New expressions for order polynomials and chromatic polynomials

Fengming Dong*
Mathematics and Mathematics Education
National Institute of Education
Nanyang Technological University, Singapore 637616

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Abstract

Let $G = (V, E)$ be a simple graph with $V = \{1, 2, \cdots, n\}$ and $\chi(G, x)$ be its chromatic polynomial. For an ordering $\pi = (v_1, v_2, \cdots, v_n)$ of elements of V, let $\delta_G(\pi)$ be the number of i's, where $1 \le i \le n-1$, with either $v_i < v_{i+1}$ or $v_i v_{i+1} \in E$. Let $\mathcal{W}(G)$ be the set of subsets $\{a, b, c\}$ of V, where $a < b < c$, which induces a subgraph with ac as its only edge. We show that $\mathcal{W}(G) = \emptyset$ if and only if $(-1)^n \chi(G, -x) = \sum_\pi \binom{x+\delta_G(\pi)}{n}$, where the sum runs over all $n!$ orderings π of V. To prove this result, we establish an analogous result on order polynomials of posets and apply Stanley's work on the relation between chromatic polynomials and order polynomials.

Keywords: graph, order polynomial, chromatic polynomial

1 Introduction

1.1 Chromatic polynomials

For a simple graph $G = (V, E)$, the *chromatic polynomial* of G is defined to be the polynomial $\chi(G, x)$ such that $\chi(G, k)$ counts the number of proper k-colourings of G for any positive integer k (for example, see [1, 2, 3, 7, 8, 16]). This concept was first introduced by Birkhoff [1] in 1912 in the hope of proving the four-color theorem (i.e., $\chi(G, 4) > 0$ holds for any loopless planar graph G). The study of chromatic polynomials is one of the most active areas in graph theory and many celebrated results on this topic have been obtained (for example, see [2, 4, 5, 6, 9, 10, 11, 12, 13, 14, 15, 17]).

One of the main purposes of this paper is to prove a new identity for $\chi(G, x)$ when G satisfies a certain condition. Assume that $V = [n]$, where $[n] = \{1, 2, \cdots, n\}$. For $u, v \in V$,

*Email: fengming.dong@nie.edu.sg.

define

$$\delta_G(u,v) = \begin{cases} 1, & u < v \text{ or } uv \in E; \\ 0, & \text{otherwise.} \end{cases} \tag{1.1}$$

Let $\mathcal{P}(V)$ denote the set of orderings of elements of V. Obviously, $|\mathcal{P}(V)| = n!$. Define

$$\Psi(G,x) = \sum_{\pi \in \mathcal{P}(V)} \binom{x + \delta_G(\pi)}{n}, \tag{1.2}$$

where for any $\pi = (u_1, u_2, \cdots, u_n) \in \mathcal{P}(V)$,

$$\delta_G(\pi) = \sum_{1 \le i \le n-1} \delta_G(u_i, u_{i+1}). \tag{1.3}$$

Clearly the function $\Psi(G,x)$ depends on the structure of G and also on the labeling of its vertices. For a bijection $\omega : V \to [n]$, let G_ω denote the graph obtained from G by relabeling each vertex v in G by $\omega(v)$. Thus $G_\omega \cong G$ but it may be not true that $\Psi(G_\omega, x) = \Psi(G,x)$. Hence, in this article, isomorphic graphs with different vertex labellings are considered to be different.

For a graph $G = (V,E)$, where $V = [n]$, let $\mathcal{W}(G)$ be the set of 3-element subsets $\{a,b,c\}$ of V with $a < b < c$ such that ac is the only edge in the subgraph of G induced by $\{a,b,c\}$. Note that $\mathcal{W}(G)$ may be different from $\mathcal{W}(G_\omega)$ for a bijection $\omega : V \to [n]$.

In Section 4, we will prove the following result on $\chi(G,x)$.

Theorem 1.1 *Let $G = (V,E)$ be a simple graph with $V = [n]$. Then*

$$(-1)^n \chi(G,-x) = \Psi(G,x) = \sum_{\pi \in \mathcal{P}(V)} \binom{x + \delta_G(\pi)}{n} \tag{1.4}$$

if and only if $\mathcal{W}(G) = \emptyset$.

To prove Theorem 1.1, we will first establish an analogous result on the order polynomial of $\bar{D}$ (i.e., Theorem 1.4), where D is an acyclic digraph and $\bar{D}$ is the poset which is the reflexive transitive closure of D, and apply Stanley's work on the relation between chromatic polynomials and order polynomials.

1.2 Order polynomials and strict order polynomials

In 1970, Stanley [13] introduced the order polynomial and the strict order polynomial of a poset (i.e. partially ordered set). Let P be a poset on n elements with a binary relation $\preceq$. For $u, v \in P$, let $u \prec v$ mean that $u \preceq v$ but $u \neq v$. A mapping $\sigma : P \to [m]$ is said to be *order-preserving* (resp., *strictly order-preserving*) if $u \preceq v$ implies that $\sigma(u) \le \sigma(v)$ (resp.,

$u \prec v$ implies that $\sigma(u) < \sigma(v)$). Let $\Omega(P, x)$ (resp., $\bar{\Omega}(P, x)$) be the function which counts the number of order-preserving (resp., strictly order-preserving) mappings $\sigma : P \to [m]$ whenever $x = m$ is a positive integer. Both $\Omega(P, x)$ and $\bar{\Omega}(P, x)$ are polynomials in x of degree n (see Theorem 1 in [13]) and are respectively called the *order polynomial* and the *strict order polynomial* of P.

An ordering $\pi = (v_1, v_2, \cdots, v_n)$ of the elements of P is said to be *P-respecting* if $v_i \prec v_j$ always implies that $i < j$ (i.e., v_i appears before v_j in π). Let $\mathcal{OP}(P)$ be the set of P-respecting orderings π of the elements of P.

Let ω be a fixed surjective order-preserving mapping $\omega : P \to [n]$. For a P-respecting ordering $\pi = (v_1, v_2, \cdots, v_n)$, a "decent" (resp. "accent") means $\omega(v_i) > \omega(v_{i+1})$ (resp. $\omega(v_i) < \omega(v_{i+1})$) for some i with $1 \le i \le n-1$. Let $\kappa_P(\pi)$ (resp., $\bar{\kappa}_P(\pi)$) denote the number of times when a "decent" (resp. an "accent") occurs in π. Clearly, $0 \le \bar{\kappa}_P(\pi), \kappa_P(\pi) \le n-1$ and $\bar{\kappa}_P(\pi) + \kappa_P(\pi) = n-1$ for each $\pi \in \mathcal{OP}(P)$. For an integer s with $0 \le s \le n-1$, let $w_s(P)$ (resp., $\bar{w}_s(P)$) be the number of $\pi \in \mathcal{OP}(P)$ with $\kappa_P(\pi) = s$ (resp., $\bar{\kappa}_P(\pi) = s$).

Stanley's Theorem 2 in [13] gives the following interpretations for $\Omega(P, m)$ and $\bar{\Omega}(P, m)$.

Theorem 1.2 (Stanley [13]) *For any integer $m \ge 1$,*

$$\Omega(P, m) = \sum_{s=0}^{n-1} w_s(P) \binom{m+n-1-s}{n} \text{ and } \bar{\Omega}(P, m) = \sum_{s=0}^{n-1} \bar{w}_s(P) \binom{m+n-1-s}{n}. \tag{1.5}$$

As $\kappa_P(\pi) + \bar{\kappa}_P(\pi) = n-1$ for each $\pi \in \mathcal{OP}(P)$, by applying Theorem 1.2, it is not difficult to deduce that

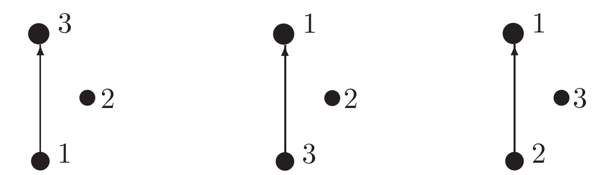
$$\Omega(P, m) = \sum_{\pi \in \mathcal{OP}(P)} \binom{m + \bar{\kappa}_P(\pi)}{n}. \tag{1.6}$$

By Theorem 1.2, a relation between $\Omega(P, m)$ and $\bar{\Omega}(P, m)$ can also be deduced easily and it appeared in Stanley's Theorem 3 in [13]: for any $m \in \mathbb{Z}^+$,

$$\bar{\Omega}(P, m) = (-1)^n \Omega(P, -m). \tag{1.7}$$

From now on we focus on the order polynomial of a poset that is reflexive transitive closure of an acyclic digraph.

A digraph $D = (V, A)$ is called *acyclic* if it does not contain any directed cycle. Let D be an acyclic digraph with $|V| = n$. For convenience of notation, we simply assume that $V = [n]$. An ordering $\pi = (u_1, u_2, \cdots, u_n)$ of elements of V is said to be *D-respecting* if $(u_i, u_j) \in A$ implies that $i < j$ holds (i.e., u_i appears before u_j in π). Let $\mathcal{OP}(D)$ be the



(a) D_1 (b) D_2 (c) D_3

Figure 1: Isomorphic digraphs D_1, D_2 and D_3

$\mathcal{OP}(D_1)$	$\delta_{D_1}(\pi_i)$	$\mathcal{OP}(D_2)$	$\delta_{D_2}(\pi'_i)$	$\mathcal{OP}(D_3)$	$\delta_{D_3}(\pi''_i)$
$\pi_1 = (2,1,3)$	1	$\pi'_1 = (2,3,1)$	2	$\pi''_1 = (3,2,1)$	1
$\pi_2 = (1,2,3)$	2	$\pi'_2 = (3,2,1)$	0	$\pi''_1 = (2,3,1)$	1
$\pi_3 = (1,3,2)$	1	$\pi'_3 = (3,1,2)$	2	$\pi''_1 = (2,1,3)$	2

Table 1: Members of $\mathcal{OP}(D_i)$ and values $\delta_{D_i}(\pi)$ for $\pi \in \mathcal{OP}(D_i)$

set of D-respecting orderings of elements of V. For example, for the digraphs in Figure 1, $\mathcal{OP}(D_i)$ has exactly three members given in Table 1 for $i = 1, 2, 3$.

Clearly, an ordering π of elements of V is D-respecting if and only if it is $\bar{D}$-respecting. Thus $\mathcal{OP}(D) = \mathcal{OP}(\bar{D})$.

For $a, b \in \mathbb{Z}^+$, let $\bar{\kappa}(a,b) = 1$ if $a < b$, and $\bar{\kappa}(a,b) = 0$ otherwise. For an ordering $\pi = (a_1, a_1, \cdots, a_n)$ of n different numbers in $\mathbb{Z}^+$, let

$$\bar{\kappa}(\pi) = \sum_{i=1}^{n-1} \bar{\kappa}(a_i, a_{i+1}).$$

Thus $\bar{\kappa}(\pi)$ is actually the number of times when an "accent" occurs in the ordering π. Note that the definition of $\bar{\kappa}(\pi)$ is only related to the numbers in the ordering π and has no relation with D.

Let $\mathcal{R}e(D) = \{(a,b) \in A : a > b\}$. Assume that $\mathcal{R}e(D) = \emptyset$. As $V = [n]$, this assumption is equivalent to a surjective mapping $\omega : V \to [n]$ with the property that $(u,v) \in A$ implies $\omega(u) < \omega(v)$. Observe that for any $\pi \in \mathcal{OP}(D)$, $\bar{\kappa}(\pi) = \bar{\kappa}_{\bar{D}}(\pi)$ holds. Thus, by (1.6), $\Omega(\bar{D}, m)$ has the following expression in terms of $\bar{\kappa}(\pi)$ under the assumption that $\mathcal{R}e(D) = \emptyset$:

$$\Omega(\bar{D}, m) = \sum_{\pi \in \mathcal{OP}(D)} \binom{m + \bar{\kappa}(\pi)}{n}. \tag{1.8}$$

Note that if $\mathcal{R}e(D) \neq \emptyset$, (1.8) may be not true, unless $\bar{\kappa}(\pi)$ is replaced by another suitable function. In the following, we remove the assumption that $\mathcal{R}e(D) = \emptyset$ and replace $\bar{\kappa}(\pi)$ by a new function $\delta_D(\pi)$. We will see for which labellings of vertices of D an identity analogous to (1.8) holds even if $\mathcal{R}e(D) \neq \emptyset$.

1.3 A new function $\Psi(D,x)$ for an acyclic digraph D

Let $D=(V,A)$ be an acyclic digraph with $V=[n]$. For $a,b\in V$, define

$$\delta_D(a,b)=\begin{cases} 1, & \text{either } a<b \text{ or } (a,b)\in A; \\ 0, & \text{otherwise.} \end{cases} \tag{1.9}$$

Clearly $\kappa(a,b)\le\delta_D(a,b)$ for every pair of members a and b of V. When $\mathcal{R}e(D)=\emptyset$, $(a,b)\in A$ implies that $a<b$. Thus, in this case, $\delta_D(a,b)=\bar{\kappa}(a,b)$ holds for every pair of numbers a and b in V, no matter whether $(a,b)\in A$ or not. However, when $\mathcal{R}e(D)\neq\emptyset$, for each $(a,b)\in A$ with $a>b$, we have $\delta_D(a,b)=1$ and $\bar{\kappa}(a,b)=0$.

Let $\Psi(D,x)$ be the function defined below:

$$\Psi(D,x)=\sum_{\pi\in\mathcal{OP}(D)}\binom{x+\delta_D(\pi)}{n}, \tag{1.10}$$

where for any $\pi=(a_1,a_2,\cdots,a_n)\in\mathcal{OP}(D)$,

$$\delta_D(\pi)=\sum_{i=1}^{n-1}\delta_D(a_i,a_{i+1}). \tag{1.11}$$

Note that $\Psi(D,x)$ is a function defined on an acyclic digraph $D=(V,A)$ with V a linearly ordered set of n vertices and its definition does not rely on a fixed mapping $\omega:V\to[n]$ with the property that $(v_i,v_j)\in A$ implies $\omega(v_i)<\omega(v_j)$.

