# Proving a conjecture on chromatic polynomials by counting the number of acyclic orientations<sup>\*</sup>

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#### Abstract

The chromatic polynomial P(G, x) of a graph G of order n can be expressed as  $\sum_{i=1}^{n} (-1)^{n-i} a_i x^i$ , where  $a_i$  is interpreted as the number of broken-cycle free spanning subgraphs of G with exactly i components. The parameter  $\epsilon(G) = \sum_{i=1}^{n} (n-i)a_i / \sum_{i=1}^{n} a_i$  is the mean size of a broken-cycle-free spanning subgraph of G. In this article, we confirm and strengthen a conjecture proposed by Lundow and Markström in 2006 that  $\epsilon(T_n) < \epsilon(G) < \epsilon(K_n)$  holds for any connected graph G of order n which is neither the complete graph  $K_n$  nor a tree  $T_n$  of order n. The most crucial step of our proof is to obtain the interpretation of all  $a_i$ 's by the number of acyclic orientations of G.

**Keywords:** chromatic polynomial; graph; acyclic orientation; combinatorial interpretation

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# 1 Introduction

All graphs considered in this paper are simple graphs. For any graph G = (V, E) and any positive integer k, a proper k-coloring f of G is a mapping  $f : V \to \{1, 2, ..., k\}$ such that  $f(u) \neq f(v)$  holds whenever  $uv \in E$ . The chromatic polynomial of G is the function P(G, x) such that P(G, k) counts the number of proper k-colorings of G for any positive integer k. In this article, the variable x in P(G, x) is a real number. The study of chromatic polynomials is one of the most active areas in graph theory. For basic concepts

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and properties on chromatic polynomials, we refer the reader to the monograph [5]. For the most celebrated results on this topic, we recommend surveys [4, 10, 14, 15].

The first interpretation of the coefficients of P(G, x) was provided by Whitney [23]: for any simple graph G of order n and size m,

$$P(G,x) = \sum_{i=1}^{n} \left( \sum_{r=0}^{m} (-1)^r N(i,r) \right) x^i,$$
(1)

where N(i, r) is the number of spanning subgraphs of G with exactly i components and r edges. Whitney further simplified (1) by introducing the notion of broken cycles. Let  $\eta: E \to \{1, 2, \ldots, |E|\}$  be a bijection. For any cycle C in G, the path C - e is called a broken cycle of G with respect to  $\eta$ , where e is the edge on C with  $\eta(e) \leq \eta(e')$  for every edge e' on C. When there is no confusion, a broken cycle of G is always assumed to be with respect to a bijection  $\eta: E \to \{1, 2, \ldots, |E|\}$ .

**Theorem 1** ([23]). Let G = (V, E) be a graph of order n and  $\eta : E \to \{1, 2, \dots, |E|\}$  be a bijection. Then,

$$P(G,x) = \sum_{i=1}^{n} (-1)^{n-i} a_i(G) x^i,$$
(2)

where  $a_i(G)$  is the number of spanning subgraphs of G with n-i edges and i components which do not contain broken cycles.

Let G be a simple graph of order n. When there is no confusion,  $a_i(G)$  is written as  $a_i$  for short. Clearly, by Theorem 1, P(G, x) is indeed a polynomial in x in which the constant term is 0, the leading coefficient  $a_n$  is 1 and all coefficients are integers alternating in signs. Thus,  $(-1)^n P(G, x) > 0$  holds for all x < 0.

The concept of broken cycles has the following connection with Tutte's work of expressing the Tutte polynomial  $\mathbf{T}_G(x, y)$  of a connected graph G in terms of spanning trees [2,22]:

$$\mathbf{T}_G(x,y) = \sum_T x^{ia_\omega(T)} y^{ea_\omega(T)},\tag{3}$$

where the sum runs over all spanning trees of G and  $ia_{\omega}(T)$  and  $ea_{\omega}(T)$  are respectively the internal and external activities of T with respect to a bijection  $\omega : E \to \{1, 2, \dots, |E|\}$ . If we take  $\omega$  to be  $\eta$ , then  $ea_{\eta}(T)$  is exactly the number of edges  $e \in E(G) \setminus E(T)$  such that  $\eta(e) \leq \eta(e')$  holds for all edges e' on the unique cycle C of  $T \cup e$ . As G is a simple graph,  $ea_{\eta}(T)$  equals the number of broken cycles contained in T with respect to  $\eta$ . In particular,  $ea_{\eta}(T) = 0$  if and only if T does not contain broken cycles with respect to  $\eta$ . By Theorem 1,  $a_1(G)$  is the number of spanning trees T of G with  $ea_{\eta}(T) = 0$ . If

$$\mathbf{T}_G(x,y) = \sum_{i \ge 0, j \ge 0} c_{i,j} x^i y^j, \tag{4}$$

then  $a_1(G) = \sum_{i \ge 0} c_{i,0} = \mathbf{T}_G(1,0).$ 

As in [11], for i = 0, 1, 2, ..., n - 1, we define  $b_i(G)$  (or simply  $b_i$ ) as the probability that a randomly chosen broken-cycle-free spanning subgraph of G has size *i*. Then

$$b_i = \frac{a_{n-i}}{a_1 + a_2 + \dots + a_n}, \quad \forall i = 0, 1, \dots, n-1.$$
(5)

Let  $\epsilon(G)$  denote the mean size of a broken-cycle-free spanning subgraph of G. Then

$$\epsilon(G) = \sum_{i=0}^{n-1} ib_i = \frac{(n-1)a_1 + (n-2)a_2 + \dots + a_{n-1}}{a_1 + a_2 + \dots + a_n}.$$
(6)

An elementary property of  $\epsilon(G)$  is given below.

**Proposition 1** ([11]). For any graph G of order n,  $\epsilon(G) = n + \frac{P'(G,-1)}{P(G,-1)}$ .

Let  $T_n$  denote a tree of order n and  $K_n$  denote the complete graph of order n. By Proposition 1,  $\epsilon(T_n) = \frac{n-1}{2}$ , while

$$\epsilon(K_n) = n - \left(1 + \frac{1}{2} + \dots + \frac{1}{n}\right) \sim n - \log n - \gamma \tag{7}$$

as  $n \to \infty$ , where  $\gamma \approx 0.577216$  is the Euler-Mascheroni constant.

Lundow and Markström [11] proposed the following conjecture on  $\epsilon(G)$ .

