# On Perfect Matchings in Matching Covered Graphs

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#### Abstract

Let G be a matching-covered graph, i.e., every edge is contained in a perfect matching. An edge subset X of G is feasible if there exists two perfect matchings  $M_1$  and  $M_2$  such that  $|M_1 \cap X| \not\equiv |M_2 \cap X|$  (mod 2). Lukot'ka and Rollová proved that an edge subset X of a regular bipartite graph is not feasible if and only if X is switching-equivalent to  $\emptyset$ , and they further ask whether a non-feasible set of a regular graph of class 1 is always switching-equivalent to either  $\emptyset$  or E(G)? Two edges of G are equivalent to each other if a perfect matching M of G either contains both of them or contains none of them. An equivalent class of G is an edge subset G with at least two edges such that the edges of G are mutually equivalent. An equivalent class is not a feasible set. Lovász proved that an equivalent class of a brick has size 2. In this paper, we show that, for every integer G is not switching-equivalent to either G or G or G is not switching-equivalent to either G or G or G is not switching-equivalent to either G or G or G is not switching-equivalent to either G or G or G is not switching-equivalent to either G or G or G is not switching-equivalent to either G or G or G is not switching-equivalent to either G or G or G is not switching-equivalent to either G or G or G is not switching-equivalent to either G or G or G is not switching-equivalent to either G or G or G or G is not switching-equivalent to either G or G or G or G or G is not switching-equivalent to either G or G or

## 1 Introduction

Let G be a graph. A perfect matching of G is a set of independent edges which covers all vertices of G. A graph with a perfect matching is called a matchable graph. A graph G is k-extendable if G has at least 2k+2 vertices and, for any k independent edges of G, there is a perfect matching containing them. It has been shown by Plummer [13] that a k-extendable graph is (k+1)-connected. A 1-extendable graph is also called matching-covered, or coverable. A 2-extendable bipartite graph is called a brace. By the result of Plummer [13], a brace is a 3-connected bipartite graph. A brick is a 3-connected graph such that, for any two vertices u and v,  $G\setminus\{u,v\}$  has a perfect matching. It is not hard to see that a brick is matching-covered but not bipartite. Plummer [13] proved that a 2-extendable graph is either a brace or a brick. But a brick is not necessarily 2-extendable. A matching-covered graph can be decomposed into a family of bricks and braces by the Lovász's Tight-Cut Decomposition [9].

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A set of edges X of a matching-covered graph G is *feasible* if G has two perfect matchings  $M_1$  and  $M_2$  such that  $|M_1 \cap X| \not\equiv |M_2 \cap X| \pmod{2}$ . Note that, every edge of G is contained by some perfect matchings but avoid by others. So a single edge of a matching-covered graph forms a trivial feasible edge set. On the other hand, if X is an edge-cut of G, the parity of  $X \cap M$  depends on the parities of the orders of components of  $G \setminus X$  and hence X is always non-feasible.

A matching-covered regular graph may have many distinct perfect matchings. It has been conjectured by Lovász and Plummer [11] that every matching-covered regular graph has exponentially many perfect matchings, which has been verified by Schrijver [15] for regular bipartite graphs and by Esperet et. al. [5] for cubic graphs. As a matching covered regular graph has many perfect matchings, it seems reasonable to believe that non-feasible edge sets are rare. It can be determined in randomized polynomial time whether a given edge set is feasible or not by using a probabilistic algorithm for exact matching (cf. Section 3.3 in [10]). Lukot'ka and Rollová [7] show that the feasible sets in cubic graphs could be used to show the existence of spanning bipartite qudrangulations (cf. [12]) and certain cycle covers in signed cubic bipartite graphs [7].

Let v be a vertex of G and E(v) be the set of all edges incident with v. For a given edge set X, the switching-operation of X on E(v) is to be defined as the symmetric difference of E(v) and X, denoted by  $E(v) \oplus X = (E(v) \cup X) \setminus (E(v) \cap X)$ . As a perfect matching always contains exactly one edge from E(v), the symmetric difference  $E(v) \oplus X$  is feasible if and only if X is feasible. Two edge sets  $X_1$  and  $X_2$  are switching-equivalent if  $X_1$  can be obtained from  $X_2$  by a series of switching-operations and vice visa. For two switching-equivalent edge sets  $X_1$  and  $X_2$ , by the definition of switching-operation,  $X_1$  is feasible if and only if  $X_2$  is feasible.

