# On non-feasible edge sets in matching-covered graphs 

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#### Abstract

Let $G=(V, E)$ be a matching-covered graph and $X$ be an edge set of $G$. $X$ is said to be feasible if there exist two perfect matchings $M_{1}$ and $M_{2}$ in $G$ such that $\left|M_{1} \cap X\right| \not \equiv\left|M_{2} \cap X\right|(\bmod 2)$. For any $V_{0} \subseteq V, X$ is said to be switching-equivalent to $X \oplus \nabla_{G}\left(V_{0}\right)$, where $\nabla_{G}\left(V_{0}\right)$ is the set of edges in $G$ each of which has exactly one end in $V_{0}$ and $A \oplus B$ is the symmetric difference of two sets $A$ and $B$. Lukot'ka and Rollová showed that when $G$ is regular and bipartite, $X$ is non-feasible if and only if $X$ is switching-equivalent to $\emptyset$. This article extends Lukot'ka and Rollová's result by showing that this conclusion holds as long as $G$ is matchingcovered and bipartite. This article also studies matching-covered graphs $G$ whose non-feasible edge sets are switching-equivalent to $\emptyset$ or $E$ and partially characterizes these matching-covered graphs in terms of their ear decompositions. Another aim of this article is to construct infinite many $r$ connected and $r$-regular graphs of class 1 containing non-feasible edge sets not switching-equivalent to either $\emptyset$ or $E$ for an arbitrary integer $r$ with $r \geq 3$, which provides negative answers to problems asked by Lukot'ka and Rollová and He , et al respectively.


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## 1 Introduction and Preliminary

This article studies finite and undirected loopless graphs. Let $G=(V, E)$ be a graph. A perfect matching of $G$ is a set of independent edges which covers all vertices of $G . G$ is said to be matching-covered if it is connected and each edge of $G$ is contained in some perfect matching of $G$. It is not difficult to verify that any regular graph of class 1 is matching-covered.

For a matching-covered graph $G$, an edge set $X$ of $G$ is said to be feasible if $G$ has two perfect matchings $M_{1}$ and $M_{2}$ such that $\left|M_{1} \cap X\right| \not \equiv\left|M_{2} \cap X\right|(\bmod 2)$ holds. Thus an edge set $X$ of $G$ is non-feasible if and only if $\left|M_{1} \cap X\right| \equiv$ $\left|M_{2} \cap X\right|(\bmod 2)$ holds for every pair of perfect matchings $M_{1}$ and $M_{2}$ of $G$. For example, $E$ and $\emptyset$ are non-feasible edge sets of $G$. In Theorem 1.4, we extend the definition of a feasible edge to connected graphs which are not matching-covered.

For any $V_{0} \subseteq V$, let $\nabla_{G}\left(V_{0}\right)$ be the set of edges in $G$ each of which has exactly one end in $V_{0}$. For any vertex $v$ in $G, \nabla_{G}(\{v\})$ is exactly the set of edges in $G$ which are incident with $v$. For any $X, Y \subseteq E, X$ and $Y$ are called switching-equivalent, denoted by $X \stackrel{s}{\sim}_{G} Y$, if $X=Y \oplus \nabla_{G}\left(V_{0}\right)$ holds for a set $V_{0}$ of vertices in $G$, where $A \oplus B$ is the symmetric difference of two sets $A$ and $B$, i.e., $A \oplus B=(A-B) \cup(B-A)$. Let $X \stackrel{s}{\sim}_{G} Y$ denote the case when edge sets $X$ and $Y$ are not switching-equivalent in $G$.

Lukot'ka and Rollová [9] proved that the property "being feasible" is invariant to switching-equivalent edge sets.

Theorem 1.1 ([9]). Let $G$ be a matching-covered graph and $X$ and $Y$ be edge subsets of $G$. If $X \stackrel{\mathcal{S}}{\sim}_{G} Y$, then $X$ is feasible if and only if $Y$ is feasible.

For a matching-covered graph $G=(V, E)$, let $\mathcal{F}(G)$ be the set of feasible edge sets of $G$ and let $\overline{\mathcal{F}}(G)$ be the set of non-feasible edge sets of $G$. Thus $\mathcal{F}(G) \cup \overline{\mathcal{F}}(G)$ is the power set of $E$. Clearly $\{\emptyset, E\} \subseteq \overline{\mathcal{F}}(G)$. Theorem 1.1 implies that $\left\{X \subseteq E: X \stackrel{\mathcal{\sim}}{\sim}_{G} \emptyset\right\} \subseteq \overline{\mathcal{F}}(G)$ and $\left\{X \subseteq E: X \stackrel{\mathcal{\sim}}{\sim}_{G} E\right\} \subseteq \overline{\mathcal{F}}(G)$. For bipartite and regular graphs, Lukot'ka and Rollová [9] got the following conclusion, described by notations in this article.

Theorem 1.2 ([9]). If $G$ is a bipartite and regular graph, then $\overline{\mathcal{F}}(G)=\{X \subseteq$ $\left.E: X \stackrel{s}{\sim}_{G} \emptyset\right\}$.

Note that any bipartite and regular graph is matching-covered, because any bipartite graph is a class 1 graph (see [4]) and any regular graph of class 1 is matching-covered.

In this article, we will extend Theorem 1.2 as stated below.

Theorem 1.3. Let $G=(V, E)$ be a matching-covered graph. Then the following statements are equivalent:
(i). $G$ is bipartite;
(ii). $\overline{\mathcal{F}}(G)=\left\{X \subseteq E: X \stackrel{s}{\sim}_{G} \emptyset\right\}$;
(iii). $\overline{\mathcal{F}}(G)=\left\{X \subseteq E: X \stackrel{s}{\sim}_{G} E\right\}$.

For any matching-covered graph $G=(V, E),\left\{X \subseteq E: X \stackrel{s}{\sim}_{G} \emptyset\right.$ or $\left.X \stackrel{s}{\sim}_{G} E\right\}$ is a subset of $\overline{\mathcal{F}}(G)$. Let $\overline{\mathcal{F}}^{*}(G)=\overline{\mathcal{F}}(G)-\left\{X \subseteq E: X \stackrel{s}{\sim}_{G} \emptyset\right.$ or $\left.X \stackrel{s}{\sim}_{G} E\right\}$. Then $\overline{\mathcal{F}}^{*}(G)=\emptyset$ holds if and only if $X \stackrel{s}{\sim}_{G} \emptyset$ or $X \stackrel{s}{\sim}_{G} E$ holds for each $X \in \overline{\mathcal{F}}(G)$. It is natural to ask when $\overline{\mathcal{F}}^{*}(G)=\emptyset$ holds. By Theorem 1.3, it holds if $G$ is bipartite and matching-covered. But there exist non-bipartite matching-covered graphs with this property. For example, $K_{4}$ is such a graph.

For a subgraph $G^{\prime}$ of $G$, a single ear of $G^{\prime}$ is a path $P$ of $G$ with an odd length such that both ends of $P$ are in $G^{\prime}$ but its internal vertices are distinct from vertices in $G^{\prime}$. A double ear of $G^{\prime}$ is a pair of vertex disjoint single ears of $G^{\prime}$. An ear of $G^{\prime}$ means a single ear or a double ear of $G^{\prime}$. An ear decomposition of a matching-covered graph $G$ is a sequence

$$
G_{0} \subset G_{1} \subset \cdots \subset G_{r}=G
$$

of matching-covered subgraphs of $G$, where (i) $G_{0}=K_{2}$, and (ii) for each $i$ with $1 \leq i \leq r, G_{i}$ is the union of $G_{i-1}$ and an ear (single or double) of $G_{i-1}$. For $i=1,2, \cdots, r$, let $\epsilon\left(G_{i-1}, G_{i}\right) \in\{1,2\}$ such that $\epsilon\left(G_{i-1}, G_{i}\right)=1$ if and only if $G_{i}$ is the union $G_{i-1}$ and a single ear. A very important result on the study of matching-covered graphs is the existence of an ear decomposition for each matching-covered graph due to Lovász and Plummer [8].

Our second aim in this article is to establish the following conclusions on matching-covered graphs $G$ with $\overline{\mathcal{F}}^{*}(G)=\emptyset$, based on ear decompositions of matching-covered graphs.

Theorem 1.4. Let $G=(V, E)$ be a matching-covered graph with an ear decomposition $G_{0} \subset G_{1} \subset \cdots \subset G_{r}=G$, where $r \geq 1$.
(i). If $\overline{\mathcal{F}}^{*}\left(G_{r-1}\right)=\emptyset$ and $\epsilon\left(G_{r-1}, G_{r}\right)=1$, then $\overline{\mathcal{F}}^{*}(G)=\emptyset$;
(ii). if $\sum_{1 \leq i \leq r} \epsilon\left(G_{i-1}, G_{i}\right) \leq r+1$, then $\overline{\mathcal{F}}^{*}(G)=\emptyset$ holds;
(iii). if $\sum_{1 \leq i \leq r} \epsilon\left(G_{i-1}, G_{i}\right) \geq r+2$ and $\epsilon\left(G_{r-1}, G_{r}\right)=2$, then $\overline{\mathcal{F}}^{*}(G) \neq \emptyset$;
(iv). if $\sum_{1 \leq i \leq r} \epsilon\left(G_{i-1}, G_{i}\right) \geq r+2$ and $\epsilon\left(G_{r-1}, G_{r}\right)=1$, then $\overline{\mathcal{F}}^{*}(G)=\emptyset$ if and only if $X \cap E\left(G_{r-1}-\{u, v\}\right)$ is feasible in the subgraph $G_{r-1}-\{u, v\}$ for each $X \in \overline{\mathcal{F}}^{*}\left(G_{r-1}\right)$, where $E(H)$ is the edge set of a graph $H$ and $u, v$ are the two ends of the single ear $P_{r}$ added to $G_{r-1}$ for obtaining $G_{r}$.

Note that the graph $G_{r-1}-\{u, v\}$ in Theorem 1.4 (iv) is the graph obtained from $G_{r-1}$ by deleting $u$ and $v$ and may not be matching-covered although it contains perfect matchings. By definition, $X^{\prime}=X \cap E\left(G_{r-1}-\{u, v\}\right)$ is feasible in $G_{r-1}-\{u, v\}$ if there exist two perfect matchings $N_{1}$ and $N_{2}$ in $G_{r-1}-\{u, v\}$ such that $\left|N_{1} \cap X^{\prime}\right| \equiv \equiv\left|N_{2} \cap X^{\prime}\right|(\bmod 2)$ holds.

Lukot'ka and Rollová [9] noticed that $\overline{\mathcal{F}}^{*}(P) \neq \emptyset$ holds for the Petersen graph $P$, which is a class 2 graph, and asked the following problem on regular graphs of class 1 , described by notations in this article.