**Conjecture 1** ([11]). For any connected graph G of order n, where  $n \ge 4$ , if G is neither  $K_n$  nor a  $T_n$ , then  $\epsilon(T_n) < \epsilon(G) < \epsilon(K_n)$ .

In this paper, we aim to prove and strengthen Conjecture 1. For any graph G, define the function  $\epsilon(G, x)$  as follows:

$$\epsilon(G, x) = \frac{P'(G, x)}{P(G, x)}.$$
(8)

By Proposition 1,  $\epsilon(G) = n + \epsilon(G, -1)$  holds for every graph G of order n. Thus, for any graphs G and H of the same order,  $\epsilon(G) < \epsilon(H)$  if and only if  $\epsilon(G, -1) < \epsilon(H, -1)$ . Conjecture 1 is equivalent to the statement that  $\epsilon(T_n, -1) < \epsilon(G, -1) < \epsilon(K_n, -1)$  holds for any connected graph G of order n which is neither  $K_n$  nor a  $T_n$ .

A graph Q is said to be *chordal* if  $Q[V(C)] \not\cong C$  for every cycle C of Q with  $|V(C)| \ge 4$ , where Q[V'] is the subgraph of Q induced by V' for  $V' \subseteq V(G)$ . In Section 2, we will establish the following result. **Theorem 2.** For any graph G, if Q is a chordal and proper spanning subgraph of G, then  $\epsilon(G, x) > \epsilon(Q, x)$  holds for all x < 0.

Note that any tree is a chordal graph and any connected graph contains a spanning tree. Thus, we have the following corollary which obviously implies the first part of Conjecture 1.

**Corollary 1.** For any connected graph G of order n which is not a tree,  $\epsilon(G, x) > \epsilon(T_n, x)$  holds for all x < 0.

The second part of Conjecture 1 is extended to the inequality  $\epsilon(K_n, x) > \epsilon(G, x)$  for any non-complete graph G of order n and all x < 0. In order to prove this inequality, we will show in Section 3 that it suffices to establish the following result.

**Theorem 3.** For any non-complete graph G = (V, E) of order n,

$$(-1)^{n}(x-n+1)\sum_{u\in V}P(G-u,x) + (-1)^{n+1}nP(G,x) > 0$$
(9)

holds for all x < 0.

Note that the left-hand side of (9) vanishes when  $G \cong K_n$ . Theorem 3 will be proved in Section 5, based on Greene & Zaslavsky's interpretation in [8] for coefficients  $a_i(G)$ 's of P(G, x) by acyclic orientations introduced in Section 4. By applying Theorem 3 and two lemmas in Section 3, we will finally prove the second main result in this article.

**Theorem 4.** For any non-complete graph G of order n,  $\epsilon(G, x) < \epsilon(K_n, x)$  holds for all x < 0.

# 2 Proof of Theorem 2

A vertex u in a graph G is called a *simplicial vertex* if  $\{u\} \cup N_G(u)$  is a clique of G, where  $N_G(u)$  is the set of vertices in G which are adjacent to u. For a simplicial vertex u of G, P(G, x) has the following property (see [5, 13, 14]):

$$P(G, x) = (x - d(u))P(G - u, x),$$
(10)

where G - u is the subgraph of G induced by  $V - \{u\}$  and d(u) is the degree of u in G. By (10), it is not difficult to show the following.

**Proposition 2.** If u is a simplicial vertex of a graph G, then

$$\epsilon(G, x) = \frac{1}{x - d(u)} + \epsilon(G - u, x). \tag{11}$$

It has been shown that a graph Q of order n is chordal if and only if Q has an ordering  $u_1, u_2, \ldots, u_n$  of its vertices such that  $u_i$  is a simplicial vertex in  $Q[\{u_1, u_2, \ldots, u_i\}]$  for all  $i = 1, 2, \ldots, n$  (see [3,6]). Such an ordering of vertices in Q is called a *perfect elimination* ordering of Q. For any perfect elimination ordering  $u_1, u_2, \ldots, u_n$  of a chordal graph Q, by Proposition 2,

$$\epsilon(Q, x) = \sum_{i=1}^{n} \frac{1}{x - d_{Q_i}(u_i)},$$
(12)

where  $Q_i$  is the subgraph  $Q[\{u_1, u_2, \ldots, u_i\}]$ .

Now we are ready to prove Theorem 2.

Proof of Theorem 2: Let G be any graph of order n and Q be any chordal and proper spanning subgraph of G. When  $n \leq 3$ , it is not difficult to verify that  $\epsilon(G, x) > \epsilon(Q, x)$ holds for all x < 0.

Suppose that Theorem 2 fails and G = (V, E) is a counter-example to this result such that |V| + |E| has the minimum value among all counter-examples. Thus the result holds for any graph H with |V(H)| + |E(H)| < |V| + |E| and any chordal and proper spanning subgraph Q' of H, but G has a chordal and proper spanning subgraph Q such that  $\epsilon(G, x) \leq \epsilon(Q, x)$  holds for some x < 0.

We will establish the following claims. Let  $u_1, u_2, \ldots, u_n$  be a perfect elimination ordering of Q and  $Q_i = Q[\{u_1, \ldots, u_i\}]$  for all  $i = 1, 2, \ldots, n$ . So  $u_i$  is a simplicial vertex of  $Q_i$  for all  $i = 1, 2, \ldots, n$ .

**Claim 1**:  $u_n$  is not a simplicial vertex of G.

Note that  $Q - u_n$  is chordal and a spanning subgraph of  $G - u_n$ . By the assumption on the minimality of |V| + |E|,  $\epsilon(G - u_n, x) \ge \epsilon(Q - u_n, x)$  holds for all x < 0, where the inequality is strict whenever  $Q - u_n \not\cong G - u_n$ .

Clearly  $d_G(u_n) \ge d_Q(u_n)$ . As Q is a proper subgraph of G,  $d_G(u_n) > d_Q(u_n)$  in the case that  $G - u_n \cong Q - u_n$ . If  $u_n$  is also a simplicial vertex of G, then by Proposition 2,

$$\epsilon(G, x) = \frac{1}{x - d_G(u_n)} + \epsilon(G - u_n, x), \quad \epsilon(Q, x) = \frac{1}{x - d_Q(u_n)} + \epsilon(Q - u_n, x), \quad (13)$$

implying that  $\epsilon(G, x) > \epsilon(Q, x)$  holds for all x < 0, a contradiction. Hence Claim 1 holds. Claim 2:  $d_G(u_n) > d_Q(u_n)$ .

