V(P'_1) \cap V(Q'_1) = \{x\}$ . Such paths exist since we can take  $P'_1 = P_1$  and  $Q'_1 = Q_1$ . Additionally assume  $|V(P'_1) \cup V(Q'_1)|$  is maximum. Suppose  $V(P'_1 \cup Q'_1) \neq V(R_1)$  and let  $K$  be a component of  $R_1 - V(P'_1 \cup Q'_1)$ . Because  $G$  is 3-connected,  $K$  must have at least three neighbors in  $G$ . Since  $V(P_1 \cup Q_1) \subseteq V(P'_1 \cup Q'_1)$ ,  $K$  contains no external vertices of  $R_1$ . Therefore, by planarity all neighbors of  $K$  are in  $R_1$  and hence in  $V(P'_1 \cup Q'_1)$ . Thus,  $K$  has at least two neighbors in one of  $P'_1$  or  $Q'_1$ .

Suppose first that  $K$  is adjacent to  $w_1, w_2 \in V(Q'_1)$ . If  $w_1w_2 \in E(Q'_1)$ , then we can lengthen  $Q'_1$  (still with  $V(Q_1) \subseteq V(Q'_1)$ ): replace the edge  $w_1w_2$  with a path from  $w_1$  to  $w_2$  through  $K$ . Hence we may assume that  $Q'_1 = t_0 \dots w_1 \dots w_3 \dots w_2 \dots x$  with  $w_3 \neq w_1, w_2$ , and we have a  $K_{2,5}$  minor  $\Theta([u_2, t_0] \cup Q'_1[t_0, w_1], Q'_1[w_2, x] \cup (P \cap R_2) \mid d, u_0, K, w_3, t_3)$ , a special case of which is shown in Figure 13. Suppose now that  $K$  is adjacent to  $w_1, w_2 \in V(P'_1)$ . If  $w_1w_2 \in E(P'_1)$  then we can lengthen  $P'_1$ , so  $P'_1 = u_0 \dots w_1 \dots w_3 \dots w_2 \dots x$  with  $w_3 \neq w_1, w_2$ , and we have a  $K_{2,5}$  minor  $\Theta([u_2, u_0] \cup P'_1[u_0, w_1], P'_1[w_2, x] \cup (P \cap R_2) \mid d, w_3, K, y, t_3)$  where  $y$  is an internal vertex of  $Q'_1$ , which exists because  $|V(Q'_1)| \geq |V(Q_1)| \geq 3$ . Thus no such component  $K$  exists,  $V(P'_1 \cup Q'_1) = V(R_1)$ , and  $P'_1$  and  $Q'_1$  are the desired paths in  $R_1$ .

By symmetric arguments,  $R_2$  has a  $u_1x^+$ -path  $P'_2$  and a  $t_3x^+$ -path  $Q'_2$  such that  $V(P'_2 \cup Q'_2) = V(R_2)$  and  $V(P'_2) \cap V(Q'_2) = \{x^+\}$ . Hence there is a longer cycle  $C[t_3, t_0] \cup Q'_1 \cup P'_1 \cup u_0du_1 \cup P'_2 \cup Q'_2$  as shown in Figure 14.

(2) Assume  $V(P) \cap V(Q) = \{x\} \subseteq [t_1, t_2]$ . The argument here will be very similar to, but not exactly the same as, that in (1). Again let  $R_1 = G[t_0, x]$  and  $R_2 = G[x^+, t_3]$ . Then  $t_0t_1 \in E(R_1)$  and  $t_2 \in V(R_2) \cup \{x\}$ . Using the argument from (1), in  $R_1$  we find a  $u_0x$ -path  $P'_1$  and a  $t_0x$ -path  $Q'_1$  with  $V(P'_1) \cup V(Q'_1) = V(R_1)$  and  $V(P'_1) \cap V(Q'_1) = \{x\}$ .

We also want to find in  $R_2$  a  $u_1x^+$ -path  $P'_2$  and a  $t_3x^+$ -path  $Q'_2$  such that  $V(P'_2 \cup Q'_2) = V(R_2)$  and

$V(P'_2) \cap V(Q'_2) = \{x^+\}$ , but this requires some changes from (1). Let  $P_2$  be the segment of the outer boundary of  $R_2$  clockwise from  $u_1$  to  $x^+$ , and let  $Q_2$  be the segment counterclockwise from  $t_3$  to  $x^+$ . Then  $P_2$  and  $Q_2$  are paths by the same argument as for  $P$  and  $Q$ . If there is an edge  $t_3x^+$  (including when  $x^+ = u_1$ ) then we can take  $P'_2 = C[x^+, u_1]$  and  $Q'_2 = Q_2 = t_3x^+$ , so we may assume there is no such edge and hence  $|V(Q_2)| \geq 3$ .