Problem 1.5. Does $\overline{\mathcal{F}}^{*}(G)=\emptyset$ hold for each regular graph $G$ of class 1?
A negative answer to this problem was provided by He , et al [3] who showed that for any $k \geq 3$, there exist infinitely many $k$-regular graphs $G$ of class 1 with an arbitrary large equivalent edge set belonging to $\overline{\mathcal{F}}^{*}(G)$, where a nonempty edge set $S$ of $G$ is called an equivalent set if $S \cap M=\emptyset$ or $S \cap M=S$ holds for all perfect matchings $M$ of $G$. The graphs constructed in [3] giving a negative answer to Problem 1.5 are not 3 -connected and the following problem was further asked in [3].

Problem 1.6. Does Problem 1.5 hold for 3 -connected and $r$-regular graph $G$ with $r \geq 3$ ?

In Section 5, we will provide negative answers to both Problems 1.5 and 1.6 by two constructions of $r$-regular graphs $G$ of class 1 with $\overline{\mathcal{F}}^{*}(G) \neq \emptyset$.

Theorem 1.7. For any integer $r \geq 3$, there are infinitely many $r$-connected and $r$-regular graphs $G$ of class 1 with $\overline{\mathcal{F}}^{*}(G) \neq \emptyset$.

## 2 Preliminary results on $X \subseteq E$ with $X \stackrel{s}{\sim}_{G} \emptyset$ or $X \stackrel{s_{\sim}^{\sim}}{G} E$

Let $G=(V, E)$ be any connected graph which may be not matching-covered. By definition, for any subset $U \subseteq V, \nabla_{G}(U)$ is the set $\{e \in E: e$ joins a vertex in in $U$ and a vertex in $V-U\}$. With the notation $\nabla_{G}(U)$, an edge set $X$ of $G$ with the property that $X \stackrel{s}{\sim}_{G} \emptyset$ or $X \stackrel{s}{\sim}_{G} E$ has the following characterization due to He , et al [3].

Proposition 2.1 ([3]). Let $G=(V, E)$ be a connected graph and $X \subseteq E$. Then
(i). $X \stackrel{s}{\sim}_{G} \emptyset$ iff $X=\nabla_{G}(U)$ for some $U \subseteq V$;
(ii). $X \stackrel{s}{\sim}_{G} E$ if and only if $E(G)-X=\nabla_{G}(U)$ for some $U \subseteq V$.

Proposition 2.1 implies the following corollary immediately. For any graph $G$ and any set $V_{0}$ of vertices in $G$, let $G\left[V_{0}\right]$ denote the subgraph of $G$ induced by $V_{0}$.

Corollary 2.2. Let $G=(V, E)$ be a connected graph and $X \subseteq E$. For any $V_{0} \subseteq V$,
(i). if $X \stackrel{s}{\sim}_{G} \emptyset$, then $X \cap E\left(G\left[V_{0}\right]\right) \stackrel{s}{\sim}_{G\left[V_{0}\right]} \emptyset$;
(ii). if $X \stackrel{s}{\sim}_{G} E$, then $X \cap E\left(G\left[V_{0}\right]\right) \stackrel{s}{\sim}_{G\left[V_{0}\right]} E\left(G\left[V_{0}\right]\right)$.

Obviously, $X \stackrel{s}{\sim}_{G} Y$ implies that $Y \stackrel{s}{\sim}_{G} X$. The transitive property of the relation $\stackrel{s}{\sim}_{G}$ " also holds.

Lemma 2.3. Let $G=(V, E)$ be a connected graph with $X, Y, Z \subseteq E$. If $X \stackrel{s}{\sim}_{G} Y$ and $Y \stackrel{s}{\sim}_{G} Z$, then $X \stackrel{s}{\sim}_{G} Z$ holds.

Proof. Assume that $X \stackrel{s}{\sim}_{G} Y$ and $Y \stackrel{s}{\sim}_{G} Z$. Then $X=Y \oplus \nabla_{G}\left(V_{1}\right)$ and $Y=Z \oplus \nabla_{G}\left(V_{2}\right)$ hold for some $V_{1}, V_{2} \subseteq V$, implying that $X=Z \oplus \nabla_{G}\left(V_{1} \oplus V_{2}\right)$. Thus $X \stackrel{s}{\sim}_{G} Z$ holds.

Assume that $G^{\prime}$ is any connected graph with two distinct vertices $v_{1}$ and $v_{2}$ and $P$ is any path with ends $u_{1}$ and $u_{2}$ such that $G^{\prime}$ and $P$ are vertex-disjoint. Let $\operatorname{Union}_{\left(v_{1}, v_{2}\right)}\left(G^{\prime}, P\right)$ (or simply Union $\left(G^{\prime}, P\right)$ ) denote the graph obtained from $G^{\prime}$ and $P$ by identifying $u_{i}$ and $v_{i}$ for $i=1,2$. For an ear decomposition $G_{0} \subset G_{1} \subset \cdots G_{r}=G$ of a matching-covered graph $G$, if $G_{i}$ is the union of $G_{i-1}$ and a single ear $P_{i}$, then $G_{i}=\operatorname{Union}\left(G_{i-1}, P_{i}\right)$. But, in this section, the results do not depend on the condition that $G^{\prime}$ is matching-covered.

Lemma 2.4. Let $G=\operatorname{Union}\left(G^{\prime}, P\right)$. For any edge set $X=X_{0} \cup X^{\prime}$ of $G$, where $X_{0} \subseteq E(P)$ and $X^{\prime} \subseteq E\left(G^{\prime}\right)$,
(i). if $|E(P)| \equiv 1(\bmod 2)$ and $X \in \overline{\mathcal{F}}(G)$, then $X^{\prime} \in \overline{\mathcal{F}}\left(G^{\prime}\right)$;
(ii). if $\left|X_{0}\right| \equiv 0(\bmod 2)$, then $X \stackrel{s}{\sim}_{G} X^{\prime}$;
(iii). if $\left|X_{0}\right| \equiv 1(\bmod 2)$, then $X \stackrel{s}{\sim}_{G} X^{\prime} \cup\{e\}$ for any $e \in E(P)$;
(iv). if $X^{\prime} \stackrel{s}{\sim}_{G^{\prime}} Y$, then $X \stackrel{s}{\sim}_{G} Y \cup Y_{0}$ for some $Y_{0} \subseteq E(P)$;
(v). if $X^{\prime} \stackrel{s}{\sim}_{G^{\prime}} \emptyset$, then either $X \stackrel{s}{\sim}_{G} \emptyset$ or $X \stackrel{s}{\sim}_{G}\{e\}$ for any $e \in E(P)$;
(vi). if $X^{\prime} \stackrel{s}{\sim}_{G^{\prime}} E\left(G^{\prime}\right)$, then either $X \stackrel{s}{\sim}_{G} E(G)$ or $X \stackrel{s}{\sim}_{G} E(G)-\{e\}$ for any $e \in E(P)$;
(vii). if $X \stackrel{s}{\sim}_{G} \emptyset$, then $X^{\prime} \stackrel{s}{\sim}_{G^{\prime}} \emptyset$; if $X \stackrel{s}{\sim}_{G} E(G)$, then $X^{\prime} \stackrel{s}{\sim}_{G^{\prime}} E\left(G^{\prime}\right)$.

Proof. (i). Assume that the edges in $P$ are $e_{1}, e_{2}, \cdots, e_{2 k-1}$ in the order of the path $P$ such that $e_{i}$ and $e_{i+1}$ have a common end for all $i=1,2, \cdots, 2 k-2$. Suppose that $X^{\prime} \in \mathcal{F}\left(G^{\prime}\right)$. Then $G^{\prime}$ has two perfect matchings $M_{1}$ and $M_{2}$ such that $\left|X^{\prime} \cap M_{1}\right|-\left|X^{\prime} \cap M_{2}\right| \equiv 1(\bmod 2)$. For $i=1,2$, the set $N_{i}$ defined below is a perfect matching of $G$ :

$$
N_{i}=M_{i} \cup\left\{e_{2 j}: j=1,2, \cdots, k-1\right\} .
$$

Observe that

$$
\left|X \cap N_{j}\right|=\left|X^{\prime} \cap M_{j}\right|+\left|X_{0} \cap\left\{e_{2 j}: j=1,2, \cdots, k-1\right\}\right|, \quad \forall j=1,2 .
$$

Thus $\left|X \cap N_{1}\right|-\left|X \cap N_{2}\right|=\left|X^{\prime} \cap M_{1}\right|-\left|X^{\prime} \cap M_{2}\right| \equiv 1(\bmod 2)$, implying that $X$ is feasible in $G$, a contradiction.

Thus (i) holds.
(ii) and (iii) will be proved by applying the following claim.

Claim 1: If $\left|X_{0}\right| \geq 2$, then $X=X_{0} \cup X^{\prime} \stackrel{\mathcal{\sim}}{\sim}_{G} X_{0}^{\prime} \cup X^{\prime}$ holds for some $X_{0}^{\prime} \subset X_{0}$ with $\left|X_{0}^{\prime}\right|=\left|X_{0}\right|-2$.

Assume that $\left|X_{0}\right| \geq 2$. Then there exists subpath $P_{0}$ of $P$ such that $X \cap$ $E\left(P_{0}\right)=\emptyset$ and $\nabla_{G}\left(V\left(P_{0}\right)\right) \subseteq X$, implying that $X \stackrel{\sim}{\sim}_{G} X \oplus \nabla_{G}\left(V\left(P_{0}\right)\right)=$ $X_{0}^{\prime} \cup X^{\prime}$, where $X_{0}^{\prime}=X_{0} \oplus \nabla_{G}\left(V\left(P_{0}\right)\right) \subset X_{0}$ and $\left|X_{0}^{\prime}\right|=\left|X_{0}\right|-2$. Thus the claim holds.
(ii). Assume that $\left|X_{0}\right|>0$ and $\left|X_{0}\right| \equiv 0(\bmod 2)$. (ii) follows by applying Claim 1 repeatedly.
(iii). Applying Claim 1 repeatedly, $X \stackrel{s}{\sim}_{G}\{e\} \cup X^{\prime}$ holds for some $e \in E(P)$. Now let $e^{\prime}$ be any edge in $P$ different from $e$. There exists a subpath $P^{\prime}$ of $P$ such that $\nabla_{G}\left(V\left(P^{\prime}\right)\right)=\left\{e, e^{\prime}\right\}$. Thus $\{e\} \cup X^{\prime} \stackrel{s}{\sim}_{G}\left(\{e\} \cup X^{\prime}\right) \oplus \nabla_{G}\left(V\left(P^{\prime}\right)\right)=\left\{e^{\prime}\right\} \cup X^{\prime}$ and the result holds.
(iv). It is trivial when $X^{\prime}=Y$. Now assume that $X^{\prime} \neq Y$. Then $Y=$ $X^{\prime} \oplus \nabla_{G^{\prime}}\left(V_{0}\right)$ for some non-empty set $V_{0} \subset V\left(G^{\prime}\right)$.