Clearly  $d_G(u_n) \ge d_Q(u_n)$ . Since  $u_n$  is a simplicial vertex of Q and Q is a subgraph of G,  $d_G(u_n) = d_Q(u_n)$  implies that  $u_n$  is a simplicial vertex of G, contradicting Claim 1. Thus Claim 2 holds.

For any edge e in G, let G - e be the graph obtained from G by deleting e. Let G/e be the graph obtained from G by contracting e and replacing multiple edges, if any arise, by single edges.

**Claim 3:** For any  $e = u_n v \in E - E(Q)$ , both  $\epsilon(G - e, x) \ge \epsilon(Q, x)$  and  $\epsilon(G/e, x) \ge \epsilon(Q - u_n, x)$  hold for all x < 0.

As  $e = u_n v \in E - E(Q)$ , Q is a spanning subgraph of G - e and  $Q - u_n$  is a spanning subgraph of G/e. As both Q and  $Q - u_n$  are chordal, by the assumption on the minimality of |V| + |E|, the theorem holds for both G - e and G/e. Thus this claim holds. **Claim 4**:  $\epsilon(G, x) > \epsilon(Q, x)$  holds for all x < 0.

By Claim 2, there exists  $e = u_n v \in E - E(Q)$ . By Claim 3,  $\epsilon(G - e, x) \ge \epsilon(Q, x)$  and  $\epsilon(G/e, x) \ge \epsilon(Q - u_n, x)$  hold for all x < 0. By (8) and (12),

$$(\epsilon(G-e,x) - \epsilon(Q,x)) \times (-1)^n P(G-e,x)$$
  
=  $(-1)^n P'(G-e,x) + (-1)^{n+1} P(G-e,x) \sum_{i=1}^n \frac{1}{x - d_{Q_i}(u_i)}.$  (14)

As  $(-1)^n P(G-e,x) > 0$  and  $\epsilon(G-e,x) \ge \epsilon(Q,x)$  for all x < 0, the left-hand side of (14) is non-negative for x < 0, implying that the right-hand side of (14) is also non-negative for x < 0, i.e.,

$$(-1)^{n} P'(G-e,x) + (-1)^{n+1} P(G-e,x) \sum_{i=1}^{n} \frac{1}{x - d_{Q_i}(u_i)} \ge 0, \quad \forall x < 0.$$
(15)

As  $u_1, \ldots, u_{n-1}$  is a perfect elimination ordering of  $Q - u_n$  and  $\epsilon(G/e, x) \ge \epsilon(Q - u_n, x)$ holds for all x < 0, similarly we have:

$$(-1)^{n-1}P'(G/e,x) + (-1)^n P(G/e,x) \sum_{i=1}^{n-1} \frac{1}{x - d_{Q_i}(u_i)} \ge 0, \quad \forall x < 0.$$
(16)

As  $(-1)^{n-1}P(G/e, x) > 0$  holds for all x < 0, (16) implies that

$$(-1)^{n-1}P'(G/e, x) + (-1)^n P(G/e, x) \sum_{i=1}^n \frac{1}{x - d_{Q_i}(u_i)}$$

$$\geq \frac{(-1)^n P(G/e, x)}{x - d_{Q_n}(u_n)} > 0, \quad \forall x < 0.$$
(17)

By the deletion-contraction formula for chromatic polynomials,

$$P(G,x) = P(G-e,x) - P(G/e,x), \quad P'(G,x) = P'(G-e,x) - P'(G/e,x).$$
(18)

Then (15), (17) and (18) imply that

$$(-1)^{n} P'(G, x) + (-1)^{n+1} P(G, x) \sum_{i=1}^{n} \frac{1}{x - d_{Q_i}(u_i)} > 0, \quad \forall x < 0.$$
<sup>(19)</sup>

By (8) and (12), inequality (19) implies that

$$(\epsilon(G, x) - \epsilon(Q, x)) (-1)^n P(G, x) > 0, \quad \forall x < 0.$$
 (20)

Since  $(-1)^n P(G, x) > 0$  holds for all x < 0, inequality (20) implies Claim 4.

As Claim 4 contradicts the assumption of G, there are no counter-examples to this result and the theorem is proved.

### 3 An approach for proving Theorem 4

In this section, we will mainly show that, in order to prove Theorem 4, it suffices to prove Theorem 3. By (12), we have

$$\epsilon(K_n, x) = \sum_{i=0}^{n-1} \frac{1}{x-i}.$$
(21)

Thus,

$$\epsilon(K_n, x) - \epsilon(G, x) = \frac{(-1)^n}{P(G, x)} \left( (-1)^n P(G, x) \sum_{i=0}^{n-1} \frac{1}{x-i} + (-1)^{n+1} P'(G, x) \right).$$
(22)

For any graph G of order n, define

$$\xi(G, x) = (-1)^n P(G, x) \sum_{i=0}^{n-1} \frac{1}{x-i} + (-1)^{n+1} P'(G, x).$$
(23)

Note that  $\xi(G, x) \equiv 0$  if G is a complete graph. For any non-complete graph G and any x < 0, we have  $(-1)^n P(G, x) > 0$  and so (22) implies that  $\epsilon(K_n, x) - \epsilon(G, x) > 0$  if and only if  $\xi(G, x) > 0$ .

**Proposition 3.** Theorem 4 holds if and only if  $\xi(G, x) > 0$  holds for every non-complete graph G and all x < 0.

It can be easily verified that  $\xi(G, x) > 0$  holds for all non-complete graphs G of order at most 3 and all x < 0. For the general case, we will prove it by induction. In the rest of this section, we will find a relation between  $\xi(G, x)$  and  $\xi(G - u, x)$  for a vertex u in Gin two cases. Lemma 1 is for the case when u is a simplicial vertex and Lemma 3 when  $d(u) \ge 1$ . We then explain why Theorem 3 implies  $\xi(G, x) > 0$  for all non-complete graphs G and all x < 0.

