# Proving a conjecture on chromatic polynomials by counting the number of acyclic orientations* 

Fengming Dong $\dagger$ Jun Ge, Helin Gong<br>Bo Ning, Zhangdong Ouyang and Eng Guan Tay


#### Abstract

The chromatic polynomial $P(G, x)$ of a graph $G$ of order $n$ can be expressed as $\sum_{i=1}^{n}(-1)^{n-i} a_{i} x^{i}$, where $a_{i}$ is interpreted as the number of broken-cycle free spanning subgraphs of $G$ with exactly $i$ components. The parameter $\epsilon(G)=\sum_{i=1}^{n}(n-i) a_{i} / \sum_{i=1}^{n} a_{i}$ is the mean size of a broken-cycle-free spanning subgraph of $G$. In this article, we confirm and strengthen a conjecture proposed by Lundow and Markström in 2006 that $\epsilon\left(T_{n}\right)<\epsilon(G)<\epsilon\left(K_{n}\right)$ holds for any connected graph $G$ of order $n$ which is neither the complete graph $K_{n}$ nor a tree $T_{n}$ of order $n$. The most crucial step of our proof is to obtain the interpretation of all $a_{i}$ 's by the number of acyclic orientations of $G$.


Keywords: chromatic polynomial; graph; acyclic orientation; combinatorial interpretation

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## 1 Introduction

All graphs considered in this paper are simple graphs. For any graph $G=(V, E)$ and any positive integer $k$, a proper $k$-coloring $f$ of $G$ is a mapping $f: V \rightarrow\{1,2, \ldots, k\}$ such that $f(u) \neq f(v)$ holds whenever $u v \in E$. The chromatic polynomial of $G$ is the function $P(G, x)$ such that $P(G, k)$ counts the number of proper $k$-colorings of $G$ for any positive integer $k$. In this article, the variable $x$ in $P(G, x)$ is a real number. The study of chromatic polynomials is one of the most active areas in graph theory. For basic concepts

[^0]and properties on chromatic polynomials, we refer the reader to the monograph [5]. For the most celebrated results on this topic, we recommend surveys [4, 10, 14, 15,

The first interpretation of the coefficients of $P(G, x)$ was provided by Whitney [23]: for any simple graph $G$ of order $n$ and size $m$,

$$
\begin{equation*}
P(G, x)=\sum_{i=1}^{n}\left(\sum_{r=0}^{m}(-1)^{r} N(i, r)\right) x^{i}, \tag{1}
\end{equation*}
$$

where $N(i, r)$ is the number of spanning subgraphs of $G$ with exactly $i$ components and $r$ edges. Whitney further simplified (11) by introducing the notion of broken cycles. Let $\eta: E \rightarrow\{1,2, \ldots,|E|\}$ be a bijection. For any cycle $C$ in $G$, the path $C-e$ is called a broken cycle of $G$ with respect to $\eta$, where $e$ is the edge on $C$ with $\eta(e) \leq \eta\left(e^{\prime}\right)$ for every edge $e^{\prime}$ on $C$. When there is no confusion, a broken cycle of $G$ is always assumed to be with respect to a bijection $\eta: E \rightarrow\{1,2, \ldots,|E|\}$.

Theorem 1 ( 23$])$. Let $G=(V, E)$ be a graph of order $n$ and $\eta: E \rightarrow\{1,2, \ldots,|E|\}$ be a bijection. Then,

$$
\begin{equation*}
P(G, x)=\sum_{i=1}^{n}(-1)^{n-i} a_{i}(G) x^{i}, \tag{2}
\end{equation*}
$$

where $a_{i}(G)$ is the number of spanning subgraphs of $G$ with $n-i$ edges and $i$ components which do not contain broken cycles.

Let $G$ be a simple graph of order $n$. When there is no confusion, $a_{i}(G)$ is written as $a_{i}$ for short. Clearly, by Theorem [1, $P(G, x)$ is indeed a polynomial in $x$ in which the constant term is 0 , the leading coefficient $a_{n}$ is 1 and all coefficients are integers alternating in signs. Thus, $(-1)^{n} P(G, x)>0$ holds for all $x<0$.

The concept of broken cycles has the following connection with Tutte's work of expressing the Tutte polynomial $\mathbf{T}_{G}(x, y)$ of a connected graph $G$ in terms of spanning trees [2, 22]:

$$
\begin{equation*}
\mathbf{T}_{G}(x, y)=\sum_{T} x^{i a_{\omega}(T)} y^{e a_{\omega}(T)} \tag{3}
\end{equation*}
$$

where the sum runs over all spanning trees of $G$ and $i a_{\omega}(T)$ and $e a_{\omega}(T)$ are respectively the internal and external activities of $T$ with respect to a bijection $\omega: E \rightarrow\{1,2, \ldots,|E|\}$. If we take $\omega$ to be $\eta$, then $e a_{\eta}(T)$ is exactly the number of edges $e \in E(G) \backslash E(T)$ such that $\eta(e) \leq \eta\left(e^{\prime}\right)$ holds for all edges $e^{\prime}$ on the unique cycle $C$ of $T \cup e$. As $G$ is a simple graph, $e a_{\eta}(T)$ equals the number of broken cycles contained in $T$ with respect to $\eta$. In
particular, $e a_{\eta}(T)=0$ if and only if $T$ does not contain broken cycles with respect to $\eta$. By Theorem $a_{1}(G)$ is the number of spanning trees $T$ of $G$ with $e a_{\eta}(T)=0$. If

$$
\begin{equation*}
\mathbf{T}_{G}(x, y)=\sum_{i \geq 0, j \geq 0} c_{i, j} x^{i} y^{j}, \tag{4}
\end{equation*}
$$

then $a_{1}(G)=\sum_{i \geq 0} c_{i, 0}=\mathbf{T}_{G}(1,0)$.
As in [11, for $i=0,1,2, \ldots, n-1$, we define $b_{i}(G)$ (or simply $b_{i}$ ) as the probability that a randomly chosen broken-cycle-free spanning subgraph of $G$ has size $i$. Then

$$
\begin{equation*}
b_{i}=\frac{a_{n-i}}{a_{1}+a_{2}+\cdots+a_{n}}, \quad \forall i=0,1, \ldots, n-1 . \tag{5}
\end{equation*}
$$

Let $\epsilon(G)$ denote the mean size of a broken-cycle-free spanning subgraph of $G$. Then

$$
\begin{equation*}
\epsilon(G)=\sum_{i=0}^{n-1} i b_{i}=\frac{(n-1) a_{1}+(n-2) a_{2}+\cdots+a_{n-1}}{a_{1}+a_{2}+\cdots+a_{n}} \tag{6}
\end{equation*}
$$

An elementary property of $\epsilon(G)$ is given below.
Proposition 1 ([11). For any graph $G$ of order $n, \epsilon(G)=n+\frac{P^{\prime}(G,-1)}{P(G,-1)}$.
Let $T_{n}$ denote a tree of order $n$ and $K_{n}$ denote the complete graph of order $n$. By Proposition $\mathbb{1} \epsilon\left(T_{n}\right)=\frac{n-1}{2}$, , while

$$
\begin{equation*}
\epsilon\left(K_{n}\right)=n-\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right) \sim n-\log n-\gamma \tag{7}
\end{equation*}
$$

as $n \rightarrow \infty$, where $\gamma \approx 0.577216$ is the Euler-Mascheroni constant.
Lundow and Markström [11] proposed the following conjecture on $\epsilon(G)$.
Conjecture 1 ( [11). For any connected graph $G$ of order $n$, where $n \geq 4$, if $G$ is neither $K_{n}$ nor a $T_{n}$, then $\epsilon\left(T_{n}\right)<\epsilon(G)<\epsilon\left(K_{n}\right)$.