As $G=\operatorname{Union}\left(G^{\prime}, P\right)$, there are three cases on the structure of $G$, i.e., $\left|\left\{v_{1}, v_{2}\right\} \cap V_{0}\right| \in\{0,1,2\}$, where $v_{1}, v_{2}$ are the two vertices in $G^{\prime}$ at which the ends of $P$ are identified with. But $\left|\left\{v_{1}, v_{2}\right\} \cap V_{0}\right|=2$ implies that $\mid\left\{v_{1}, v_{2}\right\} \cap\left(V\left(G^{\prime}\right)-\right.$ $\left.V_{0}\right) \mid=0$. Thus, we need only to consider the two cases: $\left|\left\{v_{1}, v_{2}\right\} \cap V_{0}\right|=0$ or $\left|\left\{v_{1}, v_{2}\right\} \cap V_{0}\right|=1$, as shown in Figure 1.


Figure 1: Two cases for the two ends of $P$

In both cases, $Y=X^{\prime} \oplus \nabla_{G^{\prime}}\left(V_{0}\right)$ implies that $X \oplus \nabla_{G}\left(V_{0}\right)=Y \cup Y_{0}$ holds for some $Y_{0} \subseteq E(P)$.

Thus (iv) holds.
(v). As $X^{\prime} \stackrel{s}{\sim}_{G^{\prime}} \emptyset$, the result of (iv) implies that $X \stackrel{s}{\sim}_{G} Y_{0}$ where $Y_{0} \subseteq E(P)$. The results of (ii) and (iii) imply that either $Y_{0} \stackrel{s}{\sim}_{G} \emptyset$ or $Y_{0} \stackrel{s}{\sim}_{G}\{e\}$ for any $e \in E(P)$.

Thus (v) holds.
(vi). As $X^{\prime} \stackrel{s}{\sim}_{G^{\prime}} E\left(G^{\prime}\right)$, the result of (iv) implies that $X \stackrel{s}{\sim}_{G} E\left(G^{\prime}\right) \cup Y_{0}$ where $Y_{0} \subseteq E(P)$. The results of (ii) and (iii) imply that $\left(E\left(G^{\prime}\right) \cup Y_{0}\right) \stackrel{\mathcal{S}_{\sim}^{\sim}}{G} E(G)$ when $\left|Y_{0}\right| \equiv|E(P)|(\bmod 2)$, and $\left(E\left(G^{\prime}\right) \cup Y_{0}\right) \stackrel{\mathcal{S}}{\sim}_{G} E(G)-\{e\}$ when $\left|Y_{0}\right| \not \equiv$ $|E(P)|(\bmod 2)$.

Thus (vi) holds.
(vii). Suppose that $X \stackrel{s}{\sim}_{G} \emptyset$ holds. By Proposition 2.1, $X=\nabla_{G}(U)$ holds for some $U \subseteq V(G)$. Then $X^{\prime}=\nabla_{G^{\prime}}\left(U-U_{0}\right)$, implying that $X^{\prime} \stackrel{\mathcal{S}}{\sim}_{G^{\prime}} \emptyset$, where $U_{0}$ is the set of internal vertices of $P$.

Now suppose that $X \stackrel{s}{\sim}_{G} E(G)$ holds. By Proposition 2.1, $E(G)-X=$ $\nabla_{G}(U)$ holds for some $U \subseteq V(G)$. Then $E\left(G^{\prime}\right)-X^{\prime}=\nabla_{G^{\prime}}\left(U-U_{0}\right)$, where $U_{0}$ is defined above, implying that $X^{\prime} \stackrel{s}{\sim}_{G^{\prime}} E\left(G^{\prime}\right)$.

Thus (vii) holds.
For distinct vertices $v_{1}, v_{2}, v_{3}, v_{4}$ in a graph $G^{\prime}$ and any two vertex-disjoint paths $P_{1}, P_{2}$ with $V\left(P_{i}\right) \cap V\left(G^{\prime}\right)=\emptyset$ for $i=1,2$, let $\operatorname{Union}_{\left(v_{1}, v_{2}, v_{3}, v_{4}\right)}\left(G^{\prime}, P_{1}, P_{2}\right)$ (or simply $\operatorname{Union}\left(G^{\prime}, P_{1}, P_{2}\right)$ ) be the graph $\operatorname{Union}_{\left(v_{3}, v_{4}\right)}\left(G^{\prime \prime}, P_{2}\right)$, where $G^{\prime \prime}=$ $\operatorname{Union}_{\left(v_{1}, v_{2}\right)}\left(G^{\prime}, P_{1}\right)$.

Lemma 2.5. Let $G=\operatorname{Union}\left(G^{\prime}, P_{1}, P_{2}\right)$. For any edge set $X=X_{0} \cup X^{\prime}$ of $G$, where $X^{\prime} \subseteq E\left(G^{\prime}\right)$ and $X_{0} \subseteq E\left(P_{1}\right) \cup E\left(P_{2}\right)$, if $X^{\prime}{\stackrel{\stackrel{s}{\sim}}{G^{\prime}}} \emptyset$, then $X \stackrel{\mathcal{S}}{\sim}_{G} \emptyset$, or $X \stackrel{s}{\sim}_{G}\{e\}$ for some $e \in E\left(P_{1}\right) \cup E\left(P_{2}\right)$, or $X \stackrel{s}{\sim}_{G}\left\{e_{1}, e_{2}\right\}$ where $e_{i} \in E\left(P_{i}\right)$ for
$i=1,2$.
Proof. Let $G^{\prime \prime}=\operatorname{Union}\left(G^{\prime}, P_{1}\right)$. As $X^{\prime} \stackrel{s}{\sim}_{G^{\prime}} \emptyset$, Lemma $2.4(\mathrm{v})$ implies that $X-E\left(P_{2}\right) \stackrel{s}{\sim}_{G^{\prime \prime}} \emptyset$ or $X-E\left(P_{2}\right) \stackrel{s}{\sim}_{G^{\prime \prime}}\left\{e_{1}\right\}$ for any $e_{1} \in E\left(P_{1}\right)$.

Note that $G=\operatorname{Union}\left(G^{\prime \prime}, P_{2}\right)$. If $X-E\left(P_{2}\right) \stackrel{\mathcal{S}}{\sim}_{G^{\prime \prime}} \emptyset$, then Lemma $2.4(\mathrm{v})$ implies that either $X \stackrel{\mathcal{S}}{\sim}_{G} \emptyset$ or $X \stackrel{\mathcal{\sim}}{\sim}_{G}\left\{e_{2}\right\}$ holds for any $e_{2} \in E\left(P_{1}\right)$.

If $X-E\left(P_{2}\right) \stackrel{s}{\sim}_{G^{\prime \prime}}\left\{e_{1}\right\}$ for any $e_{1} \in E\left(P_{1}\right)$, then Lemma 2.4 (iv) implies that $X \stackrel{s}{\sim}_{G}\left\{e_{1}\right\} \cup Y_{0}$ for some $Y_{0} \subseteq E\left(P_{2}\right)$. Lemma 2.4 (ii) and (iii) further imply that either $X \stackrel{s}{\sim}_{G}\left\{e_{1}\right\}$ or $X \stackrel{s}{\sim}_{G}\left\{e_{1}\right\} \cup\left\{e_{2}\right\}$ holds for any $e_{2} \in E\left(P_{2}\right)$.

Thus the result holds.

## 3 Proof of Theorem 1.3

For any matching-covered graph $G$, the following basic properties follow directly from the definitions of $\mathcal{F}(G)$ and $\overline{\mathcal{F}}(G)$.

Lemma 3.1. Let $G$ be a matching-covered graph with $|E(G)| \geq 2$ and $X \subseteq$ $E(G)$. If either $|X|=1$ or $|X|=|E|-1$, then $X \in \mathcal{F}(G)$.

By applying Lemma 2.4, we can prove that for any matching-covered graphs $G^{\prime}$ and $G=\operatorname{Union}\left(G^{\prime}, P\right), \overline{\mathcal{F}}^{*}\left(G^{\prime}\right)=\emptyset$ implies that $\overline{\mathcal{F}}^{*}(G)=\emptyset$.

Lemma 3.2. Let $G^{\prime}$ and $G=\operatorname{Union}\left(G^{\prime}, P\right)$ be matching-covered graphs, where $P$ is a single ear of $G^{\prime}$. For any $X \in \overline{\mathcal{F}}(G)$,
(i). $X \cap E\left(G^{\prime}\right) \stackrel{s}{\sim}_{G^{\prime}} \emptyset$ if and only if $X \stackrel{s}{\sim}_{G} \emptyset$;
(ii). $X \cap E\left(G^{\prime}\right) \stackrel{s}{\sim}_{G^{\prime}} E\left(G^{\prime}\right)$ if and only if $X \stackrel{s}{\sim}_{G} E(G)$;
(iii). $X \cap E\left(G^{\prime}\right) \in \overline{\mathcal{F}}^{*}\left(G^{\prime}\right)$ if and only if $X \in \overline{\mathcal{F}}^{*}(G)$.

Proof. (i). $(\Leftarrow)$ It follows directly from Lemma 2.4 (vii).
$(\Rightarrow)$ As $X \cap E\left(G^{\prime}\right) \stackrel{s}{\sim}_{G_{r-1}} \emptyset$, Lemma $2.4(\mathrm{v})$ implies that $X \stackrel{s}{\sim}_{G} \emptyset$ or $X \stackrel{s}{\sim}_{G}\{e\}$ for any $e \in E\left(P_{r}\right)$.

Suppose that $X{\underset{\sim}{\sim}}_{G}\{e\}$ for some $e \in E\left(P_{r}\right)$. As $X \in \overline{\mathcal{F}}(G)$, Theorem 1.1 implies that $\{e\} \in \overline{\mathcal{F}}(G)$. But, as $|E(G)| \geq 2$, Lemma 3.1 implies that $\{e\} \in$ $\mathcal{F}(G)$, a contradiction. Thus $X \stackrel{s}{\sim}_{G} \emptyset$.
(ii). $(\Leftarrow)$ It follows directly from Lemma 2.4 (vii).
$(\Rightarrow)$. As $X \cap E\left(G^{\prime}\right) \stackrel{s}{\sim}_{G_{r-1}} E\left(G^{\prime}\right)$, Lemma 2.4 (vi) implies that $X \stackrel{s}{\sim}_{G} E(G)$ or $X \stackrel{s}{\sim}_{G} E(G)-\{e\}$ for any $e \in E\left(P_{r}\right)$.