**Lemma 1.** Let G be a graph of order n. If u is a simplicial vertex of G with d(u) = d, then

$$\xi(G,x) = (d-x)\xi(G-u,x) + \frac{(-1)^{n-1}(n-1-d)P(G-u,x)}{n-1-x}.$$
(24)

*Proof.* As u is a simplicial vertex of G with d(u) = d, P(G, x) = (x - d)P(G - u, x) by (10). Thus P'(G, x) = P(G - u, x) + (x - d)P'(G - u, x). By (23),

$$\xi(G,x) = (-1)^{n}(x-d)P(G-u,x)\sum_{i=0}^{n-1} \frac{1}{x-i} + (-1)^{n+1}(P(G-u,x) + (x-d)P'(G-u,x))$$
  
$$= (d-x)\xi(G-u,x) + \frac{(-1)^{n}(x-d)P(G-u,x)}{x-n+1} + (-1)^{n+1}P(G-u,x)$$
  
$$= (d-x)\xi(G-u,x) + \frac{(-1)^{n-1}(n-1-d)P(G-u,x)}{n-1-x}.$$
 (25)

Note that  $d \le n - 1$  and  $(-1)^{n-1}P(G - u, x) > 0$  holds for all x < 0, implying that the second term in the right-hand side of (24) is non-negative. Thus, if u is a simplicial vertex of G and x < 0, by Lemma 1,  $\xi(G - u, x) > 0$  implies that  $\xi(G, x) > 0$ .

Now consider the case that u is a vertex in G with  $d(u) = d \ge 1$ . Assume that  $N(u) = \{u_1, u_2, \ldots, u_d\}$ . For any  $i = 1, 2, \ldots, d-1$ , let  $G_i$  denote the graph obtained from G - u by adding edges joining  $u_i$  to  $u_j$  whenever  $u_i u_j \notin E(G)$  for all j with  $i + 1 \le j \le d$ . Thus,  $u_i$  is adjacent to  $u_j$  in  $G_i$  for all j with  $i + 1 \le j \le d$ . In the case that u is a simplicial vertex of G,  $G_i \cong G - u$  for all  $i = 1, 2, \cdots, d - 1$ . By applying the deletion-contraction formula for chromatic polynomials (see [5, 13]), P(G, x) can be expressed in terms of P(G - u, x) and  $P(G_i, x)$  for  $i = 1, 2, \cdots, d - 1$ .

**Lemma 2.** Let u be a vertex in G with  $d(x) = d \ge 1$  and for  $i = 1, 2, \dots, d-1$ , let  $G_i$  be the graph defined above. Then,

$$P(G, x) = (x - 1)P(G - u, x) - \sum_{i=1}^{d-1} P(G_i, x).$$
(26)

Proof. For  $1 \leq i \leq d$ , let  $E_i$  denote the set of edges  $uu_j$  in G for  $j = 1, 2, \dots, i - 1$ . So  $|E_i| = i - 1$  and  $E_1 = \emptyset$ . For any i with  $1 \leq i \leq d - 1$ , applying the deletion-contraction formula for chromatic polynomials to edge  $uu_i$  in  $G - E_i$ , the graph obtained from G by removing all edges in  $E_i$ , we have

$$P(G - E_i, x) = P(G - E_{i+1}, x) - P((G - E_i)/uu_i, x) = P(G - E_{i+1}, x) - P(G_i, x),$$
(27)

where the last equality follows from the fact that  $(G - E_i)/uu_i \cong G_i$  by the assumption of  $G_i$ . Thus, by (27),

$$P(G, x) = P(G - E_1, x) = P(G - E_d, x) - \sum_{i=1}^{d-1} P(G_i, x).$$
(28)

As u is of degree 1 in  $G - E_d$ ,  $P(G - E_d, x) = (x - 1)P(G - u, x)$ . Hence (26) follows.  $\Box$ 

**Lemma 3.** Let G be a graph of order n and let u be a vertex of G with  $d(u) = d \ge 1$ . Then

$$\xi(G,x) = (1-x)\xi(G-u,x) + \sum_{i=1}^{d-1} \xi(G_i,x) + \frac{(-1)^n \left[ (x-n+1)P(G-u,x) - P(G,x) \right]}{n-x-1},$$
(29)

where  $G_1, \ldots, G_{d-1}$  are graphs defined above.

*Proof.* By (26), we have

$$P'(G,x) = P(G-u,x) + (x-1)P'(G-u,x) - \sum_{i=1}^{d-1} P'(G_i,x).$$
(30)

Thus

$$\begin{split} \xi(G,x) &= (-1)^n P(G,x) \sum_{j=0}^{n-1} \frac{1}{x-j} + (-1)^{n+1} P'(G,x) \\ &= (-1)^n \left[ (x-1) P(G-u,x) - \sum_{i=1}^{d-1} P(G_i,x) \right] \sum_{j=0}^{n-1} \frac{1}{x-j} \\ &+ (-1)^{n+1} \left[ P(G-u,x) + (x-1) P'(G-u,x) - \sum_{i=1}^{d-1} P'(G_i,x) \right] \\ &= (1-x) \left[ (-1)^{n-1} P(G-u,x) \sum_{j=0}^{n-2} \frac{1}{x-j} + (-1)^n P'(G-u,x) \right] \\ &+ \sum_{i=1}^{d-1} \left[ (-1)^{n-1} P(G_i,x) \sum_{j=0}^{n-2} \frac{1}{x-j} + (-1)^n P'(G_i,x) \right] + (-1)^{n+1} P(G-u,x) \\ &+ (-1)^n \left[ \frac{(x-1) P(G-u,x)}{x-(n-1)} - \frac{1}{x-(n-1)} \sum_{i=1}^{d-1} P(G_i,x) \right] \\ &= (1-x) \xi(G-u,x) + \sum_{i=1}^{d-1} \xi(G_i,x) \end{split}$$

$$+\frac{(-1)^{n}\left[(x-n+1)P(G-u,x)-P(G,x)\right]}{n-x-1},$$
(31)

where the last expression follows from (26) and the definitions of  $\xi(G-u, x)$  and  $\xi(G_i, x)$ . The result then follows.