**Theorem 1.1** (Lukot'ka and Rollová, [7]). Let G be a regular bipartite graph and  $X \subseteq E(G)$ . Then X is not feasible if and only if X is switching-equivalent to  $\emptyset$ .

Lukot'ka and Rollová [7] found that the Petersen graph has a non-feasible edge set which is not switching-equivalent to either  $\emptyset$  or E(G), and believe that an easy characterization of feasible edge sets for regular non-bipartite graphs seems not possible. More examples can be found in [12]. But all of these examples are not 3-edge-colorable cubic graphs, which are so-called snarks. For regular nonbipartite graphs of class 1, Lukot'ka and Rollová propose the following problem.

**Problem 1.2** (Lukot'ka and Rollová, [7]). Let G be a regular graph of class 1 and let X be a subset of edges of G. Is it true that X is not feasible if and only if X is switching-equivalent to either  $\emptyset$  or E(G)?

In this paper, we provide a negative answer to the above problem by showing the following result.

**Theorem 1.3.** For any integer  $k \geq 3$ , there are infinitely many k-regular nonbipartite graphs of class 1 with a non-feasible set X which is not switching-equivalent to either  $\emptyset$  or E(G).

An edge e of a matching-covered graph G is removable if  $G\setminus\{e\}$  is still matching-covered. A removable edge is also called a removable ear in Ear Decomposition of matching-covered graph [3, 8], which provides a fundamental construction of matching-covered graphs [2, 8, 16] (see also [11]). A graph G is strongly coverable if every edge of G is removable. A strongly coverable graph is also called a graph with property E(1,1) (cf. [1]). Note that a 2-extendable graph is strongly coverable [14]. Therefore, any two independent edges of a 2-extendable graph G form a feasible set of G. Aldred et. al. [1] show that a strongly coverable

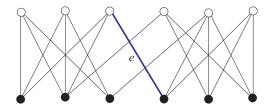


Figure 1: A 3-connected bipartite graph with a non-removable edge e.

bipartite graph is 3-connected. But a 3-connected bipartite graph is not necessarily strongly coverable. The bipartite graph in Figure 1 is 3-connected but not strongly coverable.

A matchable bipartite graph G(A, B) is always balanced, i.e. |A| = |B|. For two subsets X and Y of V(G(A, B)), let E[X, Y] denote the set of all edges joining a vertex in X and a vertex in Y. In this paper, we characterize all strongly coverable bipartite graphs as follows.

**Theorem 1.4.** Let G(A, B) be a matching-covered bipartite graph. Then G(A, B) is strongly coverable if and only if every edge-cut S separating G(A, B) into two balanced components  $G_1(A_1, B_1)$  and  $G_2(A_2, B_2)$  satisfies that  $|E[A_1, B_2]| \ge 2$  and  $|E[B_1, A_2]| \ge 2$ .

Two edges of a matching-covered graph G are equivalent to each other if a perfect matching of G either contains both of them or contains none of them. An equivalent class of G is a subset of E(G) with at least two edges such that any two edges of K are equivalent to each other. An equivalent class of a matching-covered graph is not a feasible set. A matching-covered graph with an equivalent class K is not strongly coverable because any edge of K is not removable. However, a matching-covered graph without an equivalent class may not be strongly coverable, even for bipartite graphs. For example, the graph in Figure 1 has no equivalent class but does have a non-removable edge e and hence is not strongly coverable.

**Theorem 1.5** (Lovász, [9]). Let G be a brick and K be an equivalent class. Then |K| = 2 and  $G \setminus K$  is bipartite.

In this paper, we obtain a characterization for bipartite graphs with an equivalent class as follows.

**Theorem 1.6.** Let G(A, B) be a matching-covered bipartite graph. Then G(A, B) has an equivalent class if and only if G(A, B) has a 2-edge-cut which separates G(A, B) into two balanced components.

The above result implies that a 3-connected matching-covered bipartite graph has no equivalent class. Therefore, a brace has no equivalent class. Together with Theorem 1.5, a final graph in the Lovász's Tight-Cut Decomposition either has no equivalent class or has an equivalent class of size two.