Suppose that $X \stackrel{s}{\sim}_{G} E(G)-\{e\}$ for some $e \in E\left(P_{r}\right)$. As $X \in \overline{\mathcal{F}}(G)$, Theorem 1.1 implies that $E(G)-\{e\} \in \overline{\mathcal{F}}(G)$ holds. As $|E(G)| \geq 2$, Lemma 3.1 implies that $E(G)-\{e\} \in \mathcal{F}(G)$, a contradiction. Thus $X \stackrel{s}{\sim}_{G} E(G)$.
(iii). By Lemma 2.4 (i), $X \in \overline{\mathcal{F}}(G)$ implies that $X \cap E\left(G^{\prime}\right) \in \overline{\mathcal{F}}\left(G^{\prime}\right)$. Then the result follows from (i) and (ii) directly.

An ear decomposition $G_{0} \subset G_{1} \subset \cdots \subset G_{r}$ of a matching-covered graph $G$ is called a single-ear decomposition if $\epsilon\left(G_{i-1}, G_{i}\right)=1$ holds for all $i=1, \cdots, r$. A matching-covered graph may have no single-ear decompositions. For example, the complete graph $K_{4}$ does not have. However, every matching-covered bipartite graph has a single-ear decomposition.

Theorem 3.3. Let $G$ be a matching-covered graph $G$.
(i). [8] G has an ear decomposition;
(ii). [1, 5, 2] $G$ is bipartite if and only if $G$ has a single-ear decomposition.

Now we are going to prove Theorem 1.3.

Proof of Theorem 1.3: By Proposition 2.1 (i), each edge set $X$ with $X \stackrel{\mathcal{S}}{\sim}_{G}$ $\emptyset$ induces a bipartite subgraph in $G$, implying that $E \stackrel{s}{\nsim}{ }_{G} \emptyset$ holds whenever $G$ is not bipartite. Hence, Theorem 1.3 (ii) implies Theorem 1.3 (i).

For any bipartite graph $G=(V, E), E \stackrel{s}{\sim}_{G} \emptyset$ holds. Thus, Lemma 2.3 implies that $\left\{X \subseteq E: X \stackrel{s}{\sim}_{G} \emptyset\right\}$ and $\left\{X \subseteq E: X \stackrel{s}{\sim}_{G} E\right\}$ are the same set. Thus, (ii) and (iii) in Theorem 1.3 are equivalent.

So, to prove Theorem 1.3, it suffices to show that Theorem 1.3 (i) implies Theorem 1.3 (ii).

Assume that $G$ is bipartite and matching-covered. By Theorem 3.3 (ii), $G$ has a single-ear decomposition $G_{0} \subset G_{1} \subset \cdots \subset G_{r}=G$, where $G_{0} \cong K_{2}$. Thus, for $i=1,2,3, \cdots, r, G_{i}=\operatorname{Union}\left(G_{i-1}, P_{i}\right)$ holds for some single ear $P_{i}$ of $G_{i-1}$.

If $r=0$, i.e., $G \cong K_{2}$ and (i) implies (ii) obviously. Now assume that $r \geq 1$ and the result holds for $G_{r-1}$. For any $X \in \overline{\mathcal{F}}(G)$, Lemma 2.4 (i) implies that $X \cap E\left(G_{r-1}\right) \in \overline{\mathcal{F}}\left(G_{r-1}\right)$. By the assumption, the result holds for $G_{r-1}$. Thus $X \cap E\left(G_{r-1}\right) \stackrel{\stackrel{s}{\sim}}{G_{r-1}}{ }^{\prime} \emptyset$ holds. Then, Lemma 3.2 (i) implies that $X \stackrel{s}{\sim}_{G} \emptyset$.

Hence Theorem 1.3 is proven.

## 4 Proof of Theorem 1.4

The following two lemmas will be applied for proving Theorem 1.4.
Lemma 4.1. Let $G^{\prime}$ and $G=\operatorname{Union}\left(G^{\prime}, P_{1}, P_{2}\right)$ be matching-covered graphs, where $P_{1}$ and $P_{2}$ form a double ear of $G^{\prime}$. Assume that $\operatorname{Union}\left(G^{\prime}, P_{i}\right)$ is not matching-covered for $i=1,2$. Then $\overline{\mathcal{F}}^{*}(G)=\emptyset$ if and only if $G^{\prime}$ is bipartite.

Proof. $(\Rightarrow)$ Suppose that $G^{\prime}$ is not bipartite.
Let $X_{0}=E\left(P_{1}\right) \cup E\left(P_{2}\right)$, where $E\left(P_{i}\right)=\left\{e_{i, 1}, e_{i, 2}, \cdots, e_{i, 2 k_{i}-1}\right\}$ for $i=1,2$ and $e_{i, j}$ and $e_{i, j+1}$ have a common end for all $j=1,2, \cdots, 2 k_{i}-2$. As Union $\left(G^{\prime}, P_{i}\right)$ is not matching-covered for both $i=1,2$, for each perfect matching $M$ of $G$, one of the following holds:

$$
M \cap X_{0}=\bigcup_{1 \leq i \leq 2}\left\{e_{i, 2 t-1}: t=1,2, \cdots, k_{i}\right\}
$$

or

$$
M \cap X_{0}=\bigcup_{1 \leq i \leq 2}\left\{e_{i, 2 t}: t=1,2, \cdots, k_{i}-1\right\}
$$

Thus $\left|M \cap X_{0}\right| \equiv 0(\bmod 2)$ holds for all perfect matchings $M$ of $G$, implying that $X_{0} \in \overline{\mathcal{F}}(G)$.

As $G^{\prime}$ is not bipartite, $G-X_{0}$ is not bipartite. Thus Proposition 2.1 (ii) implies that $X_{0} \stackrel{s}{\sim}_{G} E(G)$.

As $\left|E\left(P_{1}\right)\right| \equiv\left|E\left(P_{2}\right)\right| \equiv 1(\bmod 2)$, Lemma 2.4 (iii) implies that $X_{0} \stackrel{s}{\sim}_{G}$ $\left\{e_{1}, e_{2}\right\}$, where $e_{i}$ is an edge on $P_{i}$ for $i=1,2$. Clearly $G-\left\{e_{1}, e_{2}\right\}$ is connected. Then Proposition 2.1 (ii) implies that $\left\{e_{1}, e_{2}\right\} \stackrel{s}{\sim}_{G} \emptyset$. Thus $X_{0} \stackrel{s}{\sim}_{G} \emptyset$.

Hence $X_{0} \in \overline{\mathcal{F}}^{*}(G)$ and the necessity holds.
$(\Leftarrow)$ Assume that $G^{\prime}$ is bipartite and $X \in \overline{\mathcal{F}}(G)$.
As $\left|E\left(P_{1}\right)\right| \equiv\left|E\left(P_{2}\right)\right| \equiv 1(\bmod 2)$, Lemma 2.4 (i) implies that $X \cap E\left(G^{\prime}\right) \in$ $\overline{\mathcal{F}}\left(G^{\prime}\right)$. As $G^{\prime}$ is bipartite, Theorem 1.3 implies that $X \cap E\left(G^{\prime}\right) \stackrel{\mathcal{S}}{\sim}_{G^{\prime}} \emptyset$. By Lemma 2.5, $X \stackrel{s}{\sim}_{G} \emptyset$ holds or $X \stackrel{s}{\sim}_{G}\{e\}$ holds for some $e \in E\left(P_{1}\right) \cup E\left(P_{2}\right)$ or $X \stackrel{s}{\sim}_{G}\left\{e_{1}, e_{2}\right\}$ holds for some $e_{1} \in E\left(P_{1}\right)$ and $e_{2} \in E\left(P_{2}\right)$.

If $X \stackrel{s}{\sim}_{G}\{e\}$ for some $e \in E\left(P_{1}\right) \cup E\left(P_{2}\right)$, then Theorem 1.1 implies that $\{e\} \in \overline{\mathcal{F}}(G)$. But Lemma 3.1 implies that $\{e\} \in \mathcal{F}(G)$, a contradiction.

Now consider the case that $X \stackrel{s}{\sim}_{G}\left\{e_{1}, e_{2}\right\}$ for $e_{i} \in E\left(P_{i}\right)$. Lemma 2.4 (iii) implies that $\left\{e_{1}, e_{2}\right\} \stackrel{s}{\sim}_{G}\left(E\left(P_{1}\right) \cup E\left(P_{2}\right)\right)$. Thus $X \stackrel{s}{\sim}_{G}\left(E\left(P_{1}\right) \cup E\left(P_{2}\right)\right)$. Since $G^{\prime}$ is bipartite and matching-covered, $G^{\prime}$ has a bipartition $\left(U_{1}, U_{2}\right)$ with $\left|U_{1}\right|=$ $\left|U_{2}\right|$. Since $G=\operatorname{Union}\left(G^{\prime}, P_{1}, P_{2}\right)$ is not bipartite, both ends of some $P_{i}$ are within $U_{j}$ for some $j$. Assume that both ends of some $P_{1}$ are within $U_{1}$. As $\left|U_{1}\right|=\left|U_{2}\right|$ and $G$ is matching-covered, both ends of some $P_{2}$ must be in $U_{2}$. Thus $\left(E\left(P_{1}\right) \cup E\left(P_{2}\right)\right) \oplus \nabla_{G}\left(U_{1} \cup V\left(P_{1}\right)\right)=E(G)$, implying that $X \stackrel{s}{\sim}_{G}\left(E\left(P_{1}\right) \cup\right.$ $\left.E\left(P_{2}\right)\right) \stackrel{s}{\sim}_{G} E(G)$.

Hence the sufficiency holds.
Lemma 4.2. Let $G^{\prime}$ and $G=\operatorname{Union}\left(G^{\prime}, P\right)$ be matching-covered graphs, where $P$ is a single ear of $G^{\prime}$. For any $X \in \overline{\mathcal{F}}\left(G^{\prime}\right)$, both $X \in \mathcal{F}(G)$ and $X \cup E(P) \in$ $\mathcal{F}(G)$ hold if and only if $X \cap E\left(G^{o}\right) \in \mathcal{F}\left(G^{o}\right)$ holds, where $G^{o}=G^{\prime}-\{u, v\}$ and $u, v$ are the two ends of $P$ in $G^{\prime}$.

Proof. As $P$ is a single ear of $G^{\prime},|E(P)|$ is odd. Let $e_{1}, e_{2}, \cdots, e_{2 k-1}$ be the edges in $P$, where $e_{i}$ and $e_{i+1}$ have a common end for all $i=1,2, \cdots, 2 k-2$.