It is known that  $\xi(G, x) > 0$  holds for all non-complete graphs G of order at most 3 and all x < 0. For any non-complete graph G of order  $n \ge 4$ , by Lemma 1,  $\xi(G-u, x) > 0$ implies  $\xi(G, x) > 0$  for each simplicial vertex u in G and all x < 0; by Lemma 3, for any  $x < 0, \ \xi(G - u, x) > 0$  implies  $\xi(G, x) > 0$  whenever u is an non-isolated vertex in G satisfying the following inequality:

$$(-1)^{n}((x-n+1)P(G-u,x) - P(G,x)) > 0.$$
(32)

Note that the left-hand side of (32) vanishes when G is  $K_n$ . Also notice that there exist non-complete graph G and some vertex u in G such that inequality (32) does not hold for some x < 0. For example, if G is the complete bipartite graph  $K_{2,3}$  and u is a vertex of degree 3 in G, then (32) fails for all real x with -2.3 < x < 0. However, to prove that for any x < 0, there exists some vertex u in G such that inequality (32) holds, it suffices to prove the following inequality (i.e., Theorem 3):

$$(-1)^{n}(x-n+1)\sum_{u\in V}P(G-u,x) + (-1)^{n+1}nP(G,x) > 0$$
(33)

for any non-complete graph G = (V, E) of order n and all x < 0.

By Proposition 3 and inequality (32), to prove Theorem 4, we can now just focus on proving inequality (33) (i.e., Theorem 3). The proof of Theorem 3 will be given in Section 5 based on the interpretations for the coefficients of chromatic polynomials introduced in Section 4.

# 4 Combinatorial interpretations for coefficients of P(G, x)

Let G = (V, E) be any graph. In this section, we will introduce Greene & Zaslavsky's combinatorial interpretation in [8] for the coefficients of P(G, x) in terms of acyclic orientations. The result will be applied in the next section to prove Theorem 3.

An orientation D of G is called *acyclic* if D does not contain any directed cycle. Let  $\alpha(G)$  be the number of acyclic orientations of a graph G. In [18], Stanley gave a nice combinatorial interpretation of  $(-1)^n P(G, -k)$  for any positive integer k in terms of acyclic orientations of G. In particular, he proved:

**Theorem 5** ([18]). For any graph G of order n,  $(-1)^n P(G, -1) = \alpha(G)$ , *i.e.*,

$$\sum_{i=1}^{n} a_i(G) = \alpha(G). \tag{34}$$

In a digraph D, any vertex of D with in-degree (resp. out-degree) zero is called a *source* (resp. *sink*) of D. It is well known that any acyclic digraph has at least one source and at least one sink. If v is an isolated vertex of G, then v is a source and also a sink in any orientation of G.

For any  $v \in V$ , let  $\alpha(G, v)$  be the number of acyclic orientations of G with v as its unique source. Clearly  $\alpha(G, v) = 0$  if and only if G is not connected. In 1983, Greene and Zaslavsky [8] showed that  $a_1(G) = \alpha(G, v)$ .

**Theorem 6** ([8]). For any graph G = (V, E),  $a_1(G) = \alpha(G, v)$  holds for every  $v \in V$ .

This theorem was proved originally by using the theory of hyperplane arrangements. See [7] for three other nice proofs.

By Whitney's Broken-cycle Theorem (i.e., Theorem 1),  $a_i(G)$  equals the number of spanning subgraphs of G with i components and n - i edges, containing no broken cycles of G. In particular,  $a_1(G)$  is the number of spanning trees of G containing no broken cycles of G. Now we have two different combinatorial interpretations for  $a_1$ . For any  $a_i(G)$ ,  $2 \le i \le n$ , its combinatorial interpretation can be obtained by applying these two different combinatorial interpretations for  $a_1$ .

Let  $\mathcal{P}_i(V)$  be the set of partitions  $\{V_1, V_2, \ldots, V_i\}$  of V such that  $G[V_j]$  is connected for all  $j = 1, 2, \ldots, i$  and let  $\beta_i(G)$  be the number of ordered pairs  $(P_i, F)$ , where

- (a)  $P_i = \{V_1, V_2, \dots, V_i\} \in \mathcal{P}_i(V);$
- (b) F is a spanning forest of G with exactly i components  $T_1, T_2, \ldots, T_i$ , where each  $T_j$  is a spanning tree of  $G[V_j]$  containing no broken cycles of G.

For any subgraph H of G, let  $\tilde{\tau}(H)$  be the number of spanning trees of H containing no broken cycles of G. By Theorem 1,  $\tilde{\tau}(H) = a_1(H)$  holds and the next result follows.

**Theorem 7.** For any graph G and any  $1 \le i \le n$ ,

$$a_i(G) = \beta_i(G) = \sum_{\{V_1, \dots, V_i\} \in \mathcal{P}_i(V)} \prod_{j=1}^i \tilde{\tau}(G[V_j]).$$
(35)

Now let  $V = \{1, 2, ..., n\}$ . For any  $i : 1 \le i \le n$  and any vertex  $v \in V$ , let  $\mathcal{OP}_{i,v}(V)$ be the family of ordered partitions  $(V_1, V_2, ..., V_i)$  of V such that

(a)  $\{V_1, V_2, \dots, V_i\} \in \mathcal{P}_i(V)$ , where  $v \in V_1$ ;

(b) for j = 2, ..., i, the minimum number in the set  $\bigcup_{j \le s \le i} V_s$  is within  $V_j$ .

Clearly, for any  $v \in V$  and any  $\{V_1, V_2, \ldots, V_i\} \in \mathcal{P}_i(V)$ , there is exactly one permutation  $(\pi_1, \pi_2, \ldots, \pi_i)$  of  $1, 2, \ldots, i$  such that  $(V_{\pi_1}, V_{\pi_2}, \ldots, V_{\pi_i}) \in \mathcal{OP}_{i,v}(V)$ .

By Theorem 6,  $\tilde{\tau}(G[V_j]) = \alpha(G[V_j], u)$  holds for any vertex u in  $G[V_j]$  and Theorem 7 is equivalent to a result in [8] which we illustrate differently below. **Theorem 8** ([8], Theorem 7.4). For any  $v \in V$  and any  $1 \le i \le n$ ,

$$a_i(G) = \sum_{(V_1,...,V_i) \in \mathcal{OP}_{i,v}(V)} \alpha(G[V_1], v) \prod_{j=2}^i \alpha(G[V_j], m_j),$$
(36)

where  $m_j$  is the minimum number in  $V_j$  for j = 2, ..., i.

Note that the theorem above indicates that the right hand side of (36) is independent of the choice of v. Thus, for any  $1 \le i \le n$ ,

$$na_{i}(G) = \sum_{v \in V} \sum_{(V_{1},...,V_{i}) \in \mathcal{OP}_{i,v}(V)} \alpha(G[V_{1}],v) \prod_{j=2}^{i} \alpha(G[V_{j}],m_{j}).$$
(37)

Let  $P^{(i)}(G, x)$  be the *i*-th derivative of P(G, x). Very recently, Bernardi and Nadeau [1] gave an interpretation of  $P^{(i)}(G, -j)$  for any nonnegative integers *i* and *j* in terms of acyclic orientations. When i = 0, their result is exactly Theorem 5 due to Stanley [18]; and when j = 0, it is Theorem 8 due to Greene & Zaslavsky [8].