In this paper, we aim to prove and strengthen Conjecture (1) For any graph $G$, define the function $\epsilon(G, x)$ as follows:

$$
\begin{equation*}
\epsilon(G, x)=\frac{P^{\prime}(G, x)}{P(G, x)} . \tag{8}
\end{equation*}
$$

By Proposition $\epsilon(G)=n+\epsilon(G,-1)$ holds for every graph $G$ of order $n$. Thus, for any graphs $G$ and $H$ of the same order, $\epsilon(G)<\epsilon(H)$ if and only if $\epsilon(G,-1)<\epsilon(H,-1)$. Conjecture $\mathbb{\eta}$ is equivalent to the statement that $\epsilon\left(T_{n},-1\right)<\epsilon(G,-1)<\epsilon\left(K_{n},-1\right)$ holds for any connected graph $G$ of order $n$ which is neither $K_{n}$ nor a $T_{n}$.

A graph $Q$ is said to be chordal if $Q[V(C)] \neq C$ for every cycle $C$ of $Q$ with $|V(C)| \geq 4$, where $Q\left[V^{\prime}\right]$ is the subgraph of $Q$ induced by $V^{\prime}$ for $V^{\prime} \subseteq V(G)$. In Section 2, we will establish the following result.

Theorem 2. For any graph $G$, if $Q$ is a chordal and proper spanning subgraph of $G$, then $\epsilon(G, x)>\epsilon(Q, x)$ holds for all $x<0$.

Note that any tree is a chordal graph and any connected graph contains a spanning tree. Thus, we have the following corollary which obviously implies the first part of Conjecture [

Corollary 1. For any connected graph $G$ of order $n$ which is not a tree, $\epsilon(G, x)>\epsilon\left(T_{n}, x\right)$ holds for all $x<0$.

The second part of Conjecture 1 is extended to the inequality $\epsilon\left(K_{n}, x\right)>\epsilon(G, x)$ for any non-complete graph $G$ of order $n$ and all $x<0$. In order to prove this inequality, we will show in Section 3 that it suffices to establish the following result.

Theorem 3. For any non-complete graph $G=(V, E)$ of order n,

$$
\begin{equation*}
(-1)^{n}(x-n+1) \sum_{u \in V} P(G-u, x)+(-1)^{n+1} n P(G, x)>0 \tag{9}
\end{equation*}
$$

holds for all $x<0$.
Note that the left-hand side of (9) vanishes when $G \cong K_{n}$. Theorem 3 will be proved in Section [5, based on Greene \& Zaslavsky's interpretation in [8] for coefficients $a_{i}(G)$ 's of $P(G, x)$ by acyclic orientations introduced in Section [4 By applying Theorem 3 and two lemmas in Section 娄, we will finally prove the second main result in this article.

Theorem 4. For any non-complete graph $G$ of order $n, \epsilon(G, x)<\epsilon\left(K_{n}, x\right)$ holds for all $x<0$.

## 2 Proof of Theorem 2

A vertex $u$ in a graph $G$ is called a simplicial vertex if $\{u\} \cup N_{G}(u)$ is a clique of $G$, where $N_{G}(u)$ is the set of vertices in $G$ which are adjacent to $u$. For a simplicial vertex $u$ of $G$, $P(G, x)$ has the following property (see [5, 13, 14]):

$$
\begin{equation*}
P(G, x)=(x-d(u)) P(G-u, x), \tag{10}
\end{equation*}
$$

where $G-u$ is the subgraph of $G$ induced by $V-\{u\}$ and $d(u)$ is the degree of $u$ in $G$. By (10), it is not difficult to show the following.

Proposition 2. If $u$ is a simplicial vertex of a graph $G$, then

$$
\begin{equation*}
\epsilon(G, x)=\frac{1}{x-d(u)}+\epsilon(G-u, x) . \tag{11}
\end{equation*}
$$

It has been shown that a graph $Q$ of order $n$ is chordal if and only if $Q$ has an ordering $u_{1}, u_{2}, \ldots, u_{n}$ of its vertices such that $u_{i}$ is a simplicial vertex in $Q\left[\left\{u_{1}, u_{2}, \ldots, u_{i}\right\}\right]$ for all $i=1,2, \ldots, n$ (see [3, 6]). Such an ordering of vertices in $Q$ is called a perfect elimination ordering of $Q$. For any perfect elimination ordering $u_{1}, u_{2}, \ldots, u_{n}$ of a chordal graph $Q$, by Proposition 2 .

$$
\begin{equation*}
\epsilon(Q, x)=\sum_{i=1}^{n} \frac{1}{x-d_{Q_{i}}\left(u_{i}\right)}, \tag{12}
\end{equation*}
$$

where $Q_{i}$ is the subgraph $Q\left[\left\{u_{1}, u_{2}, \ldots, u_{i}\right\}\right]$.
Now we are ready to prove Theorem 2,
Proof of Theorem 图 Let $G$ be any graph of order $n$ and $Q$ be any chordal and proper spanning subgraph of $G$. When $n \leq 3$, it is not difficult to verify that $\epsilon(G, x)>\epsilon(Q, x)$ holds for all $x<0$.

Suppose that Theorem 2 fails and $G=(V, E)$ is a counter-example to this result such that $|V|+|E|$ has the minimum value among all counter-examples. Thus the result holds for any graph $H$ with $|V(H)|+|E(H)|<|V|+|E|$ and any chordal and proper spanning subgraph $Q^{\prime}$ of $H$, but $G$ has a chordal and proper spanning subgraph $Q$ such that $\epsilon(G, x) \leq \epsilon(Q, x)$ holds for some $x<0$.

We will establish the following claims. Let $u_{1}, u_{2}, \ldots, u_{n}$ be a perfect elimination ordering of $Q$ and $Q_{i}=Q\left[\left\{u_{1}, \ldots, u_{i}\right\}\right]$ for all $i=1,2, \ldots, n$. So $u_{i}$ is a simplicial vertex of $Q_{i}$ for all $i=1,2, \ldots, n$.
Claim 1: $u_{n}$ is not a simplicial vertex of $G$.
Note that $Q-u_{n}$ is chordal and a spanning subgraph of $G-u_{n}$. By the assumption on the minimality of $|V|+|E|, \epsilon\left(G-u_{n}, x\right) \geq \epsilon\left(Q-u_{n}, x\right)$ holds for all $x<0$, where the inequality is strict whenever $Q-u_{n} \neq G-u_{n}$.

Clearly $d_{G}\left(u_{n}\right) \geq d_{Q}\left(u_{n}\right)$. As $Q$ is a proper subgraph of $G, d_{G}\left(u_{n}\right)>d_{Q}\left(u_{n}\right)$ in the case that $G-u_{n} \cong Q-u_{n}$. If $u_{n}$ is also a simplicial vertex of $G$, then by Proposition 2,

$$
\begin{equation*}
\epsilon(G, x)=\frac{1}{x-d_{G}\left(u_{n}\right)}+\epsilon\left(G-u_{n}, x\right), \quad \epsilon(Q, x)=\frac{1}{x-d_{Q}\left(u_{n}\right)}+\epsilon\left(Q-u_{n}, x\right), \tag{13}
\end{equation*}
$$

implying that $\epsilon(G, x)>\epsilon(Q, x)$ holds for all $x<0$, a contradiction. Hence Claim 1 holds.
Claim 2: $d_{G}\left(u_{n}\right)>d_{Q}\left(u_{n}\right)$.
Clearly $d_{G}\left(u_{n}\right) \geq d_{Q}\left(u_{n}\right)$. Since $u_{n}$ is a simplicial vertex of $Q$ and $Q$ is a subgraph of $G, d_{G}\left(u_{n}\right)=d_{Q}\left(u_{n}\right)$ implies that $u_{n}$ is a simplicial vertex of $G$, contradicting Claim 1. Thus Claim 2 holds.