The set of perfect matchings of $G$ can be partitioned into two sets $\mathcal{M}_{0}$ and $\mathcal{M}_{1}$, where $\mathcal{M}_{0}$ is the set of perfect matchings $M$ in $G$ with $e_{1} \notin M$ and $\mathcal{M}_{1}$ is the set of perfect matchings $M$ in $G$ with $e_{1} \in M$. Then, for each $M \in \mathcal{M}_{0}$,

$$
M \cap E(P)=\left\{e_{2 r}: r=1,2, \cdots, k-1\right\}
$$

and for each $M \in \mathcal{M}_{1}$,

$$
M \cap E(P)=\left\{e_{2 r-1}: r=1,2, \cdots, k\right\}
$$

Observe that $\mathcal{M}^{\prime}=\left\{M \cap E\left(G^{\prime}\right): M \in \mathcal{M}_{0}\right\}$ is the set of perfect matchings in $G^{\prime}$. Assume that $X \in \overline{\mathcal{F}}\left(G^{\prime}\right)$ and $\left|M^{\prime} \cap X\right| \equiv a(\bmod 2)$ holds for all $M^{\prime} \in \mathcal{M}^{\prime}$, where $a$ is a fixed number in $\{0,1\}$. Thus $|M \cap X| \equiv a+k-1(\bmod 2)$ holds for all $M \in \mathcal{M}_{0}$.
$(\Rightarrow)$ Assume that both $X$ and $X \cup E(P)$ are feasible in $G$. Since $X$ is feasible in $G$ and $|M \cap X| \equiv a+k-1(\bmod 2)$ holds for all $M \in \mathcal{M}_{0},\left|M_{1} \cap X\right| \equiv$ $a+k(\bmod 2)$ holds for some $M_{1} \in \mathcal{M}_{1}$.

Claim 1: $\left|M_{2} \cap X\right| \equiv a+k-1(\bmod 2)$ holds for some $M_{2} \in \mathcal{M}_{1}$.
Suppose that Claim 1 fails. Then $|M \cap X| \equiv a+k(\bmod 2)$ holds for all $M \in \mathcal{M}_{1}$, implying that $|M \cap(X \cup E(P))| \equiv a(\bmod 2)$ holds for all $M \in \mathcal{M}_{1}$. But, for each $M \in M_{0},|M \cap(X \cup E(P))| \equiv|M \cap X|+k-1 \equiv a(\bmod 2)$ holds. Thus $|M \cap(X \cup E(P))| \equiv a(\bmod 2)$ holds for all $M \in \mathcal{M}_{0} \cup \mathcal{M}_{1}$, implying that $X \cup E(P)$ is non-feasible in $G$, a contradiction.

Thus Claim 1 holds.
Now there are two perfect matchings $M_{1}, M_{2} \in \mathcal{M}_{1}$ such that $\left|M_{i} \cap X\right| \equiv$ $a+k+i-1(\bmod 2)$ holds for $i=1,2$, implying that $\left|M_{1} \cap X\right| \not \equiv\left|M_{2} \cap X\right|(\bmod 2)$.

Let $X_{0}=X \cap E\left(G^{o}\right)$. Observe that both $M_{1}-E(P)$ and $M_{2}-E(P)$ are perfect matchings in $G^{o}$ and $\left|\left(M_{i}-E(P)\right) \cap X_{0}\right|=\left|M_{i} \cap X\right|$ holds for $i=1,2$. As $\left|M_{1} \cap X\right| \not \equiv\left|M_{2} \cap X\right|(\bmod 2),\left|\left(M_{1}-E(P)\right) \cap X_{0}\right| \not \equiv\left|\left(M_{2}-E(P)\right) \cap X_{0}\right|(\bmod 2)$ holds, implying that $X_{0}$ is feasible in $G^{o}$.
$(\Leftarrow)$ Assume that $X_{0}=X \cap E\left(G^{o}\right)$ is feasible in $G^{o}$. Then there are two perfect matchings $N_{1}$ and $N_{2}$ in $G^{o}$ such that $\left|X_{0} \cap N_{i}\right| \equiv i(\bmod 2)$ for $i=1,2$.

Clearly, $Q_{i}=N_{i} \cup\left\{e_{2 r-1}: r=1,2, \cdots, k\right\} \in \mathcal{M}_{1}$ for $i=1,2$. Observe that

$$
\left|(X \cup E(P)) \cap Q_{i}\right|=\left|X_{0} \cap N_{i}\right|+k \equiv k+i(\bmod 2), \quad \forall i=1,2
$$

implying that $X \cup E(P)$ is feasible in $G$. Also observe that

$$
\left|X \cap Q_{i}\right|=\left|X_{0} \cap N_{i}\right| \equiv i(\bmod 2), \quad \forall i=1,2
$$

implying that $X$ is feasible in $G$.
We are now ready to prove Theorem 1.4.
Proof of Theorem 1.4: (i). It follows directly from Lemma 3.2 (iii).
(ii). If $\sum_{1 \leq i \leq r} \epsilon\left(G_{i-1}, G_{i}\right)=r$, then $G_{0} \subset G_{1} \subset \cdots \subset G_{r}$ is a single ear decomposition of $G$. Thus Theorem 1.3 implies that $\overline{\mathcal{F}}^{*}(G)=\emptyset$.

Now assume that $\sum_{1 \leq i \leq r} \epsilon\left(G_{i-1}, G_{i}\right)=r+1$, implying that $\epsilon\left(G_{i-1}, G_{i}\right)=$ 2 holds for exactly one $i$ with $1 \leq i \leq r$. We first consider the case that $\epsilon\left(G_{r-1}, G_{r}\right)=2$. In this case, $\sum_{1 \leq i \leq r-1} \epsilon\left(G_{i-1}, G_{i}\right)=r-1$, implying that $G_{0} \subset G_{1} \subset \cdots \subset G_{r-1}$ is a single ear decomposition of $G_{r-1}$. Theorem 3.3 implies that $G_{r-1}$ is bipartite. Then Lemma 4.1 implies that $\overline{\mathcal{F}}^{*}\left(G_{r}\right)=\emptyset$ holds.

Now we consider the case that $\epsilon\left(G_{k-1}, G_{k}\right)=2$, where $1 \leq k<r$. Then $\sum_{1 \leq i \leq k} \epsilon\left(G_{i-1}, G_{i}\right)=k+1$. By the proven conclusion above, $\overline{\mathcal{F}}^{*}\left(G_{k}\right)=\emptyset$ holds. The result in (i) implies that $\overline{\mathcal{F}}^{*}\left(G_{k}\right)=\emptyset$ holds for all $i=k+1, \cdots, r$.

Hence (ii) holds.
(iii). As $\sum_{1 \leq i \leq r} \epsilon\left(G_{i-1}, G_{i}\right) \geq r+2$ and $\epsilon\left(G_{r-1}, G_{r}\right)=2, \sum_{1 \leq i \leq r-1} \epsilon\left(G_{i-1}, G_{i}\right) \geq$ $r$ holds. By the definition of ear decompositions, $G_{r-1}$ is not bipartite. Then Lemma 4.1 implies that $\overline{\mathcal{F}}^{*}(G) \neq \emptyset$. Hence (iii) holds.
(iv). $(\Rightarrow)$ Assume that $\overline{\mathcal{F}}^{*}(G)=\emptyset$. Suppose that there exists $X \in \overline{\mathcal{F}}^{*}\left(G_{r-1}\right)$ with $X \cap E\left(G^{o}\right) \in \overline{\mathcal{F}}\left(G^{o}\right)$, where $G^{o}=G_{r-1}-\{u, v\}$.

As $X \cap E\left(G^{o}\right) \in \overline{\mathcal{F}}\left(G^{o}\right)$, Lemma 4.2 implies that $X \in \overline{\mathcal{F}}(G)$ or $X \cup E(P) \in$ $\overline{\mathcal{F}}(G)$ holds. If $X \in \overline{\mathcal{F}}(G)$, as $X \in \overline{\mathcal{F}}^{*}\left(G_{r-1}\right)$, then Lemma 3.2 (iii) implies that $X \in \overline{\mathcal{F}}^{*}(G)$. If $X \cup E(P) \in \overline{\mathcal{F}}(G)$, it can be proved similarly that $X \cup E(P) \in$ $\overline{\mathcal{F}}^{*}(G)$ holds. Thus $\overline{\mathcal{F}}^{*}(G) \neq \emptyset$, a contradiction.
$(\Leftarrow)$ Assume that $\overline{\mathcal{F}}^{*}(G) \neq \emptyset$. Then, there exists $Z \in \overline{\mathcal{F}}^{*}(G)$ and Lemma 3.2 (iii) implies that $X=Z \cap E\left(G_{r-1}\right) \in \overline{\mathcal{F}}^{*}\left(G_{r-1}\right)$.

By Lemma 2.4 (ii) and (iii), $Z \stackrel{s}{\sim}_{G} X$ or $Z \stackrel{s}{\sim}_{G} X \cup E\left(P_{r}\right)$ holds. Then, $Z \in \overline{\mathcal{F}}(G)$ implies that $X \in \overline{\mathcal{F}}(G)$ or $X \cup E(P) \in \overline{\mathcal{F}}(G)$. Lemma 4.2 implies that $X \cap E\left(G^{o}\right) \in \overline{\mathcal{F}}\left(G^{o}\right)$ holds, where $G^{o}=G_{r-1}-\{u, v\}$, contradicting the given condition.

Thus the result holds.

## 5 Regular graphs $G$ of class 1 with $\overline{\mathcal{F}}^{*}(G) \neq \emptyset$

### 5.1 Generalize the family of graphs constructed in [3]

In this subsection, we will generalize the construction in [3] which provides a negative answer to Problem 1.5.

For two vertex-disjoint graphs $G_{1}=\left(V_{1}, E_{1}\right)$ and $G_{2}=\left(V_{2}, E_{2}\right)$ with $e_{i}=$ $x_{i} y_{i} \in E_{i}$ for $i=1,2$, let $G_{1} \#_{e_{1}, e_{2}} G_{2}$ denote the graph obtained from $G_{1}-e_{1}$ and $G_{2}-e_{2}$ by adding edges $f_{1}=x_{1} x_{2}$ and $f_{2}=y_{1} y_{2}$, as shown in Figure 2.