Let  $\mathcal{F}_{mc}$ ,  $\mathcal{F}_{sc}$ ,  $\mathcal{F}_{2-ext}$  and  $\mathcal{F}_{nec}$  denote the families of matching-covered graphs, strongly coverable graphs, 2-extendable graphs and graphs without equivalent class, respectively. Then we have the following nested relation:

$$\mathcal{F}_{2\text{-ext}} \subseteq \mathcal{F}_{sc} \subseteq \mathcal{F}_{nec} \subseteq \mathcal{F}_{mc}$$
.

In Section 2, we are going to prove Theorem 1.3. The proofs of Theorems 1.4 and 1.6 are given in Section 3.

## 2 Proof of Theorem 1.3

A signed graph  $(G, \sigma)$  is a graph associated with a mapping  $\sigma : E(G) \to \{-1, 1\}$  which is called a signature. Let  $E^-(G, \sigma) = \{e \mid \sigma(e) = -1\}$ . Two signed graphs  $(G, \sigma_1)$  and  $(G, \sigma_2)$  are switching-equivalent if  $E^-(G, \sigma_1)$  is switching-equivalent to  $E^-(G, \sigma_2)$ . A signed graph  $(G, \sigma)$  is balanced if its negative edge set is switching-equivalent to the empty set. For a subset  $U \subseteq V(G)$ , let  $\nabla U$  denote the set of all edges joining a vertex in U and a vertex in  $V(G)\setminus U$ . The following is a characterization of a balanced signed graph.

**Lemma 2.1** (Harary, [6]). A signed graph  $(G, \sigma)$  is balanced if and only if  $E^-(G, \sigma) = \nabla U$  for some  $U \subseteq V(G)$ .

Let G be a graph and  $X \subseteq E(G)$ . Define  $\sigma_X : E(G) \to \{-1,1\}$  such that  $\sigma_X(e) = -1$  if  $e \in X$  and  $\sigma_X(e) = 1$  otherwise. Then we have a signed graph  $(G, \sigma_X)$  for a graph G and a given edge subset X. The following is a straightforward observation by applying the above lemma to signed graphs  $(G, \sigma_X)$  and  $(G, \sigma_{E(G)\setminus X})$ .

**Observation 2.2.** Let G be a graph and  $X \subseteq E(G)$ . Then X is switching-equivalent to  $\emptyset$  if and only if  $X = \nabla U$  for some  $U \subseteq V(G)$ ; and X is switching-equivalent to E(G) if and only if  $E(G)\backslash X = \nabla U$  for some  $U \subseteq V(G)$ .

Now, we are going to prove our main result, Theorem 1.3.

**Proof of Theorem 1.3.** For any integer  $k \geq 3$ , take a copy of the complete bipartite graph  $K_{k,k}$ . Assume that (A, B) be the bipartition of  $K_{k,k}$ . The bipartite graph  $K_{k,k}$  is k-edge-colorable and let  $c: E(K_{k,k}) \to \{1, ..., k\}$  be a k-edge-coloring. Let  $e_1 = u_1v_1$  and  $e_2 = u_2v_2$  be two edges of  $K_{k,k}$  with the same color, say  $c(e_1) = c(e_2) = 1$ . Without loss of generality, assume that  $\{u_1, u_2\} \subseteq A$  and  $\{v_1, v_2\} \subseteq B$ . Delete  $e_1$  and  $e_2$  from  $K_{k,k}$  and let  $G_k(A, B)$  be the resulting bipartite graph. Note that  $G_k(A, B)$  has a Hamilton cycle.

Take m copies of  $G_k(A, B)$  ( $m \ge 2$ ) and denote them by  $G_k^1(A^1, B^1), G_k^2(A^2, B^2), ..., G_k^m(A^m, B^m)$ . Add the following edges to join these copies of  $G_k$  to get a new k-regular non-bipartite graph G(k, m):

$$u_1^1u_2^1,\ v_1^1u_1^2,\ v_2^1u_2^2,\ \cdots,\ v_1^iu_1^{i+1},\ v_2^iu_2^{i+1},\ \cdots,\ v_1^{m-1}u_1^m,\ v_2^{m-1}u_2^m,\ v_1^mv_2^m$$

where  $v_1^i, v_2^i, u_1^i, u_2^i \in V(G_k^i)$  with degree k-1. Let K be the set of these new edges. For example, see G(3,2) in Figure 2.

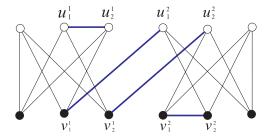


Figure 2: An example G(3,2): the set K consisting of all blue edges.