Clearly, if $\mathcal{R}e(D)=\emptyset$, then $\delta_D(\pi)=\bar{\kappa}(\pi)$ holds for every $\pi\in\mathcal{OP}(D)$, and thus (1.8) and (1.10) imply the following conclusion.

Proposition 1.1 *Let $D=([n],A)$ be an acyclic digraph. If $\mathcal{R}e(D)=\emptyset$, then*

$$\Omega(\bar{D},x)=\Psi(D,x)=\sum_{\pi\in\mathcal{OP}(D)}\binom{x+\delta_D(\pi)}{n}. \tag{1.12}$$

If $\mathcal{R}e(D)\neq\emptyset$, it is possible that $\delta_D(\pi)\neq\bar{\kappa}(\pi)$ for some $\pi\in\mathcal{OP}(D)$, and thus it is possible that $\Omega(\bar{D},x)\neq\Psi(D,x)$. For example, for the isomorphic digraphs D_1,D_2 and D_3 in Figure 1, by the data in Table 1,we have

$$\Psi(D_1,x)=\Psi(D_3,x)=\binom{x+2}{3}+2\binom{x+1}{3}\neq\Psi(D_2,x)=2\binom{x+2}{3}+\binom{x}{3}. \tag{1.13}$$

As $\mathcal{R}e(D_1)=\emptyset$, by Proposition 1.1, we have $\Psi(D_3,x)=\Psi(D_1,x)=\Omega(\bar{D}_1,x)=\Omega(\bar{D}_3,x)$. But $\Psi(D_2,x)\neq\Psi(D_1,x)=\Omega(\bar{D}_1,x)=\Omega(\bar{D}_2,x)$.

Notice that $\mathcal{R}e(D_3)\neq\emptyset$, although $\Psi(D_3,x)=\Omega(\bar{D}_3,x)$. Thus, $\Psi(D,x)=\Omega(\bar{D},x)$ does not imply $\mathcal{R}e(D)=\emptyset$. The main aim of this article is to determine exactly when the identity $\Omega(\bar{D},x)=\Psi(D,x)$ holds.

Let $D = (V, A)$ be an acyclic digraph, where $V = [n]$. For distinct $a, b \in V$, write $a \prec_D b$ if there exists a directed path in D connecting from a to b, and $a \not\prec_D b$ otherwise. Write $a \not\approx_D b$ if $a \not\prec_D b$ and $b \not\prec_D a$. Let $\mathcal{W}(D)$ be the set of 3-element subsets $\{a, b, c\}$ of V with $a < b < c$ such that $(c, a) \in A$ but $a \not\approx_D b$ and $c \not\approx_D b$. Observe that if $(c, a) \in A$, then $b \prec_D c$ implies that $b \prec_D a$, and $a \prec_D b$ implies that $c \prec_D b$. Thus, for $\{a, b, c\} \subseteq V$ with $a < b < c$ and $(c, a) \in A$, $\{a, b, c\} \in \mathcal{W}(D)$ if and only if $c \not\prec_D b$ and $b \not\prec_D a$.

For example, for the digraphs D_1, D_2 and D_3 in Figure 1, only $\mathcal{W}(D_2)$ is not empty, and for the digraph D in Figure 2 on Page 8, $\mathcal{W}(D)$ has exactly one member $\{2, 3, 5\}$.

Clearly, $\mathcal{R}e(D) = \emptyset$ implies that $\mathcal{W}(D) = \emptyset$. But the converse does not hold.In Section 2, we will show that if $\mathcal{W}(D) = \emptyset$,then there exists D' obtained from D by relabeling vertices in D such that $\mathcal{R}e(D') = \emptyset$ and $\Psi(D, x) = \Psi(D', x)$. By Proposition 1.1, we have $\Psi(D', x) = \Omega(\bar{D}', x) = \Omega(\bar{D}, x)$. Thus we establish the following result.

Theorem 1.3 *Let $D = ([n], A)$ be an acyclic graph and $\mathcal{W}(D)$ be defined as above. If $\mathcal{W}(D) = \emptyset$, then $\Psi(D, x) = \Omega(\bar{D}, x)$ holds.*

The converse of Theorem 1.3 also holds, as stated in the following result.

Theorem 1.4 *Let $D = ([n], E)$ be an acyclic graph, where $n \geq 3$. Then*

$$\Psi(D, x) - \Omega(\bar{D}, x) = \sum_{i=0}^{n-3} d_i \binom{x+i}{n-2}, \tag{1.14}$$

where $d_0, d_1, \cdots, d_{n-3}$ are non-negative integers. Furthermore, $d_i = 0$ for every $i = 0, 1, \cdots, n-3$ if and only if $\mathcal{W}(D) = \emptyset$.

Clearly, Theorem 1.4 implies that $\Psi(D, x) = \Omega(\bar{D}, x)$ if and only if $\mathcal{W}(D) = \emptyset$. To prove Theorem 1.4 in Section 3, we will first compare $\Psi(D, x)$ with $\Psi(D_{a \to r}, x)$, where $D_{a \to r}$ is the digraph obtained from D by relabeling vertex a by by a suitable number r. The new digraph $D_{a \to r}$ has the property that $\mathcal{W}(D_{a \to r}) = \mathcal{W}(D) - \{W \in \mathcal{W}(D) : a \in W\}$ and $\Psi(D, x) - \Psi(D_{a \to r}, x) = \sum\limits_{i=0}^{n-3} d_i \binom{x+i}{n-2}$, where $d_i \geq 0$ for all i, and $d_0 + \cdots + d_{n-3} = 0$ if and only if $\mathcal{W}(D) = \emptyset$.

While Theorem 1.3 is implied by Theorem 1.4, the derivation of Theorem 1.4 is independent of Theorem 1.3. For a special case, the numbers d_i in Theorem 1.4 are given an interpretation (see Proposition 5.5).

Let $\mathcal{AO}(G)$ be the set of acyclic orientations of G. The expression (1) in [12] gives a relation between $\chi(G,x)$ and $\bar{\Omega}(\bar{D},x)$:

$$\chi(G,x) = \sum_{D\in\mathcal{AO}(G)} \bar{\Omega}(\bar{D},x). \tag{1.15}$$

Thus, (1.6), (1.7) and (1.15) imply the following result.

Theorem 1.5 (Stanley [12]) *Let $G=(V,E)$ be a simple graph. Then*

$$(-1)^{|V|}\chi(G,-x) = \sum_{D\in\mathcal{AO}(G)} \Omega(\bar{D},x). \tag{1.16}$$

Note that for each $D\in\mathcal{AO}(G)$, determining $\Omega(\bar{D},x)$ by (1.8) is based on a relabeling of vertices such that $a<b$ holds for each arc (a,b) in D. Thus, the summation of (1.16) cannot be replaced by a summation over all $|V|!$ orderings of elements of V if the labeling of elements of V is fixed, although the union of $\mathcal{OP}(D)$'s for all $D\in\mathcal{AO}(G)$ is exactly the set of all $|V|!$ orderings of elements of V. This is another motivation for extending (1.8) to an analogous expression with an arbitrary relabeling of vertices in D and the result can be applied to express $\chi(G,x)$ as the summation over all $|V|!$ orderings of elements of V.

Applying Theorems 1.4 and 1.5, we can prove Theorem 1.1 in Section 4.

2 Proof of Theorem 1.3

Let $D=(V,A)$ be an acyclic digraph with vertex set V, where $V=[n]$. In this section, we shall show that $\Psi(D,x)=\Omega(\bar{D},x)$ whenever $\mathcal{W}(D)=\emptyset$.

For $S\subseteq V$, let $D[S]$ be the subdigraph of D induced by S. For $u\in V$, u is called a *sink* of D if $F_D(u)=\emptyset$, where $F_D(u)=\{v:(u,v)\in A\}$. We first define a bijection $L:V\to[n]$ by the following algorithm:

Algorithm A:

Step 1. Set $S:=V$;

Step 2. Let u be the largest number among all sinks of $D[S]$;

Step 3. Set $L(u):=|S|$ and $S:=S\setminus\{u\}$;

Step 4. If $S\neq\emptyset$, go to Step 2; otherwise, output $L(v)$ for all $v\in V$.

The bijection L defined above will be written as L_D when there is a possibility of confusion.

Example 2.1 *If D is the acyclic digraph in Figure 2, then*

$$L(3) = 5, L(2) = 4, L(5) = 3, L(4) = 2, L(1) = 1.$$

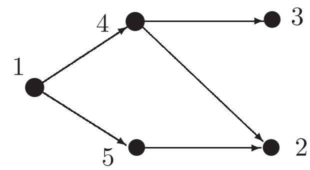


Figure 2: An acyclic digraph

Recall that for distinct $u, v \in V$, $u \prec_D v$ if D has a directed path from u to v; and for $u \in V$, $R_D(u)$ (or simply $R(u)$) denote the set $\{v \in V : u \prec_D v\}$. Let $R_D[u] = \{u\} \cup R_D(u)$. Then $u \in R_D[u]$ but $u \notin R_D(u)$.

By definitions of $\prec_D$ and L_D, we have the following basic properties of $\prec_D$ and L_D.

Proposition 2.1 *Let a, b and c be distinct vertices in D.*

(i) *If $a \prec_D b$ and $b \prec_D c$, then $a \prec_D c$.*

(ii) *If $a \prec_D b$, then $L(a) < L(b)$.*

For distinct vertices b, c in D, let $N_D[c, b] = \{c' \in R_D[c] \setminus R_D[b] : \forall y \in R_D(c) \cap R_D(b), L(c') < L(y)\}$.

Example 2.2 *For the digraph D in Figure 2, $N_D[5, 3] = \{5, 2\}$ and $N_D[5, 4] = \{5\}$.*

Proposition 2.2 *Let b and c be distinct vertices in D with $c \notin R_D(b)$. Then*

(i) *$c \in N_D[c, b]$;*

(ii) *when $R_D(c) \subseteq R_D(b)$, $N_D[c, b] = \{c\}$ holds.*

Proof. (i). Clearly $c \in R_D[c] \setminus R_D[b]$. As $R_D(c) \cap R_D(b) \subseteq R_D(c)$, we have $L(c) < L(y)$ for all $y \in R_D(c) \cap R_D(b)$ by Proposition 2.1 (ii), implying that $c \in N_D[c, b]$. Thus (i) holds.

(ii). By the result in (i), $c \in N_D[c, b]$. As $R_D(c) \subseteq R_D(b)$, $R_D[c] \setminus R_D[b] = \{c\}$. Thus (ii) holds. $\square$

For an non-empty finite set S of $\mathbb{Z}^+$, let $\min S$ and $\max S$ denote the minimum value and the maximum value of S respectively. In case of any confusion, $\min S$ and $\max S$ are respectively written as $\min(S)$ and $\max(S)$.

The bijection $L_D : V \to \{1, 2, \cdots, n\}$ has the following property.

Proposition 2.3 *Let a, b and c be distinct vertices in D.*

(i) *If $c \not\approx_D b$, then $L(c) < L(b)$ if and only if* $\min(N_D[c,b]) < \min(N_D[b,c])$*;*

(ii) *If $c \not\approx_D b$, $L(c) < L(b)$ and $b < c$, then there exist $a, c' \in R_D[c] \setminus R_D[b]$ such that $\{a, b, c'\} \in \mathcal{W}(D)$;*

(iii) *If $\mathcal{W}(D) = \emptyset$, $b < c$ and $c \not\to_D b$, then $L(b) < L(c)$.*

Proof. (i). Assume that $c \not\approx_D b$. It suffices to prove that if $\min(N_D[c,b]) < \min(N_D[b,c])$, then $L(c) < L(b)$, as exchanging b and c yields that if $\min(N_D[b,c]) < \min(N_D[c,b])$, then $L(b) < L(c)$.

By Proposition 2.2 (i), $c \in N_D[c,b]$ and $b \in N_D[b,c]$. Let $c_0 = \min(N_D[c,b])$. By Proposition 2.1 (ii), $L(c) \le L(c_0)$.

Let S' be the set of sinks of D and let $w = \max S'$. Then $L(w) = |V|$. Now we want to prove the two following claims under the assumption that $c_0 < \min(N_D[b,c])$.

Claim 1: $w \neq c_0$.

Assume that $w = c_0$. As $L(c_0) = |V|$, c_0 is the largest sink of D. Note that $S' \cap R_D[b] \neq \emptyset$. Let $b_0 = \max(S' \cap R_D[b])$. As $c_0 \in R_D[c] \setminus R_D[b]$, we have $b_0 \neq c_0$ and so $b_0 < c_0$ and $L(b_0) < L(c_0) = |V|$. As $b_0 < c_0 < \min(N_D[b,c])$ and $b_0 \in R_D[b]$, we have $b_0 \in R_D[b] \setminus N_D[b,c]$. By the assumption on $N_D[b,c]$, $b_0 \in R_D[b] \setminus N_D[b,c]$ implies that $b_0 \in R_D(c) \cap R_D(b)$ or $L(b_0) > L(y)$ for some $y \in R_D(c) \cap R_D(b)$. Thus $L(b_0) \ge L(y)$ for some $y \in R_D(c) \cap R_D(b)$. As $L(c_0) < L(y)$ for all $y \in R_D(c) \cap R_D(b)$, we have $L(c_0) < L(b_0)$, a contradiction.

Claim 2: $L(c_0) < L(b)$.

This claim is trivial when $|V| = 2$. Now assume $|V| \ge 3$ and that this claim fails. Thus $L(b) < L(c_0) \le |V|$.

By Claim 1, $w \neq c_0$. Then $L(c) \le L(c_0) < L(w) = |V|$. As Claim 1 holds for $D - w$, by induction, and

$$\min(N_{D-w}[c,b]) = \min(N_D[c,b]) = c_0 < \min(N_D[b,c]) = \min(N_{D-w}[b,c]),$$

we have $L_{D-w}(c_0) < L_{D-w}(b)$. Since $L_{D-w}(c_0) = L_D(c_0)$ and $L_D(b) = L_{D-w}(b)$, we have $L_D(c_0) < L_D(b)$, a contradiction. Thus Claim 2 holds.