Figure 2: A graph constructed from $G_{1}$ and $G_{2}$

Lemma 5.1. For $i=1,2$, assume that $G_{i}=\left(V_{i}, E_{i}\right)$ is a matching-covered graph with $\left|E_{i}\right| \geq 2$ and $S_{i}$ is an equivalent set of $G_{i}$ with $e_{i} \in S_{i}$, where $e_{i}=x_{i} y_{i}$. Let $G$ denote the graph $G_{1} \# e_{e_{1}, e_{2}} G_{2}$ and let $S=\left(S_{1}-\left\{e_{1}\right\}\right) \cup\left(S_{2}-\right.$ $\left.\left\{e_{2}\right\}\right) \cup\left\{f_{1}, f_{2}\right\}$. Then
(i). $G$ is matching-covered;
(ii). $S$ is an equivalent set in $G$;
(iii). when $G_{1}$ and $G_{2}$ are 2-connected, $G$ is also 2-connected;
(iv). when $G_{1}$ and $G_{2}$ are r-regular graphs of class $1, G$ is also a r-regular graph of class 1;
(v). for any $S^{\prime} \subseteq S$, when $G_{i}-e_{i}-\left(S^{\prime} \cap E_{i}\right)$ is not bipartite for some $i \in\{1,2\}$, $S^{\prime} \stackrel{s}{\sim}_{G} E(G)$ holds;
(vi). for any $S^{\prime} \subseteq S$, when $S^{\prime} \cap E_{j} \stackrel{s}{\not}_{G_{j}-e_{j}} \emptyset$ for some $j \in\{1,2\}, S^{\prime} \stackrel{s}{\not}_{G} \emptyset$ holds.

Proof. For $i=1,2$ and $j=0,1$, let $\mathcal{M}_{i, j}$ be the set of perfect matchings $M$ in $G_{i}$ with $\left|M \cap\left\{e_{i}\right\}\right|=j$. Since $G_{i}$ is matching-covered and $\left|E_{i}\right| \geq 2$ holds for $i=1,2, \mathcal{M}_{i, j} \neq \emptyset$ for all $i=1,2$ and $j=0,1$. Let $\mathcal{M}$ be the set of perfect matchings of $G$.
(i). The following facts imply that $G$ is matching-covered:
(a) for any $M_{i} \in \mathcal{M}_{i, 1}, i=1,2,\left(M_{1}-\left\{e_{1}\right\}\right) \cup\left(M_{2}-\left\{e_{2}\right\}\right) \cup\left\{x_{1} y_{1}, x_{2} y_{2}\right\}$ is a member in $\mathcal{M}$;
(b) if $N_{i} \in \mathcal{M}_{i, 0}$ for $i=1,2$, then $N_{1} \cup N_{2} \in \mathcal{M}$;
(c) for $i=1,2$ and any $e \in E_{i}, e \in M_{i}$ holds for some $M_{i} \in \mathcal{M}_{i, 1} \cup \mathcal{M}_{i, 0}$.
(ii). To show that $S$ is an equivalent set of $G$, we need only to prove the two claims below:
Claim 1: For $\left\{f_{1}, f_{2}\right\}$ is an equivalent set of $G$.
Suppose the claim fails. Then there exists $M \in \mathcal{M}$ with $\left|\left\{f_{1}, f_{2}\right\} \cap M\right|=1$. Assume that $f_{1} \in M$ but $f_{2} \notin M$. Then $M \cap E_{1}$ is a perfect matching of $G-x_{1}$, implying that $\left|V\left(G_{1}\right)\right| \equiv 1(\bmod 2)$, contradicting the condition that $G_{1}$ is matching-covered. Thus the claim holds.
Claim 2: both $\left\{f_{1}, e\right\}$ is an equivalent set of $G$ for any $e \in\left(S_{1}-\left\{e_{1}\right\}\right) \cup\left(S_{2}-\right.$ $\left\{e_{2}\right\}$ ).

We may assume that $e \in S_{1}-\left\{e_{1}\right\}$. Suppose the claim fails. Then there exists $M \in \mathcal{M}$ with $\left|\left\{f_{1}, e\right\} \cap M\right|=1$.

If $e \in M$ but $f_{1} \notin M$, then Claim 1 implies that $f_{2} \notin M$. Thus $M_{1}=$ $M \cap E_{1} \in \mathcal{M}_{1,0}$. Clearly, $e \in M_{1}$ but $e_{1} \notin M_{1}$. Thus $\left\{e, e_{1}\right\}$ is not an equivalent set of $G_{1}$, contradicting the assumption that $S_{1}$ is an equivalent set of $G_{1}$ with $e, e_{1} \in S_{1}$.

If $e \notin M$ but $f_{1} \in M$, then Claim 1 implies that $f_{2} \in M$. Thus $M_{1}^{\prime}=\left\{e_{1}\right\} \cup$ $\left(M \cap E_{1}\right) \in \mathcal{M}_{1,1}$. Clearly, $e \notin M_{1}^{\prime}$ but $e_{1} \in M_{1}^{\prime}$, implying that $\left\{e, e_{1}\right\}$ is not an equivalent set of $G_{1}$, contradicting the assumption that $S_{1}$ is an equivalent set of $G_{1}$ with $e, e_{1} \in S_{1}$.

Hence Claim 2 holds and (ii) follows.
(iii). It is trivial to verify.
(iv). Clearly, when both $G_{1}$ and $G_{2}$ are $r$-regular, $G$ is also $r$-regular. Assume that both $G_{1}$ and $G_{2}$ are $r$-regular graphs of class 1 . Then the edge set of each $G_{i}$ can be partitioned into $r$ independent sets $E_{i, 1}, \cdots, E_{i, r}$. Assume that $e_{i} \in E_{i, 1}$ for $i=1,2$. Then $E(G)$ has a partition $E_{1}, E_{2}, \cdots, E_{r}$ in which each subset is an independents set of $G$, where

$$
E_{1}=\left(E_{1,1}-\left\{e_{1}\right\}\right) \cup\left(E_{2,1}-\left\{e_{2}\right\}\right) \cup\left\{f_{1}, f_{2}\right\}, E_{j}=E_{1, j} \cup E_{2, j}, \quad \forall j=2,3, \cdots, r
$$

implying that $G$ is of class 1 . Thus the result holds.
(v). Suppose that $S^{\prime} \stackrel{s}{\sim}_{G} E(G)$. Corollary 2.2 (ii) implies that $G_{i}-e_{i}-$ ( $S^{\prime} \cap E_{i}$ ) is bipartite for $i=1,2$, a contradiction. Thus the result holds.
(vi). Suppose that $S^{\prime} \stackrel{s}{\sim}_{G} \emptyset$. Corollary 2.2 (i) implies that $S^{\prime} \cap E\left(G_{i}-\right.$ $\left.e_{i}\right) \stackrel{s}{\sim}_{G_{i}-e_{i}} \emptyset$ for $i=1,2$, a contradiction. Thus the result holds.

By applying Lemma 5.1, the following conclusion follows.


Figure 3: $H_{1}=G_{1}$ and $H_{j+1}$ is the graph $H_{j} \#_{e_{j}^{\prime}, e_{j+1}} G_{j+1}$ for $j=1,2, \cdots, k-1$

Theorem 5.2. Let $G_{1}, G_{2}, \cdots, G_{k}$ be vertex-disjoint 2-connected and r-regular graphs of class 1 and let $S_{i}$ be an equivalent set of $G_{i}$ with $\left\{e_{i}, e_{i}^{\prime}\right\} \subseteq S_{i}$, where $e_{i}=x_{i} y_{i}$ and $e_{i}^{\prime}=x_{i}^{\prime} y_{i}^{\prime}$, for all $i=1,2, \cdots, k$. Let $H_{1}=G_{1}$ and let $H_{j+1}$ be the graph $H_{j} \#_{e_{j}^{\prime}, e_{j+1}} G_{j+1}$ for $j=1,2, \cdots, k-1$, as shown in Figure 3. Then
(i). $H_{k}$ is a 2-connected and r-regular graph of class 1;
(ii). for any subset $S$ of $\left\{S_{1}-\left\{e_{1}^{\prime}\right\}\right\} \cup\left\{S_{k}-\left\{e_{k}\right\}\right\} \cup \bigcup_{i=2}^{k-1}\left\{S_{i}-\left\{e_{i}, e_{i}^{\prime}\right\}\right\}$ with $|S| \equiv 0(\bmod 2)$, when $G_{i}^{\prime}-\left(S \cap E\left(G_{i}^{\prime}\right)\right)$ is not bipartite for some $i$ with $1 \leq i \leq k$ and $\left(S \cap E\left(G_{j}^{\prime}\right)\right) \stackrel{s}{\diamond}_{G_{j}-e_{j}} \emptyset$ holds for some $j$ with $1 \leq j \leq k$, $S$ is an equivalent set of $H_{k}$ which belongs to $\overline{\mathcal{F}}^{*}\left(H_{k}\right)$, where $G_{1}^{\prime}=G_{1}-\left\{e_{1}^{\prime}\right\}$, $G_{k}^{\prime}=G_{k}-\left\{e_{k}\right\}$ and $G_{s}^{\prime}=G_{s}-\left\{e_{s}, e_{s}^{\prime}\right\}$ for $2 \leq s \leq k-1$.

Proof. (i). It follows directly from Lemma 5.1 (iii) and (iv).
(ii). Let $Q=\left\{S_{1}-\left\{e_{1}^{\prime}\right\}\right\} \cup\left\{S_{k}-\left\{e_{k}\right\}\right\} \cup \bigcup_{i=2}^{k-1}\left\{S_{i}-\left\{e_{i}, e_{i}^{\prime}\right\}\right\}$. Applying

Lemma 5.1 (ii) repeatedly shows that $Q$ is an equivalents set of $H_{k}$. As $S \subseteq Q$ and $|S| \equiv 0(\bmod 2), S \in \overline{\mathcal{F}}\left(H_{k}\right)$ holds. As $G_{i}^{\prime}-\left(S \cap E\left(G_{i}^{\prime}\right)\right)$ is not bipartite for some $i$ with $1 \leq i \leq k, S \cap E\left(G_{i}^{\prime}\right) \stackrel{s}{\sim} E\left(G_{i}^{\prime}\right)$ holds, implying that $S \stackrel{s}{\sim}_{H_{k}} E\left(H_{k}\right)$ by Corollary 2.2 (ii). As $S \cap E\left(G_{j}^{\prime}\right) \stackrel{s}{\diamond}_{G_{j}^{\prime}} \emptyset$ for some $j$ with $1 \leq j \leq k$, Corollary 2.2 (ii) implies that $S \stackrel{s}{\sim}_{H_{k}} \emptyset$. Hence $S \in \overline{\mathcal{F}}^{*}\left(H_{k}\right)$.

By Theorem 5.2, it can be verified easily that the graphs constructed in [3] give a negative answer to Problem 1.5.

### 5.2 4-connected and $r$-regular graphs $G$ of class 1 with $\overline{\mathcal{F}}^{*}(G) \neq \emptyset$

In this subsection, we construct infinitely many 4-connected $r$-regular graphs $G$ of class 1 with $\overline{\mathcal{F}}^{*}(G) \neq \emptyset$, where $r$ is an integer with $r \geq 4$.