Assume there is  $v \in (V(P_2) \cap V(Q_2)) - \{x^+\}$ . Using Claim 4, every edge leaving  $(x, v)$  (which contains  $x^+$ ) goes to  $x$  or  $v$ , or is the edge  $t_2u_2$ . But since  $t_2$  is adjacent to  $t_3$ ,  $t_2 \notin (x, v)$  so  $\{x, v\}$  is a 2-cut in  $G$ , a contradiction. Thus,  $V(P_2) \cap V(Q_2) = \{x^+\}$ . Now we have a  $u_1x^+$ -path  $P_2$  and a  $t_3x^+$ -path  $Q_2$  so that (a) all external vertices of  $R_2$  belong to  $V(P_2 \cup Q_2)$ , (b)  $V(P_2) \cap V(Q_2) = \{x^+\}$ , and (c)  $|V(Q_2)| \geq 3$ . This allows us to apply the argument for  $|V(Q_2)| \geq 3$  from (1) to find the required  $P'_2$  and  $Q'_2$  in  $R_2$ .

As in (1), we use  $P'_1, Q'_1, P'_2, Q'_2$  to find a longer cycle.

(3) Finally suppose  $V(P) \cap V(Q) = \emptyset$ . Let  $P'$  be a  $u_0u_1$ -path in  $R$  and  $Q'$  a  $t_0t_3$ -path in  $R$  so that  $V(P) \subseteq V(P')$ ,  $V(Q) \subseteq V(Q')$  and  $V(P') \cap V(Q') = \emptyset$ . Such paths exist because we can take  $P' = P$  and  $Q' = Q$ . Assume additionally that  $|V(P') \cup V(Q')|$  is maximum. Suppose  $V(P' \cup Q') \neq V(R)$  and let  $K$  be a component of  $R - V(P' \cup Q')$ . Because  $G$  is 3-connected,  $K$  must have at least three neighbors in  $G$ . Since  $V(P \cup Q) \subseteq V(P' \cup Q')$ ,  $K$  contains no external vertices of  $R$ . Therefore, by planarity all neighbors of  $K$  are in  $R$  and hence in  $V(P' \cup Q')$ . Thus,  $K$  has at least two neighbors in one of  $P'$  or  $Q'$ .

First suppose  $K$  is adjacent to  $w_1, w_2 \in V(P')$ . If  $w_1w_2 \in E(P')$ , then we can lengthen  $P'$  (still with  $V(P) \subseteq V(P')$ ): replace the edge  $w_1w_2$  with a path from  $w_1$  to  $w_2$  through  $K$ . Hence we may assume that  $P' = u_0 \dots w_1 \dots w_3 \dots w_2 \dots u_1$  with  $w_3 \neq w_1, w_2$ , and we have a  $K_{2,5}$  minor  $\Theta(u_0t_0 \cup P'[u_0, w_1], P'[w_2, u_1] \cup u_1t_3 \mid u_2, d, w_3, K, y)$  as shown in Figure 15, where  $y$  is an internal vertex of  $Q'$ , which exists because  $|V(Q')| \geq |V(Q)| \geq 3$ . We can reason similarly if  $K$  is adjacent to  $w_1, w_2 \in V(Q')$ . Thus no such component  $K$  exists and  $V(P' \cup Q') = V(R)$ .

Suppose there is a  $K_{2,2}$  minor  $\Theta(t_0u_0, u_1t_3 \mid s_1, s_2)$  in  $G[u_1, u_0]$ . Then there is a  $K_{2,5}$  minor  $\Theta(t_0u_0, u_1t_3 \mid s_1, s_2, d, p, q)$  as shown in Figure 16, where  $p, q$  are arbitrary internal vertices of  $P, Q$  respectively. So we may assume there is no such minor. Therefore, there is no  $K_{2,2}$  minor along  $[t_3, t_0]$  or any of its subintervals.