As $L(c) \le L(c_0)$, Claim 2 implies $L(c) < L(b)$ under the condition that $\min(N_D(c,b)) < \min(N_D(b,c))$. Thus (i) holds.

(ii). Assume that $b \not\approx_D c$, $b < c$ and $L(c) < L(b)$. By (i), $\min(N_D[c,b]) < \min(N_D[b,c])$. Let $c_1 = \min(N_D[c,b])$. Then $c_1 < \min(N[b,c]) \le b < c$. As $c_1 \in N_D[c,b] \subseteq R_D[c]$, there is a path in D from c to c_1: $c \to a_1 \to \cdots \to a_k$, where $a_k = c_1$ and $a_i \to a_{i+1}$ is short for $(a_i, a_{i+1}) \in A$. As $a_k = c_1 < b < c$, there exists $i : 1 \le i \le k-1$ such that $a_i > b > a_{i+1}$. As $c_1 \in N_D[c,b] \subseteq R_D[c] \setminus R_D[b]$, we have $a_i, a_{i+1} \in R_D[c] \setminus R_D[b]$, implying that $b \not\approx_D a_i$ and $b \not\approx_D a_{i+1}$. Thus $\{a_{i+1}, b, a_i\} \in \mathcal{W}(D)$ and the result holds.

(iii). Assume that $\mathcal{W}(D) = \emptyset$, $b < c$ and $c \not\prec_D b$. If $b \prec_D c$, then Proposition 2.1 (ii) implies that $L(b) < L(c)$. Now assume that $b \not\prec_D c$. Thus $b \not\approx_D c$. As $\mathcal{W}(D) = \emptyset$ and $b < c$, by (ii), we have $L(b) < L(c)$ in this case. □

Let D_L be the digraph obtained from D by relabeling each vertex y in D as $L(y)$. Clearly, D_L is isomorphic to D and Proposition 2.1 (ii) implies that $\mathcal{R}e(D_L) = \emptyset$. By Proposition 1.1, $\Psi(D_L, x) = \Omega(\bar{D}_L, x) = \Omega(\bar{D}, x)$.

For $\pi = (a_1, a_2, \cdots, a_n) \in \mathcal{OP}(D)$, let $L(\pi) = (L(a_1), L(a_2), \cdots, L(a_n))$.

Proposition 2.4 *Let $\pi = (a_1, a_2, \cdots, a_n) \in \mathcal{OP}(D)$. If $\mathcal{W}(D) = \emptyset$, then*

(i) $\delta_D(a_i, a_{i+1}) = \delta_{D_L}(L(a_i), L(a_{i+1}))$ *holds for* $i = 1, 2, \cdots, n-1$; *and*

(ii) $\delta_D(\pi) = \delta_{D_L}(L(\pi))$ *holds.*

Proof. (i). As $\pi = (a_1, a_2, \cdots, a_n) \in \mathcal{OP}(D)$, we have $a_{i+1} \not\prec a_i$. Thus, either $a_i \prec_D a_{i+1}$ or $a_i \not\approx_D a_{i+1}$.

First consider the case that $a_i \prec_D a_{i+1}$. As $\pi = (a_1, a_2, \cdots, a_n) \in \mathcal{OP}(D)$, if $a_{j_1} \to a_{j_2} \to \cdots \to a_{j_k}$ is a path in D, then $j_1 < j_2 < \cdots < j_k$. Thus $a_i \prec_D a_{i+1}$ implies that $(a_i, a_{i+1}) \in A$, and so $\delta_D(a_i, a_{i+1}) = 1$. As $(a_i, a_{i+1}) \in A$, we have $(L(a_i), L(a_{i+1})) \in A(D_L)$ and so $\delta_{D_L}(L(a_i), L(a_{i+1})) = 1$.

Now assume that $a_i \not\approx_D a_{i+1}$. As $\mathcal{W}(D) = 0$, by Proposition 2.3 (iii), if $a_i < a_{i+1}$ then $L(a_i) < L(a_{i+1})$; if $a_{i+1} < a_i$ then $L(a_{i+1}) < L(a_i)$. As $a_i \not\approx_D a_{i+1}$, we have $(a_i, a_{i+1}) \notin A(D)$ and $(L(a_i), L(a_{i+1})) \notin A(D_L)$. By definition of $\delta_D(a_i, a_{i+1})$, $\delta_{D_L}(L(a_i), L(a_{i+1})) = \delta_D(a_i, a_{i+1})$ holds in this case.

Thus (i) holds. By the result in (i), (ii) follows directly from the definition of $\delta_D(\pi)$. □

Corollary 2.1 *If $\mathcal{W}(D) = \emptyset$, then $\Psi(D, x) = \Psi(D_L, x)$*

Proof. Note that $\pi \in \mathcal{OP}(D)$ if and only if $L(\pi) \in \mathcal{OP}(D_L)$. Thus

$$\mathcal{OP}(D_L) = \{L(\pi) : \pi \in \mathcal{OP}(D)\}.$$

By Proposition 2.4 (ii), $\delta_D(\pi) = \delta_{D_L}(L(\pi))$ holds for each $\pi \in \mathcal{OP}(D)$. By definition of $\Psi(D, x)$, $\Psi(D, x) = \Psi(D_L, x)$ holds. $\square$

Since $\mathcal{R}e(D_L) = \emptyset$, Proposition 1.1 implies that $\Psi(D_L, x) = \Omega(\bar{D}_L, x) = \Omega(\bar{D}, x)$. Thus Theorem 1.3 follows from Corollary 2.1.

3 Proof of Theorem 1.4

In this section, we assume that $D = (V, A)$ is an acyclic digraph with $V \subset \mathbb{Z}^+$ and $|V| = n$, where $n \geq 3$. For $a \in V$ and $r \in \mathbb{Z}^+ \setminus V$, let $D_{a \to r}$ be the digraph obtained from D by relabeling a by r. We will compare $\Psi(D, x)$ with $\Psi(D_{a\to r}, x)$ and apply the result on $\Psi(D, x) - \Psi(D_{a\to r}, x)$ to prove Theorem 1.4.

Clearly, if $V = [n]$, then $r \geq n + 1$ and the vertex set of $D_{a\to r}$ is $([n] \setminus \{a\}) \cup \{r\}$ which is no longer $[n]$. Thus, for the purpose of comparing $\Psi(D, x)$ with $\Psi(D_{a\to r}, x)$, in this section the vertex set V is allowed to be any subset of $\mathbb{Z}^+$ and it is possible that $V \neq [n]$.

Note that if $V = \{v_1, v_2, \cdots, v_n\}$ with $1 \leq v_1 < v_2 < \cdots < v_n$, then $\Psi(D, x) = \Psi(D', x)$ holds, where D' is obtained from D by relabeling each v_i by i. So the function $\Psi(D, x)$ is not affected even if $V \neq [n]$.

3.1 Relabel a vertex in D by a sufficiently large number

Define

$$\Delta(D, z) = \sum_{\pi \in \mathcal{OP}(D)} z^{\delta_D(\pi)}. \tag{3.17}$$

By definitions of $\Psi(D, x)$ and $\Delta(D, z)$, for any two acyclic digraphs D_1 and D_2 of the same order, $\Delta(D_1, z) = \Delta(D_2, z)$ if and only if $\Psi(D_1, x) = \Psi(D_2, x)$.

In this subsection, we always assume that a is a fixed vertex in D and m is a number in $\mathbb{Z}^+ \setminus V$ with $m > y$ for all $y \in V \setminus R_D[a]$. We compare $\Delta(D, z)$ with $\Delta(D_{a\to m}, z)$ under this assumption. This result will be applied in the next subsection for relabeling vertex a by a suitable number r so that D can be replaced by $D_{a\to r}$ for the purpose of proving Theorem 1.4.

3.1.1 A function $\Delta_{D,\pi_0}(z)$

Let $\pi_0 = (a_1, a_2, \cdots, a_{n-1})$ be a fixed member of $\mathcal{OP}(D - a)$, where $D - a$ is the digraph obtained from D by removing vertex a. Let $\mathcal{OP}(D, \pi_0)$ be the set of those members $\pi \in \mathcal{OP}(D)$ such that $\pi - a = \pi_0$, where $\pi - a$ is obtained from π by removing a. For example, if $\pi = (2, 1, 3, 4)$, then $\pi - 2 = (1, 3, 4)$. Observe that $(a_1, a_2, \cdots, a_{n-1}, a) \in \mathcal{OP}(D, \pi_0)$ if and only if a is a sink of D, and $(a_1, \cdots, a_i, a, a_{i+1}, \cdots, a_{n-1}) \in \mathcal{OP}(D, \pi_0)$ if and only if $(a_j, a) \notin A$ for all $j = i + 1, \cdots, n - 1$ and $(a, a_j) \notin A$ for all $j = 1, \cdots, i$.

A vertex u of D is called a *source* if $(v, u) \notin A$ for all $v \in V$. Throughout this section, let s and t be the two numbers defined below:

(i) let $s = 0$ if a is a source of D, and let $s = \max\{1 \leq k \leq n - 1 : (a_k, a) \in A\}$ otherwise;

(ii) let $t = n$ if a is a sink of D, and let $t = \min\{1 \leq k \leq n - 1 : (a, a_k) \in A\}$ otherwise.

If $s = 0$ or $t = n$, then clearly $s < t$. Otherwise, $(a_s, a) \in A$ and $(a, a_t) \in A$ imply that $a_s \prec_D a_t$, and so $s < t$ by the assumption that $\pi_0 \in \mathcal{OP}(D - a)$. Hence we always have $s < t$.

By definition of $\mathcal{OP}(D)$ and the assumptions on s and t, we have

$$\mathcal{OP}(D, \pi_0) = \{(\cdots, a_i, a, a_{i+1}, \cdots) : s \leq i \leq t - 1\}. \tag{3.18}$$

For $\pi \in \mathcal{OP}(D)$, let $\pi_{a\to m}$ be the ordering obtained from π by replacing a by m. Then,

$$\mathcal{OP}(D_{a\to m}, \pi_0) = \{\pi_{a\to m} : \pi \in \mathcal{OP}(D, \pi_0)\} = \{(\cdots, a_i, m, a_{i+1}, \cdots) : s \leq i \leq t - 1\}. \tag{3.19}$$

Define

$$\Delta_{D,\pi_0}(z) = \sum_{\pi \in \mathcal{OP}(D,\pi_0)} z^{\delta_D(\pi) - \delta_{D-a}(\pi_0)}. \tag{3.20}$$

By (3.18), we have

$$\Delta_{D,\pi_0}(z) = \sum_{s \leq i \leq t-1} z^{\delta_D(a_i,a) + \delta_D(a,a_{i+1}) - \delta_D(a_i,a_{i+1})}, \tag{3.21}$$

where the following numbers are assumed in case that $s = 0$ or $t = n$:

$$1 = \delta_D(a_0, a_1) = \delta_D(a_0, a) = \delta_D(a_{n-1}, a_n) = \delta_D(a, a_n). \tag{3.22}$$

3.1.2 Expression for $\Delta(D,z)-\Delta(D_{a\to m},z)$

Let U_1 and U_2 be the two disjoint subsets of $\{i : s+1\le i\le t-2\}$ defined below:

$$\left\{\begin{array}{l} U_1=\{s+1\le i\le t-2 : a_i>a>a_{i+1}\},\\ U_2=\{s+1\le i\le t-2 : a_i<a<a_{i+1}\}.\end{array}\right. \tag{3.23}$$

Lemma 3.1 *(i) $\Delta_{D,\pi_0}(z)$ has the following expression:*

$$\begin{aligned}\Delta_{D,\pi_0}(z) &= z^{1+\delta_D(a,a_{s+1})-\delta_D(a_s,a_{s+1})}+z^{1+\delta_D(a_{t-1},a)-\delta_D(a_{t-1},a_t)}+\sum_{i\in U_1} z^{-\delta_D(a_i,a_{i+1})}\\ &\quad +\sum_{i\in U_2} z^{2-\delta_D(a_i,a_{i+1})}+\sum_{\substack{s+1\le i\le t-2\\ i\notin U_1\cup U_2}} z^{1-\delta_D(a_i,a_{i+1})}.\end{aligned} \tag{3.24}$$

(ii) If $m\in\mathbb{Z}^+\setminus V$ and $m>y$ for all $y\in V\setminus R_D[a]$, then

$$\Delta_{D_{a\to m},\pi_0}(z)=z^{2-\delta_D(a_{t-1},a_t)}+\sum_{s\le i\le t-2} z^{1-\delta_D(a_i,a_{i+1})}. \tag{3.25}$$

Proof. (i). We will prove this result by applying (3.21). Note that $\delta_D(a_s,a)=\delta_D(a,a_t)=1$ as $a_s\to a$ and $a\to a_t$ in D. For any i with $s+1\le i\le t-2$, by (3.23), we have

$$\delta_D(a_i,a)+\delta_D(a,a_{i+1})=\left\{\begin{array}{ll} 0, & \text{if } i\in U_1;\\ 2, & \text{if } i\in U_2;\\ 1, & \text{otherwise.}\end{array}\right. \tag{3.26}$$

Thus (3.24) follows from (3.21).

