Graphs of bounded depth-2 rank-brittleness

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Abstract

We characterize classes of graphs closed under taking vertex-minors and having no P_n and no disjoint union of n copies of the 1-subdivision of $K_{1,n}$ for some n. Our characterization is described in terms of a tree of radius 2 whose leaves are labelled by the vertices of a graph G, and the width is measured by the maximum possible cut-rank of a partition of $V(G)$ induced by splitting an internal node of the tree to make two components. The minimum width possible is called the depth-2 rank-brittleness of G. We prove that for all n, every graph with sufficiently large depth-2 rank-brittleness contains P_n or disjoint union of n copies of the 1-subdivision of $K_{1,n}$ as a vertex-minor.

1 Introduction

Tree-depth is a graph parameter in the theory of sparse graph classes, which measures how far a graph is from being a star, introduced by Nešetřil and Ossona de Mendez [19]. An equivalent concept has been introduced a few times under the names like the vertex ranking number and the minimum

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height elemination tree [3, 5, 25]. It is known that a graph has large tree-depth if and only if it has a long path, see [20, Section 6.2].

For some applications, it is desirable to say that complete graphs are also very similar to stars. However, complete graphs have unbounded tree-depth. To design a graph parameter similar to tree-depth but more suitable for dense graph classes, DeVos, Kwon, and Oum [6] introduced the *rank-depth* of a graph. Roughly speaking, the rank-depth of a graph G is defined in terms of a *decomposition*, which is a tree whose leaves are labelled by the vertices of G. A decomposition has two qualities, one of which is the radius of the tree, and the other is the maximum width of internal nodes, measured by some connectivity function of G. The rank-depth of a graph G is defined as the minimum integer k such that G admits a decomposition of radius at most k and width at most k. The detailed definition of rank-depth will be reviewed in Section 2. In fact, there was an equivalent concept called the *shrub-depth* of classes of graphs, introduced by Ganian, Hliněný, Nešetřil, Obdržálek, Ossona de Mendez, and Ramadurai [11, 12]. The definition of shrub-depth uses logical terms similar to the definition of clique-width [4], while the definition of rank-depth uses a tree-like decomposition similar to that of rank-width [24]. DeVos, Kwon, and Oum [6] showed that a class of graphs has bounded rank-depth if and only if it has bounded shrub-depth.

Hliněný, Kwon, Obdržálek, and Ordyniak [14] proposed the following conjecture, which we state in terms of rank-depth. To state their conjecture, we first introduce vertex-minors. The *local complementation* at a vertex v of a graph G is an operation to obtain a new graph $G*v$ from G by removing all edges xy between two adjacent pairs x, y of neighbors of v and adding edges xy for all non-adjacent pairs x, y of neighbors of v. A graph H is a *vertex-minor* of a graph G if H can be obtained from G by a sequence of local complementations and vertex deletions. It is known that the rank-depth of a vertex-minor of G is at most the rank-depth of G and so it is natural to think of an obstruction for graphs of bounded rank-depth in terms of vertex-minors. The following conjecture states that paths are obstructions for having bounded rank-depth. This conjecture was verified for graphs of rank-width 1 by Novotný [21, Theorem 6.3.2].

Conjecture 1.1 (Hliněný, Kwon, Obdržálek, and Ordyniak [14])**.** *A class $\mathcal{C}$ of graphs has bounded rank-depth if and only if there exists an integer t such that no graph $G \in \mathcal{C}$ contains a path of length t as a vertex-minor.*

As a step towards Conjecture 1.1, we define a new parameter called *depth-d rank-brittleness* for an integer d by restricting the radius of the tree in the decomposition to be at most d in the definition of rank-depth. The

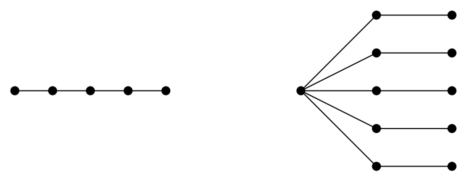

Figure 1: Graphs P_5 and $T_{2,5}$.

depth-d rank-brittleness of a graph G is the minimum integer k such that G admits a decomposition of radius at most d and width at most k. We denote this parameter by $\mathrm{rbrit}_d(G)$. By definition, the rank-depth of a graph G is at most $\max\{d, \mathrm{rbrit}_d(G)\}$ for all $d \geqslant 1$ and

$$\mathrm{rbrit}_1(G) \geqslant \mathrm{rbrit}_2(G) \geqslant \mathrm{rbrit}_3(G) \geqslant \cdots .$$

In Section 6, we will show that a graph of rank-depth k has linear rank-width at most k^2.

A class $\mathcal{C}$ of graphs is a *vertex-minor ideal* if for every graph $G \in \mathcal{C}$, $\mathcal{C}$ contains all graphs isomorphic to vertex-minors of G. For a graph H, we write nH for the disjoint union of n copies of H. It is straightforward to deduce the following proposition by using Ramsey-type results. To see this, one can use Theorem 2.3, Ramsey's theorem, and Lemma 2.2. It can be also seen as a special case of a theorem due to Kwon and Oum [18, Theorem 1.4], which is stated in Theorem 4.2.

Proposition 1.2. *A vertex-minor ideal $\mathcal{C}$ has bounded depth-1 rank-brittleness if and only if $\{K_2, 2K_2, 3K_2, \ldots\} \nsubseteq \mathcal{C}$.*

In this paper, we characterize classes of graphs of bounded depth-2 rank-brittleness in terms of forbidden vertex-minors. Let $T_{2,n}$ be the 1-subdivision of $K_{1,n}$, see Figure 1. Here is our main theorem.

Theorem 1.3. *A vertex-minor ideal $\mathcal{C}$ has bounded depth-2 rank-brittleness if and only if*

$$\{P_1, P_2, P_3, P_4, \ldots\} \nsubseteq \mathcal{C} \text{ and } \{T_{2,1}, 2T_{2,2}, 3T_{2,3}, 4T_{2,4}, \ldots\} \nsubseteq \mathcal{C}.$$

Since $T_{2,n}$ contains P_5 if $n \geqslant 2$, we obtain the following corollary, confirming a weaker statement of Conjecture 1.1.