Suppose that  $(u_2, t_0) = \emptyset$  or all jumps out of  $(u_2, t_0)$  go to  $u_0$ . Apply Corollary 2.3 to  $G[u_2, t_0]$  to find a path  $J = t_0 \dots t$  such that  $V(J) = (u_2, t_0]$  and either  $t = t_0$  if  $(u_2, t_0) = \emptyset$ , or  $t$  is a vertex of degree 2 in  $G[u_2, t_0]$ , from which there must be a jump to  $u_0$ . In either case,  $t_0u_0 \in E(G)$  and there is a longer cycle  $P' \cup u_1du_2 \cup C[t_3, u_2] \cup Q' \cup J \cup tu_0$ ; the case when  $(u_2, t_0) \neq \emptyset$  is shown in Figure 17.

So we may assume that not all jumps out of  $(u_2, t_0)$  go to  $u_0$  and so there is a jump  $x_1 - x_2$  with  $x_1 \in (u_2, t_0)$  and  $x_2 \in [t_3, u_2)$ . By a symmetric argument there is also a jump  $x'_2 - x'_1$  with  $x'_2 \in (t_3, u_2)$  and  $x'_1 \in (u_2, t_0]$ . These jumps cannot cross because they are just edges, so we cannot have both  $x_2 = t_3$  and  $x'_1 = t_0$ . Without loss of generality,  $x_2 \neq t_3$ , so  $x_1 - x_2$  is a jump from  $(u_2, t_0)$  to  $(t_3, u_2)$ . Out of all such jumps we may assume that  $x_1 - x_2$  has  $x_1$  closest to  $t_0$  and  $x_2$  closest to  $t_3$ .

If there is a jump  $y_1 - y_2$  from  $(u_2, x_1)$  to  $(x_1, u_0]$ , then  $x_1 - x_2$  and  $y_1 - y_2$  give a  $K_{2,2}$  minor in  $G[u_1, u_0]$  that we excluded above, namely  $\Theta([u_1, x_2], [y_2, u_0] \mid x_1, y_1)$  if  $y_2 \neq u_0$ , or  $\Theta([u_1, x_2], t_0u_0 \mid x_1, y_1)$  if  $y_2 = u_0$ . A symmetric minor exists if there is a jump from  $(x_2, u_2)$  to  $[u_1, x_2)$ . Hence edges of  $G$  leaving  $G[x_2, x_1]$  leave at  $x_1, x_2$  or  $u_2$ . Since  $G[x_2, x_1]$  is bounded by the cycle  $C[x_2, x_1] \cup x_1x_2$ ,  $G[x_2, x_1]$  is 2-connected by Lemma 2.4. Apply Lemma 2.2 to  $G[x_2, x_1]$  to find a path  $J_1 = x_2x_1 \dots t$  where  $V(J_1) = [x_2, x_1]$  and  $t$  is a degree 2 vertex in  $G[x_2, x_1]$  and hence must be  $u_2$ . Apply Corollary 2.3 to  $G[x_1, t_0]$  to find a path  $J_2 = t_0 \dots s$  where  $V(J_2) = (x_1, t_0]$  and either  $s = t_0$  or  $s$  is a degree 2 vertex in  $G[x_1, t_0]$ . In either case  $su_0 \in E(G)$  and there is a longer cycle  $P' \cup u_1du_2 \cup J_1 \cup C[t_3, x_2] \cup Q' \cup J_2 \cup su_0$ , as shown in Figure 18.

This completes the proof of Claim 5.