(ii). Recall that $F_D(a)=\{v : (a,v)\in A\}$. By the assumption on t, $F_D(a)\subseteq\{a_j : t\le j\le n-1\}$. As $\pi_0=(a_1,a_2,\cdots,a_{n-1})\in\mathcal{OP}(D-a)$, we have $R_D(a)\subseteq\{a_j : t\le j\le n-1\}$. Thus $V(D)\setminus R_D[a]\subseteq\{a_j : 1\le j\le t-1\}$. By the assumption on m, $m>a_i$ holds for all $i : 1\le i\le t-1$, implying that

$$\delta_{D_{a\to m}}(a_i,m)+\delta_{D_{a\to m}}(m,a_{i+1})=\left\{\begin{array}{ll} 1, & \text{if } s\le i\le t-2;\\ 2, & \text{if } i=t-1.\end{array}\right. \tag{3.27}$$

As $\delta_{D_{a\to m}}(a_i,a_{i+1})=\delta_D(a_i,a_{i+1})$, (3.25) follows from (3.21) by replacing D by $D_{a\to m}$. $\square$

Let

$$\left\{\begin{array}{l} Q(a,\pi_0)=\{s+1\le i\le t-2 : a_i>a>a_{i+1}, (a_i,a_{i+1})\in A\};\\ p(a,\pi_0)=(1-\delta_D(a_s,a_{s+1}))\delta_D(a,a_{s+1})-(1-\delta_D(a_{t-1},a_t))\delta_D(a,a_{t-1}).\end{array}\right. \tag{3.28}$$

When there is no confusion, $Q(a,\pi_0)$ and $p(a,\pi_0)$ are simply written as Q and p respectively. Applying Lemma 3.1, we can express $\Delta_{D,\pi_0}(z)-\Delta_{D_{a\to m},\pi_0}(z)$ in terms of Q and p.

Proposition 3.1 *If $m \in \mathbb{Z}^+ \setminus V$ and $m > y$ holds for all $y \in V \setminus R_D[a]$, then*

$$\Delta_{D,\pi_0}(z) - \Delta_{D_{a\to m},\pi_0}(z) = \left(p + |Q|z^{-1}\right)(z-1)^2.$$

Proof. By (3.24) and (3.25) in Lemma 3.1,

$$\begin{aligned}
& \Delta_{D,\pi_0}(z) - \Delta_{D_{a\to m},\pi_0}(z) \\
= \ & z^{1-\delta_D(a_s,a_{s+1})}(z^{\delta_D(a,a_{s+1})} - 1) + z^{1-\delta_D(a_{t-1},a_t)}(z^{\delta_D(a_{t-1},a)} - z) \\
& +(z^{-1}-1)\sum_{i\in U_1} z^{1-\delta_D(a_i,a_{i+1})} + (z-1)\sum_{i\in U_2} z^{1-\delta_D(a_i,a_{i+1})} \\
= \ & z^{1-\delta_D(a_s,a_{s+1})}(z^{\delta_D(a,a_{s+1})} - 1) + z^{1-\delta_D(a_{t-1},a_t)}(z^{\delta_D(a_{t-1},a)} - z) \\
& +|Q|(z^{-1}-1) + (|U_1| - |Q|)(1-z) + |U_2|(z-1) \\
= \ & z^{1-\delta_D(a_s,a_{s+1})}(z^{\delta_D(a,a_{s+1})} - 1) + z^{1-\delta_D(a_{t-1},a_t)}(z^{\delta_D(a_{t-1},a)} - z) \\
& +(|U_2| - |U_1|)(z-1) + |Q|z^{-1}(z-1)^2,
\end{aligned} \tag{3.29}$$

where the second last equality follows from the fact that for any i with $s+1 \le i \le t-2$,

$$\delta_D(a_i, a_{i+1}) = \begin{cases} 1, & \text{if } i \in Q \cup U_2; \\ 0, & \text{if } i \in U_1 \setminus Q. \end{cases}$$

By definitions of U_1 and U_2, it can be verified that

$$|U_2| - |U_1| = \begin{cases} 0, & \text{if } a > a_{s+1} \text{ and } a > a_{t-1}; \\ 1, & \text{if } a > a_{s+1} \text{ and } a < a_{t-1}; \\ -1, & \text{if } a < a_{s+1} \text{ and } a > a_{t-1}; \\ 0, & \text{if } a < a_{s+1} \text{ and } a < a_{t-1}. \end{cases} \tag{3.30}$$

Then, by (3.30),

$$\begin{aligned}
& z^{1-\delta_D(a_s,a_{s+1})}(z^{\delta_D(a,a_{s+1})} - 1) + z^{1-\delta_D(a_{t-1},a_t)}(z^{\delta_D(a_{t-1},a)} - z) + (|U_2| - |U_1|)(z-1) \\
= & \begin{cases} 0, & \text{if } a > a_{s+1} \text{ and } a > a_{t-1}; \\ z^{1-\delta_D(a_{t-1},a_t)}(1-z) + (z-1), & \text{if } a > a_{s+1} \text{ and } a < a_{t-1}; \\ z^{1-\delta_D(a_s,a_{s+1})}(z-1) - (z-1), & \text{if } a < a_{s+1} \text{ and } a > a_{t-1}; \\ z^{1-\delta_D(a_s,a_{s+1})}(z-1) + z^{1-\delta_D(a_{t-1},a_t)}(1-z), & \text{if } a < a_{s+1} \text{ and } a < a_{t-1} \end{cases} \\
= & \begin{cases} 0, & \text{if } a > a_{s+1} \text{ and } a > a_{t-1}; \\ (\delta_D(a_{t-1},a_t) - 1)(z-1)^2, & \text{if } a > a_{s+1} \text{ and } a < a_{t-1}; \\ (1-\delta_D(a_s,a_{s+1}))(z-1)^2, & \text{if } a < a_{s+1} \text{ and } a > a_{t-1}; \\ (\delta_D(a_{t-1},a_t) - \delta_D(a_s,a_{s+1}))(z-1)^2, & \text{if } a < a_{s+1} \text{ and } a < a_{t-1}. \end{cases}
\end{aligned} \tag{3.31}$$

By (3.28), (3.29) and (3.31), the result holds. □

By applying Proposition 3.1, an expression for $\Delta(D,z) - \Delta(D_{a\to m}, z)$ can be obtained.

Theorem 3.1 *If $m \in \mathbb{Z}^+ \setminus V$ and $m > y$ holds for all $y \in V \setminus R_D[a]$, then*

$$\Delta(D,z) - \Delta(D_{a\to m}, z) = (z-1)^2 \sum_{\pi_0 \in \mathcal{OP}(D-a)} \left[p(a,\pi_0) + |Q(a,\pi_0)|z^{-1}\right] z^{\delta_{D-a}(\pi_0)}. \tag{3.32}$$

Proof. Observe that

$$\begin{aligned}
& \Delta(D,z)-\Delta(D_{a\to m},z) \\
= & \sum_{\pi_1\in\mathcal{OP}(D)} z^{\delta_D(\pi_1)} - \sum_{\pi_2\in\mathcal{OP}(D_{a\to m})} z^{\delta_{D_{a\to m}}(\pi_2)} \\
= & \sum_{\pi_0\in\mathcal{OP}(D-a)} \sum_{\pi_1\in\mathcal{OP}(D,\pi_0)} z^{\delta_D(\pi_1)} - \sum_{\pi_0\in\mathcal{OP}(D-a)} \sum_{\pi_2\in\mathcal{OP}(D_{a\to m},\pi_0)} z^{\delta_{D_{a\to m}}(\pi_2)} \\
= & \sum_{\pi_0\in\mathcal{OP}(D-a)} z^{\delta_{D-a}(\pi_0)}\Delta_{D,\pi_0}(z) - \sum_{\pi_0\in\mathcal{OP}(D-a)} z^{\delta_{D-a}(\pi_0)}\Delta_{D_{a\to m},\pi_0}(z) \\
= & \sum_{\pi_0\in\mathcal{OP}(D-a)} \left[\Delta_{D,\pi_0}(z)-\Delta_{D_{a\to m},\pi_0}(z)\right] z^{\delta_{D-a}(\pi_0)} \\
= & (z-1)^2 \sum_{\pi_0\in\mathcal{OP}(D-a)} \left[p(a,\pi_0)+|Q(a,\pi_0)|z^{-1}\right] z^{\delta_{D-a}(\pi_0)}, \qquad (3.33)
\end{aligned}$$

where the last equality follows from Proposition 3.1. □

3.2 Compare D with $D_{a\to r}$ for some $r > a$

Let $D=(V,A)$ be an acyclic digraph with $|V|=n$. Recall that for $u\in V(D)$, $F_D(u)=\{v\in V:(u,v)\in A\}$. Let $B_D(u)=\{v\in V:(v,u)\in A\}$ and $B_D[u]=B_D(u)\cup\{u\}$. Thus u is a sink of D if and only if $F_D(u)=\emptyset$, and u is a source of D if and only if $B_D(u)=\emptyset$.

A vertex u of D is called a *turning vertex* if either $F_D(u)=\emptyset$ or $\min F_D(u)\ge 2+\max(\mathcal{P}_D(u))$ holds, where

$$\mathcal{P}_D(u)=B_D[u]\cup\{c\in V:\exists b<c,(c,b)\in A\}. \qquad (3.34)$$

In this subsection, we always assume that a is a turning vertex of D and r is a number in $\mathbb{Z}^+\setminus V$ such that $r>\max\mathcal{P}_D(a)$ whenever $F_D(a)=\emptyset$, and $\min F_D(a)>r>\max\mathcal{P}_D(a)$ otherwise. Thus $y_1>r>y_2$ holds for all $y_1\in F_D(a)$ and $y_2\in\mathcal{P}_D(a)$. Clearly $r>a$ holds, as $a\in B_D[a]\subseteq\mathcal{P}_D(a)$. In this section, the assumptions on a and r will not be mentioned again and we shall compare D with $D_{a\to r}$ under this assumption.

For $u\in V$, let $\mathcal{W}(D,u)=\{W\in\mathcal{W}(D):u\in W\}$. So $\mathcal{W}(D,u)=\mathcal{W}(D)\setminus\mathcal{W}(D-u)$, and $\mathcal{W}(D,u)=\emptyset$ iff $\mathcal{W}(D)=\mathcal{W}(D-u)$.

Lemma 3.2 *$\mathcal{W}(D_{a\to r},r)=\emptyset$ and so $\mathcal{W}(D_{a\to r})=\mathcal{W}(D-a)$.*

Proof. Clearly $\mathcal{W}(D_{a\to r})=\mathcal{W}(D-a)\cup\mathcal{W}(D_{a\to r},r)$. Thus it suffices to prove that $\mathcal{W}(D_{a\to r},r)=\emptyset$, i.e., $r\notin W$ for every $W\in\mathcal{W}(D_{a\to r})$.

Suppose that $W = \{r, b, c\} \in \mathcal{W}(D_{a\to r})$, where $b < c$. Assume that $r = \max W$. Then $r \to b$ in $D_{a\to r}$ by definition of $\mathcal{W}(D_{a\to r})$. But $r \to b$ in $D_{a\to r}$ implies that $a \to b$ in D and so $b \in F_D(a)$. By the given condition on r, $r < \min F_D(a) \le b$, contradicting the assumption that $r = \max W > b$. Hence $r < \max W$ and so $\max W = c$.

If $r = \min W$, then, by definition of $\mathcal{W}(D_{a\to r})$, $c > b > r$ and $c \to r$ in $D_{a\to r}$, where the later implies that $c \to a$ in D. So $c \in B_D(a) \subseteq \mathcal{P}_D(a)$. By the given condition on r, we have $r > \max \mathcal{P}_D(a) \ge c$, a contradiction.

By the above conclusions, we have $\min W < r < \max W$, i.e., $b < r < c$. As $W \in \mathcal{W}(D_{a\to r})$, we have $c \to b$ in both $D_{a\to r}$ and D. Thus $c \in \mathcal{P}_D(a)$. But $r > \max \mathcal{P}_D(a)$ implies that $r > c$, a contradiction again.

Hence the result holds. □

Lemma 3.3 *Let $\pi_0 = (a_1, a_2, \cdots, a_{n-1}) \in \mathcal{OP}(D-a)$ and s and t be the numbers defined in Subsubsection 3.1.1 with respect to a and $\pi_0 \in \mathcal{OP}(D-a)$. Then*

(i) *$Q(r, \pi_0) = \emptyset$;*

(ii) *$p(a, \pi_0) - p(r, \pi_0) = 1$ if $\{a, a_{s+1}, a_s\} \in \mathcal{W}(D)$, and $p(a, \pi_0) - p(r, \pi_0) = 0$ otherwise.*

Proof. (i) By definition,

$$Q(r, \pi_0) = \{s+1 \le i \le t-2 : a_i > r > a_{i+1}, a_i \to a_{i+1}\}.$$

Assume that $k \in Q(r, \pi_0)$. Then $a_k > a_{k+1}$ and $a_k \to a_{k+1}$, implying that $a_k \in \mathcal{P}_D(a)$. By the assumption on r, we have $r > \max \mathcal{P}_D(a) \ge a_k$. However, $k \in Q(r, \pi_0)$ implies that $r < a_k$, a contradiction. Thus $Q(r, \pi_0) = \emptyset$.

(ii) By definition of $p(a, \pi_0)$, we have

$$\begin{aligned} p(a, \pi_0) - p(r, \pi_0) &= (1 - \delta_D(a_s, a_{s+1}))\,[\delta_D(a, a_{s+1}) - \delta_{D_{a\to r}}(r, a_{s+1})] \\ &\quad + (1 - \delta_D(a_{t-1}, a_t))\,[\delta_{D_{a\to r}}(r, a_{t-1}) - \delta_D(a, a_{t-1})]\,. \end{aligned} \tag{3.35}$$

Claim 1: $p(a, \pi_0) - p(r, \pi_0) = (1 - \delta_D(a_s, a_{s+1}))\,[\delta_D(a, a_{s+1}) - \delta_{D_{a\to r}}(r, a_{s+1})]$.

By (3.35), it suffices to show that $(1 - \delta_D(a_{t-1}, a_t))\,[\delta_{D_{a\to r}}(r, a_{t-1}) - \delta_D(a, a_{t-1})] = 0$. Suppose that it does not hold. Then $\delta_D(a_{t-1}, a_t) = 0$. Thus $t < n$ and $a_{t-1} > a_t$. By the assumption on t, we have $(a, a_t) \in A$, implying that $a_t \in F_D(a)$. Since $a < r < \min F_D(a)$, we have $a < r < a_t$. As $a_t < a_{t-1}$, we have $a < r < a_{t-1}$ and

$$\delta_{D_{a\to r}}(r, a_{t-1}) = \delta_D(a, a_{t-1}) = 1.$$

So $(1-\delta_D(a_{t-1},a_t))\,[\delta_{D_{a\to r}}(r,a_{t-1})-\delta_D(a,a_{t-1})] = 0$, a contradiction. Hence Claim 1 holds.