Corollary 1.4. *For every positive integer n, graphs with no vertex-minors isomorphic to nP_5 have bounded depth-2 rank-brittleness, bounded rank-depth, and bounded linear rank-width.*

We sketch the proof of Theorem 1.3. It is straightforward to show that P_n and $nT_{2,n}$ have large depth-2 rank-brittleness. We mainly show that for every fixed n, if a graph G has sufficiently large depth-2 rank-brittleness, then it has a vertex-minor isomorphic to P_n or $nT_{2,n}$. A theorem of Kwon and Oum [18, Theorem 1.4] will imply that every graph of large depth-2 rank-brittleness has a vertex-minor isomorphic to aK_b for large a and b. By taking a graph locally equivalent to G, we may assume that G has an induced subgraph isomorphic to aK_b.

In Section 3, we prove that if a graph H contains 3 pairwise twins, then one of them can be removed without decreasing the depth-2 rank-brittleness. Using that, each component C of aK_b can be partitioned into at least $b/2$ sets such that vertices in distinct sets are not twins. By the Ramsey-type result on bipartite graphs, we will extract a large (induced) matching or an anti-matching or a half graph between C and the rest. We find this for each component of aK_b. Then using the sunflower lemma and Ramsey's theorem, we will clean up all the structures and find a vertex-minor isomorphic to $nT_{2,n}$ or P_n. Section 4 is devoted to describe all the intermediate structures. The proof of Theorem 1.3 is given in Section 5. Section 6 shows an inequality between linear rank-width and rank-depth and presents a corollary of Theorem 1.3 for graphs with no vertex-minors isomorphic to nP_5.

2 Preliminaries

All graphs in this paper are simple and undirected. For a graph G, we denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively. Let G be a graph. For $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S, and for two disjoint vertex subsets S and T of G, we denote by $G[S,T]$ the bipartite graph with bipartition (S,T) such that for $a \in S$ and $b \in T$, a, b are adjacent in $G[S,T]$ if and only if they are adjacent in G. For $v \in V(G)$, we denote by $G-v$ the graph obtained from G by removing v and all edges incident with v. For a set X of vertices, we denote by $G-X$ the graph obtained from G by deleting all vertices in X and all edges incident with those vertices. For $v \in V(G)$, the set of *neighbors* of v in G is denoted by $N_G(v)$, and the *degree* of v is the size of $N_G(v)$. We denote by $A(G)$ the *adjacency matrix* of G.

For two disjoint vertex subsets A and B of G, we say that A is *complete* to B if every vertex in A is adjacent to all vertices in B. Similarly, A is *anti-complete* to B, if every vertex in A is non-adjacent to all vertices in B.

A *clique* is a set of pairwise adjacent vertices and an *independent* set is a set of pairwise non-adjacent vertices.

Two vertices v and w in a graph G are called *twins* if $N_G(v)\backslash\{v,w\} = N_G(w)\backslash\{v,w\}$. Note that a set of pairwise twins is either a clique or an independent set.

Let K_n denote the complete graph on n vertices, and let $K_{1,n}$ denote the star with n leaves. Let P_n denote the path on n vertices. For a graph G, we denote by $\overline{G}$ the *complement* of G, that is, two vertices v and w in G are adjacent if and only if they are not adjacent in $\overline{G}$.

We write $R(n;k)$ to denote the minimum number N such that every coloring of the edges of K_N into k colors induces a monochromatic complete subgraph on n vertices. The classical theorem of Ramsey implies that $R(n;k)$ exists.

We also use the sunflower lemma. Let $\mathcal{F}$ be a family of sets. A subset $\{M_1, M_2, \ldots, M_p\}$ of $\mathcal{F}$ is a *sunflower* with *core* A (possibly an empty set) and p *petals* if for all distinct $i,j \in \{1,2,\ldots,p\}$, $M_i \cap M_j = A$.

Theorem 2.1 (Sunflower Lemma [9, Erdős and Rado])**.** *Let k and p be positive integers, and $\mathcal{F}$ be a family of sets each of cardinality k. If $|\mathcal{F}| > k!(p-1)^k$, then $\mathcal{F}$ contains a sunflower with p petals.*

2.1 Vertex-minors

For a vertex v in a graph G, to perform *local complementation* at v, replace the subgraph of G induced on $N_G(v)$ by its complement graph. We write $G*v$ to denote the graph obtained from G by applying local complementation at v. Two graphs G and H are *locally equivalent* if G can be obtained from H by a sequence of local complementations. A graph H is a *vertex-minor* of a graph G if H is an induced subgraph of a graph which is locally equivalent to G.

2.2 Rank-depth and rank-brittleness

The *cut-rank* function of a graph G, denoted by $\rho_G(S)$ for a subset S of $V(G)$, is defined as the rank of an $S \times (V(G)\backslash S)$ 0-1 matrix over the binary field whose (a,b)-entry for $a \in S$, $b \notin S$ is 1 if a, b are adjacent and 0 otherwise. The cut-rank function is invariant under the local complementation, see Oum [22]. The cut-rank function satisfies the *submodular inequality*, that is, for all $X, Y \subseteq V(G)$, $\rho_G(X) + \rho_G(Y) \geqslant \rho_G(X \cap Y) + \rho_G(X \cup Y)$.

The ρ_G-*width* of a partition $\mathcal{P} = (X_1, X_2, \ldots, X_m)$ of $V(G)$, for some m, is

$$\max\left\{\rho_G\big(\bigcup_{i\in I} X_i\big) : I \subseteq \{1, 2, \ldots, m\}\right\}.$$

A *decomposition* of a graph G is a pair (T, σ) of a tree T with at least one internal node and a bijection σ from $V(G)$ to the set of leaves of T. The *radius* of a decomposition (T, σ) is defined to be the radius of the tree T. For an internal node $v \in V(T)$, the components of the graph $T - v$ give rise to a partition $\mathcal{P}_v$ of $V(G)$ by σ and the *width* of v is defined to be the ρ_G-width of $\mathcal{P}_v$. The *width* of the decomposition (T, σ) is the maximum width of an internal node of T. We say that a decomposition (T, σ) is a (k, r)-*decomposition* of G if the width is at most k and the radius is at most r. The *rank-depth* of a graph G, denoted by $\mathrm{rd}(G)$, is the minimum integer k such that G admits a (k, k)-decomposition. If $|V(G)| < 2$, then there exists no decomposition and rank-depth is defined to be 0. Note that every tree in a decomposition has radius at least 1 and therefore the rank-depth of a graph is at least 1 if $|V(G)| \geqslant 2$.