**Claim 6.** *Either  $t_1 = u_0^+$  or  $t_2 = u_1^-$  (at least one of  $X_1$  and  $X_2$  is empty).*

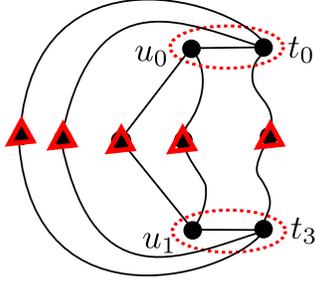


Figure 16

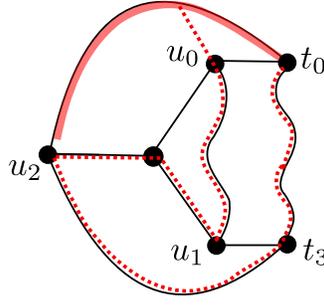


Figure 17

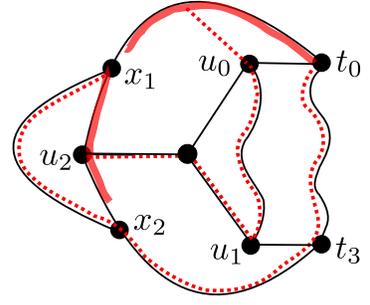


Figure 18

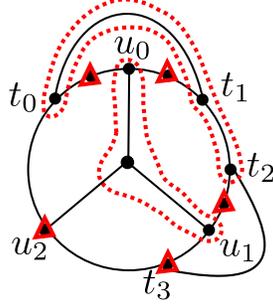


Figure 19

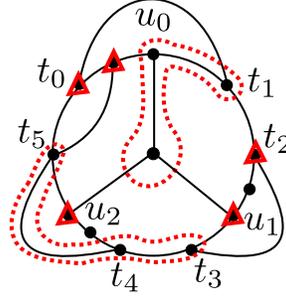


Figure 20

Assume  $t_1 \neq u_0^+$  and  $t_2 \neq u_1^-$ . By Claim 5, either  $t_0 \neq u_0^-$  or  $t_3 \neq u_1^+$ . Without loss of generality, suppose  $t_0 \neq u_0^-$ . Then there is a  $K_{2,5}$  minor  $\Theta(u_0du_1, t_0t_1 \cup [t_1, t_2] \mid u_0^-, u_0^+, u_1^-, t_3, u_2)$  as shown in Figure 19.

**Claim 7.** *At most two pairs of sectors have jumps between them.*

Assume that there are three sector jumps  $t_0 - t_1$ ,  $t_2 - t_3$ , and  $t_4 - t_5$  where possibly  $t_0 = t_5$ ,  $t_1 = t_2$ , or  $t_3 = t_4$ . By Claim 5,  $X_0$  and  $X_3$  cannot both be empty and symmetrically,  $X_1$  and  $X_4$  cannot both be empty and  $X_2$  and  $X_5$  cannot both be empty. Hence  $X_i \neq \emptyset$  for at least three  $i$ . By Claim 6, at least one of  $X_1$  and  $X_2$  is empty and symmetrically, at least one of  $X_3$  and  $X_4$  is empty and at least one of  $X_5$  and  $X_0$  is empty. Hence  $X_i \neq \emptyset$  for exactly three  $i$ . Furthermore, the nonempty  $X_i$  must be rotationally symmetric about  $C$ . Without loss of generality, suppose  $X_0$ ,  $X_2$ , and  $X_4$  are nonempty and  $X_1$ ,  $X_3$ , and  $X_5$  are empty.

If  $t_1 = t_2$ , then there is a standard longer cycle  $L(u_0^+ - u_1^+)$ . Thus  $t_1 \neq t_2$ , and symmetrically  $t_3 \neq t_4$  and  $t_5 \neq t_0$ .

Consider a jump  $r_0 - r'_0$  out of  $X_0$ . There are three options for  $r'_0$ :  $r'_0 \in [t_5, t_0)$ ,  $r'_0 = t_1$ , or  $r'_0 = u_2$ . If  $r'_0 \in [t_5, t_0)$  then, since  $t_1 \neq t_2$ , there is a  $K_{2,5}$  minor  $\Theta(du_0t_1, [t_3, t_4] \cup t_4t_5 \cup [t_5, r'_0] \mid t_0, r_0, t_2, u_1, u_2)$ ; the case  $r'_0 = t_5$  is shown in Figure 20. Thus,  $r'_0 \in \{u_2, t_1\}$ , and symmetrically  $r'_2 \in \{u_0, t_3\}$  for a jump  $r_2 - r'_2$  out of  $X_2$ , and  $r'_4 \in \{u_1, t_5\}$  for a jump  $r_4 - r'_4$  out of  $X_4$ .

If at least two of  $r'_0$ ,  $r'_2$ , and  $r'_4$  belong to  $U = \{u_0, u_1, u_2\}$  then without loss of generality we may assume that  $r'_0 = u_2$  and  $r'_2 = u_0$ . We have a  $K_{2,5}$  minor  $M([t_0, t_1], [u_1, t_4])$  as shown in Figure 21. If only one of  $r'_0$ ,  $r'_2$ , and  $r'_4$  belongs to  $U$ , then without loss of generality  $r'_0 = u_2$  and there is a  $K_{2,5}$  minor  $\Theta(u_2du_1t_3, [t_5, t_0] \cup t_0t_1 \cup [t_1, t_2] \mid r_0, u_0, r_2, t_4, r_4)$  as shown in Figure 22. Hence we may assume that all jumps out of  $X_0$  go to  $t_1$ , out of  $X_2$  go to  $t_3$ , and out of  $X_4$  go to  $t_5$ .