Claim 2: $p(a,\pi_0)-p(r,\pi_0)\ge 0$.

As $r>a$, $\delta_D(a,a_{s+1})-\delta_{D_{a\to r}}(r,a_{s+1})\ge 0$. Then Claim 2 follows from Claim 1.

Claim 3: $p(a,\pi_0)-p(r,\pi_0)=1$ if and only if $\{a,a_s,a_{s+1}\}\in\mathcal{W}(D)$.

By Claims 1 and 2, $p(a,\pi_0)-p(r,\pi_0)\in\{0,1\}$.

Assume that $\{a,a_s,a_{s+1}\}\in\mathcal{W}(D)$. By definition of s, $a_s\to a$ in D. By definition of $\mathcal{W}(D)$, $a_s>a_{s+1}>a$, $a\not\to a_{s+1}$ and $a_s\not\to a_{s+1}$ in D. So $\delta_D(a_s,a_{s+1})=0$ and $\delta_D(a,a_{s+1})=1$. As $a_s\in B_D[a]\subseteq\mathcal{P}_D(a)$, by the assumption on r, $r>a_s$ holds, implying that $r>a_s>a_{s+1}$. Since $a\not\to a_{s+1}$ in D, we have $r\not\to a_{s+1}$ in $D_{a\to r}$. Thus $\delta_{D_{a\to r}}(r,a_{s+1})=0$. By Claim 1, we have $p(a,\pi_0)-p(r,\pi_0)=1$

Now assume that $p(a,\pi_0)-p(r,\pi_0)=1$. By Claim 1, $\delta_D(a_s,a_{s+1})=0$ and $\delta_D(a,a_{s+1})-\delta_{D_{a\to r}}(r,a_{s+1})=1$, where the later implies that $\delta_D(a,a_{s+1})=1$. Observe that $\delta_D(a_s,a_{s+1})=0$ implies that $a_s>a_{s+1}$ and $a_s\not\to a_{s+1}$, and $\delta_D(a,a_{s+1})=1$ implies that $a<a_{s+1}$ or $a\to a_{s+1}$ in D. However, if $a\to a_{s+1}$ in D, then $r\to a_{s+1}$ in $D_{a\to r}$, implying that $\delta_D(a,a_{s+1})-\delta_{D_{a\to r}}(r,a_{s+1})=1-1=0$, a contradiction. Thus $a_s>a_{s+1}>a$, but $a_s\not\to a_{s+1}$ and $a\not\to a_{s+1}$ in D. By definition of s, $a_s\to a$. Hence $\{a,a_s,a_{s+1}\}\in\mathcal{W}(D)$ and the claim holds. □

For an integer j with $0\le j\le n-1$,

(i) let $c_j(D,a)$ be the number of $\pi=(a_1,\cdots,a_i,a,a_{i+1},\cdots,a_{n-1})\in\mathcal{OP}(D)$ such that $\delta_D(\pi)=j$ and $\{a,a_i,a_{i+1}\}\in\mathcal{W}(D)$ for some i with $1\le i\le n-1$, where $(a_i,a)\in A$;

(ii) let $c'_j(D,a)$ be the number of $\pi=(a_1,\cdots,a_i,a,a_{i+1},\cdots,a_{n-1})$ such that $\delta_D(\pi)=j$ and $\{a,a_i,a_{i+1}\}\in\mathcal{W}(D)$ for some i with $1\le i\le n-1$, where $(a_i,a_{i+1})\in A$.

Clearly $c_j(D,a)+C'_j(D,a)$ is not more than the number of π's in $\mathcal{OP}(D)$ with $\delta_D(\pi)=j$, and $c_j(D,a)=C'_j(D,a)=0$ whenever $\mathcal{W}(D,a)=0$.

Lemma 3.4 *$c_j(D,a)=0$ for $j=0,1$, and $c'_j(D,a)=0$ for $j\ge n-2$.*

Proof. For any $\pi=(a_1,\cdots,a_i,a,a_{i+1},\cdots,a_{n-1})\in\mathcal{OP}(D)$, if $\{a,a_i,a_{i+1}\}$ is a member of $\mathcal{W}(D)$ with $a<a_{i+1}<a_i$ and $(a_i,a)\in A$, then $\delta_D(a_i,a)=\delta_D(a,a_{i+1})=1$, implying that $\delta_D(\pi)\ge 2$. Thus $c_j(D,a)=0$ for $j\le 1$ by definition of $c_j(D,a)$.

For any $\pi=(a_1,\cdots,a_i,a,a_{i+1},\cdots,a_{n-1})\in\mathcal{OP}(D)$, if $\{a,a_i,a_{i+1}\}$ is a member of $\mathcal{W}(D)$ with $a_{i+1}<a<a_i$ and $(a_i,a_{i+1})\in A$, then $a_i\not\approx a$ and $a\not\approx a_{i+1}$, implying that $\delta_D(a_i,a)=$

$\delta_D(a, a_{i+1}) = 0$. Thus $\delta_D(\pi) \le n-3$. By definition of $c'_j(D,a)$, $c'_j(D,a) = 0$ for $j \ge n-2$. $\Box$

Theorem 3.2 *Assume that $n = |V| \ge 3$. Then*

$$\Delta(D,z) - \Delta(D_{a\to r}, z) = (z-1)^2 \sum_{0\le j\le n-3} (c_{j+2}(D,a) + c'_j(D,a))z^j. \tag{3.36}$$

Furthermore, $\Delta(D,z) = \Delta(D_{a\to r}, z)$ if and only if $\mathcal{W}(D,a) = \emptyset$.

Proof. Let m be a number in $\mathbb{Z}^+ \setminus V$ such that $m > y$ for all $y \in V \setminus R_D[a]$. By Theorem 3.1, we have

$$\Delta(D,z) - \Delta(D_{a\to m}, z) = (z-1)^2 \sum_{\pi_0 \in \mathcal{OP}(D-a)} \left[p(a,\pi_0) + |Q(a,\pi_0)|z^{-1}\right] z^{\delta_{D-a}(\pi_0)}. \tag{3.37}$$

By Lemma 3.3 (i), $Q(r,\pi_0) = \emptyset$. Replacing D by $D_{a\to r}$ in (3.37) gives that

$$\Delta(D_{a\to r},z) - \Delta(D_{a\to m}, z) = (z-1)^2 \sum_{\pi_0 \in \mathcal{OP}(D-a)} p(r,\pi_0) z^{\delta_{D-a}(\pi_0)}. \tag{3.38}$$

By (3.37) and (3.37), $\Delta(D,z) - \Delta(D_{a\to r}, z)$ has the following expression:

$$(z-1)^2 \sum_{\pi_0 \in \mathcal{OP}(D-a)} \left[p(a,\pi_0) - p(r,\pi_0) + |Q(a,\pi_0)|z^{-1}\right] z^{\delta_{D-a}(\pi_0)}. \tag{3.39}$$

The proof will be completed by establishing the following claims.

Claim 1: For each $\pi_0 = (a_1, a_2, \cdots, a_{n-1}) \in \mathcal{OP}(D-a)$, $p(a,\pi_0) - p(r,\pi_0) \in \{0,1\}$, and $p(a,\pi_0) - p(r,\pi_0) = 1$ if and only if $\{a, a_s, a_{s+1}\} \in \mathcal{W}(D)$, where $(a_s, a) \in A$.

Claim 1 follows from Lemma 3.3 (ii).

Claim 2: $\sum\limits_{\pi_0 \in \mathcal{OP}(D-a)} (p(a,\pi_0) - p(r,\pi_0)) z^{\delta_{D-a}(\pi_0)} = \sum\limits_{j=0}^{n-3} c_{j+2}(D,a)x^j$.

Let $\mathcal{OP}^*(D-a)$ be the set of those $\pi_0 \in \mathcal{OP}(D-a)$ with $p(a,\pi_0) - p(r,\pi_0) = 1$, and let q_j be the number of π_0's in $\mathcal{OP}^*(D-a)$ with $\delta_{D-a}(\pi_0) = j$, where $0 \le j \le n-2$. Then, by Claim 1,

$$\sum_{\pi_0 \in \mathcal{OP}(D-a)} (p(a,\pi_0) - p(r,\pi_0)) z^{\delta_{D-a}(\pi_0)} = \sum_{j=0}^{n-2} \sum_{\substack{\pi_0 \in \mathcal{OP}^*(D-a) \\ \delta_{D-a}(\pi_0) = j}} z^j = \sum_{j=0}^{n-2} q_j z^j. \tag{3.40}$$

For each $\pi_0 \in \mathcal{OP}^*(D-a)$ with $\delta_{D-a}(\pi_0) = j$, $\pi = (a_1, \cdots, a_s, a, a_{s+1}, \cdots, a_{n-1})$ is a member of $\mathcal{OP}(D)$. By Claim 1, $\{a, a_s, a_{s+1}\} \in \mathcal{W}(D)$ with $(a_s, a) \in A$. Thus, $\delta_D(a_s, a) = \delta(a, a_{s+1}) = 1$ but $\delta_{D-a}(a_s, a_{s+1}) = 0$, implying that $\delta_D(\pi) = \delta_{D-a}(\pi_0) + 2 = j+2$. As

$\delta_{D-a}(a_s, a_{s+1}) = 0$, we have $\delta_{D-a}(\pi_0) \leq n-3$ and so $q_{n-2} = 0$. On the other hand, for any $\pi = (a_1, \cdots, a_i, a, a_{i+1}, \cdots, a_{n-1}) \in \mathcal{OP}(D)$ with $\delta_D(\pi) = j+2$ and $\{a_i, a, a_{i+1}\} \in \mathcal{W}(D)$, where $a_i > a_{i+1} > a$, $\pi_0 = (a_1, \cdots, a_i, a_{i+1}, \cdots, a_{n-1})$ is a $(D-a)$-respecting ordering with $s = i$ and $\delta_{D-a}(\pi) = j$. Thus, by definition, $q_j = c_{j+2}(D, a)$ holds and so Claim 2 holds.

Claim 3: $\sum_{\pi_0 \in \mathcal{OP}(D-a)} |Q(a, \pi_0)| z^{\delta_{D-a}(\pi_0)-1} = \sum_{j=0}^{n-3} c'_j(D, a) z^j.$

By definition, for each $\pi_0 = (a_1, a_2, \cdots, a_{n-1}) \in \mathcal{OP}(D-a)$, $|Q(a, \pi_0)|$ is the number of integers i with $s+1 \leq i \leq t-2$ such that $a_i > a > a_{i+1}$ and $(a_i, a_{i+1}) \in A$. As $(a_i, a_{i+1}) \in A$, we have $\delta_{D-a}(\pi_0) \geq 1$. As $s+1 \leq i \leq t-2$, the definitions of s and t imply that $D[\{a_i, a, a_{i+1}\}]$ has only one arc, i.e., (a_i, a_{i+1}). Thus $\{a_i, a, a_{i+1}\} \in \mathcal{W}(D)$. Clearly, $Q(a, \pi_0) > 0$ implies that $\delta_{D-a}(\pi_0) \geq 1$. Thus,

$$\sum_{\pi_0 \in \mathcal{OP}(D-a)} |Q(a, \pi_0)| z^{\delta_{D-a}(\pi_0)-1} = \sum_{j=0}^{n-2} \sum_{\substack{\pi_0 \in \mathcal{OP}(D-a) \\ \delta_{D-a}(\pi_0)=j}} Q(a, \pi_0) z^{j-1} = \sum_{j=0}^{n-3} q'_j z^j, \tag{3.41}$$

where q'_j is the number of order pairs (π_0, i), where $\pi_0 \in \mathcal{OP}(D-a)$ with $\delta_{D-a}(\pi_0) = j+1$ and i is an integer with $s+1 \leq i \leq t-2$ such that $\{a_i, a, a_{i+1}\} \in \mathcal{W}(D)$, where $(a_i, a_{i+1}) \in A$.

For each $\pi_0 = (a_1, a_2, \cdots, a_{n-1}) \in \mathcal{OP}(D-a)$ with $\delta_{D-a}(\pi_0) = j+1$, if i is an integer with $s+1 \leq i \leq t-2$ such that $\{a_i, a, a_{i+1}\} \in \mathcal{W}(D)$, where $(a_i, a_{i+1}) \in A$, then $\pi = (a_1, \cdots, a_i, a, a_{i+1}, \cdots, a_{n-1})$ is a member of $\mathcal{OP}(D)$. As $\delta_D(a_i, a) = \delta_D(a, a_{i+1}) = 0$ but $\delta_{D-a}(a_i, a_{i+1}) = 1$, we have $\delta_D(\pi) = \delta_{D-a}(\pi_0) - 1 = j$.

On the other hand, for each $\pi = (a_1, \cdots, a_i, a, a_{i+1}, \cdots, a_{n-1})$, if $\{a_i, a, a_{i+1}\} \in \mathcal{W}(D)$, where $(a_i, a_{i+1}) \in A$, by definitions of s and t, we have $s+1 \leq i \leq t-2$ and $\pi_0 = (a_1, \cdots, a_i, a_{i+1}, \cdots, a_{n-1})$ is a member of $\mathcal{OP}(D-a)$. As $\delta_D(a_i, a) = \delta_D(a, a_{i+1}) = 0$ and $\delta_{D-a}(a_i, a_{i+1}) = 1$, we have $\delta_D(\pi) = \delta_{D-a}(\pi_0) - 1 = j$ whenever $\delta_{D-a}(\pi_0) = j+1$.

By the assumption on q'_j and the above arguments, q'_j equals the number of members $\pi = (a_1, \cdots, a_i, a, a_{i+1}, \cdots, a_{n-1})$ of $\mathcal{OP}(D)$ with $\delta_D(\pi) = j$ such that $\{a_i, a, a_{i+1}\} \in \mathcal{W}(D)$, where $(a_i, a_{i+1}) \in A$. By definition of $c'_j(D, a)$, we have $q'_j = c'_j(D, a)$. Then, by (3.41), Claim 3 holds.

By (3.39) and Claims 2 and 3, (3.36) holds.