The *depth-d rank-brittleness* of a graph G, denoted by $\mathrm{rbrit}_d(G)$, is the minimum integer k such that G admits a (k, d)-decomposition. If $|V(G)| < 2$, then we define $\mathrm{rbrit}_d(G) = 0$. Note that the depth-1 rank-brittleness of a graph G is equal to $\max_{A\subseteq V(G)} \rho_G(A)$.

2.3 Constructions of common graphs

For two graphs G and H on the disjoint vertex sets, each having n vertices, we would like to introduce operations to construct graphs on $2n$ vertices by making the disjoint union of them and adding some edges between two graphs. Roughly speaking, $G \boxminus H$ will add a perfect matching, $G \boxtimes H$ will add the complement of a perfect matching, and $G \boxslash H$ will add a half graph. Formally, for two n-vertex graphs G and H with fixed ordering on the vertex sets $\{v_1, v_2, \ldots, v_n\}$ and $\{w_1, w_2, \ldots, w_n\}$ respectively, let $G \boxminus H$, $G \boxtimes H$, $G \boxslash H$ be graphs on the vertex set $V(G) \cup V(H)$ whose subgraph induced by $V(G)$ or $V(H)$ is G or H, respectively such that for all $i, j \in \{1, 2, \ldots, n\}$,

(i) $v_i w_j \in E(G \boxminus H)$ if and only if $i = j$,

(ii) $v_i w_j \in E(G \boxtimes H)$ if and only if $i \neq j$,

(iii) $v_i w_j \in E(G \boxslash H)$ if and only if $i \geqslant j$.

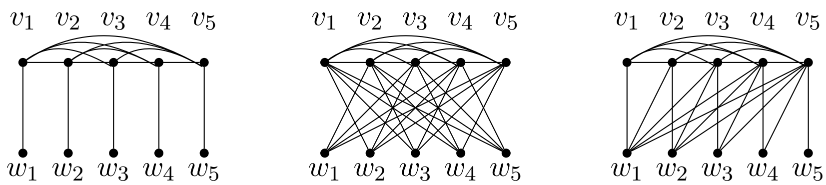

Figure 2: $K_5 \boxminus \overline{K_5}$, $K_5 \boxtimes \overline{K_5}$, and $K_5 \boxslash \overline{K_5}$.

See Figure 2 for illustrations of $K_5 \boxminus \overline{K_5}$, $K_5 \boxtimes \overline{K_5}$, and $K_5 \boxslash \overline{K_5}$.

We will use the following lemma. Similar lemmas appeared in [17, Lemma 2.8], [14, Proposition 6.2], and [16, Lemma 5.6].

Lemma 2.2 (Kwon and Oum [18, Lemma 6.5])**.** *Let n be a positive integer.*

(1) $\overline{K_n} \boxslash \overline{K_n}$ is locally equivalent to P_{2n}.

(2) $K_n \boxslash \overline{K_n}$ is locally equivalent to P_{2n}.

(3) If $n \geqslant 2$, then $K_n \boxslash K_n$ has a vertex-minor isomorphic to P_{2n-2}.

We also use the Ramsey-type result on bipartite graphs without twins.

Theorem 2.3 (Ding, Oporowski, Oxley, and Vertigan [8])**.** *For every positive integer n, there exists an integer $B(n)$ such that for every bipartite graph G with a bipartition (S,T), if no two vertices in S have the same set of neighbors and $|S| \geqslant B(n)$, then S and T have n-element subsets S' and T', respectively, such that $G[S',T']$ is isomorphic to $\overline{K_n} \boxminus \overline{K_n}$, $\overline{K_n} \boxslash \overline{K_n}$, or $\overline{K_n} \boxtimes \overline{K_n}$.*

For a positive integer t, a 2×2 matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, and a binary operator $\odot \in \{\boxminus, \boxtimes, \boxslash\}$ on two graphs of the same number of vertices, we define $(G \odot H)^t_A$ as the graph on the disjoint union of t copies of $G \odot H$ such that for all $1 \leqslant i < j \leqslant t$,

(i) the i-th copy of G is complete to the j-th copy of G if $a = 1$ and anti-complete if $a = 0$,

(ii) the i-th copy of G is complete to the j-th copy of H if $b = 1$, and anti-complete if $b = 0$,

(iii) the i-th copy of H is complete to the j-th copy of G if $c = 1$, and anti-complete if $c = 0$,

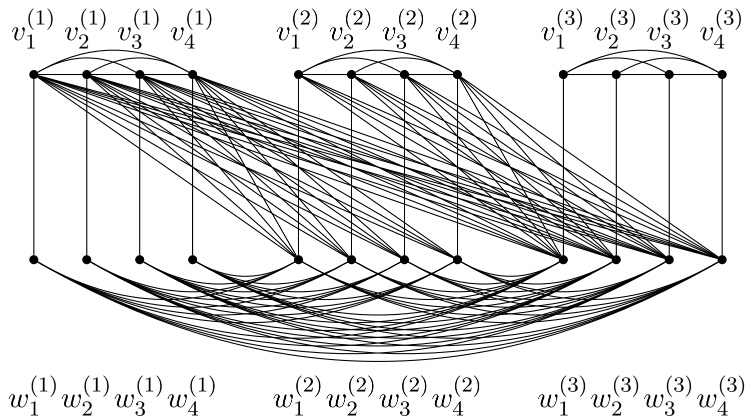

Figure 3: The graph $(K_4 \boxminus \overline{K_4})^3_A$ for $A = \binom{0\ 1}{0\ 1}$.

(iv) the i-th copy of H is complete to the j-th copy of H if $d = 1$, and anti-complete if $d = 0$.

See Figure 3 for an illustration.

3 Lemma on three twins

In this section, we prove that if a graph has three pairwise twins, then one of them can be removed without decreasing its depth-d rank-brittleness for $d \geqslant 2$. It holds for all $d \geqslant 2$ but we will only use it for $d = 2$ later.