For any edge $e$ in $G$, let $G-e$ be the graph obtained from $G$ by deleting $e$. Let $G / e$ be the graph obtained from $G$ by contracting $e$ and replacing multiple edges, if any arise, by single edges.

Claim 3: For any $e=u_{n} v \in E-E(Q)$, both $\epsilon(G-e, x) \geq \epsilon(Q, x)$ and $\epsilon(G / e, x) \geq$ $\epsilon\left(Q-u_{n}, x\right)$ hold for all $x<0$.

As $e=u_{n} v \in E-E(Q), Q$ is a spanning subgraph of $G-e$ and $Q-u_{n}$ is a spanning subgraph of $G / e$. As both $Q$ and $Q-u_{n}$ are chordal, by the assumption on the minimality of $|V|+|E|$, the theorem holds for both $G-e$ and $G / e$. Thus this claim holds.

Claim 4: $\epsilon(G, x)>\epsilon(Q, x)$ holds for all $x<0$.
By Claim 2, there exists $e=u_{n} v \in E-E(Q)$. By Claim 3, $\epsilon(G-e, x) \geq \epsilon(Q, x)$ and $\epsilon(G / e, x) \geq \epsilon\left(Q-u_{n}, x\right)$ hold for all $x<0$. By (8) and (12),

$$
\begin{align*}
& (\epsilon(G-e, x)-\epsilon(Q, x)) \times(-1)^{n} P(G-e, x) \\
= & (-1)^{n} P^{\prime}(G-e, x)+(-1)^{n+1} P(G-e, x) \sum_{i=1}^{n} \frac{1}{x-d_{Q_{i}}\left(u_{i}\right)} . \tag{14}
\end{align*}
$$

As $(-1)^{n} P(G-e, x)>0$ and $\epsilon(G-e, x) \geq \epsilon(Q, x)$ for all $x<0$, the left-hand side of (14) is non-negative for $x<0$, implying that the right-hand side of (14) is also non-negative for $x<0$, i.e.,

$$
\begin{equation*}
(-1)^{n} P^{\prime}(G-e, x)+(-1)^{n+1} P(G-e, x) \sum_{i=1}^{n} \frac{1}{x-d_{Q_{i}}\left(u_{i}\right)} \geq 0, \quad \forall x<0 . \tag{15}
\end{equation*}
$$

As $u_{1}, \ldots, u_{n-1}$ is a perfect elimination ordering of $Q-u_{n}$ and $\epsilon(G / e, x) \geq \epsilon\left(Q-u_{n}, x\right)$ holds for all $x<0$, similarly we have:

$$
\begin{equation*}
(-1)^{n-1} P^{\prime}(G / e, x)+(-1)^{n} P(G / e, x) \sum_{i=1}^{n-1} \frac{1}{x-d_{Q_{i}}\left(u_{i}\right)} \geq 0, \quad \forall x<0 . \tag{16}
\end{equation*}
$$

As $(-1)^{n-1} P(G / e, x)>0$ holds for all $x<0$, (16) implies that

$$
\begin{align*}
& (-1)^{n-1} P^{\prime}(G / e, x)+(-1)^{n} P(G / e, x) \sum_{i=1}^{n} \frac{1}{x-d_{Q_{i}}\left(u_{i}\right)} \\
\geq & \frac{(-1)^{n} P(G / e, x)}{x-d_{Q_{n}}\left(u_{n}\right)}>0, \quad \forall x<0 . \tag{17}
\end{align*}
$$

By the deletion-contraction formula for chromatic polynomials,

$$
\begin{equation*}
P(G, x)=P(G-e, x)-P(G / e, x), \quad P^{\prime}(G, x)=P^{\prime}(G-e, x)-P^{\prime}(G / e, x) . \tag{18}
\end{equation*}
$$

Then (15), (17) and (18) imply that

$$
\begin{equation*}
(-1)^{n} P^{\prime}(G, x)+(-1)^{n+1} P(G, x) \sum_{i=1}^{n} \frac{1}{x-d_{Q_{i}}\left(u_{i}\right)}>0, \quad \forall x<0 . \tag{19}
\end{equation*}
$$

By (8) and (12), inequality (19) implies that

$$
\begin{equation*}
(\epsilon(G, x)-\epsilon(Q, x))(-1)^{n} P(G, x)>0, \quad \forall x<0 . \tag{20}
\end{equation*}
$$

Since $(-1)^{n} P(G, x)>0$ holds for all $x<0$, inequality (20) implies Claim 4.
As Claim 4 contradicts the assumption of $G$, there are no counter-examples to this result and the theorem is proved.

## 3 An approach for proving Theorem 4

In this section, we will mainly show that, in order to prove Theorem 4 , it suffices to prove Theorem 3, By (12), we have

$$
\begin{equation*}
\epsilon\left(K_{n}, x\right)=\sum_{i=0}^{n-1} \frac{1}{x-i} . \tag{21}
\end{equation*}
$$

Thus,

$$
\begin{equation*}
\epsilon\left(K_{n}, x\right)-\epsilon(G, x)=\frac{(-1)^{n}}{P(G, x)}\left((-1)^{n} P(G, x) \sum_{i=0}^{n-1} \frac{1}{x-i}+(-1)^{n+1} P^{\prime}(G, x)\right) \tag{22}
\end{equation*}
$$

For any graph $G$ of order $n$, define

$$
\begin{equation*}
\xi(G, x)=(-1)^{n} P(G, x) \sum_{i=0}^{n-1} \frac{1}{x-i}+(-1)^{n+1} P^{\prime}(G, x) . \tag{23}
\end{equation*}
$$

Note that $\xi(G, x) \equiv 0$ if $G$ is a complete graph. For any non-complete graph $G$ and any $x<0$, we have $(-1)^{n} P(G, x)>0$ and so (22) implies that $\epsilon\left(K_{n}, x\right)-\epsilon(G, x)>0$ if and only if $\xi(G, x)>0$.

Proposition 3. Theorem $\boldsymbol{Q}^{2}$ holds if and only if $\xi(G, x)>0$ holds for every non-complete graph $G$ and all $x<0$.

It can be easily verified that $\xi(G, x)>0$ holds for all non-complete graphs $G$ of order at most 3 and all $x<0$. For the general case, we will prove it by induction. In the rest of this section, we will find a relation between $\xi(G, x)$ and $\xi(G-u, x)$ for a vertex $u$ in $G$ in two cases. Lemma 1 is for the case when $u$ is a simplicial vertex and Lemma 3 when $d(u) \geq 1$. We then explain why Theorem 3 implies $\xi(G, x)>0$ for all non-complete graphs $G$ and all $x<0$.