Claim 4: If $\mathcal{W}(D, a) \neq \emptyset$, then $c_{j+2}(D, a) + c'_j(D, a) > 0$ for some j.

Assume that $W = \{a, b, c\} \in \mathcal{W}(D)$, where $b < c$. If $a > c$, then $(a, b) \in A$, implying that $b \in F_D(a)$. But a is a turning vertex of D, implying that $a < y$ for all $y \in F_D(a)$, a contradiction. Thus, either $a < b < c$ or $b < a < c$.

Suppose that $a < b < c$. As $\{a, b, c\} \in W(D)$, by definition of $\mathcal{W}(D)$, $(c, a) \in A$ and $b \not\approx_D a$ and $b \not\approx_D c$. It is easy to check that there exists $\pi = (a_1, \cdots, a_s, a, a_{s+1}, \cdots, a_{n-1}) \in \mathcal{OP}(D)$, where $a_s = c$ and $a_{s+1} = b$. Thus $\{a, a_s, a_{s+1}\} \in \mathcal{W}(D)$. Let $\pi_0 = \pi - a$, i.e., $\pi_0 = (a_1, \cdots, a_s, a_{s+1}, \cdots, a_{n-1})$. Clearly $\pi_0 \in \mathcal{OP}(D - a)$, $\delta_D(\pi) \geq \delta_D(a_s, a) + \delta_D(a, a_{s+1}) = 2$ and $\delta_D(\pi) = \delta_{D-a}(\pi_0) + 2 \geq 2$. By definition, $c_{j+2}(D, a) > 0$ for some j with $0 \leq j \leq n-3$.

Now suppose that $b < a < c$. As $\{a, b, c\} \in W(D)$, by definition of $\mathcal{W}(D)$, $(c, b) \in A$ and $a \not\approx_D b$ and $a \not\approx_D c$. It is easy to check that there exists $\pi = (a_1, \cdots, a_i, a, a_{i+1}, \cdots, a_{n-1}) \in \mathcal{OP}(D)$, where $a_i = c$ and $a_{i+1} = b$. Let $\pi_0 = (a_1, \cdots, a_i, a_{i+1}, \cdots, a_{n-1})$. Clearly $\pi_0 \in \mathcal{OP}(D - a)$ and $\delta_D(\pi) = \delta_{D-a}(\pi_0) - 1 \leq n - 3$. By definition, $c'_j(D, a) > 0$ for some j with $0 \leq j \leq n - 3$.

Thus Claim 4 holds. If $\mathcal{W}(D, a) = \emptyset$, by definition of $c_j(D, a)$ and $c'_j(D, a)$, we have $c_j(D, a) = c'_j(D, a) = 0$ for all $i = 0, 1, \cdots, n - 1$. By this fact and Claim 4, $\mathcal{W}(D, a) = \emptyset$ if and only if $\Delta(D, z) = \Delta(D_{a\to r}, z)$. □

Applying Theorem 3.2 and the following result, we will obtain an expression for $\Psi(D, x) - \Psi(D_{a\to r}, x)$ in terms of $c_{j+2}(D, a) + c'_j(D, a)$ for $j = 0, 1, \cdots, n - 3$.

Lemma 3.5 *Let D_1 and D_2 be any two acyclic digraphs of order n.*

(i) *If $\Delta(D_1, z) - \Delta(D_2, z) = t_0 + t_1 z + \cdots + t_{n-1} z^{n-1}$, then*

$$\Psi(D_1, x) - \Psi(D_2, x) = \sum_{i=0}^{n-1} t_i \binom{x+i}{n}; \tag{3.42}$$

(ii) *if $\Delta(D_1, z) - \Delta(D_2, z) = (z-1)^2 P(z)$, where $P(z) = d_0 + d_1 z + \cdots + d_{n-3} z^{n-3}$, then*

$$\Psi(D_1, x) - \Psi(D_2, x) = \sum_{i=0}^{n-3} d_i \binom{x+i}{n-2}. \tag{3.43}$$

Proof. (i). Assume that

$$\Delta(D_2, z) = \sum_{i=0}^{n-1} b_i z^i.$$

Then, by the given condition,

$$\Delta(D_1, z) = \sum_{i=0}^{n-1} (b_i + t_i) z^i.$$

By the relation between $\Delta(D_i, z)$ and $\Psi(D_i, x)$, we have

$$\Psi(D_1, x) = \sum_{i=0}^{n-1} (b_i + t_i) \binom{x+i}{n}, \quad \Psi(D_2, x) = \sum_{i=0}^{n-1} b_i \binom{x+i}{n}.$$

Thus the result holds.

(ii). Note that

$$\Delta(D_1,z)-\Delta(D_2,z)=(z-1)^2\sum_{i=0}^{n-3}d_iz^i=\sum_{i=0}^{n-3}\left(d_iz^{i+2}-2d_iz^{i+1}+d_iz^i\right).$$

Then, the result in (i) implies that

$$\begin{aligned}\Psi(D_1,x)-\Psi(D_2,x) &= \sum_{i=0}^{n-3}d_i\left[\binom{x+i+2}{n}-2\binom{x+i+1}{n}+\binom{x+i}{n}\right]\\ &= \sum_{i=0}^{n-3}d_i\binom{x+i}{n-2}. \end{aligned}\tag{3.44}$$

□

Theorem 3.3 *Assume that $n=|V|\geq 3$. Then*

$$\Psi(D,x)-\Psi(D_{a\to r},x)=\sum_{j=0}^{n-3}(c_{j+2}(D,a)+c'_j(D,a))\binom{x+j}{n-2}.\tag{3.45}$$

Furthermore, $\Psi(D,x)=\Psi(D_{a\to r},x)$ if and only if $\mathcal{W}(D,a)=\emptyset$.

Proof. The result follows directly from Theorem 3.2 and Lemma 3.5(ii). □

Let $D_1=(V_1,A_1)$ be an acyclic digraph and $V'\subseteq V_1$. Let $D_2=(V_2,A_2)$ be an acyclic digraph obtained from D_1 by relabeling each $u\in V'$ by $\mu(u)$, where μ is a bijection from V' to V'', where V'' is some subset of $\mathbb{Z}^+\setminus V_1$ with $|V''|=|V'|$. Write $D_1\succeq D_2$ if conditions (a) and (b) below are satisfied:

(a) for any 3-element subset W of V_1, if $W\notin\mathcal{W}(D_1)$, then $W'\notin\mathcal{W}(D_2)$, where $W'=(W\setminus V')\cup\{\mu(u):u\in W\cap V'\}$;

(b) $\Delta(D_1,z)-\Delta(D_2,z)=(z-1)^2P(z)$, where $P(z)=0$ or $P(z)$ is a polynomial of degree at most n_1-3 without negative coefficients, where $n_1=|V_1|$; furthermore, $P(z)=0$ if and only if $|\mathcal{W}(D_1)|=|\mathcal{W}(D_2)|$.

Proposition 3.2 *If $D_1\succeq D_2$ and $D_2\succeq D_3$, then $D_1\succeq D_3$.*

Proposition 3.3 *Assume that $D_1\succeq D_2$. Then*

(i) *$|\mathcal{W}(D_1)|\geq|\mathcal{W}(D_2)|$;*

(ii) *if* $|\mathcal{W}(D_1)| = |\mathcal{W}(D_2)|$, *then* $\Delta(D_1, z) = \Delta(D_2, z)$ *and* $\Psi(D_1, x) = \Psi(D_2, x)$*;*

(iii) *if* $|\mathcal{W}(D_1)| > |\mathcal{W}(D_2)|$, *then there exists non-negative integers* $d_0, d_1, \cdots, d_{n_1-3}$ *such that*

$$\Delta(D_1, z) - \Delta(D_2, z) = (z-1)^2 \sum_{i=0}^{n_1-3} d_i z^i$$

and

$$\Psi(D_1, x) - \Psi(D_2, x) = \sum_{i=0}^{n_1-3} d_i \binom{x+i}{n_1-2},$$

where $d_i > 0$ *for some* i*.*

By applying Lemma 3.2 and Theorem 3.2, we get the following conclusion on D and $D_{a\to r}$.

Corollary 3.1 $D \succeq D_{a\to r}$.

3.3 Complete the proof of Theorem 1.4

Let $D = (V, A)$ be an acyclic digraph. For $S \subseteq V$, S is said to be *ideal* in D if either $S = \emptyset$ or the following conditions are satisfied:

(i.1) for each $y \in S$, $R_D(y) \subseteq S$;

(i.2) for each $y \in S$, either $F_D(y) = \emptyset$ or $y < \min F_D(y)$; and

(i.3) either $S = V$ and $\min V \geq 2$ or $\min S \geq 2 + \max(V \setminus S)$.

Proposition 3.4 *Let* $S \subseteq V$ *be ideal in* D*. Then* $\mathcal{R}e(D) = \mathcal{R}e(D-S)$ *and* $\mathcal{W}(D) = \mathcal{W}(D-S)$*.*

Proof. We just need to consider the case that $S \neq \emptyset$. As S is ideal in D, it is easy to verify that $\mathcal{R}e(D) = \mathcal{R}e(D-S)$.

It is clear that $\mathcal{W}(D-S) \subseteq \mathcal{W}(D)$. Assume that $W \in \mathcal{W}(D)$ and $W \cap S \neq \emptyset$. As $\min S \geq 2 + \max(V \setminus S)$, we have $\max W \in S$. Let $c = \max W$ and $a = \min W$. So $c > a$. By definition of $\mathcal{W}(D)$, $(c, a) \in A$ and so $a \in F_D(c)$. As S is ideal, $c < \min F_D(c) \leq a$, a contradiction. □

Assume that $\min S = +\infty$ whenever $S = \emptyset$.

Proposition 3.5 *Let* $S \subseteq V$ *be ideal in* D *and let* $u \in V \setminus S$ *with* $F_D(u) \subseteq S$*. Then*

(i) $\mathcal{P}_D(u) \subseteq V \setminus S$ *and* u *is a turning vertex of* D*;*

(ii) *if* $V = S \cup \{u\}$*, then* $S \cup \{u'\}$ *is ideal in* $D_{u \to u'}$ *for any* $u' \in \mathbb{Z}^+$ *with* $0 < u' < \min S$*;*

(iii) *if* $V \neq S \cup \{u\}$ *and* $\min S \geq 3 + \max(V \setminus S)$*, then* $S \cup \{u'\}$ *is ideal in* $D_{u \to u'}$ *for any* $u' \in \mathbb{Z}^+$ *with* $2 + \max(V \setminus S) \leq u' < \min S$*.*

Proof. (i) The result is trivial if $S = \emptyset$. So we assume that $S \neq \emptyset$. As S is ideal in D and $u \notin S$, we have $B_D(u) \subseteq V \setminus S$. For any $(c, b) \in A$, if $c \in S$, then $b \in S$ by condition (i.1) and so $c < b$ by condition (i.2). Thus, $(c, b) \in A$ and $c > b$ imply that $c \notin S$. Therefore,

$$\mathcal{P}_D(u) \quad = \quad B_D[u] \cup \{c \in V : \exists b < c, (c, b) \in A\} \subseteq V \setminus S. \tag{3.46}$$

As $F_D(u) \subseteq S$ and S is ideal in D, we have

$$\min F_D(u) \geq \min S \geq 2 + \max(V \setminus S) \geq 2 + \max P_D(u). \tag{3.47}$$

Thus, u is a turning vertex of D.

(ii) This is trivial to verify.

(iii) The result is trivial when $S = \emptyset$. Now assume that $S \neq \emptyset$. Let $S' = S \cup \{u'\}$. By the given condition, to verify if S' is ideal in $D_{u \to u'}$, it suffices to show that condition (i.3) is satisfied. As $u' = \min S - 1$ and $\min S \geq 3 + \max(V \setminus S)$, we have

$$\begin{aligned} \min S' &= u' \geq 2 + \max(V \setminus S) \geq 2 + \max(V \setminus (S \cup \{u\})) \\ &= 2 + \max(V(D_{u \to u'}) \setminus S'). \end{aligned} \tag{3.48}$$

Thus S' is ideal in $D_{u \to u'}$. □

Proposition 3.6 *Let* $S \subset V$ *be ideal in* D *and* u *be a vertex in* $V \setminus S$ *with* $F_D(u) \subseteq S$*. For any* $u' \in \mathbb{Z}^+$ *with* $\max(V \setminus S) < u' < \min S$*,* $D \succeq D_{u \to u'}$ *holds.*

Proof. By Proposition 3.5 (i), $\mathcal{P}_D(u) \subseteq V \setminus S$ and u is a turning vertex of D. Thus, if $\max(V \setminus S) < u' < \min S$, then $\max \mathcal{P}_D(u) \leq \max(V \setminus S) < u' < \min S \leq \min F_D(u)$. Replacing r by u' in Corollary 3.1 implies that $D \succeq D_{u \to u'}$. □

For an acyclic digraph $D = (V, A)$, an ordering $\alpha = (u_1, u_2, \cdots, u_n)$ of its vertices is said to be a *sink-elimination ordering*, if u_i is a sink of the subdigraph $D[V_i]$ for all $i = 1, 2, \cdots, n-1$, where $V_i = \{u_i, u_{i+1}, \cdots, u_n\}$. Now assume that $\alpha = (u_1, u_2, \cdots, u_n)$ is a sink-elimination ordering of D and $M = n + 1 + \max V$. Let $\Gamma_{D,\alpha}$ denote the sequence $(D_0, D_1, \cdots, D_{n-1})$ of digraphs produced from D according to α: D_0 is D, and for $i = 1, 2, \cdots, n-1$, D_i is the digraph $(D_{i-1})_{u_i \to M-i}$ (i.e., D_i is obtained from D_{i-1} by relabeling vertex u_i as $M-i$). For example, if D is the digraph in Figure 2, then $\alpha=(3,2,4,5,1)$ is a sink-elimination ordering of its vertices, $M=11$ and $\Gamma_{D,\alpha}=(D_0,D_1,\cdots,D_4)$, where D_0 is the digraph in Figure 2, D_1,D_2,D_3 and D_4 are shown in Figure 3.