Let $\Psi_{r}$ be the set of 4-connected and $r$-regular graphs of class 1 , each of which contains an equivalent set of size 2 . Let $Q_{r}$ denote the graph obtained
from the complete bipartite graph $K_{r, r}$ by removing two independent edges $a_{1} b_{1}$ and $a_{2} b_{2}$ and adding two new edges $a_{1} a_{2}$ and $b_{1} b_{2}$, where $a_{1}$ and $a_{2}$ are vertices in one partite set of $K_{r, r}$. Observe that $Q_{r}$ is a $r$-connected and $r$-regular graph of class 1 with an equivalent set $\left\{a_{1} a_{2}, b_{1} b_{2}\right\}$. Thus $Q_{r} \in \Psi_{r}$.

Let $\Psi_{r}^{*}$ be the set of graphs $H \in \Psi_{r}$ containing an equivalent set $\left\{e, e^{\prime}\right\}$ such that $H-\left\{e, e^{\prime}\right\}$ is not bipartite. From the remark in Page 20, it is known that $\Psi_{r}^{*} \neq \emptyset$.

For a list $L=\left(G_{1}, G_{2}, \cdots, G_{k}\right)$ of vertex-disjoint graphs in $\Psi_{r}$, where $k \geq 3$ and $\left\{e_{i}, e_{i}^{\prime}\right\}$ is an equivalent set of $G_{i}$ with $e_{i}=x_{i} y_{i}$ and $e_{i}^{\prime}=x_{i}^{\prime} y_{i}^{\prime}$ for $i=$ $1,2, \cdots, k$, let $\mathcal{C}_{L}$ denote the graph obtained from $G_{1}, G_{2}, \cdots, G_{k}$ by deleting edges $e_{i}$ and $e_{i}^{\prime}$ and adding new edges $f_{i}$ and $f_{i}^{\prime}$ for all $i=1,2, \cdots, k$, where $f_{i}=x_{i} y_{i+1}, f_{i}^{\prime}=x_{i}^{\prime} y_{i+1}^{\prime}, y_{k+1}=y_{1}$ and $y_{k+1}^{\prime}=y_{1}^{\prime}$. For any $i$ with $1 \leq i \leq k$, assume that $G_{i}-\left\{e_{i}, e_{i}^{\prime}\right\}$ is not bipartite whenever $G_{i} \in S_{r}^{*}$. An example of $\mathcal{C}_{L}$ for $k=3$ is shown in Figure 4.


Figure 4: Graph $\mathcal{C}_{L}$, where $L=\left(G_{1}, G_{2}, G_{3}\right)$

Lemma 5.3. Let $L=\left(G_{1}, G_{2}, \cdots, G_{k}\right)$ be any list of graphs in $\Psi_{r}$, where $k$ is an odd number with $k \geq 3$. The graph $\mathcal{C}_{L}$ defined above has the following properties:
(i). $\mathcal{C}_{L} \in \Psi_{r}$ with equivalent sets $\left\{f_{i}, f_{i}^{\prime}\right\}$ for all $i=1,2, \cdots, k$;
(ii). if $G_{j}-\left\{e_{j}, e_{j}^{\prime}\right\}$ is not bipartte for some $j$ with $1 \leq j \leq k$, then $\left\{f_{i}, f_{i}^{\prime}\right.$ : $i=1,2, \cdots, k\} \in \overline{\mathcal{F}}^{*}\left(\mathcal{C}_{L}\right)$ holds.

Proof. (i). As $G_{i}$ is 4-connected for all $i=1,2, \cdots, k$, it is not difficult to show that any two non-adjacent vertices in $\mathcal{C}_{L}$ are joined by 4 internally vertex-disjoint paths, implying that $\mathcal{C}_{L}$ is 4-connected.

Clearly $\mathcal{C}_{L}$ is $r$-regular. As $G_{i}$ is a $r$-regular graph of class 1 and with an equivalent set $\left\{e_{i}, e_{i}^{\prime}\right\}, E\left(G_{i}\right)$ can be partitioned into perfect matchings $E_{i, 1}, E_{i, 2}, \cdots, E_{i, r}$ with $\left\{e_{i}, e_{i}^{\prime}\right\} \subseteq E_{i, 1}$. Thus, $\mathcal{C}_{r}$ is of class 1 , as its edge set can be partitioned into $r$ perfect matchings $\mathcal{E}_{1}, \mathcal{E}_{2}, \cdots, \mathcal{E}_{r}$, where

$$
\mathcal{E}_{1}=\bigcup_{i=1}^{k}\left(\left\{f_{i}, f_{i}^{\prime}\right\} \cup\left(E_{i, 1}-\left\{e_{i}, e_{i}^{\prime}\right\}\right)\right), \quad \mathcal{E}_{j}=\bigcup_{i=1}^{k} E_{i, j}, \quad \forall j=2,3, \cdots, r
$$

To show that $\left\{f_{i}, f_{i}^{\prime}\right\}$ is an equivalent set of $\mathcal{C}_{L}$, we need to apply the following claim.
Claim 1: For any perfect matching $M$ of $\mathcal{C}_{L}$ and any $i$ with $1 \leq i \leq k$, $M \cap\left\{f_{i}, f_{i}^{\prime}\right\}=\left\{f_{i}\right\}$ implies that $M \cap\left\{f_{i+1}, f_{i+1}^{\prime}\right\}=\left\{f_{i+1}^{\prime}\right\}$, and $M \cap\left\{f_{i}, f_{i}^{\prime}\right\}=$ $\left\{f_{i}^{\prime}\right\}$ implies that $M \cap\left\{f_{i+1}, f_{i+1}^{\prime}\right\}=\left\{f_{i+1}\right\}$.

Without loss of generality, it suffices to prove that $M \cap\left\{f_{1}, f_{1}^{\prime}\right\}=\left\{f_{1}\right\}$ implies $M \cap\left\{f_{2}, f_{2}^{\prime}\right\}=\left\{f_{2}^{\prime}\right\}$. As $G_{2}$ is matching-covered, $\left|V_{2}\right| \equiv 0(\bmod 2)$. Thus $M \cap\left\{f_{1}, f_{1}^{\prime}\right\}=\left\{f_{1}\right\}$ implies that $\left|M \cap\left\{f_{2}, f_{2}^{\prime}\right\}\right|=1$. Suppose that $M \cap\left\{f_{2}, f_{2}^{\prime}\right\}=$ $\left\{f_{2}\right\}$. Then, $M_{2}=\left\{e_{2}\right\} \cup\left(M \cap E\left(G_{2}\right)\right)$ is a perfect matching of $G_{2}$. But $e_{2}^{\prime} \notin M_{2}$ contradicting the assumption that $\left\{e_{2}, e_{2}^{\prime}\right\}$ is an equivalent set of $G_{2}$. Thus the claim holds.

Suppose that $\left\{f_{i}, f_{i}^{\prime}\right\}$ is not an equivalent set of $\mathcal{C}_{L}$, say $i=1$. Then $\mid M \cap$ $\left\{f_{1}, f_{1}^{\prime}\right\} \mid=1$ holds for some perfect matching $M$ of $\mathcal{C}_{L}$, say $f_{1} \in M$ but $f_{1}^{\prime} \notin M$. Claim 1 implies $M \cap\left\{f_{2}, f_{2}^{\prime}\right\}=\left\{f_{2}^{\prime}\right\}, M \cap\left\{f_{3}, f_{3}^{\prime}\right\}=\left\{f_{3}\right\}$ and so on. As $k$ is odd, we have $M \cap\left\{f_{k}, f_{k}^{\prime}\right\}=\left\{f_{k}\right\}$. However, by Claim $1, M \cap\left\{f_{k}, f_{k}^{\prime}\right\}=\left\{f_{k}\right\}$ implies that $M \cap\left\{f_{1}, f_{1}^{\prime}\right\}=\left\{f_{1}^{\prime}\right\}$, a contradiction. Hence (i) holds.
(ii). Suppose that $G_{j}-\left\{e_{j}, e_{j}^{\prime}\right\}$ is not bipartite for some $j$ with $1 \leq j \leq k$.

Let $S=\left\{f_{i}, f_{i}^{\prime}: i=1,2, \cdots, k\right\}$. As $\left\{f_{i}, f_{i}^{\prime}\right\}$ is an equivalent set of $\mathcal{C}_{L}$ for all $i=1,2, \cdots, k,|S \cap M|$ is even for all perfect matchings $M$ of $\mathcal{C}_{L}$, implying that $S \in \overline{\mathcal{F}}\left(\mathcal{C}_{L}\right)$ holds.

As $G_{j}-\left\{e_{j}, e_{j}^{\prime}\right\}$ is not bipartite for some $j$ with $1 \leq j \leq k$, Corollary 2.2 (ii) implies that $S \stackrel{s}{\sim}_{C_{L}} E\left(\mathcal{C}_{L}\right)$. Suppose that $S \stackrel{s}{\sim}_{\mathcal{C}_{L}} \emptyset$. Then Proposition 2.1 (i) implies that $S=\nabla_{\mathcal{C}_{L}}(U)$ for some $U \subset V\left(\mathcal{C}_{L}\right)$. As $G_{i}-\left\{e_{i}, e_{i}^{\prime}\right\}$ is connected, we have $V\left(G_{i}\right) \subseteq U$ or $V\left(G_{i}\right) \subseteq V\left(\mathcal{C}_{L}\right)-U$ for all $i=1,2, \cdots, k$. Assume that $V\left(G_{1}\right) \subseteq U$. Then $S=\nabla_{\mathcal{C}_{L}}(U)$ implies $V\left(G_{2}\right) \subseteq V\left(\mathcal{C}_{L}\right)-U, V\left(G_{3}\right) \subseteq U$ and so on. Since $k$ is odd, $V\left(G_{k}\right) \subseteq U$, contradicting the assumption that $f_{k}, f_{k}^{\prime} \in S=\nabla_{\mathcal{C}_{L}}(U)$.

Hence $S \in \overline{\mathcal{F}}^{*}\left(\mathcal{C}_{L}\right)$ and (ii) holds.
By Lemma 5.3, we can prove the following result.
Corollary 5.4. $\Psi_{r}^{*}$ is an infinite set.

Proof. Let $\mathcal{L}$ be the family of lists $L=\left(G_{1}, G_{2}, \cdots, G_{k}\right)$, where $k \geq 3$ is odd, $G_{i} \in \Psi_{r}$ for $i=1,2, \cdots, k$ and $G_{j} \in \Psi_{r}^{*}$ for at least one $j$ with $1 \leq j \leq k$. By the remark in Page 20, $\Psi_{r}^{*} \neq \emptyset$. Thus $\mathcal{L} \neq \emptyset$. By Lemma 5.3, $\mathcal{C}_{L} \in \Psi_{r}$ holds for any list $L \in \mathcal{L}$. Furthermore, as $G_{j} \in \Psi_{r}^{*}$ holds for at least one $j, G_{j}-\left\{e_{j}, e_{j}^{\prime}\right\}$ is not bipartite for an equivalent set $\left\{e_{j}, e_{j}^{\prime}\right\}$, implying that $\mathcal{C}_{L}-\left\{f_{i}, f_{i}^{\prime}: 1 \leq i \leq k\right\}$ is not bipartite. By Lemma 5.3 (i), $\left\{f_{i}, f_{i}^{\prime}\right\}$ is an equivalent set of $\mathcal{C}_{L}$ for any $i$ with $1 \leq i \leq k$, implying that $\mathcal{C}_{L} \in \Psi_{r}^{*}$ holds. Clearly, $\mathcal{C}_{L}$ is different from anyone in the list of $L$. Applying Lemma 5.3 repeatedly implies that the result holds.