# 5 Proofs of Theorems 3 and 4

By the explanation in Section 3, to prove Theorem 4, it suffices to prove Theorem 3. In this section, we will prove Theorem 3 by showing that the coefficient of  $x^i$  in the expansion of the left-hand side of (9) in Theorem 3 is of the form  $(-1)^i d_i$  with  $d_i \ge 0$  for all i = 1, 2, ..., n. Furthermore,  $d_i > 0$  holds for some i when G is not complete.

We first establish the following result.

**Lemma 4.** Let G = (V, E) be a non-complete graph of order  $n \ge 3$  and component number c.

- (a). If c = 1 and G is not the n-cycle  $C_n$ , then there exist non-adjacent vertices  $u_1, u_2$  of G such that  $G \{u_1, u_2\}$  is connected.
- (b). If  $2 \le c \le n-1$ , then for any integer *i* with  $c \le i \le n-1$ , there exists a partition  $V_1, V_2, \ldots, V_i$  of *V* such that  $G[V_j]$  is connected for all  $j = 2, \ldots, i$  and  $G[V_1]$  has exactly two components one of which is an isolated vertex.

*Proof.* (a). As c = 1, G is connected. As G is non-complete, the result is trivial when G is 3-connected.

If G is not 2-connected, choose vertices  $u_1$  and  $u_2$  from distinct blocks  $B_1$  and  $B_2$  of G such that both  $u_1$  and  $u_2$  are not cut-vertices of G. Then  $u_1u_2 \notin E(G)$  and  $G - \{u_1, u_2\}$ is connected. Now consider the case that G is 2-connected but not 3-connected. Since G is not  $C_n$ , there exists a vertex w such that  $d(w) \ge 3$ . If d(w) = n - 1, then  $G - \{u_1, u_2\}$  is connected for any two non-adjacent vertices  $u_1$  and  $u_2$  in G. If G - w is 2-connected and  $d(w) \le n - 2$ , then  $G - \{w, u\}$  is connected for any  $u \in V - N_G(w)$ . If G - w is not 2-connected, then G - w contains two non-adjacent vertices  $u_1, u_2$  such that  $G - \{w, u_1, u_2\}$  is connected, implying that  $G - \{u_1, u_2\}$  is connected as  $d(w) \ge 3$ .

(b). Let  $G_1, G_2, \ldots, G_c$  be the components of G with  $|V(G_1)| \ge |V(G_j)|$  for all  $j = 1, 2, \ldots, c$ . As  $c \le n - 1$ ,  $|V(G_1)| \ge 2$ . Choose  $u \in V(G_1)$  such that  $G_1 - u$  is connected. Then  $V(G_2) \cup \{u\}, V(G_1) - \{u\}, V(G_3), \ldots, V(G_c)$  is a partition of V satisfying the condition in (b) for i = c.

Assume that (b) holds for i = k, where  $c \le k < n - 1$ , and  $V_1, V_2, \ldots, V_k$  is a partition of V satisfying the condition in (a). Then  $G[V_1]$  has an isolated vertex u and  $G[V'_1]$  is connected, where  $V'_1 = V_1 - \{u\}$ . Since  $k \le n - 2$ , either  $|V'_1| \ge 2$  or  $|V_j| \ge 2$  for some  $j \ge 2$ .

If  $|V'_1| \ge 2$ , then  $V'_1$  has a partition  $V'_{1,1}, V'_{1,2}$  such that both  $G[V'_{1,1}]$  and  $G[V'_{1,2}]$  are connected, implying that  $V'_{1,1} \cup \{u\}, V'_{1,2}, V_2, V_3, \ldots, V_k$  is a partition of V satisfying the condition in (b) for i = k + 1.

Similarly, if  $|V_j| \ge 2$  for some  $j \ge 2$  (say j = 2), then  $V_2$  has a partition  $V_{2,1}, V_{2,2}$ such that both  $G[V_{2,1}]$  and  $G[V_{2,2}]$  are connected, implying that  $V_1, V_{2,1}, V_{2,2}, V_3, \ldots, V_k$  is a partition of V satisfying the condition in (b) for i = k + 1.

For any graph G = (V, E) of order n, write

$$(-1)^{n} \left[ (x - n + 1) \sum_{u \in V(G)} P(G - u, x) - nP(G, x) \right] = \sum_{i=1}^{n} (-1)^{i} d_{i} x^{i}.$$
 (38)

By comparing coefficients, it can be shown that

$$d_i = \sum_{u \in V(G)} \left[ a_{i-1}(G-u) + (n-1)a_i(G-u) \right] - na_i(G), \quad \forall i = 1, 2, \dots, n.$$
(39)

It is obvious that when G is the complete graph  $K_n$ , the left-hand side of (38) vanishes and thus  $d_i = 0$  for all i = 1, 2, ..., n. Now we consider the case that G is not complete.

**Proposition 4.** Let G = (V, E) be a non-complete graph of order n and component number c. Then, for any i = 1, 2, ..., n,  $d_i \ge 0$  and equality holds if and only if one of the following cases happens:

(a). i = n;

- (b).  $1 \le i \le c 2;$
- (c). i = c 1 and G does not have isolated vertices;
- (d). i = c = 1 and G is  $C_n$ .

*Proof.* We first show that  $d_i = 0$  in any one of the four cases above.

By (39),  $d_n = \sum_{u \in V} [1 + (n-1) \cdot 0] - n \cdot 1 = 0.$ 

It is known that for  $1 \le i \le n$ ,  $a_i(G) = 0$  if and only if i < c (see [5, 13, 14]). Similarly,  $a_i(G-u) = 0$  for all i with  $1 \le i < c-1$  and all  $u \in V$ , and  $a_{c-1}(G-u) = 0$  if u is not an isolated vertex of G. By (39),  $d_i = 0$  for all i with  $1 \le i \le c-2$ , and  $d_{c-1} = 0$  when Gdoes not have isolated vertices.

If G is  $C_n$ , then  $a_1(G) = n - 1$ ,  $a_0(G - u) = 0$  and  $a_1(G - u) = 1$  for each  $u \in V$ , implying that  $d_1 = 0$  by (39).