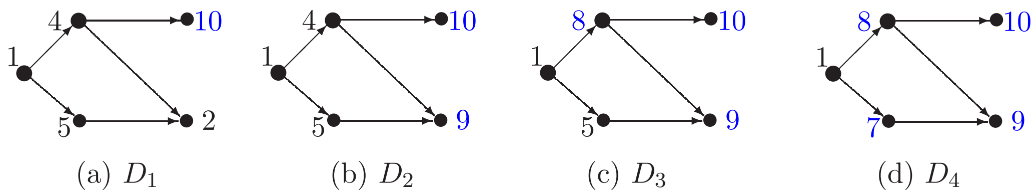


Figure 3: $\Gamma_{D,\alpha}=(D_0,D_1,\cdots,D_4)$ for D in Figure 2 and $\alpha=(3,2,4,5,1)$

Theorem 3.4 *Assume that* $\Gamma_{D,\alpha}=(D_0,D_1,\cdots,D_{n-1})$. *Then* $\mathcal{R}e(D_{n-1})=\emptyset$ *and* $D_i\succeq D_{i+1}$ *for all* $i=0,1,\cdots,n-2$.

Proof. Let $M=n+1+\max V$. By definition, D_i is obtained from D by relabeling vertex u_j as $M-j$ for all $j=1,2,\cdots,i$, where $\alpha=(u_1,u_2,\cdots,u_n)$ is a sink-elimination ordering of D. Then $V(D_i)$ is the disjoint union of S_i and V_{i+1}, where $S_i=\{M-j:1\le j\le i\}$ and $V_{i+1}=\{u_j:i+1\le j\le n\}$.

We first prove two claims below.

Claim 1: $F_{D_i}(u_{i+1})\subseteq S_i$ for all $i=0,1,\cdots,n-1$.

As α is a sink-elimination ordering of D, u_{i+1} is a sink of $D[V_{i+1}]$ and so $F_D(u_{i+1})\subseteq\{u_1,\cdots,u_i\}$. By definition of D_i, we have $F_{D_i}(u_{i+1})\subseteq\{M-j:1\le j\le i\}=S_i$. Hence Claim 1 holds.

Claim 2: S_i is ideal in D_i for all $i=0,1,\cdots,n-1$.

As $S_0=\emptyset$, S_0 is ideal in D_0. It is also trivial that $S_1=\{M-1\}$ is ideal in D_1, as $M-1=n+\max V\ge 3+\max V\ge \max(V_2)+3$ and $M-1$ is a sink in D_1.

Now assume that S_{i-1} is ideal in D_{i-1}, where $2\le i\le n-2$. We will apply Proposition 3.5 to show that S_i is ideal in D_i.

Note that $u_i\in V(D_{i-1})\setminus S_{i-1}=V_i$ and D_i is obtained from D_{i-1} by relabeling u_i as $M-i$. Observe that $M-i<M-i+1=\min S_{i-1}$ and

$$M-i=n+1+\max V-i\ge 3+\max V\ge 3+\max V_i.$$

By Claim 1, $F_{D_{i-1}}(u_i)\subseteq S_{i-1}$. By Proposition 3.5 (ii) and (iii), $S_i=S_{i-1}\cup\{M-i\}$ is ideal in $D_i=(D_{i-1})_{u_i\to M-i}$.

Hence Claim 2 holds.

By Claim 2 and Proposition 3.4,$\mathcal{R}e(D_i) = \mathcal{R}e(D_i - S_i)$ for all $i = 0, 1, \cdots, n-1$. Hence $\mathcal{R}e(D_{n-1}) = \mathcal{R}e(D[\{u_n\}]) = \emptyset$.

Note that $u_j < M - (i+1) < M - i = \min S_i$ for all $j : i+1 \le j \le n$. Thus, by Claims 1, 2 and Proposition 3.6, $D_i \succeq D_{i+1}$, as $D_{i+1} = (D_i)_{u_{i+1} \to M-(i+1)}$. □

Corollary 3.2 *$\mathcal{R}e(D_{n-1}) = \emptyset$ and*

$$\Psi(D, x) - \Psi(D_{n-1}, x) = \sum_{i=0}^{n-3} d_i \binom{x+i}{n-2}, \tag{3.49}$$

where $d_i \ge 0$ for all $i = 0, 1, \cdots, n-3$. Furthermore, $\mathcal{W}(D) = \emptyset$ if and only if $d_i = 0$ for all $i = 0, 1, \cdots, n-3$.

Proof. By Theorem 3.4, $\mathcal{R}e(D_{n-1}) = \emptyset$. By Theorem 3.4 again, $D_i \succeq D_{i+1}$ for all $i = 0, 1, \cdots, n-2$. By Proposition 3.2, $D_0 \succeq D_{n-1}$. Thus, the result follows from Proposition 3.3. □

Proof of Theorem 1.4: As $\mathcal{R}e(D_{n-1}) = \emptyset$, $\Psi(D_{n-1}, x) = \Omega(\bar{D}_{n-1}, x)$ by Proposition 1.1. As $\Omega(\bar{D}_{n-1}, x) = \Omega(\bar{D}, x)$, Theorem 1.4 follows from Corollary 3.2. □

We end this section with a special sink-elimination ordering of D determined by the injective mapping $L : V \to \{1, 2, \cdots, n\}$ defined in Section 2. Assume that $V = \{v_1, v_2, \cdots, v_n\}$ and $L(v_i) = n+1-i$ for $i = 1, 2, \cdots, n$. Then $\alpha = (v_1, v_2, \cdots, v_n)$ is a sink-elimination ordering of D. Assume that $\Gamma_{D,\alpha} = (D_0, D_1, \cdots, D_{n-1})$. So D_{n-1} is obtained from D by relabeling each v_i by $n+1-i+\max V$ for all $i = 1, 2, \cdots, n-1$. Recall that D_L denotes the digraph obtained from D by relabeling each vertex v_i by $L(v_i) = n+1-i$. By definition, D^*_{n-1} is exactly the digraph D_L, implying that $\Psi(D_L, x) = \Psi(D^*_{n-1}, x) = \Psi(D_{n-1}, x)$. By Corollary 3.2, we have the following conclusion.

Corollary 3.3 *$\Psi(D, x) = \Psi(D_L, x)$ if and only if $\mathcal{W}(D) = \emptyset$.*

4 Proof of Theorem 1.1

Let $G = (V, E)$ be a simple graph, where $V = [n]$. Recall that $\mathcal{P}(V)$ is the set of orderings of members of V. So $|\mathcal{P}(V)| = n!$. Recall that $\mathcal{AO}(G)$ is the set of acyclic orientations of G. Then $\mathcal{P}(V)$ can be partitioned according to members D of $\mathcal{AO}(G)$ as stated in the following lemma.

Lemma 4.1 *(i) $\mathcal{P}(V) = \bigcup_{D \in \mathcal{AO}(G)} \mathcal{OP}(D)$;*

(ii) $\mathcal{OP}(D_1) \cap \mathcal{OP}(D_2) = \emptyset$ *for any pair of distinct orientations* $D_1, D_2 \in \mathcal{AO}(G)$.

Proof. (i). Clearly, $\mathcal{OP}(D) \subseteq \mathcal{P}(V)$. For an ordering $\pi = (a_1, a_2, \cdots, a_n)$ of the elements of $V = [n]$, if D is the orientation of G such that $(a_i, a_j) \in A(D)$ whenever $i < j$ and $a_i a_j \in E$, then $\pi \in \mathcal{OP}(D)$. Thus (i) holds.

(ii). Suppose that $D_1, D_2 \in \mathcal{AO}(G)$ and $\pi = (a_1, a_2, \cdots, a_n) \in \mathcal{OP}(D_1) \cap \mathcal{OP}(D_2)$. For any edge $a_i a_j$ in G, $i < j$ implies that $(a_i, a_j) \in A(D_1) \cap A(D_2)$. Thus D_1 and D_2 are the same. Hence (ii) holds. $\square$

Lemma 4.2 *For any simple graph* G,

$$\Psi(G, x) = \sum_{D \in \mathcal{AO}(G)} \Psi(D, x). \tag{4.50}$$

Proof. Let $D \in \mathcal{AO}(G)$. For any $\pi = (a_1, a_2, \cdots, a_n) \in \mathcal{OP}(D)$ and any $i = 1, 2, \cdots, n-1$, a_i and a_{i+1} are adjacent in G if and only if $a_i \to a_{i+1}$ in D, implying that $\delta_G(a_i, a_{i+1}) = \delta_D(a_i, a_{i+1})$. Thus $\delta_G(\pi) = \delta_D(\pi)$ holds for any $\pi \in \mathcal{OP}(D)$, implying that

$$\Psi(D, x) = \sum_{\pi \in \mathcal{OP}(D)} \binom{x + \delta_D(\pi)}{n} = \sum_{\pi \in \mathcal{OP}(D)} \binom{x + \delta_G(\pi)}{n}. \tag{4.51}$$

Then, by Lemma 4.1,

$$\Psi(G, x) = \sum_{\pi \in \mathcal{P}(V)} \binom{x + \delta_G(\pi)}{n} = \sum_{D \in \mathcal{AO}(G)} \sum_{\pi \in \mathcal{OP}(D)} \binom{x + \delta_G(\pi)}{n}. \tag{4.52}$$

Thus (4.50) follows from (4.52) and (4.51). $\square$

Proposition 4.1 *For any simple graph* $G = (V, E)$ *with* $V = [n]$*, where* $n \geq 3$,

$$\Psi(G, x) = (-1)^n \chi(G, -x) + \sum_{i=0}^{n-3} d_i \binom{x + i}{n}, \tag{4.53}$$

where $d_i \geq 0$ *for all* $i = 0, 1, \cdots, n-3$*. Furthermore,* $d_i = 0$ *for all* $i = 0, 1, \cdots, n-3$ *if and only if* $\mathcal{W}(G) = \emptyset$.

Proof. By Theorem 1.5,

$$(-1)^n \chi(G, -x) = \sum_{D \in \mathcal{AO}(G)} \Omega(\bar{D}, x). \tag{4.54}$$

Then, Lemma 4.2, (4.54) and Theorem 1.4 imply that

$$\begin{aligned}\Psi(G,x)-(-1)^n\chi(G,-x) &= \sum_{D\in\mathcal{AO}(G)} \big(\Psi(D,x)-\Omega(\bar{D},x)\big)\\ &= \sum_{D\in\mathcal{AO}(G)} \sum_{i=0}^{n-3} d_{D,i}\binom{x+i}{n-2}, \end{aligned} \tag{4.55}$$

where $d_{D,i}\geq 0$ for all $i=0,1,\cdots,n-3$, and $d_{D,i}=0$ for all $i=0,1,\cdots,n-3$ if and only if $\mathcal{W}(D)=\emptyset$. Thus $d_i=\sum\limits_{D\in\mathcal{AO}(G)} d_{D,i}\geq 0$ for all $i=0,1,\cdots,n-3$. If $\mathcal{W}(G)=\emptyset$, then $\mathcal{W}(D)=\emptyset$ for all $D\in\mathcal{AO}(G)$, implying that $d_i=0$ for all $i=0,1,\cdots,n-3$. Thus, it remains to show that if $\mathcal{W}(G)\neq\emptyset$, then $d_i>0$ for some i.

Now assume that $\mathcal{W}(G)\neq\emptyset$. Then $\mathcal{W}(D)\neq\emptyset$ for some $D\in\mathcal{AO}(G)$. By Theorem 1.4, $d_{D,i}>0$ for some i, implying that $d_i>0$.

Hence Proposition 4.1 holds. □

Theorem 1.1 follows directly from Proposition 4.1.

5 Further study

We end this article with some problems that may merit further study. We assume that $G=(V,E)$ is a simple graph with $V=[n]$, unless otherwise stated.

5.1 Possible extensions of Theorems 1.1 and 1.4

Theorem 1.1 gives an expression for $\chi(G,x)$ in terms of a summation over all $n!$ orderings of elements of V whenever $\mathcal{W}(G)=\emptyset$. This result is established by applying Theorem 1.4 and Stanley's result, Theorem 1.5. Is it possible to find new results analogous to Theorems 1.1 and 1.4 by revising $\delta_D(\pi)$ and $\delta_G(\pi)$ for orderings π of elements of V such that the new results hold for a larger family of acyclic digraphs D and a larger family of simple graphs G respectively?

5.2 Graphs G with $\mathcal{W}(G)=\emptyset$

Recall that $\mathcal{W}(G)=\{\{a,b,c\}:1\leq a<b<c\leq n, ac\in E, ab,bc\notin E\}$. By definition of $\mathcal{W}(G)$, the following observation follows directly. Let $N_G(a)$ denote the set of vertices in G which are adjacent to a.

Proposition 5.1 *$\mathcal{W}(G) = \emptyset$ if and only if for every edge $ac \in E$ with $a < c$, $\{b : a < b < c\} \subseteq N_G(a) \cup N_G(c)$ holds.*

For a bijection $\omega : V \to [n]$, let G_ω be the graph obtained from G by relabeling each vertex $v \in V$ by $\omega(v)$. Let $\mathbb{W}$ denote the set of simple graphs $G = (V, E)$ such that $\mathcal{W}(G_\omega) = \emptyset$ for some bijection $\omega : V \to [n]$.

If G is a complete multi-partite graph and ac is an edge in G, then $u \in N_G(a) \cup N_G(c)$ holds for every $u \in [n] - \{a, c\}$. Thus, by Proposition 5.1, $\mathcal{W}(G_\omega) = \emptyset$ holds for an arbitrary bijection $\omega : V \to [n]$. If G is not a complete multi-partite graph, this property does not hold.

The observations in the following proposition can be verified easily.

Proposition 5.2

(i) *If $G = (V, E)$ is a complete multi-partite graph and $\omega : [n] \to [n]$ is a bijection, then $\mathcal{W}(G_\omega) = \emptyset$ holds and thus $G \in \mathbb{W}$;*

(ii) *if G is disconnected, then $G \in \mathbb{W}$ if and only if each component of G belongs to $\mathbb{W}$;*

(iii) *if $G \in \mathbb{W}$, then the subgraph of G induced by any subset $S \subseteq V(G)$ belongs to $\mathbb{W}$.*

By Proposition 5.1, we have the following relation between $\mathcal{W}(G)$ and $\mathcal{W}(G - u)$, where $u \in V$.