If there is a  $K_{2,2}$  minor  $\Theta(t_0, u_0 \mid s_1, s_2)$  along  $[t_0, u_0]$ , then there is a  $K_{2,5}$  minor  $\Theta(\{d, u_0, u_1, u_2\}, t_2t_3 \cup [t_3, t_4] \cup t_4t_5 \cup [t_5, t_0] \mid s_1, s_2, t_1, r_2, r_4)$  as shown in Figure 23. Hence there is no  $K_{2,2}$  minor along  $[t_0, u_0]$ , or symmetrically, along  $[t_2, u_1]$  or  $[t_4, u_2]$ . Because all jumps out of  $X_4$  go to  $t_5$  we can apply Corollary 2.3 to  $G[t_4, u_2]$  and find a path  $P = t_4 \dots t$  where  $V(P) = [t_4, u_2]$  and  $t$  has degree 2 in  $G[t_4, u_2]$ , so  $tt_5 \in E(G)$ . If  $(t_5, t_0) = \emptyset$ , then there is a longer cycle  $C[t_0, u_0] \cup t_0t_1 \cup C[t_1, t_4] \cup P \cup tt_5u_2du_0$  as shown in Figure 24.

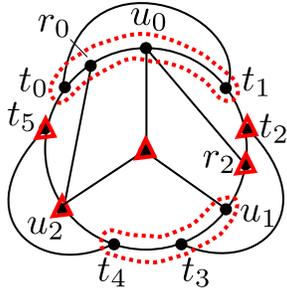


Figure 21

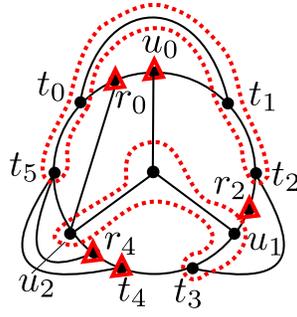


Figure 22

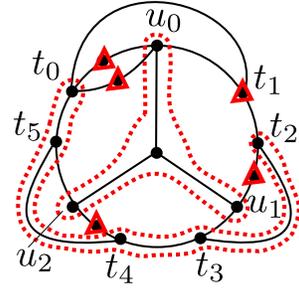


Figure 23

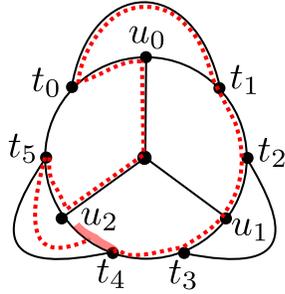


Figure 24

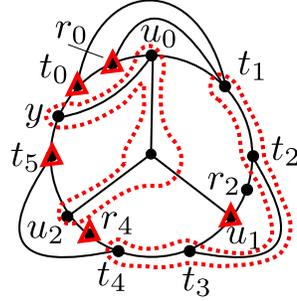


Figure 25

Hence  $(t_5, t_0) \neq \emptyset$ . Let  $y - y'$  be a jump out of  $(t_5, t_0)$ . Since all jumps out of  $X_0$  go to  $t_1$ ,  $y' \notin X_0$ , so  $y' = u_0$  or  $u_2$ . Then there is a  $K_{2,5}$  minor  $\Theta(yy' \cup u_0 u_2, [t_1, t_2] \cup t_2 t_3 \cup [t_3, t_4] \mid t_0, r_0, u_1, r_4, t_5)$ ; the case  $y' = u_0$  is shown in Figure 25. This completes the proof of Claim 7.

Henceforth we assume there are jumps  $t_0 - t_1$  and  $t_2 - t_3$ , but not  $t_4 - t_5$ . By Claim 6, at least one of  $X_1$  and  $X_2$  is empty. Without loss of generality, assume  $X_1 = \emptyset$  and hence  $t_1 = u_0^+$ . We claim that there are  $K_{2,2}$  minors  $M_1$  in  $G[u_2, t_1]$  and  $M_2$  in  $G[u_0, u_2]$ , both rooted at  $u_2$  and  $[u_0, t_1]$ .