Lemma 1. Let $G$ be a graph of order $n$. If $u$ is a simplicial vertex of $G$ with $d(u)=d$, then

$$
\begin{equation*}
\xi(G, x)=(d-x) \xi(G-u, x)+\frac{(-1)^{n-1}(n-1-d) P(G-u, x)}{n-1-x} . \tag{24}
\end{equation*}
$$

Proof. As $u$ is a simplicial vertex of $G$ with $d(u)=d, P(G, x)=(x-d) P(G-u, x)$ by (10). Thus $P^{\prime}(G, x)=P(G-u, x)+(x-d) P^{\prime}(G-u, x)$. By (23),

$$
\begin{align*}
\xi(G, x) & =(-1)^{n}(x-d) P(G-u, x) \sum_{i=0}^{n-1} \frac{1}{x-i}+(-1)^{n+1}\left(P(G-u, x)+(x-d) P^{\prime}(G-u, x)\right) \\
& =(d-x) \xi(G-u, x)+\frac{(-1)^{n}(x-d) P(G-u, x)}{x-n+1}+(-1)^{n+1} P(G-u, x) \\
& =(d-x) \xi(G-u, x)+\frac{(-1)^{n-1}(n-1-d) P(G-u, x)}{n-1-x} . \tag{25}
\end{align*}
$$

Note that $d \leq n-1$ and $(-1)^{n-1} P(G-u, x)>0$ holds for all $x<0$, implying that the second term in the right-hand side of (24) is non-negative. Thus, if $u$ is a simplicial vertex of $G$ and $x<0$, by Lemmarit $\xi(G-u, x)>0$ implies that $\xi(G, x)>0$.

Now consider the case that $u$ is a vertex in $G$ with $d(u)=d \geq 1$. Assume that $N(u)=\left\{u_{1}, u_{2}, \ldots, u_{d}\right\}$. For any $i=1,2, \ldots, d-1$, let $G_{i}$ denote the graph obtained from $G-u$ by adding edges joining $u_{i}$ to $u_{j}$ whenever $u_{i} u_{j} \notin E(G)$ for all $j$ with $i+1 \leq j \leq d$. Thus, $u_{i}$ is adjacent to $u_{j}$ in $G_{i}$ for all $j$ with $i+1 \leq j \leq d$. In the case that $u$ is a simplicial vertex of $G, G_{i} \cong G-u$ for all $i=1,2, \cdots, d-1$. By applying the deletioncontraction formula for chromatic polynomials (see [5, 13), $P(G, x)$ can be expressed in terms of $P(G-u, x)$ and $P\left(G_{i}, x\right)$ for $i=1,2, \cdots, d-1$.

Lemma 2. Let $u$ be a vertex in $G$ with $d(x)=d \geq 1$ and for $i=1,2, \cdots, d-1$, let $G_{i}$ be the graph defined above. Then,

$$
\begin{equation*}
P(G, x)=(x-1) P(G-u, x)-\sum_{i=1}^{d-1} P\left(G_{i}, x\right) . \tag{26}
\end{equation*}
$$

Proof. For $1 \leq i \leq d$, let $E_{i}$ denote the set of edges $u u_{j}$ in $G$ for $j=1,2, \cdots, i-1$. So $\left|E_{i}\right|=i-1$ and $E_{1}=\emptyset$. For any $i$ with $1 \leq i \leq d-1$, applying the deletion-contraction formula for chromatic polynomials to edge $u u_{i}$ in $G-E_{i}$, the graph obtained from $G$ by removing all edges in $E_{i}$, we have

$$
\begin{equation*}
P\left(G-E_{i}, x\right)=P\left(G-E_{i+1}, x\right)-P\left(\left(G-E_{i}\right) / u u_{i}, x\right)=P\left(G-E_{i+1}, x\right)-P\left(G_{i}, x\right), \tag{27}
\end{equation*}
$$

where the last equality follows from the fact that $\left(G-E_{i}\right) / u u_{i} \cong G_{i}$ by the assumption of $G_{i}$. Thus, by (27),

$$
\begin{equation*}
P(G, x)=P\left(G-E_{1}, x\right)=P\left(G-E_{d}, x\right)-\sum_{i=1}^{d-1} P\left(G_{i}, x\right) . \tag{28}
\end{equation*}
$$

As $u$ is of degree 1 in $G-E_{d}, P\left(G-E_{d}, x\right)=(x-1) P(G-u, x)$. Hence (26) follows.

Lemma 3. Let $G$ be a graph of order $n$ and let $u$ be a vertex of $G$ with $d(u)=d \geq 1$. Then

$$
\begin{equation*}
\xi(G, x)=(1-x) \xi(G-u, x)+\sum_{i=1}^{d-1} \xi\left(G_{i}, x\right)+\frac{(-1)^{n}[(x-n+1) P(G-u, x)-P(G, x)]}{n-x-1} \tag{29}
\end{equation*}
$$

where $G_{1}, \ldots, G_{d-1}$ are graphs defined above.
Proof. By (26), we have

$$
\begin{equation*}
P^{\prime}(G, x)=P(G-u, x)+(x-1) P^{\prime}(G-u, x)-\sum_{i=1}^{d-1} P^{\prime}\left(G_{i}, x\right) . \tag{30}
\end{equation*}
$$

Thus

$$
\begin{align*}
\xi(G, x)= & (-1)^{n} P(G, x) \sum_{j=0}^{n-1} \frac{1}{x-j}+(-1)^{n+1} P^{\prime}(G, x) \\
= & (-1)^{n}\left[(x-1) P(G-u, x)-\sum_{i=1}^{d-1} P\left(G_{i}, x\right)\right] \sum_{j=0}^{n-1} \frac{1}{x-j} \\
& +(-1)^{n+1}\left[P(G-u, x)+(x-1) P^{\prime}(G-u, x)-\sum_{i=1}^{d-1} P^{\prime}\left(G_{i}, x\right)\right] \\
= & (1-x)\left[(-1)^{n-1} P(G-u, x) \sum_{j=0}^{n-2} \frac{1}{x-j}+(-1)^{n} P^{\prime}(G-u, x)\right] \\
& +\sum_{i=1}^{d-1}\left[(-1)^{n-1} P\left(G_{i}, x\right) \sum_{j=0}^{n-2} \frac{1}{x-j}+(-1)^{n} P^{\prime}\left(G_{i}, x\right)\right]+(-1)^{n+1} P(G-u, x) \\
& +(-1)^{n}\left[\frac{(x-1) P(G-u, x)}{x-(n-1)}-\frac{1}{x-(n-1)} \sum_{i=1}^{d-1} P\left(G_{i}, x\right)\right] \\
& =(1-x) \xi(G-u, x)+\sum_{i=1}^{d-1} \xi\left(G_{i}, x\right) \\
& +\frac{(-1)^{n}[(x-n+1) P(G-u, x)-P(G, x)]}{n-x-1} \tag{31}
\end{align*}
$$

where the last expression follows from (26) and the definitions of $\xi(G-u, x)$ and $\xi\left(G_{i}, x\right)$. The result then follows.

It is known that $\xi(G, x)>0$ holds for all non-complete graphs $G$ of order at most 3 and all $x<0$. For any non-complete graph $G$ of order $n \geq 4$, by Lemma $\xi(G-u, x)>0$ implies $\xi(G, x)>0$ for each simplicial vertex $u$ in $G$ and all $x<0$; by Lemma 3, for any
$x<0, \xi(G-u, x)>0$ implies $\xi(G, x)>0$ whenever $u$ is an non-isolated vertex in $G$ satisfying the following inequality:

$$
\begin{equation*}
(-1)^{n}((x-n+1) P(G-u, x)-P(G, x))>0 . \tag{32}
\end{equation*}
$$