Lemma 3.1. *Let $d \geqslant 2$ be an integer. Let v, w, z be vertices of a graph G that are pairwise twins. Then* $\operatorname{rbrit}_d(G) = \operatorname{rbrit}_d(G - v)$.

Proof. The inequality $\operatorname{rbrit}_d(G) \geqslant \operatorname{rbrit}_d(G - v)$ is trivial by definition. We will show that if $G - v$ has a (k, d)-decomposition (T, σ), then G also has a (k, d)-decomposition.

Let r be a node of T, called a *root* of T, which has distance at most d to every node of T. We may assume that r is not a leaf node. Let a be the leaf of T with $a = \sigma(w)$ and b be the parent of a in T, which is the unique neighbor of a in T. We obtain a decomposition (T_1, σ_1) of G as follows: T_1 is the tree obtained from T by adding a new node a' adjacent to b, and assign $\sigma_1(v) := a'$ and $\sigma_1(x) := \sigma(x)$ for all $x \in V(G) \backslash \{v\}$. We claim that (T_1, σ_1) is a (k, d)-decomposition of G. Clearly, T_1 has radius at most d. So, it is sufficient to show that every internal node of T_1 has width at most k. For each internal node t of T_1, let $\mathcal{P}_t$ be the partition of $V(G)$ derived from the components of $T_1 - t$ by σ_1^{-1}.

For an internal node $t \neq b$ in T_1, the width of t in (T_1, σ_1) is the same as its width in the decomposition (T, σ) because v and w are twins of G and v and w lie on the same part of $\mathcal{P}_t$.

We claim that the width of b in (T_1, σ_1) is at most k. Let $\mathcal{P}' \subseteq \mathcal{P}_b$ and $A := \bigcup_{X \in \mathcal{P}'} X$. In the bipartition $(A, V(G) \backslash A)$, if v is contained in a part together with w or z, then the bipartition obtained by removing v arises in the decomposition (T, σ) as well. So, without loss of generality, we may assume that $w, z \in A$ and $v \in V(G) \backslash A$. But in this case, as v, w, z are pairwise twins, the bipartition obtained by exchanging v and w has the same cut-rank. As w is a single-vertex part of $\mathcal{P}_b$, the bipartition $(A \backslash \{w\}, V(G) \backslash (A \backslash \{w\}))$ arises in the decomposition (T, σ). So,

$$\rho_G(A) = \rho_G((A \backslash \{w\}) \cup \{v\}) = \rho_{G-v}(A \backslash \{w\}) \leqslant k.$$

We conclude that the width of every internal node of T_1 is at most k. □

4 Reducing to two cases

We recall the definition of rank k-brittleness [18]. The *rank k-brittleness* of a graph G, denoted by $\beta_k^\rho(G)$, is the minimum ρ_G-width of all partitions of $V(G)$ into parts of size at most k.

Lemma 4.1. $\mathrm{rbrit}_2(G) \leqslant \max(2, k, \beta_k^\rho(G))$ *for every positive integer k.*

Proof. Suppose that $(X_1, X_2, \ldots, X_m)$ is a partition of $V(G)$ whose ρ_G-width is $\beta_k^\rho(G)$. We create a decomposition (T, σ) of G as follows. Let r be the root of T, and let $r_1, r_2, \ldots, r_m$ be the children of T, and each r_i has exactly $|X_i|$ leaves adjacent to r_i, and we assign X_i to these leaves by σ. It is easy to see that each r_i has width at most k, and the root r has width at most $\beta_k^\rho(G)$. Thus, $\mathrm{rbrit}_2(G) \leqslant \max(2, k, \beta_k^\rho(G))$. □

Kwon and Oum [18] proved the following.

Theorem 4.2 (Kwon and Oum [18, Theorem 1.4])**.** *For every positive integer n, there exists N such that every graph G with $\beta_k^\rho(G) \geqslant N$ contains a vertex-minor isomorphic to nH for some connected graph H on $k+1$ vertices.*

Every large connected graph has a long induced path or a vertex of large degree.

Proposition 4.3 (See Diestel [7, Proposition 1.3.3])**.** *For integers $k > 3$ and $\ell > 0$, every connected graph on at least $(k-1)(k-2)^{\ell-2}/(k-3)$ vertices contains a vertex of degree at least k or an induced path on ℓ vertices.*

As a corollary we deduce the following. Essentially its proof is almost identical to the proof of [18, Theorem 1.6].

Corollary 4.4. *For all positive integers k and n, there exists $N = h(n,k)$ such that every graph G with depth-2 rank-brittleness at least N has a vertex-minor isomorphic to nK_k.*

Proof. We may assume that $k > 3$ by increasing k if necessary. Let

$$M := \lceil (R(k-1;2)-1)(R(k-1;2)-2)^{2(k-1)-2}/(R(k-1;2)-3) \rceil.$$

By Proposition 4.3, every connected graph with at least M vertices has a vertex of degree at least $R(k-1;2)$ or an induced subgraph isomorphic to $P_{2(k-1)}$. By Theorem 4.2, there exists N such that $N \geqslant M$ and every graph G with $\beta^{\rho}_{M-1}(G) \geqslant N$ contains nH for some connected graph H on M vertices. By Lemma 2.2(2), if H contains P_{2k} as an induced subgraph, then H contains K_k as a vertex-minor. If H contains a vertex of degree at least $R(k-1;2)$, then it contains $K_{1,k-1}$ or K_k as an induced subgraph. In all cases, H contains K_k as a vertex-minor and so nH contains nK_k as a vertex-minor.