Assume that  $M_1$  does not exist. If  $t_0 = u_2^+$ , then there is a standard longer cycle  $L(u_2^+ - u_0^+)$ . Hence  $t_0 \neq u_2^+$ , and there must be a jump  $r - r'$  from  $(u_2, t_0)$  to  $(t_0, u_0)$ . If  $r' \neq u_0$  then we may take  $M_1$  to be  $\Theta([u_2, r], (r', t_1] \mid r', t_0)$ , so all jumps from  $(u_2, t_0)$  must go to  $u_0$ . If there is a  $K_{2,2}$  minor along  $[u_2, t_0]$  or along  $[t_0, u_0]$  then we also have  $M_1$ , so neither of these minors exist. All jumps out of  $(t_0, u_0)$  must go to  $t_1$  since jumps to  $u_2$  are blocked by planarity. By Corollary 2.3 applied to  $G[u_2, t_0]$ , there is a path  $P_1 = t_0 \dots t$  such that  $V(P_1) = (u_2, t_0]$  and  $t$  is a degree 2 vertex in  $G[u_2, t_0]$ , or  $t = t_0$  if  $(t_0, u_0) = \emptyset$ , so that  $t$  is adjacent to  $u_0$ . Similarly by Corollary 2.3 there is a path  $P_2 = t_0 \dots s$  such that  $V(P_2) = [t_0, u_0)$  and  $s$  is a degree 2 vertex in  $G[t_0, u_0]$  or  $s = t_0$ , so that  $s$  is adjacent to  $t_1$ . Then there is a longer cycle  $P_2 \cup s t_1 \cup C[t_1, u_2] \cup u_2 d u_0 t \cup P_1$  as shown in Figure 26. This is a contradiction, so  $M_1$  exists.

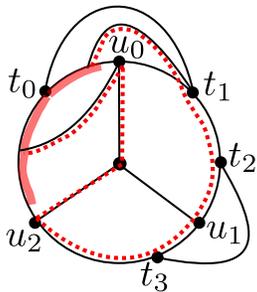


Figure 26

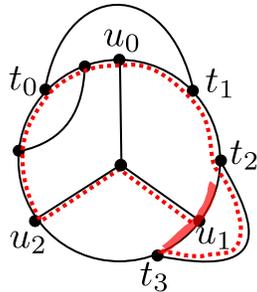


Figure 27

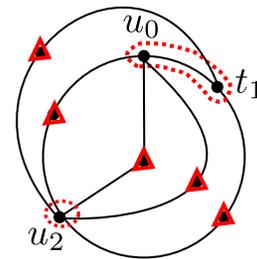


Figure 28

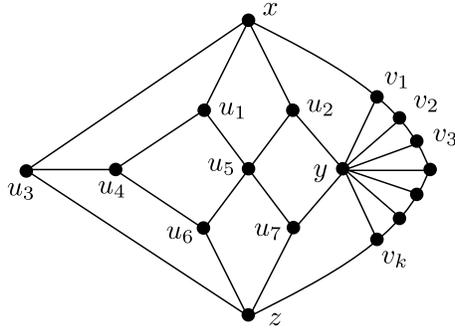


Figure 29

Assume that  $M_2$  does not exist. If there is an inside jump out of  $(t_2, u_1)$  or  $(u_1, t_3)$ , or any jump out of  $(t_3, u_2)$ , then this jump and  $t_2 - t_3$  give us  $M_2$ . So all edges of  $G$  leaving  $G[t_2, t_3]$  leave at  $t_2, t_3$  or  $u_1$ , and  $(t_3, u_2) = \emptyset$ . Any  $K_{2,2}$  minor along  $[t_2, t_3]$  would also provide  $M_2$ , so there is no such minor. Therefore, by Lemma 2.2 there is a Hamilton path  $P = t_2 t_3 \dots t$  in  $G[t_2, t_3]$  with  $t$  of degree 2 in  $G[t_2, t_3]$ . Then  $t = u_1$  and we have a longer cycle  $C[u_2, t_2] \cup P \cup u_1 d u_2$  as shown in Figure 27. This is a contradiction, so  $M_2$  exists.

Together  $M_1$  and  $M_2$  give a  $K_{2,5}$  minor  $M([u_0, t_1], u_2)$  as in Figure 28. This contradiction concludes the proof of Theorem 3.1.  $\square$

## 4 Sharpness

A natural next step is to consider the same result for  $K_{2,6}$ -minor-free graphs. It is not true, however, that all 3-connected planar  $K_{2,6}$ -minor-free graphs are Hamiltonian. In fact, we can construct an infinite family of non-Hamiltonian 3-connected planar  $K_{2,6}$ -minor-free graphs. Let  $G_k$  be the graph shown in Figure 29, where  $k \geq 1$ . We begin by analyzing  $K_{2,5}$  minors in  $G_1$ , which is the Herschel graph, mentioned earlier.

**Lemma 4.1.** *Suppose  $G_1$  has a  $K_{2,5}$  minor with standard model  $\Sigma(R_1, R_2 | S)$ . Then*

- (a)  $R_1 \cup R_2 \cup S = V(G_1)$ ,
- (b) each of  $R_1$  and  $R_2$  contains exactly one degree 4 vertex of  $G_1$ , and
- (c)  $G_1$  has no edge between  $R_1$  and  $R_2$ .