Note that the left-hand side of (32) vanishes when $G$ is $K_{n}$. Also notice that there exist non-complete graph $G$ and some vertex $u$ in $G$ such that inequality (32) does not hold for some $x<0$. For example, if $G$ is the complete bipartite graph $K_{2,3}$ and $u$ is a vertex of degree 3 in $G$, then (32) fails for all real $x$ with $-2.3<x<0$. However, to prove that for any $x<0$, there exists some vertex $u$ in $G$ such that inequality (32) holds, it suffices to prove the following inequality (i.e., Theorem 3):

$$
\begin{equation*}
(-1)^{n}(x-n+1) \sum_{u \in V} P(G-u, x)+(-1)^{n+1} n P(G, x)>0 \tag{33}
\end{equation*}
$$

for any non-complete graph $G=(V, E)$ of order $n$ and all $x<0$.
By Proposition 3 and inequality (32), to prove Theorem 4, we can now just focus on proving inequality (33) (i.e., Theorem(3). The proof of Theorem 3 will be given in Section 5 based on the interpretations for the coefficients of chromatic polynomials introduced in Section 4

## 4 Combinatorial interpretations for coefficients of $P(G, x)$

Let $G=(V, E)$ be any graph. In this section, we will introduce Greene \& Zaslavsky's combinatorial interpretation in [8] for the coefficients of $P(G, x)$ in terms of acyclic orientations. The result will be applied in the next section to prove Theorem 3 ,

An orientation $D$ of $G$ is called acyclic if $D$ does not contain any directed cycle. Let $\alpha(G)$ be the number of acyclic orientations of a graph $G$. In [18], Stanley gave a nice combinatorial interpretation of $(-1)^{n} P(G,-k)$ for any positive integer $k$ in terms of acyclic orientations of $G$. In particular, he proved:

Theorem $5([18])$. For any graph $G$ of order $n,(-1)^{n} P(G,-1)=\alpha(G)$, i.e.,

$$
\begin{equation*}
\sum_{i=1}^{n} a_{i}(G)=\alpha(G) \tag{34}
\end{equation*}
$$

In a digraph $D$, any vertex of $D$ with in-degree (resp. out-degree) zero is called a source (resp. sink) of $D$. It is well known that any acyclic digraph has at least one source and at least one sink. If $v$ is an isolated vertex of $G$, then $v$ is a source and also a sink in any orientation of $G$.

For any $v \in V$, let $\alpha(G, v)$ be the number of acyclic orientations of $G$ with $v$ as its unique source. Clearly $\alpha(G, v)=0$ if and only if $G$ is not connected. In 1983, Greene and Zaslavsky [8] showed that $a_{1}(G)=\alpha(G, v)$.

Theorem 6 ( [8]). For any graph $G=(V, E), a_{1}(G)=\alpha(G, v)$ holds for every $v \in V$.

This theorem was proved originally by using the theory of hyperplane arrangements. See [7] for three other nice proofs.

By Whitney's Broken-cycle Theorem (i.e., Theorem 1), $a_{i}(G)$ equals the number of spanning subgraphs of $G$ with $i$ components and $n-i$ edges, containing no broken cycles of $G$. In particular, $a_{1}(G)$ is the number of spanning trees of $G$ containing no broken cycles of $G$. Now we have two different combinatorial interpretations for $a_{1}$. For any $a_{i}(G), 2 \leq i \leq n$, its combinatorial interpretation can be obtained by applying these two different combinatorial interpretations for $a_{1}$.

Let $\mathcal{P}_{i}(V)$ be the set of partitions $\left\{V_{1}, V_{2}, \ldots, V_{i}\right\}$ of $V$ such that $G\left[V_{j}\right]$ is connected for all $j=1,2, \ldots, i$ and let $\beta_{i}(G)$ be the number of ordered pairs $\left(P_{i}, F\right)$, where
(a) $P_{i}=\left\{V_{1}, V_{2}, \ldots, V_{i}\right\} \in \mathcal{P}_{i}(V)$;
(b) $F$ is a spanning forest of $G$ with exactly $i$ components $T_{1}, T_{2}, \ldots, T_{i}$, where each $T_{j}$ is a spanning tree of $G\left[V_{j}\right]$ containing no broken cycles of $G$.

For any subgraph $H$ of $G$, let $\widetilde{\tau}(H)$ be the number of spanning trees of $H$ containing no broken cycles of $G$. By Theorem 1, $\widetilde{\tau}(H)=a_{1}(H)$ holds and the next result follows.

Theorem 7. For any graph $G$ and any $1 \leq i \leq n$,

$$
\begin{equation*}
a_{i}(G)=\beta_{i}(G)=\sum_{\left\{V_{1}, \ldots, V_{i}\right\} \in \mathcal{P}_{i}(V)} \prod_{j=1}^{i} \widetilde{\tau}\left(G\left[V_{j}\right]\right) \tag{35}
\end{equation*}
$$

Now let $V=\{1,2, \ldots, n\}$. For any $i: 1 \leq i \leq n$ and any vertex $v \in V$, let $\mathcal{O} \mathcal{P}_{i, v}(V)$ be the family of ordered partitions $\left(V_{1}, V_{2}, \ldots, V_{i}\right)$ of $V$ such that
(a) $\left\{V_{1}, V_{2}, \ldots, V_{i}\right\} \in \mathcal{P}_{i}(V)$, where $v \in V_{1}$;
(b) for $j=2, \ldots, i$, the minimum number in the set $\bigcup_{j \leq s \leq i} V_{s}$ is within $V_{j}$.

Clearly, for any $v \in V$ and any $\left\{V_{1}, V_{2}, \ldots, V_{i}\right\} \in \mathcal{P}_{i}(V)$, there is exactly one permutation $\left(\pi_{1}, \pi_{2}, \ldots, \pi_{i}\right)$ of $1,2, \ldots, i$ such that $\left(V_{\pi_{1}}, V_{\pi_{2}}, \ldots, V_{\pi_{i}}\right) \in \mathcal{O} \mathcal{P}_{i, v}(V)$.

By Theorem 6, $\widetilde{\tau}\left(G\left[V_{j}\right]\right)=\alpha\left(G\left[V_{j}\right], u\right)$ holds for any vertex $u$ in $G\left[V_{j}\right]$ and Theorem 7 is equivalent to a result in [8] which we illustrate differently below.

Theorem 8 ( 8], Theorem 7.4). For any $v \in V$ and any $1 \leq i \leq n$,

$$
\begin{equation*}
a_{i}(G)=\sum_{\left(V_{1}, \ldots, V_{i}\right) \in \mathcal{O} \mathcal{P}_{i, v}(V)} \alpha\left(G\left[V_{1}\right], v\right) \prod_{j=2}^{i} \alpha\left(G\left[V_{j}\right], m_{j}\right), \tag{36}
\end{equation*}
$$

where $m_{j}$ is the minimum number in $V_{j}$ for $j=2, \ldots, i$.
Note that the theorem above indicates that the right hand side of (36) is independent of the choice of $v$. Thus, for any $1 \leq i \leq n$,

$$
\begin{equation*}
n a_{i}(G)=\sum_{v \in V} \sum_{\left(V_{1}, \ldots, V_{i}\right) \in \mathcal{O} \mathcal{P}_{i, v}(V)} \alpha\left(G\left[V_{1}\right], v\right) \prod_{j=2}^{i} \alpha\left(G\left[V_{j}\right], m_{j}\right) . \tag{37}
\end{equation*}
$$

Let $P^{(i)}(G, x)$ be the $i$-th derivative of $P(G, x)$. Very recently, Bernardi and Nadeau [1] gave an interpretation of $P^{(i)}(G,-j)$ for any nonnegative integers $i$ and $j$ in terms of acyclic orientations. When $i=0$, their result is exactly Theorem 5 due to Stanley [18] and when $j=0$, it is Theorem [8 due to Greene \& Zaslavsky [8].