Since  $G_k(A, B)$  has a Hamiltonian cycle, the copy of  $G_k^1(A^1, B^1)$  has a Hamilton cycle C which together with  $u_1^1u_2^1$  contains two odd cycles. Hence G(k, m) is not bipartite. On the other hand, G(k, m) has a k-edge-coloring which comes from a k-edge-coloring of  $G_k$  together coloring all new edges by the color  $c(e_1) = c(e_2)$ . Hence G(k, m) is a k-regular non-bipartite graph of class 1. Let K be the set of all new edges.

**Claim:** The edge set K is an equivalent class of G(k, m).

Proof of Claim. In the graph G(k,m), two edges  $v_1^i u_1^{i+1}$  and  $v_2^i u_2^{i+1}$  form a 2-edge-cut which separates G(k,m) into two components with an even number of vertices. Hence a perfect matching of G(k,m) contains either none of them or both of them. So  $v_1^i u_1^{i+1}$  is equivalent to  $v_2^i u_2^{i+1}$  for i = 1, ..., m-1.

Let M be a perfect matching of G(k,m) containing both  $v_1^iu_1^{i+1}$  and  $v_2^iu_2^{i+1}$ . Consider the copy  $G_k^{i+1}(A^{i+1},B^{i+1})$ . The perfect matching M matches all vertices  $A^{i+1}\setminus\{u_1^{i+1},u_2^{i+1}\}$  to k-2 vertices of  $B^{i+1}$ . So the remaining two vertices of  $B^{i+1}$  are matched to two vertices of  $A^{i+2}$  where  $i+1\leq m-1$ . Hence  $v_1^{i+1}u_1^{i+2}\in M$  and  $v_2^{i+1}u_2^{i+2}\in M$ . A similar argument shows that  $K\subseteq M$ . So all edges in K are dependent on  $v_j^iu_j^{i+1}$  for any  $j\in\{1,2\}$  and  $i\in\{1,...,m-1\}$ , which implies that  $K\setminus\{u_1^1u_1^2,v_1^mv_2^m\}$  is an equivalent class.

On the other hand, a perfect matching M of G(k,m) containing  $u_1^1u_2^1$  matches  $v_1^1$  and  $v_2^1$  to  $u_1^2$  and  $u_2^2$  respectively. So all edges of K are dependent on  $u_1^1u_2^1$ . By symmetry, all edges of K are dependent on  $v_1^mv_2^m$  too. It follows that K is an equivalent class of G(K,m). This completes the proof of Claim.

Let  $X = \{u_1^1 u_2^1, v_1^1 u_1^2, ..., v_1^i u_1^{i+1}, ..., v_1^{m-1} u_1^m\} \subset K$ . So X is an equivalent class by Claim. Hence not a feasible set. In the following, it suffices to show that X is not switching-equivalent to either  $\emptyset$  or E(G(k, m)).

First, note that  $G(k,m)\backslash X$  is connected. Therefore, there is no  $U\subseteq V(G(k,m))$  such that  $X=\nabla U$ . On the other hand,  $G(k,m)\backslash X$  is not a bipartite graph because the edge  $v_1^mv_2^m$  together with a Hamilton cycle C of  $G_k(A^m,B^m)$  contains two odd cycles which belong to  $G(k,m)\backslash X$ . So G(k,m) does not have a vertex subset U such that  $E(G(k,m)\backslash X=\nabla U)$ . Hence X is not switching-equivalent to  $\emptyset$  or E(G) by Observation 2.2. Hence G(k,m) is a k-regular non-bipartite graph of class 1 which has a non-feasible set X not switching-equivalent to  $\emptyset$  or E(G(k,m)).

As  $m \geq 2$  could be any integer, there are infinitely many such graphs G(k,m) for any  $k \geq 3$  with a non-feasible set which is not switching-equivalent to  $\emptyset$  or E(G(k,m)). This completes the proof of the theorem.

**Remark.** In the above construction, the complete bipartite graph  $K_{k,k}$  could be replaced by any k-regular bipartite graph G with a Hamilton cycle C. For a k-edge-coloring of G, choose two edges with the same color but not from the cycle C to be deleted. Let G' be the resulting bipartite graph and then take m copies of G'. Then the construction generates infinitely many other examples.