By Lemma 5.3 and Corollary 5.4, we get the following result.
Theorem 5.5. For any $r \geq 4$, there are infinitely many 4-connected and $r$ regular graphs $H$ of class 1 with $\overline{\mathcal{F}}^{*}(H) \neq \emptyset$.

## $5.3 r$-connected and $r$-regular graphs $G$ of class 1 with $\overline{\mathcal{F}}^{*}(G) \neq \emptyset$

For any integer $r$ with $r \geq 3$, let $\Phi_{r}$ be the set of $r$-connected and $r$-regular graphs of class 1. Clearly, $\Phi_{r}$ includes the complete bipartite graph $K_{r, r}$, the graph $Q_{r}$ defined in Page 15 and the complete graph $k_{r+1}$ when $r$ is odd.

For any set $S=\left\{G_{1}, G_{2}, \cdots, G_{r}\right\}$ of $r$ vertex-disjoint graphs in $\Phi_{r}$ with $w_{i} \in V\left(G_{i}\right)$ and $N_{G_{i}}\left(w_{i}\right)=\left\{v_{i, j}: j=1,2, \cdots, r\right\}$ for $i=1,2, \cdots, r$, let $\mathcal{X}_{S}$ denote the graph obtained from $G_{1}-w_{1}, G_{2}-w_{2}, \cdots, G_{r}-w_{r}$ by adding vertices $u_{1}, u_{2}, \cdots, u_{r}$ and adding edges joining $u_{j}$ to vertex $v_{i, j}$ for all $i=1,2, \cdots, r$ and $j=1,2, \cdots, r$, without referring to vertices $w_{i}$ in $G_{i}$ for $i=1,2, \cdots, r$. An example of $\mathcal{X}_{S}$ when $r=3$ is given in Figure 5, where $S=\left\{G_{1}, G_{2}, G_{3}\right\}$ and $G_{i} \cong K_{4}$ for all $i=1,2,3$.

Lemma 5.6. For any set $S=\left\{G_{1}, G_{2}, \cdots, G_{r}\right\}$ of graphs in $\Phi_{r}$, the graph $\mathcal{X}_{S}$ constructed above has the following properties:
(i). $\mathcal{X}_{S} \in \Phi_{r}$;
(ii). for any $i=1,2, \cdots, r$, if both $G_{i}-w_{i}$ and $G_{j}-w_{j}$ are not bipartite for some $j \in\{1,2, \cdots, r\}-\{i\}$, then $E\left(G_{i}-w_{i}\right) \in \overline{\mathcal{F}}^{*}\left(\mathcal{X}_{S}\right)$ holds.

Proof. (i). Observe that $\mathcal{X}_{S}$ is $r$-connected by the two facts below:
(a) If both graphs $H_{1}$ and $H_{2}$ are $r$-connected and vertex-disjoint with $x_{i} \in$ $V\left(H_{i}\right)$ and $N_{H_{i}}\left(x_{i}\right)=\left\{z_{i, j}: j=1,2, \cdots, r\right\}$ for $i=1,2$, then the graph obtained from $H_{1}-x_{1}$ and $H_{2}-x_{2}$ by adding edges joining $x_{1, j}$ and $x_{2, j}$ for all $j=1,2, \cdots, r$ is also $r$-connected;


Figure 5: Graph $\mathcal{X}_{S}$ with $S=\left\{G_{1}, G_{2}, G_{3}\right\}$ and $G_{i} \cong K_{4}$ for $i=1,2,3$
(b) for any $r$-connected graph $H$ and any $r$ independent edges $e_{1}, e_{2}, \cdots, e_{r}$, the graph obtained from $H$ by subdividing each $e_{i}=y_{i, 1} y_{i, 2}$ with a vertex, denoted by $q_{i}$, adding $r-2$ new vertices $z_{1}, z_{2}, \cdots, z_{r-2}$ and adding new edges joining $z_{j}$ to $q_{i}$ for all $j=1,2, \cdots, r-2$ and all $i=1,2, \cdots, r$ is also $r$-connected.

The two facts above can be verified by proving that each pair of non-adjacent vertices are joined by $r$ internally vertex-disjoint paths.

As $G_{i}$ is $r$-regular, by the definition, $\mathcal{X}_{S}$ is also $r$-regular.
For $i=1,2, \cdots, r$, as $G_{i}$ is a $r$-regular graph of class $1, G_{i}$ has a $r$-edgecoloring which partitions $E\left(G_{i}\right)$ into $r$ perfect matchings $E_{i, 1}, E_{i, 2}, \cdots, E_{i, r}$ of $G_{i}$. Assume that $w_{i} v_{i, j} \in E_{i, j}$ for all $i=1,2, \cdots, r$ and $j=1,2, \cdots, r$. Let $\pi_{1}, \pi_{2}, \cdots, \pi_{r}$ be permutations of $1,2, \cdots, r$ such that $\left\{\pi_{s}(i): s=1,2, \cdots, r\right\}=$ $\{1,2, \cdots, r\}$ holds for all $i=1,2, \cdots, r$. Certainly such permutations exist. Then $\mathcal{E}_{1}, \mathcal{E}_{2}, \cdots, \mathcal{E}_{r}$ defined below form a partition of $E\left(\mathcal{X}_{S}\right)$ each of which is a matching of $\mathcal{X}_{S}$ :

$$
\mathcal{E}_{s}=\bigcup_{i=1}^{r}\left(\left(E_{i, \pi_{s}(i)}-\left\{w_{i} v_{i, \pi_{s}(i)}\right\}\right) \cup\left\{u_{i} v_{i, \pi_{s}(i)}\right\}\right), \quad \forall s=1,2, \cdots, r .
$$

Hence $\mathcal{X}_{S}$ is of class 1 and $\mathcal{X}_{S} \in \Phi_{r}$.
(ii). For $i=1,2, \cdots, r$, as $G_{i}$ is a $r$-regular graph of class $1, G_{i}$ is matchingcovered, implying that $\left|V\left(G_{i}\right)\right| \equiv 0(\bmod 2)$. Thus $\left|V\left(G_{i}-w_{i}\right)\right| \equiv 1(\bmod 2)$ for all $i=1,2, \cdots, r$.

For $i=1,2, \cdots, r$, let $W_{i}=E\left(G_{i}-w_{i}\right)$ and $N_{i}=\left\{u_{j} v_{i, j}: j=1,2, \cdots, r\right\}$. As $\left|V\left(G_{i}-w_{i}\right)\right| \equiv 1(\bmod 2),\left|M \cap N_{i}\right| \geq 1$ holds for each perfect matching $M$ of $\mathcal{X}_{S}$ and all $i=1,2, \cdots, r$. But $\left|M \cap\left(N_{1} \cup N_{2} \cup \cdots \cup N_{r}\right)\right|=r$, implying that
$\left|M \cap N_{i}\right|=1$ holds for each perfect matching $M$ of $\mathcal{X}_{S}$ and all $i=1,2, \cdots, r$. Thus $\left|M \cap W_{i}\right|=\left|V\left(G_{i}\right)\right| / 2-1$ holds for each perfect matching $M$ of $\mathcal{X}_{S}$, implying that $W_{i} \in \overline{\mathcal{F}}\left(\mathcal{X}_{S}\right)$ for all $i=1,2, \cdots, r$.

If both $G_{i}-w_{i}$ and $G_{j}-w_{j}$ are not bipartite, where $j \neq i$, Corollary 2.2 implies that $W_{i} \stackrel{s}{\sim} \mathcal{X}_{S} \emptyset$ and $W_{i} \stackrel{s}{\sim} \mathcal{X}_{S} E\left(\mathcal{X}_{S}\right)$. Thus $W_{i} \in \overline{\mathcal{F}}^{*}\left(\mathcal{X}_{S}\right)$.

For any $r \geq 3$, let $\Phi_{r}^{*}$ be the set of graphs $G \in \Phi_{r}$ such that $G-w$ is not bipartite for every vertex $w$ in $G$. Clearly, $Q_{r} \in \Phi_{r}^{*}$ and when $r$ is odd, $K_{r+1} \in \Phi_{r}^{*}$.

Lemma 5.7. For any integer $r$ with $r \geq 3, \Phi_{r}^{*}$ is an infinite set.
Proof. Note that $\Phi_{r}^{*} \neq \emptyset$ for any $r \geq 3$.
If $S=\left\{G_{1}, G_{2}, \cdots, G_{r}\right\}$ is a set of vertex-disjoint graphs in $\Phi_{r}$ and $G_{i} \in \Phi_{r}^{*}$ holds for some pair $i, j$ with $1 \leq i<j \leq r$, then $\mathcal{X}_{S}-w$ is not bipartite for each vertex $w$ in $\mathcal{X}_{S}$. By Lemma 5.6, $\mathcal{X}_{S} \in \Phi_{r}^{*}$ holds. Note that $\mathcal{X}_{S}$ is different from any one in $S$. Thus the result holds by applying Lemma 5.6 repeatedly.

Remark: By the definition in Page $15, Q_{r}$ is a graph in $\Phi_{r}$ with an equivalent set $\left\{a_{1} a_{2}, b_{1} b_{2}\right\}$. For any $S=\left\{G_{1}, G_{2}, \cdots, G_{r}\right\}$, where $G_{i} \in \Phi_{r}$ for all $i=$ $1,2, \cdots, r$, if $G_{1}$ is the graph $Q_{r}$ and $w_{1} \notin\left\{a_{1}, a_{2}, b_{1}, b_{2}\right\}$, then it is not difficult to verify that $\left\{a_{1} a_{2}, b_{1} b_{2}\right\}$ is an equivalent set of $\mathcal{X}_{S}$. Furthermore, if $G_{j}-w_{j}$ is not bipartite graph for some $j$ with $2 \leq j \leq r$, then $\mathcal{X}_{S}-\left\{a_{1} a_{2}, b_{1} b_{2}\right\}$ is not bipartite, implying that $\mathcal{X}_{S} \in \Psi_{r}^{*}$ when $r \geq 4$.

Theorem 1.7 follows directly from Lemmas 5.6 and 5.7.

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