## 5 Proofs of Theorems 3 and 4

By the explanation in Section 3, to prove Theorem [4 it suffices to prove Theorem 3 In this section, we will prove Theorem 3 by showing that the coefficient of $x^{i}$ in the expansion of the left-hand side of (9) in Theorem 3 is of the form $(-1)^{i} d_{i}$ with $d_{i} \geq 0$ for all $i=1,2, \ldots, n$. Furthermore, $d_{i}>0$ holds for some $i$ when $G$ is not complete.

We first establish the following result.
Lemma 4. Let $G=(V, E)$ be a non-complete graph of order $n \geq 3$ and component number c.
(a). If $c=1$ and $G$ is not the $n$-cycle $C_{n}$, then there exist non-adjacent vertices $u_{1}, u_{2}$ of $G$ such that $G-\left\{u_{1}, u_{2}\right\}$ is connected.
(b). If $2 \leq c \leq n-1$, then for any integer $i$ with $c \leq i \leq n-1$, there exists a partition $V_{1}, V_{2}, \ldots, V_{i}$ of $V$ such that $G\left[V_{j}\right]$ is connected for all $j=2, \ldots, i$ and $G\left[V_{1}\right]$ has exactly two components one of which is an isolated vertex.

Proof. (a). As $c=1, G$ is connected. As $G$ is non-complete, the result is trivial when $G$ is 3 -connected.

If $G$ is not 2-connected, choose vertices $u_{1}$ and $u_{2}$ from distinct blocks $B_{1}$ and $B_{2}$ of $G$ such that both $u_{1}$ and $u_{2}$ are not cut-vertices of $G$. Then $u_{1} u_{2} \notin E(G)$ and $G-\left\{u_{1}, u_{2}\right\}$ is connected.

Now consider the case that $G$ is 2 -connected but not 3-connected. Since $G$ is not $C_{n}$, there exists a vertex $w$ such that $d(w) \geq 3$. If $d(w)=n-1$, then $G-\left\{u_{1}, u_{2}\right\}$ is connected for any two non-adjacent vertices $u_{1}$ and $u_{2}$ in $G$. If $G-w$ is 2 -connected and $d(w) \leq n-2$, then $G-\{w, u\}$ is connected for any $u \in V-N_{G}(w)$. If $G-w$ is not 2-connected, then $G-w$ contains two non-adjacent vertices $u_{1}, u_{2}$ such that $G-\left\{w, u_{1}, u_{2}\right\}$ is connected, implying that $G-\left\{u_{1}, u_{2}\right\}$ is connected as $d(w) \geq 3$.
(b). Let $G_{1}, G_{2}, \ldots, G_{c}$ be the components of $G$ with $\left|V\left(G_{1}\right)\right| \geq\left|V\left(G_{j}\right)\right|$ for all $j=1,2, \ldots, c$. As $c \leq n-1,\left|V\left(G_{1}\right)\right| \geq 2$. Choose $u \in V\left(G_{1}\right)$ such that $G_{1}-u$ is connected. Then $V\left(G_{2}\right) \cup\{u\}, V\left(G_{1}\right)-\{u\}, V\left(G_{3}\right), \ldots, V\left(G_{c}\right)$ is a partition of $V$ satisfying the condition in (b) for $i=c$.

Assume that (b) holds for $i=k$, where $c \leq k<n-1$, and $V_{1}, V_{2}, \ldots, V_{k}$ is a partition of $V$ satisfying the condition in (a). Then $G\left[V_{1}\right]$ has an isolated vertex $u$ and $G\left[V_{1}^{\prime}\right]$ is connected, where $V_{1}^{\prime}=V_{1}-\{u\}$. Since $k \leq n-2$, either $\left|V_{1}^{\prime}\right| \geq 2$ or $\left|V_{j}\right| \geq 2$ for some $j \geq 2$.

If $\left|V_{1}^{\prime}\right| \geq 2$, then $V_{1}^{\prime}$ has a partition $V_{1,1}^{\prime}, V_{1,2}^{\prime}$ such that both $G\left[V_{1,1}^{\prime}\right]$ and $G\left[V_{1,2}^{\prime}\right]$ are connected, implying that $V_{1,1}^{\prime} \cup\{u\}, V_{1,2}^{\prime}, V_{2}, V_{3}, \ldots, V_{k}$ is a partition of $V$ satisfying the condition in (b) for $i=k+1$.

Similarly, if $\left|V_{j}\right| \geq 2$ for some $j \geq 2$ (say $j=2$ ), then $V_{2}$ has a partition $V_{2,1}, V_{2,2}$ such that both $G\left[V_{2,1}\right]$ and $G\left[V_{2,2}\right]$ are connected, implying that $V_{1}, V_{2,1}, V_{2,2}, V_{3}, \ldots, V_{k}$ is a partition of $V$ satisfying the condition in (b) for $i=k+1$.

For any graph $G=(V, E)$ of order $n$, write

$$
\begin{equation*}
(-1)^{n}\left[(x-n+1) \sum_{u \in V(G)} P(G-u, x)-n P(G, x)\right]=\sum_{i=1}^{n}(-1)^{i} d_{i} x^{i} . \tag{38}
\end{equation*}
$$

By comparing coefficients, it can be shown that

$$
\begin{equation*}
d_{i}=\sum_{u \in V(G)}\left[a_{i-1}(G-u)+(n-1) a_{i}(G-u)\right]-n a_{i}(G), \quad \forall i=1,2, \ldots, n . \tag{39}
\end{equation*}
$$

It is obvious that when $G$ is the complete graph $K_{n}$, the left-hand side of (38) vanishes and thus $d_{i}=0$ for all $i=1,2, \ldots, n$. Now we consider the case that $G$ is not complete.

Proposition 4. Let $G=(V, E)$ be a non-complete graph of order $n$ and component number $c$. Then, for any $i=1,2, \ldots, n, d_{i} \geq 0$ and equality holds if and only if one of the following cases happens:
(a). $i=n$;
(b). $1 \leq i \leq c-2$;
(c). $i=c-1$ and $G$ does not have isolated vertices;
(d). $i=c=1$ and $G$ is $C_{n}$.

Proof. We first show that $d_{i}=0$ in any one of the four cases above.
By (39), $d_{n}=\sum_{u \in V}[1+(n-1) \cdot 0]-n \cdot 1=0$.
It is known that for $1 \leq i \leq n, a_{i}(G)=0$ if and only if $i<c$ (see [5,13, 14]). Similarly, $a_{i}(G-u)=0$ for all $i$ with $1 \leq i<c-1$ and all $u \in V$, and $a_{c-1}(G-u)=0$ if $u$ is not an isolated vertex of $G$. By (39), $d_{i}=0$ for all $i$ with $1 \leq i \leq c-2$, and $d_{c-1}=0$ when $G$ does not have isolated vertices.

If $G$ is $C_{n}$, then $a_{1}(G)=n-1, a_{0}(G-u)=0$ and $a_{1}(G-u)=1$ for each $u \in V$, implying that $d_{1}=0$ by (39).

In the following, we will show that $d_{i}>0$ when $i$ does not belong to any one of the four cases.

If $G$ has isolated vertices, then $a_{c-1}(G-u)>0$ for any isolated vertex $u$ of $G$ and

$$
\begin{equation*}
\sum_{u \in V} a_{c-1}(G-u)=\sum_{\substack{u \in V \\ u \text { isolated }}} a_{c-1}(G-u)>0 . \tag{40}
\end{equation*}
$$

As $a_{c-1}(G)=0$, by (39), we have $d_{c-1}>0$ in this case. Now it remains to show that $d_{i}>0$ holds for all $i$ with $c \leq i \leq n-1$, except when $i=c=1$ and $G$ is $C_{n}$.