The graph G(k,m) from the above construction is a matching-covered graph with an equivalent class of size 2m. So the equivalent class of a matching-covered graph could goes to arbitrarily large. However, the edge-connectivity of G(k,m) is 2. We do not know whether there are highly connected matching-covered graphs with a large equivalent class. Theorem 1.5 shows that bricks do not have a large equivalent class. In the next section, we show that the edge-connectivity of a matching-covered bipartite graph G is 2 if it has an equivalent class.

## 3 Matchable bipartite graphs

Let G(A, B) be a matchable bipartite graph with bipartition (A, B), and let M be a perfect matching of G(A, B). A cycle C of G(A, B) is M-alternating if  $E(C) \cap M$  is a perfect matching of C. Similarly, a path P of G(A, B) is M-alternating if  $E(P) \cap M$  is a perfect matching of P. Hall's Theorem provides a characterization of matchable graph, which says that a bipartite graph G(A, B) is matchable if and only if |A| = |B| and for any  $U \subseteq A$ ,  $|N(U)| \ge |U|$ . The following is a similar result for matching-covered bipartite graph.

**Lemma 3.1** (Theorem 4.1.1 in [11]). Let G(A, B) be a bipartite graph. Then G(A, B) is matching-covered if and only if |A| = |B| and for any proper subset  $U \neq \emptyset$  of A,  $|N(U)| \geq |U| + 1$ .

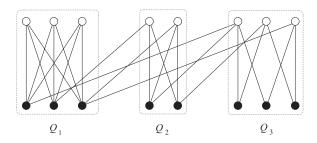
Let G(A, B) be a matching-covered graph. For any two vertex  $x \in A$  and  $y \in B$  such that  $xy \notin E(G(A, B))$ ,  $G \cup \{xy\}$  is matching-covered by Lemma 3.1. Hence,  $G \cup \{xy\}$  has a perfect matching M containing xy, and another perfect matching M' containing an edge of G incident with x. Therefore, the symmetric difference  $M \oplus M'$  has a cycle C containing xy. Further, G has an M'-alternating path joining xy, which is  $C \setminus \{xy\}$ . So the following lemma holds.

**Lemma 3.2.** Let G(A, B) be a matching-covered bipartite graph. Then for any vertex  $x \in A$  and  $y \in B$ , there is an M-alternating joining x and y for some perfect matching M.

For matchable bipartite graphs, the Dulmage-Mendelsohn Decomposition [4] provides a structure characterization as follows.

**Lemma 3.3** (Dulmage and Mendelsohn, [4]). Let G(A, B) be a matchable bipartite graph. Then G(A, B) has a decomposition into disjoint matching-covered subgraphs  $Q_1, ..., Q_k$  such that:

- (1) every  $Q_i$  is vertex induced and,
- (2) for any  $e \in E[Q_i, Q_j]$  with  $i, j \in \{1, 2, ..., k\}$ , e is not contained by any perfect matching of G.



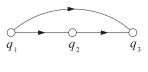


Figure 3: The Dulmage-Mendelsohn Decomposition of G(A, B) and the Dulmage-Mendelsohn digraph D (right).

For a matchable bipartite graph, the Dulmage-Mendelsohn Decomposition is unique. Let G(A, B) be a matchable bipartite graph and let  $\mathcal{G} = \{Q_1, ..., Q_k\}$  be the Dulmage-Mendelsohn Decomposition. For any  $1 \leq i \leq k$ , identify all vertices in  $A \cap Q_i$  to a vertex  $u_i$  and all vertices in  $B \cap Q_i$  to a vertex  $v_i$  and delete all multiple-edges to get a simple bipartite graph. For an edge  $u_i v_j$ , orient it from  $u_i$  to  $v_j$  if i = j and from  $v_j$  to  $u_i$  if  $i \neq j$ . Since the Dulmage-Mendelsohn Decomposition is unique, the digraph generated this way is unique and is denoted by D'. The Dulmage-Mendelsohn digraph D is obtained from

D' by contracting all arcs  $u_iv_i$  to a single vertex  $q_i$  for all  $1 \leq i \leq k$ . (For example, see Figure 3.) So if G(A, B) is matching-covered, then  $\mathcal{G}$  has only one graph and hence D has one vertex but no arcs. The following is a property of the Dulmage-Mendelsohn digraph D of a matchable bipartite graph G(A, B).