Proposition 5.3 *Let $u \in \{1, n\}$. If $\{w : \min\{u, v\} < w < \max\{u, v\}\} \subseteq N_G(u) \cup N_G(v)$ holds for every $v \in N_G(u)$, then $\mathcal{W}(G) = \emptyset$ if and only if $\mathcal{W}(G - u) = \emptyset$.*

By Proposition 5.3, the following corollary follows.

Corollary 5.1 *Let $u \in \{1, n\}$. If either $u = 1$ and $N_G(u) = \{2, 3, \cdots, k\}$, or $u = n$ and $N_G(u) = \{k, k + 1, \cdots, n - 1\}$, where $1 \leq k \leq n$, then $\mathcal{W}(G) = \emptyset$ if and only if $\mathcal{W}(G - u) = \emptyset$.*

Applying Proposition 5.3 or Corollary 5.1, we find a family of graphs in $\mathbb{W}$ which are not complete multi-partite graphs.

Proposition 5.4 *Let $G = (V, E)$ be a simple graph on n vertices.*

(i) *If u is a vertex in G such that $G - u$ is a complete multi-partite graph, then $G \in \mathbb{W}$;*

(ii) *Assume that $\{u,v\}$ is an independent set of G such that $G-\{u,v\}$ is a complete multi-partite graph. If either $\{u,v\}$ is a dominating set of G or $N_G(u)\cap N_G(v)=\emptyset$, then $G\in\mathbb{W}$.*

Proof. (i). Assume that $k=|N_G(u)|$. Let ω be a bijection from V to $[n]$ such that $\omega(u)=1$ and $2\le\omega(w)\le k+1$ for all $w\in N_G(u)$. As $G-u$ is a complete multi-partite graph, by Proposition 5.2 (i), $\mathcal{W}((G-u)_{\omega'})=\emptyset$, where ω' is the mapping of ω restricted to $V-\{u\}$. By Corollary 5.1, $\mathcal{W}(G_\omega)=\emptyset$ and so $G\in\mathbb{W}$. Thus (i) holds.

(ii). We first the case that $\{u,v\}$ is a dominating set of G. Assume that $d_G(u)=n_1$ and $d_G(v)=n_2$. Then $|N_G(u)\cap N_G(v)|=n_1+n_2-n+2$, $|N_G(u)\setminus N_G(v)|=n-2-n_2$ and $|N_G(v)\setminus N_G(u)|=n-2-n_1$. Let ω be a bijection from V to $[n]$ such that

(a) $\omega(u)=1$, $\omega(v)=n$;

(b) $2\le\omega(w)\le n-1-n_2$ for all $w\in N_G(u)\setminus N_G(v)$;

(c) $n-n_2\le\omega(w)\le n_1+1$ for all $w\in N_G(u)\cap N_G(v)$; and

(d) $n_1+2\le\omega(w)\le n-1$ for all $w\in N_G(v)\setminus N_G(u)$.

As $G-\{u,v\}$ is a complete multi-partite graph, by Proposition 5.2 (i), $\mathcal{W}((G-\{u,v\})_{\omega''})=\emptyset$, where ω'' is the mapping of ω restricted to $V-\{u,v\}$. As ω satisfies the above conditions (a), (b), (c) and (d), by Corollary 5.1, $\mathcal{W}((G-u)_{\omega'})=\emptyset$ and $\mathcal{W}(G_\omega)=\emptyset$. Thus $G\in\mathbb{W}$.

Now consider the case that $N_G(u)\cap N_G(v)=\emptyset$. Assume that $d_G(u)=n_1$ and $d_G(v)=n_2$. Then $n_1+n_2\le n-2$. Let ω be a bijection from V to $[n]$ such that

(a') $\omega(u)=1$, $\omega(v)=n$;

(b') $2\le\omega(w)\le n_1+1$ for all $w\in N_G(u)$; and

(c') $n-n_2\le\omega(w)\le n-1$ for all $w\in N_G(v)$.

As $G-\{u,v\}$ is a complete multi-partite graph, by Proposition 5.2 (i), $\mathcal{W}((G-\{u,v\})_{\omega''})=\emptyset$, where ω'' is the mapping of ω restricted to $V-\{u,v\}$. As ω satisfies the above conditions (a'), (b') and (c'), by Corollary 5.1, $\mathcal{W}((G-u)_{\omega'})=\emptyset$ and $\mathcal{W}(G_\omega)=\emptyset$. Thus $G\in\mathbb{W}$. □

In general, it seems not easy to determine all graphs in $\mathbb{W}$. We now propose the following problem.

Problem 5.1 *Characterize the family $\mathbb{W}$.*

As an example of studying Problem 5.1, we now consider trees. For any tree T on n vertices, if T is a star or a path, then it can be verified easily that $T\in\mathbb{W}$.

Now assume that $n \geq 5$. Let T' denote the tree obtained from T by removing all vertices of degree 1. If T' is a path, we can prove that $T \in \mathbb{W}$. Assume that T' is a path of order k: $u_1u_2\cdots u_k$. Then T is a tree obtained from T' by adding c_i new vertices and adding a new edge joining u_i to each of them for all $i = 1, 2, \cdots, k$, where $c_1, c_2, \cdots, c_k$ are some non-negative integers. We first label each u_i by $c_1+\cdots+c_i+i$. Then we label the c_i leaves adjacent to u_i by numbers in the set $\{j : c_1 + \cdots + c_{i-1} + i \leq j \leq c_1 + \cdots + c_i + i - 1\}$. An example is shown in Figure 4.

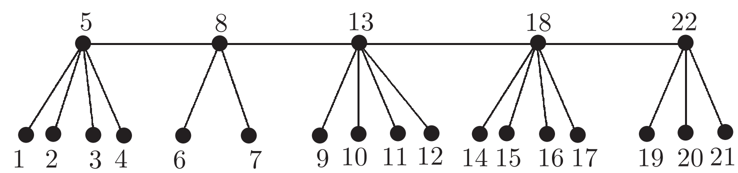


Figure 4: Labeling vertices in a tree T whose non-leaf vertices induce a path

If T' is not a path, we believe $T \notin \mathbb{W}$. For example, for the tree T in Figure 5, T' is not a path. It is left to the readers to verify that $T \notin \mathbb{W}$.

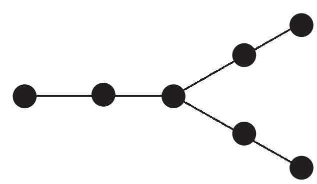


Figure 5: A tree which does not belong to $\mathbb{W}$

Conjecture 5.1 *For any tree T on n vertices, $T \in \mathbb{W}$ if and only if either $n \leq 2$ or the tree T' obtained from T by deleting all vertices of degree 1 is a path.*

5.3 Interpretations of d_i's in Theorem 1.4

Let $D = (V, A)$ be an acyclic digraph with $V = [n]$, where $n \geq 3$, and let $d_0, d_1, \cdots, d_{n-3}$ be the numbers in Theorem 1.4. In the following, we give an interpretation of d_k's for a special case.

Let $\mathcal{S}ink(D)$ be the set of sinks of D. Recall that for $u \in V$, $R_D(u)$ is the set of $v \in V - \{u\}$ such that D has a directed path from u to v. Thus $u \in \mathcal{S}ink(D)$ if and only if $R_D(u) = \emptyset$.

Proposition 5.5 *Let $u \in V$ with either $u \in \mathcal{S}ink(D)$ or $\max(V \setminus R_D(u)) < \min R_D(u)$ and $y < \min R_D(y)$ for each $y \in R_D(u) - \mathcal{S}ink(D)$. If $\mathcal{W}(D) = \mathcal{W}(D, u)$, then for $j = 0, 1, 2, \cdots, n-3$, d_j is the number of D-respecting orderings $\pi = (a_1, \cdots, a_i, u, a_{i+1}, \cdots, a_{n-1})$ such that* *(i) either* $\delta_D(\pi) = j + 2$ *and* $\{a_i, u, a_{i+1}\} \in \mathcal{W}(D)$ *with* $a_i > a_{i+1} > u$*; or*

(ii) $\delta_D(\pi) = j$ *and* $\{a_i, u, a_{i+1}\} \in \mathcal{W}(D)$ *with* $a_i > u > a_{i+1}$.

Proof. Let $S = R_D(u)$. It is known that when $u \notin \mathcal{S}ink(D)$, $\min S \geq 1 + \max(V \setminus S)$. Assume that $u \notin \mathcal{S}ink(D)$ and $\min S = 1 + \max(V - S)$. Let $D' = (V', A')$ be the digraph obtained from D by relabeling each vertex v by $2v$ to get a new digraph D'. It is easy to verify that $W = \{a, b, c\} \in \mathcal{W}(D)$ if and only if $W' = \{2a, 2b, 2c\} \in \mathcal{W}(D')$, $\delta_D(\pi) = \delta_{D'}(\pi')$, where $\pi' = (2a_1, 2a_2, \cdots, 2a_n)$ for $\pi = (a_1, a_2, \cdots, a_n)$, and so $\Psi(D', x) = \Psi(D, x)$. Observe that $\min S' \geq 2 + \max(V' - S')$. We can replace D by D' for the proof of this result.

Thus we may assume that either $u \in \mathcal{S}ink(D)$ or $\min S \geq 2 + \max(V \setminus S)$. We are now going to complete the proof by showing the following claims.

Claim 1: Let $r = \max(V) + 1$ if $u \in \mathcal{S}ink(D)$, and let $r = \min(S) - 1$ otherwise. Then $\Psi(D_{u \to r}, x) = \Omega(\bar{D}, x)$.

Clearly, $r > \max \mathcal{P}_D(u)$ when $u \in \mathcal{S}ink(D)$, and $\max \mathcal{P}_D(u) < r < \min F_D(u)$ otherwise. It is also clear that $r \notin V$. By Lemma 3.2, $\mathcal{W}(D_{u \to r}) = \mathcal{W}(D - u)$. By the given condition, $\mathcal{W}(D - u) = \emptyset$, implying that $\mathcal{W}(D_{u \to r}) = \emptyset$. Thus, by Theorem 1.3, $\Psi(D_{u \to r}), x) = \Omega(\bar{D}_{u \to r}, x) = \Omega(\bar{D}, x)$.

Claim 2: Let r be the number given in the previous Claim. Then $\Psi(D, x) - \Psi(D_{u \to r}, x) = \sum_{j=0}^{n-3} d_j \binom{x+j}{n-2}$, where d_j is the number defined in Proposition 5.5.

By Theorem 3.3, $\Psi(D, x) - \Psi(D_{u \to r}, x) = \sum_{j=0}^{n-3} d_j \binom{x+j}{n-2}$ holds with $d_j = c_{j+2}(D, u) + c'_j(D, u)$ for $j = 0, 1, \cdots, n-3$. By definitions of $c_j(D, u)$ and $c'_j(D, u)$, Claim 2 holds.

By Claims 1 and 2, the result holds. □

Note that Proposition 5.5 gives an interpretation of d_i for a special case only. In general, we would like to propose the following problem.

Problem 5.2 *Interpret the numbers* d_i *in Theorem 1.4.*

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References

[1] G.D. Birkhoff, A determinantal formula for the number of ways of coloring a map, *Ann. of Math.* **14** (1912), 42-46.

[2] G.D. Birkhoff and D.C. Lewis, Chromatic polynomials, *Trans. Amer. Math. Soc.* **60** (1946), 355-451.

[3] F.M. Dong, K.M. Koh and K.L. Teo, *Chromatic polynomials and chromaticity of graphs*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005.

[4] F.M. Dong, Proof of a chromatic polynomial conjecture, *J. Combin. Theory Ser. B* **78** (2000), 35-44.

[5] B. Jackson, A zero-free interval for chromatic polynomials of graphs, *Combin. Probab. Comput.* **2** (1993), 325–336.

[6] B. Jackson, Chromatic polynomials, L.W. Beineke, R.J. Wilson (Eds.), Topics in Chromatic Graph Theory, vol. 156, Cambridge University Press, 2015, pp. 56–72.

[7] R.C. Read, An introduction to chromatic polynomials, *J. Combin. Theory* **4** (1968), 52–71.

[8] R.C. Read and W.T. Tutte, Chromatic polynomials, in *Selected Topics in Graph Theory 3*, Academic Press, 1988, 15–42.

[9] G. Royle, Recent results on chromatic and flow roots of graphs and matroids, in: Surveys in combinatorics, London Math. Soc. Lecture Note Ser., vol. 365, Cambridge Univ. Press, Cambridge, 2009, pp. 289–327.

[10] A.D. Sokal, Bounds on the complex zeros of (di)chromatic polynomials and Potts-model partition functions, *Combin. Probab. Comput.* **10** (2001), 41–77.

[11] A.D. Sokal, Chromatic roots are dense in the whole complex plane, *Combin. Probab. Comput.* **13** (2004), 221–261.

[12] R.P. Stanley, Acyclic orientations of graphs, *Discrete Math.* **5** (1973), 171–178.

[13] R.P. Stanley, A chromatic-like polynomial for ordered sets, in: *Proc. second Chapel Hill conference on combinatorial mathematics and its applications* (1970), 421-427.

[14] C. Thomassen, The zero-free intervals for chromatic polynomials of graphs, *Combin. Probab. Comput.* **6** (1997), 497–506.

[15] C. Thomassen, Chromatic roots and Hamiltonian paths, *J. Combin. Theory Ser. B* **80** (2000), 218–224.

[16] H. Whitney, A logical expansion in mathematics, *Bull. Amer. Math. Soc.* **38** (1932), 572–579.

[17] D.R. Woodall, Zeros of chromatic polynomials, in *Combinatorial Survey, Proc. Sixth British Combinat. Conf.* (ed. P.J. Cameron), Academic Press, 1977, 199-223.