If G has depth-2 rank-brittleness at least N, then by Lemma 4.1,

$$\beta^{\rho}_{M-1}(G) \geqslant N$$

and therefore G has a vertex-minor isomorphic to nK_k. $\square$

Proposition 4.5. *For every integer $n \geqslant 2$, there exists an integer $\sigma(n)$ such that every graph G of depth-2 rank-brittleness at least $\sigma(n)$ contains a vertex-minor G' satisfying one of the following.*

(i) *$V(G') = X_1^* \cup X_2^* \cup \cdots \cup X_n^* \cup Q^*$ for disjoint sets $X_1^*, X_2^*, \ldots, X_n^*, Q^*$ of n vertices such that each X_i^* is a clique in G', $G'[X_1^* \cup X_2^* \cup \cdots \cup X_n^*]$ is isomorphic to nK_n, and either $K_n \boxminus \overline{K_n}$ or $K_n \boxtimes \overline{K_n}$ is isomorphic to all $G'[X_i^* \cup Q^*]$.*

(ii) *$V(G') = X_1^* \cup X_2^* \cup \cdots \cup X_n^* \cup Y_1^* \cup Y_2^* \cup \cdots \cup Y_n^*$ for disjoint sets $X_1^*, X_2^*, \ldots, X_n^*, Y_1^*, Y_2^*, \ldots, Y_n^*$ of n vertices such that each X_i^* is a clique in G', $G'[X_1^* \cup X_2^* \cup \cdots \cup X_n^*]$ is isomorphic to nK_n, and one of $K_n \boxminus \overline{K_n}$, $K_n \boxminus K_n$, $K_n \boxtimes \overline{K_n}$, and $K_n \boxtimes K_n$ is isomorphic to all $G'[X_i^* \cup Y_i^*]$.*

(iii) G' is isomorphic to P_n.

Proof. If G contains a component with at least 3 vertices, then it contains a vertex-minor isomorphic to P_3. Thus, we may assume that $n \geqslant 4$.

Let B be the function defined in Theorem 2.3, and h be the function defined in Corollary 4.4. Let

$$\begin{aligned} f_2(n) &:= R(2n; 2), \\ f_1(n) &:= (2n)! n^{2n} + 1, \\ \sigma(n) &:= h(4f_1(n), 2B(f_2(n))). \end{aligned}$$

We may assume that no proper vertex-minor of G has depth-2 rank-brittleness at least $\sigma(n)$. By Lemma 3.1, every graph locally equivalent to G has no three vertices that are pairwise twins.

By Corollary 4.4, G has a vertex-minor isomorphic to

$$H := (4f_1(n))K_{2B(f_2(n))}.$$

We may assume that $(4f_1(n))K_{2B(f_2(n))}$ is an induced subgraph of G by applying local complementations. Let $C_1, C_2, \ldots, C_{4f_1(n)}$ be the set of connected components of H, and let $U := V(G) \backslash V(H)$.

Observe that G has no three vertices that are pairwise twins. It means that in each C_i, there are no three vertices that have the same neighborhood on U in G, and thus each C_i contains a subset S_i with

$$|S_i| = \lceil |C_i|/2 \rceil = B(f_2(n))$$

that have pairwise distinct sets of neighbors on U.

Now, we consider the bipartite graph $G[S_i, U]$ for each i. In this bipartite graph, since vertices in S_i have distinct neighborhoods on U and $|S_i| = B(f_2(n))$, by Theorem 2.3, there exist $X_i' \subseteq S_i$ and $Y_i' \subseteq U$ such that $G[X_i', Y_i']$ is isomorphic to $\overline{K_{f_2(n)}} \boxminus \overline{K_{f_2(n)}}$, $\overline{K_{f_2(n)}} \boxslash \overline{K_{f_2(n)}}$, or $\overline{K_{f_2(n)}} \boxtimes \overline{K_{f_2(n)}}$ for each $i \in \{1, 2, \ldots, 4f_1(n)\}$.

As $f_2(n) = R(2n; 2)$, by Ramsey's theorem, there exist $X_i \subseteq X_i'$ and $Y_i \subseteq Y_i'$ with $|X_i| = |Y_i| = 2n$ where

- Y_i is a clique or an independent set in G,
- $G[X_i, Y_i]$ is isomorphic to $\overline{K_{2n}} \boxminus \overline{K_{2n}}$, $\overline{K_{2n}} \boxslash \overline{K_{2n}}$, or $\overline{K_{2n}} \boxtimes \overline{K_{2n}}$.

This can be done by selecting Y_i from Y_i' by using Ramsey's theorem and then selecting X_i by using the relation between X_i' and Y_i'. If $G[X_i, Y_i]$

is isomorphic to $\overline{K_{2n}} \boxslash \overline{K_{2n}}$ for some i, then G contains a vertex-minor isomorphic to a path on $4n-2 \geqslant n$ vertices by Lemma 2.2. Thus, we may assume that $G[X_i, Y_i]$ is isomorphic to $\overline{K_{2n}} \boxminus \overline{K_{2n}}$ or $\overline{K_{2n}} \boxtimes \overline{K_{2n}}$ for all $i \in \{1, 2, \ldots, 4f_1(n)\}$. So $G[X_i \cup Y_i]$ for each i is isomorphic to $K_{2n} \boxminus \overline{K_{2n}}$, $K_{2n} \boxtimes \overline{K_{2n}}$, $K_{2n} \boxminus K_{2n}$, or $K_{2n} \boxtimes K_{2n}$. By the pigeonhole principle, we may assume that for all $i \in \{1, 2, \ldots, f_1(n)\}$, all graphs $G[X_i \cup Y_i]$ are isomorphic to exactly one of $K_{2n} \boxminus \overline{K_{2n}}$, $K_{2n} \boxtimes \overline{K_{2n}}$, $K_{2n} \boxminus K_{2n}$, and $K_{2n} \boxtimes K_{2n}$.

We now apply Theorem 2.1, the sunflower lemma, to sets $Y_1, Y_2, \ldots, Y_{f_1(n)}$. As we choose $f_1(n) > (2n)!n^{2n}$, $\{Y_1, Y_2, \ldots, Y_{f_1(n)}\}$ contains a sunflower $\mathcal{F}$ with $n+1$ petals. We may assume that $\mathcal{F} = \{Y_1, Y_2, \ldots, Y_{n+1}\}$. Let Q be the core of $\mathcal{F}$, that is $\bigcap_{i=1}^{n+1} Y_i$. Note that either Q has at least $n+1$ vertices, or $Y_i \backslash Q$ has at least n vertices for all $i = 1, 2, \ldots, n+1$. We divide into two cases depending on the size of the core.

First, suppose that $|Q| \geqslant n+1$. Let Q^* be a subset of Q with $|Q^*| = n$. For $i = 1, 2, \ldots, n$, let X_i^* be the set of vertices in X_i paired with vertices in $Q^* \subseteq Y_i$ in the graph $G[X_i \cup Y_i]$. Then each X_i^* is a clique and $G[X_1^* \cup X_2^* \cup \cdots \cup X_n^*]$ is isomorphic to nK_n. Let $w \in Q \backslash Q^*$.