*Proof.* For  $i = 1$  and 2 let  $H_i = G_1[R_i]$ , and let  $N(R_i)$  be the set of neighbors of  $R_i$  in  $G_1$ ; then  $S \subseteq N(R_i)$ . We use the fact that  $G_1$  is highly symmetric: besides the 2-fold symmetries generated by reflecting Figure 29 about horizontal and vertical axes, there is a 3-fold symmetry generated by the automorphism  $(u_4)(u_1 u_6 u_3)(x u_5 z)(u_2 u_7 v_1)(y)$ . Thus all degree 4 vertices in  $G_1$  are similar,  $u_4$  and  $y$  are similar, and all other degree 3 vertices are similar.

Assume without loss of generality that  $|R_1| \leq |R_2|$ . Since all vertices of  $G_1$  have degree 3 or 4, we have  $|R_1| \geq 2$ . Since  $|R_1| + |R_2| \leq 6$ , we have  $|R_1| \leq 3$ . Since  $G_1$  has no triangles,  $H_1$  is a path  $w_1 w_2$  or  $w_1 w_2 w_3$ . Define the *type* of a path  $w_1 w_2 \dots w_k$  to be the sequence  $d_1 d_2 \dots d_k$  where  $d_i = \deg(w_i)$ . We break into cases according to the type of  $H_1$ . The possible types are restricted by the fact that no two degree 4 vertices of  $G_1$  are adjacent. When  $|R_1| = 3$  we must also have  $|R_2| = 3$  so  $H_2$  is a path  $x_1 x_2 x_3$ , and  $V(G_1) - R_1 - S = R_2$  so that  $V(G_1) - R_1 - N(R_1) \subseteq R_2$ .

If  $H_1$  has type 33 then  $|N(R_1)| < 5$ . If  $H_1$  has type 333 then by symmetry we may assume  $H_1 = u_1 u_4 u_6$ , and again  $|N(R_1)| < 5$ . So neither of these cases happen.

If  $H_1$  has type 34 (or 43) then up to symmetry  $H_1 = u_3 x$ . Then  $S = N(R_1) = \{u_1, u_2, u_4, v_1, z\}$ . Now  $R_2$  contains  $u_6$  (so that  $u_4 \in N(R_2)$ ) and  $y$  (so that  $v_1 \in N(R_2)$ ) so  $H_2$  is the path  $u_6 u_5 u_7 y$  of type 3433.

If  $H_1$  has type 334 (or 433) then up to symmetry  $H_1 = u_4u_1x$ . Then  $S = N(R_1) = \{u_2, u_3, u_5, u_6, v_1\}$  and  $H_2$  is the path  $yu_7z$  of type 334.

If  $H_1$  has type 343 then  $w_1$  and  $w_3$  may be either opposite or adjacent neighbors of  $w_2$ . If they are opposite neighbors, then up to symmetry  $H_1 = u_3xu_2$ . Then  $V(G_1) - R_1 - N(R_1) = \{u_6, u_7\} \subseteq R_2$  and so either  $H_2 = u_6u_5u_7$  and  $R_2$  is not adjacent to  $v_1 \in S$ , or  $H_2 = u_6zu_7$  and  $R_2$  is not adjacent to  $u_1 \in S$ . So this does not occur. If  $w_1$  and  $w_3$  are adjacent neighbors of  $w_2$ , then up to symmetry  $H_1 = u_1xu_2$ . Then  $S = N(R_1) = \{u_3, u_4, u_5, y, v_1\}$  and  $H_2$  is the path  $u_6zu_7$  of type 343.

If  $H_1$  has type 434 then up to symmetry  $H_1 = xu_1u_5$ . Then  $V(G_1) - R_1 - N(R_1) = \{y, z\} \subseteq R_2$  and so  $H_2$  is either  $yu_7z$  or  $yv_1z$ . But in either case  $R_2$  is not adjacent to  $u_4 \in S$ , so this case does not occur.

Whenever the minor exists (types 34, 334, and 343 with adjacent neighbors) all of (a), (b) and (c) hold.  $\square$

**Proposition 4.2.** *For all  $k \geq 1$ ,  $G_k$  is a 3-connected planar non-Hamiltonian  $K_{2,6}$ -minor-free graph.*

*Proof.* In the plane embedding of  $G_k$  shown in Figure 29 every pair of faces intersect at most once (at a vertex or along an edge), so  $G_k$  is 3-connected. Let  $S = \{x, y, z, u_4, u_5\}$ . Then  $|S| = 5$  but  $G_k - S$  has six components, so  $G_k$  cannot be Hamiltonian ( $G_k$  is not 1-tough).