In the following, we will show that  $d_i > 0$  when *i* does not belong to any one of the four cases.

If G has isolated vertices, then  $a_{c-1}(G-u) > 0$  for any isolated vertex u of G and

$$\sum_{u \in V} a_{c-1}(G-u) = \sum_{\substack{u \in V \\ u \text{ isolated}}} a_{c-1}(G-u) > 0.$$
(40)

As  $a_{c-1}(G) = 0$ , by (39), we have  $d_{c-1} > 0$  in this case. Now it remains to show that  $d_i > 0$  holds for all i with  $c \le i \le n-1$ , except when i = c = 1 and G is  $C_n$ .

For any  $v \in V$ , let  $\mathcal{OP}'_{i,v}(V)$  be the set of ordered partitions  $(V_1, \ldots, V_i) \in \mathcal{OP}_{i,v}(V)$ with  $V_1 = \{v\}$ . As  $\alpha(G[V_1], v) = 1$ , for any *i* with  $c \leq i \leq n$ , by Theorem 8,

$$a_{i-1}(G-v) = \sum_{(V_1,\dots,V_i)\in\mathcal{OP}'_{i,v}(V)} \alpha(G[V_1],v) \prod_{j=2}^{i} \alpha(G[V_j],m_j),$$
(41)

where  $m_j$  is the minimum number in  $V_j$  for all  $j = 2, \ldots, i$ .

Let s and v be distinct members in V. For any  $V_1 \subseteq V - \{s\}$  with  $v \in V_1$ , let  $\alpha(G[V_1 \cup \{s\}], v, s)$  be the number of those acyclic orientations of  $G[V_1 \cup \{s\}]$  with v as the unique source and s as one sink. Then  $\alpha(G[V_1 \cup \{s\}], v, s) \leq \alpha(G[V_1], v)$  holds, where the inequality is strict if and only if  $G[V_1]$  is connected but  $G[V_1 \cup \{s\}]$  is not. Observe that

$$a_{i}(G-s) = \sum_{(V_{1},...,V_{i})\in\mathcal{OP}_{i,v}(V-\{s\})} \alpha(G[V_{1}],v) \prod_{j=2}^{i} \alpha(G[V_{j}],m_{j})$$

$$\geq \sum_{(V_{1},...,V_{i})\in\mathcal{OP}_{i,v}(V-\{s\})} \alpha(G[V_{1}\cup\{s\}],v,s) \prod_{j=2}^{i} \alpha(G[V_{j}],m_{j})$$
(42)

$$= \sum_{(V'_1, \dots, V'_i) \in \mathcal{OP}_{i,v,s}(V)} \alpha(G[V'_1], v, s) \prod_{j=2}^i \alpha(G[V'_j], m_j),$$
(43)

where  $\mathcal{OP}_{i,v,s}(V)$  is the set of ordered partitions  $(V'_1, \ldots, V'_i) \in \mathcal{OP}_{i,v}(V)$  with  $s, v \in V'_1$ . By the explanation above, inequality (42) is strict whenever  $V - \{s\}$  has a partition  $V_1, V_2, \ldots, V_i$  with  $v \in V_1$  such that each  $G[V_j]$  is connected for all  $j = 1, 2, \ldots, i$  but  $G[V_1 \cup \{s\}]$  is not connected.

By (37), we have

$$na_{i}(G) = \sum_{v \in V} \sum_{(V_{1},...,V_{i}) \in \mathcal{OP}_{i,v}(V)} \alpha(G[V_{1}],v) \prod_{j=2}^{i} \alpha(G[V_{j}],m_{j})$$

$$= \sum_{v \in V} \sum_{(V_{1},...,V_{i}) \in \mathcal{OP}'_{i,v}(V)} \alpha(G[V_{1}],v) \prod_{j=2}^{i} \alpha(G[V_{j}],m_{j})$$

$$+ \sum_{v \in V} \sum_{(V_{1},...,V_{i}) \in \mathcal{OP}_{i,v}(V) - \mathcal{OP}'_{i,v}(V)} \alpha(G[V_{1}],v) \prod_{j=2}^{i} \alpha(G[V_{j}],m_{j}). \quad (44)$$

By (41),

$$\sum_{v \in V} \sum_{(V_1, \dots, V_i) \in \mathcal{OP}'_{i,v}(V)} \alpha(G[V_1], v) \prod_{j=2}^i \alpha(G[V_j], m_j) = \sum_{v \in V} a_{i-1}(G - v),$$
(45)

and by (43),

$$\sum_{v \in V} \sum_{(V_1, \dots, V_i) \in \mathcal{OP}_{i,v}(V) - \mathcal{OP}'_{i,v}(V)} \alpha(G[V_1], v) \prod_{j=2}^{i} \alpha(G[V_j], m_j)$$

$$\sum_{v \in V} \sum_{(V_1, \dots, V_i) \in \mathcal{OP}_{i,v}(V) - \mathcal{OP}'_{i,v}(V)} \alpha(G[V_i], v, s) \prod_{j=2}^{i} \alpha(G[V_j], m_j)$$
(46)

$$\leq \sum_{v \in V} \sum_{s \in V - \{v\}} \sum_{(V_1, \dots, V_i) \in \mathcal{OP}_{i,v,s}(V)} \alpha(G[V_1], v, s) \prod_{j=2} \alpha(G[V_j], m_j)$$
(46)

$$\leq \sum_{v \in V} \sum_{s \in V - \{v\}} a_i(G - s) \tag{47}$$

$$= (n-1)\sum_{v \in V} a_i(G-v),$$
(48)

where inequality (46) is strict if there exists  $(V_1, \ldots, V_i) \in \mathcal{OP}_{i,v}(V)$  for some  $v \in V$  such that  $G[V_j]$  is connected for all  $j = 1, \ldots, i$  and  $G[V_1]$  has acyclic orientations with v as the unique source but with at least two sinks, and by (42) and (43), inequality (47) is strict if V can be partitioned into  $V_1, \ldots, V_i$  such that  $G[V_j]$  is connected for all  $j = 2, \ldots, i$  but  $G[V_1]$  has exactly two components, one of which is an isolated vertex in  $G[V_1]$ .