**Lemma 3.4.** Let G(A, B) be a connected matchable bipartite graph. If G(A, B) is not matching-covered, then the Dulmage-Mendelsohn digraph D of G(A, B) is acyclic.

Proof. Let G(A, B) be a matchable bipartite graph and let  $\mathcal{G} = \{Q_1, ..., Q_k\}$  be the Dulmage-Mendelsohn Decomposition. Since G(A, B) is not matching-covered, then  $k \geq 2$ . Let D be the Dulmage-Mendelsohn digraph. Since G(A, B) is connected, D has at least one arc. Suppose to the contrary that D has a directed cycle C. Without loss of generality, assume that  $C = q_1q_2 \cdots q_mq_1$  for some  $2 \leq m \leq k$  (relabeling if necessary).

By the definition of D, for each arc  $q_iq_{i+1}$  where i and i+1 are taken modulo m, G(A,B) has an edge joining a vertex  $u_{i+1} \in Q_{i+1} \cap A$  and a vertex  $v_i \in Q_i \cap B$  which is not contained by any perfect matching of G(A,B) by (2) in Lemma 3.3. In each  $Q_i$  with  $1 \le i \le m$ , there exists an  $M_i$ -alternating path  $P_i$  joining  $u_i$  and  $v_i$  for some perfect matching  $M_i$  of  $Q_i$  by Lemma 3.2. For  $m+1 \le i \le k$ , let  $M_i$  be a perfect matching of  $Q_i$  which is matching-covered. Let  $M = \bigcup_{i=1}^k M_i$  and

$$C' := (\bigcup_{i=1}^{m} P_i) \cup \{v_i u_{i+1} | i, i+1 \in \{1, ..., m\} \pmod{m}\}.$$

Then M is a perfect matching of G and G' is an M-alternating cycle of G. So the symmetric difference  $M \oplus E(G')$  is another perfect matching containing edges  $v_i u_{i+1}$ , which contradicts that  $v_i u_{i+1}$  is not contained in any perfect matching of G(A, B). This completes the proof.

By Lemma 3.4 and the definition of the Dulmage-Mendelsohn digraph, if D has an arc  $q_iq_j$ , then all edges of  $E(Q_i,Q_j)$  join vertices of  $Q_i\cap B$  and the vertices of  $Q_j\cap A$ . In other words,  $E[Q_i\cap A,Q_j\cap B]=\emptyset$ . On the other hand, if  $E[Q_i\cap B,Q_j\cap A]\neq\emptyset$ , then  $q_iq_j$  is an arc of D.

Let G(A, B) be a matchable bipartite graph, but not matching-covered. Then, by Lemma 3.4, the Dulmage-Mendelsohn digraph D of G(A, B) is acyclic. A directed cut S of D is a subset of arcs of D which separates D into two components and all arcs of S are oriented from the one component to the other. A family of directed paths  $\mathcal{P}$  intersects all directed cuts of D if for any directed cut S of D, there exists a path  $P \in \mathcal{P}$  such that  $E(P) \cap S \neq \emptyset$ . The following result shows how many new edges should be added to a non-matching-covered bipartite graph to obtain a matching-covered bipartite graph.

**Theorem 3.5.** Let G(A, B) be a matchable bipartite graph and let G'(A, B) be a smallest matching-covered bipartite graph such that  $G(A, B) \subseteq G'(A, B)$ . Then

$$|E(G'(A,B))| \le |E(G(A,B))| + \ell,$$

where  $\ell$  is the smallest size of a family of directed paths intersecting all directed cuts of the Dulmage-Mendelsohn digraph D of G(A,B).

Proof. Let G(A, B) be a matchable bipartite graph and let D be the Dulmage-Mendelsohn digraph. If G(A, B) is a matching-covered graph, then D is a single vertex and  $P = \emptyset$ . The theorem holds trivially. So in the following, assume that G(A, B) is not matching-covered. Therefore, the Dulmage-Mendelsohn Decomposition  $\mathcal{G} = \{Q_1, ..., Q_k\}$  of G(A, B) has at least two graphs, i.e.,  $k \geq 2$ . By Lemma 3.4, D is acyclic. Let P be a family of directed paths intersecting all directed cuts of D such that  $|P| = \ell$ .

For any  $P \in \mathcal{P}$ , add an arc  $e_P$  from the terminal vertex of P to the initial vertex of P, and let the new digraph be D'. Since  $\mathcal{P}$  intersects all directed cuts of D, D' has no directed cut and hence is strongly-connected. Hence, for any arc e of D, D' has a directed cycle containing e.