If all $G[X_i^* \cup Q^*]$ are isomorphic to $K_n \boxtimes K_n$, then X_{n+1} has a vertex v adjacent to all vertices in Q^*. In this case, we take $G' := G * v$. Then for every $i \in \{1, 2, \ldots, n\}$, $G'[X_i^* \cup Q^*]$ is isomorphic to $K_n \boxtimes \overline{K_n}$.

If all $G[X_i^* \cup Q^*]$ are isomorphic to $K_n \boxminus K_n$, then we take $G' := G * w$. Then all $G'[X_i^* \cup Q^*]$ are isomorphic to $K_n \boxminus \overline{K_n}$.

If all $G[X_i^* \cup Q^*]$ are isomorphic to $K_n \boxminus \overline{K_n}$ or $K_n \boxtimes \overline{K_n}$, then we take $G' := G$.

We conclude that G has a vertex-minor G' on $X_1^* \cup X_2^* \cup \cdots \cup X_n^* \cup Q^*$ such that each X_i^* is a clique in G', X_i^* is anti-complete to X_j^* for all $i \neq j$ in G', and one of $K_n \boxminus \overline{K_n}$ or $K_n \boxtimes \overline{K_n}$ is isomorphic to $G'[X_i^* \cup Q^*]$ for all $i \in \{1, 2, \ldots, n\}$. So, G' provides a desired vertex-minor of the first type.

Now it remains to consider the case that $|Q| \leqslant n$. Then for all $i \in \{1, 2, \ldots, n\}$, $|Y_i \backslash Q| \geqslant n$. For each $i = 1, 2, \ldots, n$, let Y_i^* be a subset of $Y_i \backslash Q$ with $|Y_i^*| = n$. For $i = 1, 2, \ldots, n$, let X_i^* be the set of vertices in X_i paired with vertices in Y_i^* in the graph $G[X_i \cup Y_i]$. Then we deduce that $G[X_1^* \cup X_2^* \cup \cdots \cup X_n^*]$ is isomorphic to nK_n and for all $i = 1, 2, \ldots, n$, all $G[X_i^* \cup Y_i^*]$ are isomorphic to exactly one of $K_n \boxminus \overline{K_n}$, $K_n \boxtimes \overline{K_n}$, $K_n \boxminus K_n$, or $K_n \boxtimes K_n$. and $X_1^*, X_2^*, \ldots, X_n^*, Y_1^*, Y_2^*, \ldots, Y_n^*$ are disjoint. So, $(X_1^*, X_2^*, \ldots, X_n^*)$ and $(Y_1^*, Y_2^*, \ldots, Y_n^*)$ provide a desired induced subgraph of the second type. $\square$

In the rest, we will find a vertex-minor isomorphic to P_n or $nT_{2,n}$ when a given graph satisfies (i) or (ii) of Proposition 4.5.

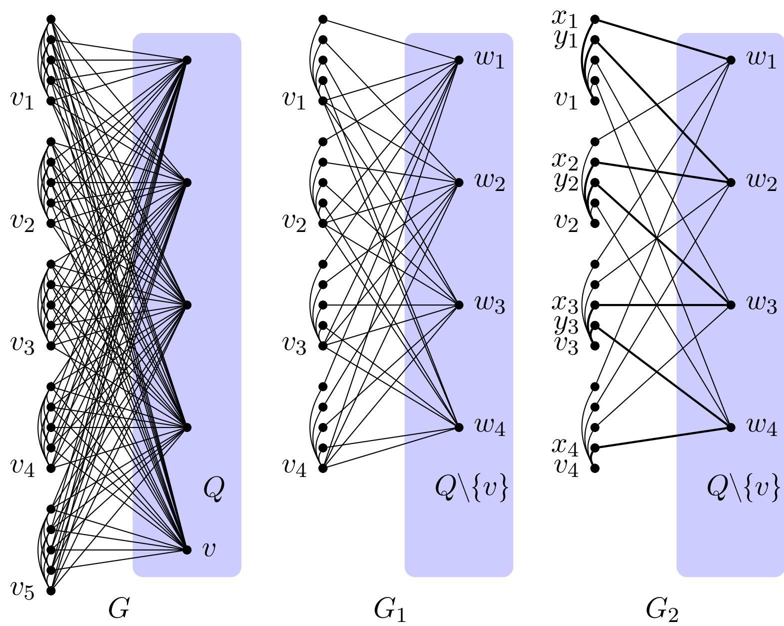

Figure 4: Three graphs in the proof of Lemma 4.7 when $n = 5$.

4.1 The first case

Lemma 4.6. *Let $n \geqslant 2$ be an integer. Let G be a graph on $n^2 + n$ vertices such that $V(G) = X_1 \cup X_2 \cup \cdots \cup X_n \cup Q$ for disjoint sets $X_1, X_2, \ldots, X_n, Q$ of n vertices, each X_i is a clique in G, $G[X_1 \cup X_2 \cup \cdots \cup X_n]$ is isomorphic to nK_n, and all $G[X_i \cup Q]$ are isomorphic to $K_n \boxminus \overline{K_n}$. Then G has a vertex-minor isomorphic to P_{3n-1}.*

Proof. Let $v_1, v_2, \ldots, v_n$ be an arbitrary enumeration of vertices in Q. For each $i \in \{1, 2, \ldots, n-1\}$, there are two vertices x_i, y_i in X_i such that $N_G(x_i) \cap Q = \{v_i\}$, $N_G(y_i) \cap Q = \{v_{i+1}\}$. Let x_n be the neighbor of v_n in X_n. Then $v_1x_1y_1v_2x_2y_2v_3 \cdots v_nx_n$ is an induced path on $3n-1$ vertices. $\square$

Lemma 4.7. *Let $n \geqslant 2$ be an integer. Let G be a graph on $n^2 + n$ vertices such that $V(G) = X_1 \cup X_2 \cup \cdots \cup X_n \cup Q$ for disjoint sets $X_1, X_2, \ldots, X_n, Q$ of n vertices, each X_i is a clique in G, $G[X_1 \cup X_2 \cup \cdots \cup X_n]$ is isomorphic to nK_n, and all $G[X_i \cup Q]$ are isomorphic to $K_n \boxtimes \overline{K_n}$. Then G has a vertex-minor isomorphic to P_{4n-5}.*