As G is not complete, by Lemma 4 and the above explanation, the inequality of (48) is strict for all i with  $c \leq i \leq n-1$ , except when i = c = 1 and G is  $C_n$ . Then, by (44), (45) and (48), we conclude that

$$d_i = \sum_{v \in V} \left[ a_{i-1}(G-u) + (n-1)a_i(G-u) \right] - na_i(G) > 0, \quad \forall c \le i \le n-1,$$
(49)

except that i = c = 1 and G is  $C_n$ . Hence the proof is complete.

Now everything is ready for proving Theorems 3 and 4.

Proof of Theorem 3: Let G be a non-complete graph of order n. Recall (38) that

$$(-1)^n \left[ (x - n + 1) \sum_{u \in V(G)} P(G - u, x) - nP(G, x) \right] = \sum_{i=1}^n (-1)^i d_i x^i.$$
(50)

By Proposition 4, we know that  $d_i \ge 0$  for all i with  $1 \le i \le n$  and  $d_{n-1} > 0$ . Thus  $\sum_{i=1}^{n} (-1)^i d_i x^i > 0$  holds for all x < 0, which completes the proof of Theorem 3.

**Proposition 5.** For any non-complete graph G,  $\xi(G, x) > 0$  holds for all x < 0.

*Proof.* We will prove this result by induction on the order n of G. When n = 2, the empty graph  $N_2$  of order 2 is the only non-complete graph of order 2. As  $P(N_2, x) = x^2$ , by (23), we have

$$\xi(N_2, x) = (-1)^2 x^2 \left(\frac{1}{x} + \frac{1}{x-1}\right) + (-1)^3 2x = \frac{x}{x-1} > 0$$
(51)

for all x < 0.

Assume that this result holds for any non-complete graph G of order less than n, where  $n \ge 3$ . Now let G be any non-complete graph of order n.

**Case 1**: G contains an isolated vertex u.

By the inductive assumption,  $\xi(G-u, x) \ge 0$  holds for all x < 0, where equality holds when G-u is a complete graph. By Lemma 1,  $\xi(G, x) > 0$  holds for all x < 0. **Case 2**: G has no isolated vertex.

By Theorem 3, (9) holds for all x < 0. Thus, for any x < 0, there exists some  $u \in V(G)$ such that  $(-1)^n(x-n+1)P(G-u,x) + (-1)^{n+1}P(G,x) > 0$  holds. Then, by Lemma 3 and by the inductive assumption,  $\xi(G,x) > 0$  holds for any x < 0.

Hence the result holds.

*Proof of Theorem* 4: It follows directly from Propositions 3 and 5.

# 6 Remarks and problems

First we give some remarks here.

(a). Theorem 4 implies that for any non-complete graph G of order n,  $\frac{P(G,x)}{P(K_n,x)}$  is strictly decreasing when x < 0.

(b). Let G be a non-complete graph of order n and  $P(G, x) = \sum_{i=1}^{n} (-1)^{n-i} a_i x^i$ . Then  $\epsilon(G) < \epsilon(K_n)$  implies that

$$\frac{a_1 + 2a_2 + \dots + na_n}{a_1 + a_2 + \dots + a_n} > 1 + \frac{1}{2} + \dots + \frac{1}{n}.$$
(52)

(c). When x = -1, Theorem 3 implies that for any graph G of order n,

$$(-1)^{n-1} \sum_{u \in V} P(G-u, -1) \ge (-1)^n P(G, -1),$$
(53)

where the inequality holds if and only if G is complete. By Stanley's interpretation for  $(-1)^n P(G, -1)$  in [18], the inequality above implies that for any graph G = (V, E), the number of acyclic orientations of G is at most the total number of acyclic orientations of G - u for all  $u \in V$ , where the equality holds if and only if G is complete.

Now we raise some problems for further study.

It is clear that for any graph G of order n,

$$\frac{d}{dx}\left(\ln[(-1)^n P(G, x)]\right) = \frac{P'(G, x)}{P(G, x)} < 0$$
(54)

holds for all x < 0. We surmise that this property holds for higher derivatives of the function  $\ln[(-1)^n P(G, x)]$  in the interval  $(-\infty, 0)$ .

**Conjecture 2.** Let G be a graph of order n. Then  $\frac{d^k}{dx^k} (\ln[(-1)^n P(G, x)]) < 0$  holds for all  $k \ge 2$  and  $x \in (-\infty, 0)$ .

Observe that  $\epsilon(G, x) = \frac{d}{dx} (\ln[(-1)^n P(G, x)])$ . We believe that Theorems 2 and 4 can be extended to higher derivatives of the function  $\ln[(-1)^n P(G, x)]$ .

**Conjecture 3.** Let G be any non-complete graph of order n and Q be any chordal and proper spanning subgraph Q of G. Then

$$\frac{d^k}{dx^k} \left( \ln[(-1)^n P(Q, x)] \right) < \frac{d^k}{dx^k} \left( \ln[(-1)^n P(G, x)] \right) < \frac{d^k}{dx^k} \left( \ln[(-1)^n P(K_n, x)] \right)$$
(55)

holds for any integer  $k \ge 2$  and all x < 0.

It is not difficult to show that Conjecture 2 holds for  $G \cong K_n$ . Thus the second inequality of Conjecture 3 implies Conjecture 2.

It is natural to extend the second part of Conjecture 1 (i.e.,  $\epsilon(G) < \epsilon(K_n)$  for any non-complete graph G of order n) to the inequality  $\epsilon(G) \leq \epsilon(G')$  for any graph G' which contains G as a subgraph. However, this inequality is not always true. Let  $G_n$  denote the graph obtained from the complete bipartite graph  $K_{2,n}$  by adding a new edge joining the two vertices in the partite set of size 2. Lundow and Markström [11] stated that  $\epsilon(K_{2,n}) > \epsilon(G_n)$  holds for all  $n \ge 3$ . In spite of this, we believe that for any non-complete graph G, we can add a new edge to G to obtain a graph G' with the property that  $\epsilon(G) < \epsilon(G')$ , as stated below.

**Conjecture 4.** For any non-complete graph G, there exist non-adjacent vertices u and v in G such that  $\epsilon(G) < \epsilon(G + uv)$ .

Obviously, Conjecture 4 implies  $\epsilon(G) < \epsilon(K_n)$  for any non-complete graph G of order n (i.e., Theorem 4). Conjecture 4 is similar to but may be not equivalent to the following conjecture due to Lundow and Markström [11].

**Conjecture 5** ([11]). For any 2-connected graph G, there exists an edge e in G such that  $\epsilon(G-e) < \epsilon(G)$ .

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