For each new arc  $e_P = x_i x_j$ , then add a new edge to G joining a vertex  $v_i \in B \cap Q_i$  and a vertex  $u_j \in A \cap Q_j$ . Let the new bipartite graph be G'(A, B). Let e be an edge of G'(A, B). If e is an edge of some  $Q_i$ , then e is contained in a perfect matching of G(A, B) which is also a perfect matching of G'(A, B). If e is an edge of  $E[Q_i, Q_j]$ , the digraph D' has a directed cycle C containing the arc  $q_i q_j$  or  $q_j q_i$ . By a similar argument as in Lemma 3.4, the directed cycle C of D' corresponds to an M-alternating cycle in G'(A, B) for some perfect matching M of G'(A, B). Therefore, e is contained in a perfect matching of G'(A, B). So G'(A, B) is matching-covered. Hence, the number of edges of a smallest matching-covered graph containing G(A, B) is at most  $|E(G'(A, B))| = |E(G(A, B))| + |\mathcal{P}| = |E(G(A, B))| + \ell$ .

Now, we are going to prove our main results, Theorems 1.4 and 1.6.

## **Proof of Theorem 1.4.** Let G(A, B) be a matching-covered bipartite graph.

First, assume that G(A, B) is strongly coverable. Let S be an edge-cut of G(A, B), which separates G(A, B) into two balanced components  $G_1(A_1, B_1)$  and  $G_2(A_2, B_2)$ . Then  $S = E[A_1, B_2] \cup E[A_2, B_1]$ . We need to show that  $|E[A_1, B_2]| \geq 2$  and  $|E[B_1, A_2]| \geq 2$ . If not, we may assume that  $|E[A_1, B_2]| \leq 1$  by symmetry. Let  $e \in E[A_1, B_2]$ . Then  $G(A, B) \setminus e$  has no edges joining vertices of  $A_1$  to vertices  $B_2$ . Since  $G_1(A_1, B_1)$  is balanced, any perfect matching of  $G(A, B) \setminus e$  does not contain edges from  $E[B_1, A_2]$ . Therefore,  $G(A, B) \setminus e$  is not matching-covered. Hence G(A, B) is not strongly coverable, a contradiction to the assumption that G(A, B) is strongly coverable.

In the following, assume that every edge-cut S separating G(A, B) into two balanced components  $G_1(A_1, B_1)$  and  $G_2(A_2, B_2)$  satisfies  $|E[A_1, B_2]| \geq 2$  and  $|E[A_2, B_1]| \geq 2$ . We need to show that G(A, B) is strongly coverable. In other words, for any edge e,  $G(A, B) \setminus e$  is matching-covered. If not, then G(A, B) has an edge e such that  $G(A, B) \setminus e$  is not matching-covered. Let  $\mathcal{G} = \{Q_1, Q_2, ..., Q_k\}$  be the Dulmage-Mendelsohn Decomposition of  $G(A, B) \setminus e$ , and let D be the Dulmage-Mendelsohn digraph. By Lemma 3.4, D is a cyclic. By Theorem 3.5, adding one more arc to D generates a strongly connected digraph D'. Therefore, D has only exactly one sink and one source. Without loss of generality, assume  $q_1$  and  $q_k$  be the source and sink of D, respectively, where  $q_1$  and  $q_k$  correspond to the graphs  $Q_1$  and  $Q_k$ . By the definition of D, all edges of  $G(A, B) \setminus e$  joining vertices of  $Q_1$  to vertices  $Q_i$  with  $i \neq 1$  are incident with vertices in  $Q_1 \cap B$ . So the edge e joins a vertex in  $Q_1 \cap A$  and a vertex in  $Q_k \cap B$ . Let  $S = \nabla V(Q_1)$ , the set of all edges joining vertices of  $Q_1$  and vertices of its component in G(A, B). Then S is an edge-cut separating G(A, B) into  $G_1(A_1, B_1) = Q_1$  and  $G_2(A_2, B_2) = G(A, B) \setminus Q_1$ , where  $A_1 = V(Q_1) \cap A$  and  $B_1 = V(Q_1) \cap B$ . Note that both  $G_1(A_1, B_1)$  and  $G_2(A_2, B_2)$  are matchable and therefore balanced. However,  $|E(A_1, B_2)| = |\{e\}| = 1$ , a contradiction to the assumption. This completes the proof.