Proof. Let v be a vertex in Q. Let v_i be the vertex in X_i non-adjacent to v. Let

$$G_1 = \begin{cases} G * v_1 * v_2 * \cdots * v_n - v - X_n & \text{if } n \text{ is even,} \\ G * v_1 * v_2 * \cdots * v_{n-1} - v - X_n & \text{if } n \text{ is odd.} \end{cases}$$

Observe that every vertex in $X_i \backslash \{v_i\}$ has degree 2 in G_1 and $N_{G_1}(v_i) = (X_i \backslash \{v_i\}) \cup (Q \backslash \{v\})$ for all $i \in \{1, 2, \ldots, n-1\}$. See Figure 4 for an illustration. Let G_2 be the graph obtained from G_1 by applying local complementations at all vertices in $\bigcup_{i=1}^{n-1} (X_i \backslash \{v_i\})$. It is easy to see that G_2 is obtained from G_1 by deleting all edges from v_i to $Q \backslash \{v\}$ for all $i \leqslant n-1$. Then $N_{G_2}(v_i) = X_i \backslash \{v_i\}$ and $G_2[(X_i \backslash \{v_i\}) \cup (Q \backslash \{v\})]$ is isomorphic to $\overline{K_{n-1}} \boxminus \overline{K_{n-1}}$. Let $w_1, w_2, \ldots, w_{n-1}$ be an arbitrary enumeration of vertices in $Q \backslash \{v\}$. For each $i \in \{1, 2, \ldots, n-2\}$, there are vertices $x_i, y_i \in X_i \backslash \{v_i\}$ such that $N_{G_2}(x_i) \cap Q = \{w_i\}$ and $N_{G_2}(y_i) \cap Q = \{w_{i+1}\}$. Let x_{n-1} be the neighbor of w_{n-1} in X_{n-1}. Then $w_i x_i v_i y_i w_{i+1}$ is an induced path and so $w_1 x_1 v_1 y_1 w_2 x_2 v_2 y_2 \cdots w_{n-2} x_{n-2} v_{n-2} y_{n-2} w_{n-1} x_{n-1} v_{n-1}$ is an induced path on $n - 1 + 3(n-2) + 2 = 4n - 5$ vertices in G_2. Thus, G has a vertex-minor isomorphic to P_{4n-5}. $\square$

4.2 The second case

We will use the product Ramsey theorem described below.

Theorem 4.8 ([26, Theorem 11.5]; See also [13])**.** *Let r, t be positive integers, and let $k_1, k_2, \ldots, k_t$ be nonnegative integers, and let $m_1, m_2, \ldots, m_t$ be integers with $m_i \geqslant k_i$ for each $i \in \{1, 2, \ldots, t\}$. Then there exists an integer $R = R_{prod}(r, t; k_1, k_2, \ldots, k_t; m_1, m_2, \ldots, m_t)$ such that if $X_1, X_2, \ldots, X_t$ are sets with $|X_i| \geqslant R$ for each $i \in \{1, 2, \ldots, t\}$, then for every function $f : \binom{X_1}{k_1} \times \binom{X_2}{k_2} \times \cdots \times \binom{X_t}{k_t} \to \{1, 2, \ldots, r\}$, there exist an element $\alpha \in \{1, 2, \ldots, r\}$ and subsets $Y_1, Y_2, \ldots, Y_t$ of $X_1, X_2, \ldots, X_t$, respectively, so that $|Y_i| \geqslant m_i$ for each $i \in \{1, 2, \ldots, t\}$, and f maps every element of $\binom{Y_1}{k_1} \times \binom{Y_2}{k_2} \times \cdots \times \binom{Y_t}{k_t}$ to α.*

Lemma 4.9. *For integers s and t, there exist $M = f(s, t)$ and $N = g(s, t)$ such that for $m \geqslant M$ and $n \geqslant N$, if a graph G has $2m$ disjoint n-vertex sets $X_1, X_2, \ldots, X_m, Y_1, Y_2, \ldots, Y_m$, each X_i is a clique of G, $G[X_1 \cup X_2 \cup \cdots \cup X_m]$ is isomorphic to mK_n, and one of $K_n \boxminus \overline{K_n}$, $K_n \boxminus K_n$, $K_n \boxtimes \overline{K_n}$, and $K_n \boxtimes K_n$ is isomorphic to all $G[X_i \cup Y_i]$, then there exist indices $1 \leqslant i_1 < i_2 < \cdots < i_s \leqslant m$ and subsets $X_1^*, X_2^*, \ldots, X_s^*$ of $X_{i_1}, X_{i_2}, \ldots, X_{i_s}$ respectively and subsets $Y_1^*, Y_2^*, \ldots, Y_s^*$ of $Y_{i_1}, Y_{i_2}, \ldots, \ldots, Y_{i_s}$ respectively such that the following hold.*

(i) $|X_i^*| = |Y_i^*| = t$ *for all* $1 \leqslant i \leqslant s$,

(ii) *one of* $K_t \boxminus \overline{K_t}$, $K_t \boxminus K_t$, $K_t \boxtimes \overline{K_t}$, *and* $K_t \boxtimes K_t$ *is isomorphic to all* $G[X_i^* \cup Y_i^*]$ *for all* $1 \leqslant i \leqslant s$,

(iii) X_i^* *is complete to* Y_j^* *for all* $i < j$ *or* X_i^* *is anti-complete to* Y_j^* *for all* $i < j$,

(iv) Y_i^* *is complete to* X_j^* *for all* $i < j$ *or* Y_i^* *is anti-complete to* X_j^* *for all* $i < j$,

(v) Y_i^* *is complete to* Y_j^* *for all* $i < j$ *or* Y_i^* *is anti-complete to* Y_j^* *for all* $i < j$.

In other words, G *has an induced subgraph isomorphic to one of*

$$(K_t \boxminus K_t)^s_A, (K_t \boxminus \overline{K_t})^s_A, (K_t \boxtimes K_t)^s_A, \text{ and } (K_t \boxtimes \overline{K_t})^s_A$$

for some 0-1 *matrix* $A = \left(\begin{smallmatrix} 0 & b \\ c & d \end{smallmatrix}\right)$.