For any $v \in V$, let $\mathcal{O} \mathcal{P}_{i, v}^{\prime}(V)$ be the set of ordered partitions $\left(V_{1}, \ldots, V_{i}\right) \in \mathcal{O} \mathcal{P}_{i, v}(V)$ with $V_{1}=\{v\}$. As $\alpha\left(G\left[V_{1}\right], v\right)=1$, for any $i$ with $c \leq i \leq n$, by Theorem 8 ,

$$
\begin{equation*}
a_{i-1}(G-v)=\sum_{\left(V_{1}, \ldots, V_{i}\right) \in \mathcal{P}_{i, v}^{\prime}(V)} \alpha\left(G\left[V_{1}\right], v\right) \prod_{j=2}^{i} \alpha\left(G\left[V_{j}\right], m_{j}\right) \tag{41}
\end{equation*}
$$

where $m_{j}$ is the minimum number in $V_{j}$ for all $j=2, \ldots, i$.
Let $s$ and $v$ be distinct members in $V$. For any $V_{1} \subseteq V-\{s\}$ with $v \in V_{1}$, let $\alpha\left(G\left[V_{1} \cup\{s\}\right], v, s\right)$ be the number of those acyclic orientations of $G\left[V_{1} \cup\{s\}\right]$ with $v$ as the unique source and $s$ as one sink. Then $\alpha\left(G\left[V_{1} \cup\{s\}\right], v, s\right) \leq \alpha\left(G\left[V_{1}\right], v\right)$ holds, where the inequality is strict if and only if $G\left[V_{1}\right]$ is connected but $G\left[V_{1} \cup\{s\}\right]$ is not. Observe that

$$
\begin{align*}
a_{i}(G-s) & =\sum_{\left(V_{1}, \ldots, V_{i}\right) \in \mathcal{O P} \mathcal{P}_{i, v}(V-\{s\})} \alpha\left(G\left[V_{1}\right], v\right) \prod_{j=2}^{i} \alpha\left(G\left[V_{j}\right], m_{j}\right) \\
& \geq \sum_{\left(V_{1}, \ldots, V_{i}\right) \in \mathcal{O P}_{i, v}(V-\{s\})} \alpha\left(G\left[V_{1} \cup\{s\}\right], v, s\right) \prod_{j=2}^{i} \alpha\left(G\left[V_{j}\right], m_{j}\right)  \tag{42}\\
& =\sum_{\left(V_{1}^{\prime}, \ldots, V_{i}^{\prime}\right) \in \mathcal{O} \mathcal{P}_{i, v, s}(V)} \alpha\left(G\left[V_{1}^{\prime}\right], v, s\right) \prod_{j=2}^{i} \alpha\left(G\left[V_{j}^{\prime}\right], m_{j}\right), \tag{43}
\end{align*}
$$

where $\mathcal{O} \mathcal{P}_{i, v, s}(V)$ is the set of ordered partitions $\left(V_{1}^{\prime}, \ldots, V_{i}^{\prime}\right) \in \mathcal{O} \mathcal{P}_{i, v}(V)$ with $s, v \in V_{1}^{\prime}$. By the explanation above, inequality (42) is strict whenever $V-\{s\}$ has a partition $V_{1}, V_{2}, \ldots, V_{i}$ with $v \in V_{1}$ such that each $G\left[V_{j}\right]$ is connected for all $j=1,2, \ldots, i$ but $G\left[V_{1} \cup\{s\}\right]$ is not connected.

By (37), we have

$$
\begin{align*}
n a_{i}(G)= & \sum_{v \in V} \sum_{\left(V_{1}, \ldots, V_{i}\right) \in \mathcal{O} \mathcal{P}_{i, v}(V)} \alpha\left(G\left[V_{1}\right], v\right) \prod_{j=2}^{i} \alpha\left(G\left[V_{j}\right], m_{j}\right) \\
= & \sum_{v \in V} \sum_{\left(V_{1}, \ldots, V_{i}\right) \in \mathcal{O} \mathcal{P}_{i, v}^{\prime}(V)} \alpha\left(G\left[V_{1}\right], v\right) \prod_{j=2}^{i} \alpha\left(G\left[V_{j}\right], m_{j}\right) \\
& +\sum_{v \in V} \sum_{\left(V_{1}, \ldots, V_{i}\right) \in \mathcal{O} \mathcal{P}_{i, v}(V)-\mathcal{O P}_{i, v}^{\prime}(V)} \alpha\left(G\left[V_{1}\right], v\right) \prod_{j=2}^{i} \alpha\left(G\left[V_{j}\right], m_{j}\right) . \tag{44}
\end{align*}
$$

By (41),

$$
\begin{equation*}
\sum_{v \in V} \sum_{\left(V_{1}, \ldots, V_{i}\right) \in \mathcal{O} \mathcal{P}_{i, v}^{\prime}(V)} \alpha\left(G\left[V_{1}\right], v\right) \prod_{j=2}^{i} \alpha\left(G\left[V_{j}\right], m_{j}\right)=\sum_{v \in V} a_{i-1}(G-v), \tag{45}
\end{equation*}
$$

and by (43),

$$
\begin{align*}
& \sum_{v \in V} \sum_{\left(V_{1}, \ldots, V_{i}\right) \in \mathcal{O P}_{i, v}(V)-\mathcal{O P}_{i, v}^{\prime}(V)} \alpha\left(G\left[V_{1}\right], v\right) \prod_{j=2}^{i} \alpha\left(G\left[V_{j}\right], m_{j}\right) \\
\leq & \sum_{v \in V} \sum_{s \in V-\{v\}} \sum_{\left(V_{1}, \ldots, V_{i}\right) \in \mathcal{O}_{i, v, s}(V)} \alpha\left(G\left[V_{1}\right], v, s\right) \prod_{j=2}^{i} \alpha\left(G\left[V_{j}\right], m_{j}\right)  \tag{46}\\
\leq & \sum_{v \in V} \sum_{s \in V-\{v\}} a_{i}(G-s)  \tag{47}\\
= & (n-1) \sum_{v \in V} a_{i}(G-v), \tag{48}
\end{align*}
$$

where inequality (46) is strict if there exists $\left(V_{1}, \ldots, V_{i}\right) \in \mathcal{O P}_{i, v}(V)$ for some $v \in V$ such that $G\left[V_{j}\right]$ is connected for all $j=1, \ldots, i$ and $G\left[V_{1}\right]$ has acyclic orientations with $v$ as the unique source but with at least two sinks, and by (42) and (43), inequality (47) is strict if $V$ can be partitioned into $V_{1}, \ldots, V_{i}$ such that $G\left[V_{j}\right]$ is connected for all $j=2, \ldots, i$ but $G\left[V_{1}\right]$ has exactly two components, one of which is an isolated vertex in $G\left[V_{1}\right]$.

As $G$ is not complete, by Lemma 4 and the above explanation, the inequality of (48) is strict for all $i$ with $c \leq i \leq n-1$, except when $i=c=1$ and $G$ is $C_{n}$. Then, by (44), (45) and (48), we conclude that

$$
\begin{equation*}
d_{i}=\sum_{v \in V}\left[a_{i-1}(G-u)+(n-1) a_{i}(G-u)\right]-n a_{i}(G)>0, \quad \forall c \leq i \leq n-1, \tag{49}
\end{equation*}
$$

except that $i=c=1$ and $G$ is $C_{n}$. Hence the proof is complete.