We prove that  $G_k$  is  $K_{2,6}$ -minor-free by induction on  $k$ . For  $G_1$  this follows from Lemma 4.1(a). So suppose that  $k \geq 2$ , all  $G_j$  for  $j \leq k - 1$  are  $K_{2,6}$ -minor-free, and  $G_k$  has a  $K_{2,6}$  minor with standard model  $\Sigma(R_1, R_2 | S)$ .

Let  $F = v_1v_2 \dots v_k$ . Let  $R'_j = R_j - V(F)$  for  $j = 1$  and  $2$ ,  $S' = S - V(F)$ ,  $S'' = S \cap V(F)$  and  $T = V(G_k) - R_1 - R_2 - S$ . We cannot have  $R_j \subseteq V(F)$  because any subset of  $V(F)$  that induces a connected subgraph in  $G_k$  has only three neighbors in  $G_k$ . Therefore, each  $R'_j$  is nonempty. If  $v_i \in R_j \cup S$  for some  $v_i \in V(F)$ , then there is a path  $P_j(v_i)$  from  $v_i$  to a vertex of  $R'_j$ , all of whose internal vertices belong to  $R_j \cap V(F)$ . The other end of  $P_j(v_i)$  is one of  $x, y$  or  $z$ .

We claim that (\*)  $V(F) \subseteq R_1 \cup R_2 \cup S$  and no two consecutive vertices of  $F$  belong to the same  $R_j$ . If not, there is  $e \in E(F)$  with one end in  $T$ , or both ends in the same  $R_j$ . Contracting  $e$  preserves the existence of a  $K_{2,6}$  minor and gives a graph isomorphic to  $G_{k-1}$ , contradicting our inductive hypothesis.

Suppose  $y \in S \cup T$ . If some  $v_a \in S$  then  $P_j(v_a) = v_a v_{a-1} \dots v_1 x$  and  $P_{3-j}(v_a) = v_a v_{a+1} \dots z$  for  $j = 1$  or  $2$ . Thus  $\Sigma(R'_1, R'_2 | S' \cup \{v_1\})$  is a  $K_{2,6}$  minor in  $G_1$ , a contradiction. Otherwise, by (\*),  $v_1 \in R_j$  and  $v_2 \in R_{3-j}$  for some  $j$ . We must have  $P_j(v_1) = v_1 x$  and  $P_{3-j}(v_2) = v_2 v_3 \dots v_k z$ . Then  $\Sigma(R'_1, R'_2 | S - \{y\} \cup \{v_1\})$  is a  $K_{2,6}$  or  $K_{2,7}$  minor in  $G_1$ , again a contradiction.

So we may assume without loss of generality that  $y \in R_2$ . If  $|S''| \geq 2$  we can choose  $v_a, v_b \in S''$  with  $a < b$  so that there is no  $v_i \in S''$  with  $a < i < b$ . Then  $P_1(v_a) = v_a v_{a-1} \dots v_1 x$  and  $P_1(v_b) = v_b v_{b+1} \dots v_k z$ , so  $S'' = \{v_a, v_b\}$  and  $x, z \in R'_1$ . Then  $\Sigma(R'_1, R'_2 | S' \cup \{v_1\})$  is a  $K_{2,5}$  minor in  $G_1$  that contradicts Lemma 4.1(b). If  $|S''| \leq 1$  then there is either  $v_a \in S$ , or since  $k \geq 2$  by (\*) there is  $v_a \in R_1$ . Without loss of generality  $P_1(v_a) = v_a v_{a-1} \dots v_1 x$ . Now  $\Sigma(R'_1, R'_2 \cup \{v_1\} | S')$  is a  $K_{2,5}$  or  $K_{2,6}$  minor in  $G_1$  with  $x \in R'_1$  and  $v_1 \in R'_2 \cup \{v_1\}$ , contradicting Lemma 4.1(c).  $\square$

Based on computer results of Gordon Royle (personal communication), we suspect that it may be possible to characterize all exceptions to the statement that all 3-connected planar  $K_{2,6}$ -minor-free graphs are Hamiltonian. All known exceptions are closely related to the family shown in Figure 29.

## Acknowledgements

The first author thanks Zachary Gaslowitz and Kelly O'Connell for helpful discussions.

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