#### **Proof of Theorem 1.6.** Let G(A, B) be a matching-covered bipartite graph.

First, assume that G(A, B) has a 2-edge-cut S which separates G(A, B) into two balanced components  $G_1(A_1, B_1)$  and  $G_2(A_2, B_2)$ . Then  $G_1(A_1, B_1)$  has an even number of vertices because  $|A_1| = |B_1|$ . Therefore, every perfect matching M of G(A, B) has an even number of edges of S. Hence,  $|S \cap M| = 0$  or  $|S \cap M| = 2$ . In other words,  $S \cap M = \emptyset$  or  $S \subseteq M$ . So S is an equivalent class of G(A, B).

In the following, assume that G(A, B) has an equivalent class K. Let  $e, e' \in K$ . It suffices to show that e is contained by a 2-edge-cut S which separates G(A, B) into two balanced components. Since e is

equivalent to e', it follows that  $G(A, B) \setminus e$  has no perfect matching containing e'. Let  $\mathcal{G} = \{Q_1, ..., Q_k\}$  be the Dulmage-Mendelsohn Decomposition of  $G(A, B) \setminus e$  and let D be the Dulmage-Mendelsohn digraph. By Lemma 3.3, every  $Q_i$  is matching-covered and e' joins two vertices from different components, say  $Q_i$  and  $Q_j$  with  $i \neq j$ . Without loss of generality, assume that  $q_iq_j$  is an arc of D where  $q_i$  and  $q_j$  correspond to  $Q_i$  and  $Q_j$ . By the definition of the Dulmage-Mendelsohn digraph, e' joins a vertex of  $Q_i \cap B$  and a vertex of  $Q_j \cap A$ .

Claim: The arc  $q_iq_j$  is a cut-edge of D.

Proof of Claim. If not, let T be a directed cut containing  $q_iq_j$ . Then T contains another arc, say e''. By Lemma 3.4, D is acyclic. Since G(A, B) is matching-covered, by Theorem 3.5, D has one directed path P intersecting all directed cuts. So D has exactly one source and one sink, say  $q_1$  and  $q_k$  respectively, where  $q_1$  and  $q_k$  correspond to  $Q_1$  and  $Q_k$ . By Theorem 3.5, adding an arc from  $q_k$  to  $q_1$  generates a strongly connected digraph D'. So there is a directed cycle C containing e''. Note that C contains exactly one arc in T. It follows that C is still a directed cycle of  $D' \setminus q_i q_j$ . The directed cycle C corresponds to an M-alternating cycle of  $G(A, B) \setminus e'$  containing the edge e for some perfect matching M of G(A, B). Therefore, G(A, B) has a perfect matching containing e but not e', contradicting that e and e' are equivalent to each other. This completes the proof of Claim.

By Claim,  $E(Q_i, Q_j) \cup \{e\}$  is an edge-cut of G(A, B). All edges in  $E(Q_i, Q_j) \setminus \{e\}$  join a vertex of  $Q_i \cap B$  and a vertex of  $Q_j \cap A$ . If  $E(Q_i, Q_j)$  contains an edge f other than e and e', then  $G(A, B) \setminus e'$  is matching-covered because the Dulmage-Mendelsohn digraph of  $G(A, B) \setminus \{e, e'\}$  is the same as D. Therefore, adding the edge e makes  $G(A, B) \setminus \{e'\}$  matching-covered. So G(A, B) has a perfect matching containing e but not e', contradicting  $e, e' \in K$  again. The contradiction implies that  $\{e, e'\}$  is a 2-edge-cut, which separates G(A, B) into two components such that, for any  $Q_m$  with  $1 \leq m \leq k$ , a component of  $G(A, B) \setminus \{e, e'\}$  either contains  $Q_m$  or does not intersect  $Q_m$ . Hence, every component of  $G(A, B) \setminus \{e, e'\}$  is balanced. This completes the proof.

**Remark.** In [2], Carvalho et. al. proved that two equivalent edges e and e' of a matching-covered bipartite graph form an edge cut. Theorem 1.6 can be proved by the result of Carvalho et. al. easily. The proofs of Theorems 1.4 and 1.6 in this paper are based on the Dulmage-Mendelsohn Decomposition which provides insight into the structure of matchable bipartite graphs.

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