Now everything is ready for proving Theorems 3 and 4
Proof of Theorem 圆 Let $G$ be a non-complete graph of order $n$. Recall (38) that

$$
\begin{equation*}
(-1)^{n}\left[(x-n+1) \sum_{u \in V(G)} P(G-u, x)-n P(G, x)\right]=\sum_{i=1}^{n}(-1)^{i} d_{i} x^{i} . \tag{50}
\end{equation*}
$$

By Proposition 4, we know that $d_{i} \geq 0$ for all $i$ with $1 \leq i \leq n$ and $d_{n-1}>0$. Thus $\sum_{i=1}^{n}(-1)^{i} d_{i} x^{i}>0$ holds for all $x<0$, which completes the proof of Theorem 3,

Proposition 5. For any non-complete graph $G, \xi(G, x)>0$ holds for all $x<0$.
Proof. We will prove this result by induction on the order $n$ of $G$. When $n=2$, the empty graph $N_{2}$ of order 2 is the only non-complete graph of order 2. As $P\left(N_{2}, x\right)=x^{2}$, by (23), we have

$$
\begin{equation*}
\xi\left(N_{2}, x\right)=(-1)^{2} x^{2}\left(\frac{1}{x}+\frac{1}{x-1}\right)+(-1)^{3} 2 x=\frac{x}{x-1}>0 \tag{51}
\end{equation*}
$$

for all $x<0$.
Assume that this result holds for any non-complete graph $G$ of order less than $n$, where $n \geq 3$. Now let $G$ be any non-complete graph of order $n$.
Case 1: $G$ contains an isolated vertex $u$.
By the inductive assumption, $\xi(G-u, x) \geq 0$ holds for all $x<0$, where equality holds when $G-u$ is a complete graph. By Lemma $\{(G, x)>0$ holds for all $x<0$.
Case 2: $G$ has no isolated vertex.
By Theorem 3(9) holds for all $x<0$. Thus, for any $x<0$, there exists some $u \in V(G)$ such that $(-1)^{n}(x-n+1) P(G-u, x)+(-1)^{n+1} P(G, x)>0$ holds. Then, by Lemma 3 and by the inductive assumption, $\xi(G, x)>0$ holds for any $x<0$.

Hence the result holds.
Proof of Theorem 4 It follows directly from Propositions 3 and .

## 6 Remarks and problems

First we give some remarks here.
(a). Theorem 4 implies that for any non-complete graph $G$ of order $n, \frac{P(G, x)}{P\left(K_{n}, x\right)}$ is strictly decreasing when $x<0$.
(b). Let $G$ be a non-complete graph of order $n$ and $P(G, x)=\sum_{i=1}^{n}(-1)^{n-i} a_{i} x^{i}$. Then $\epsilon(G)<\epsilon\left(K_{n}\right)$ implies that

$$
\begin{equation*}
\frac{a_{1}+2 a_{2}+\cdots+n a_{n}}{a_{1}+a_{2}+\cdots+a_{n}}>1+\frac{1}{2}+\cdots+\frac{1}{n} . \tag{52}
\end{equation*}
$$

(c). When $x=-1$, Theorem 3 implies that for any graph $G$ of order $n$,

$$
\begin{equation*}
(-1)^{n-1} \sum_{u \in V} P(G-u,-1) \geq(-1)^{n} P(G,-1), \tag{53}
\end{equation*}
$$

where the inequality holds if and only if $G$ is complete. By Stanley's interpretation for $(-1)^{n} P(G,-1)$ in [18, the inequality above implies that for any graph $G=$ $(V, E)$, the number of acyclic orientations of $G$ is at most the total number of acyclic orientations of $G-u$ for all $u \in V$, where the equality holds if and only if $G$ is complete.

Now we raise some problems for further study.
It is clear that for any graph $G$ of order $n$,

$$
\begin{equation*}
\frac{d}{d x}\left(\ln \left[(-1)^{n} P(G, x)\right]\right)=\frac{P^{\prime}(G, x)}{P(G, x)}<0 \tag{54}
\end{equation*}
$$

holds for all $x<0$. We surmise that this property holds for higher derivatives of the function $\ln \left[(-1)^{n} P(G, x)\right]$ in the interval $(-\infty, 0)$.

Conjecture 2. Let $G$ be a graph of order $n$. Then $\frac{d^{k}}{d x^{k}}\left(\ln \left[(-1)^{n} P(G, x)\right]\right)<0$ holds for all $k \geq 2$ and $x \in(-\infty, 0)$.

Observe that $\epsilon(G, x)=\frac{d}{d x}\left(\ln \left[(-1)^{n} P(G, x)\right]\right)$. We believe that Theorems 2 and 4 can be extended to higher derivatives of the function $\ln \left[(-1)^{n} P(G, x)\right]$.

Conjecture 3. Let $G$ be any non-complete graph of order $n$ and $Q$ be any chordal and proper spanning subgraph $Q$ of $G$. Then

$$
\begin{equation*}
\frac{d^{k}}{d x^{k}}\left(\ln \left[(-1)^{n} P(Q, x)\right]\right)<\frac{d^{k}}{d x^{k}}\left(\ln \left[(-1)^{n} P(G, x)\right]\right)<\frac{d^{k}}{d x^{k}}\left(\ln \left[(-1)^{n} P\left(K_{n}, x\right)\right]\right) \tag{55}
\end{equation*}
$$

holds for any integer $k \geq 2$ and all $x<0$.
It is not difficult to show that Conjecture 2 holds for $G \cong K_{n}$. Thus the second inequality of Conjecture 3 implies Conjecture 2,

It is natural to extend the second part of Conjecture $\mathbb{1}$ (i.e., $\epsilon(G)<\epsilon\left(K_{n}\right)$ for any non-complete graph $G$ of order $n$ ) to the inequality $\epsilon(G) \leq \epsilon\left(G^{\prime}\right)$ for any graph $G^{\prime}$ which contains $G$ as a subgraph. However, this inequality is not always true. Let $G_{n}$ denote
the graph obtained from the complete bipartite graph $K_{2, n}$ by adding a new edge joining the two vertices in the partite set of size 2. Lundow and Markström [11 stated that $\epsilon\left(K_{2, n}\right)>\epsilon\left(G_{n}\right)$ holds for all $n \geq 3$. In spite of this, we believe that for any non-complete graph $G$, we can add a new edge to $G$ to obtain a graph $G^{\prime}$ with the property that $\epsilon(G)<\epsilon\left(G^{\prime}\right)$, as stated below.

Conjecture 4. For any non-complete graph $G$, there exist non-adjacent vertices $u$ and $v$ in $G$ such that $\epsilon(G)<\epsilon(G+u v)$.

Obviously, Conjecture 4 implies $\epsilon(G)<\epsilon\left(K_{n}\right)$ for any non-complete graph $G$ of order $n$ (i.e., Theorem (4). Conjecture 4 is similar to but may be not equivalent to the following conjecture due to Lundow and Markström [11.

Conjecture 5 ( [11]). For any 2-connected graph $G$, there exists an edge $e$ in $G$ such that $\epsilon(G-e)<\epsilon(G)$.

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(F. Dong and E. Tay) Mathematics and Mathematics Education, National Institute of Education, Nanyang Technological University, Singapore. Email (Tay): engguan.tay@nie.edu.sg.
(J. Ge) School of Mathematical Sciences, Sichuan Normal University, Chengdu, P. R. China. Email: mathsgejun@163.com.
(H. Gong) Department of Mathematics, Shaoxing University, Shaoxing, P. R. China. Email: helingong@126.com.
(B. Ning) College of Computer Science, Nankai University, Tianjin 300071, P.R. China. Email: ningbo-maths@163.com.
(Z. Ouyang) Department of Mathematics, Hunan First Normal University, Changsha, P. R. China. Email: oymath@163.com.


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    ${ }^{\dagger}$ Corresponding author. Email: fengming.dong@nie.edu.sg and donggraph@163.com.