Proof. Let $m \geqslant M := R(s; 8)$ and let

$$n \geqslant N := R_{prod}(8^{\binom{m}{2}}, m; 1, 1, \ldots, 1; t, t, \ldots, t).$$

The first step of the proof is to clean up edges between $X_i \cup Y_i$ and $X_j \cup Y_j$ for distinct i and j. We consider a function that maps $(v_1, v_2, \ldots, v_m)$ for $v_i \in Y_i$ to an edge-coloring of K_m with colors on the edges ij based on the three possible adjacencies between a pair of v_i and its unique neighbor or non-neighbor in X_i and a pair of v_j and its unique neighbor or non-neighbor in X_j. Each edge of K_m will receive one of 2^3 colors and the range of this function has at most $8^{\binom{m}{2}}$ edge-colorings of K_m. By Theorem 4.8, there exist subsets X'_1, X'_2, …, X'_m, Y'_1, Y'_2, …, Y'_m of $X_1, X_2, \ldots, X_m, Y_1, Y_2, \ldots, Y_m$, respectively, such that

(i) $|X'_i| = |Y'_i| = t$,

(ii) for each $i \neq j$, X'_i is complete or anti-complete to Y'_j, and Y'_i is complete or anti-complete to Y'_j,

(iii) one of $K_t \boxminus \overline{K_t}$, $K_t \boxminus K_t$, $K_t \boxtimes \overline{K_t}$, and $K_t \boxtimes K_t$ is isomorphic to all $G[X'_i \cup Y'_i]$.

Now our next goal is to take a subset of $\{1, 2, \ldots, m\}$ by using Ramsey's theorem. Let us color the edges ij of K_m $(i < j)$ by the one of 8 colors determined by the following:

- X_i' is complete to Y_j' or not.
- Y_i' is complete to X_j' or not.
- Y_i' is complete to Y_j' or not.

Then by Ramsey's theorem, there exists a subset I of $\{1, 2, \dots, m\}$ with $|I| = s$ such that every edge of K_m has the same color. Let $I = \{i_1, i_2, \dots, i_s\}$ for $i_1 < i_2 < \cdots < i_s$ and $X_j^* = X_{i_j}$, $Y_j^* = Y_{i_j}$ for $1 \leqslant j \leqslant s$. This provides our conclusion. $\square$

Now we will see that in many cases, we will have a vertex-minor isomorphic to $nT_{2,n}$.

Lemma 4.10. *Let n be a positive integer.*

(1) $K_{n+1} \boxminus \overline{K_{n+1}}$ contains a vertex-minor isomorphic to $T_{2,n}$.

(2) $K_{n+2} \boxminus K_{n+2}$ contains a vertex-minor isomorphic to $T_{2,n}$.

(3) $K_{n+2} \boxtimes \overline{K_{n+2}}$ contains a vertex-minor isomorphic to $T_{2,n}$.

(4) $K_{n+1} \boxtimes K_{n+1}$ contains a vertex-minor isomorphic to $T_{2,n}$.

Therefore, if $A = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, then all of $(K_{n+1} \boxminus \overline{K_{n+1}})_A^n$, $(K_{n+2} \boxminus K_{n+2})_A^n$, $(K_{n+2} \boxtimes \overline{K_{n+2}})_A^n$, and $(K_{n+1} \boxtimes K_{n+1})_A^n$ have vertex-minors isomorphic to $nT_{2,n}$.

Proof. (1) Let $V(K_{n+1}) = \{v_i : 1 \leqslant i \leqslant n+1\}$ and $V(\overline{K_{n+1}}) = \{w_i : 1 \leqslant i \leqslant n+1\}$. The graph $(K_{n+1} \boxminus \overline{K_{n+1}} - w_1) * v_1$ is isomorphic to $T_{2,n}$.

(2) Let $\{v_i : 1 \leqslant i \leqslant n+2\}$ and $\{w_i : 1 \leqslant i \leqslant n+2\}$ be the vertex sets of two copies of K_{n+2}. The graph $((K_{n+2} \boxminus K_{n+2} - \{v_1, w_2\}) * v_2 * w_1) - \{w_1\}$ is isomorphic to $T_{2,n}$.

(3) Let $V(K_{n+2}) = \{v_i : 1 \leqslant i \leqslant n+2\}$ and $V(\overline{K_{n+2}}) = \{w_i : 1 \leqslant i \leqslant n+2\}$. The graph $((K_{n+2} \boxtimes \overline{K_{n+2}} - \{w_1, v_2\}) * v_1 * w_2) - \{w_2\}$ is isomorphic to $T_{2,n}$.

(4) Let $\{v_i : 1 \leqslant i \leqslant n+1\}$ and $\{w_i : 1 \leqslant i \leqslant n+1\}$ be the vertex sets of two copies of K_{n+1}. The graph $(K_{n+1} \boxtimes K_{n+1} - w_1) * v_1$ is isomorphic to the graph obtained from $\overline{K_n} \boxminus \overline{K_n}$ by adding a vertex v_1 adjacent to all other vertices. Thus, the graph $(K_{n+1} \boxtimes K_{n+1} - w_1) * v_1 * v_2 * \cdots * v_{n+1}$ is isomorphic to $T_{2,n}$. $\square$

In the following lemma, we show that if A is not symmetric, then we obtain P_n as a vertex-minor.

Lemma 4.11. *Let $n \geqslant 2$ be an integer. If $A = \left(\begin{smallmatrix} 0 & b \\ c & d \end{smallmatrix}\right)$ is a 0-1 matrix such that $b \neq c$, then both $(K_1 \boxminus K_1)_A^n$ and $(K_1 \boxtimes K_1)_A^{n+1}$ have vertex-minors isomorphic to P_n.*

Proof. If $d = 0$, then $(K_1 \boxminus K_1)_A^n$ is isomorphic to $K_n \boxslash \overline{K_n}$ and $(K_1 \boxtimes K_1)_A^{n+1}$ contains an induced subgraph isomorphic to $K_n \boxslash \overline{K_n}$. If $d = 1$, then $(K_1 \boxminus K_1)_A^n$ is isomorphic to $K_n \boxslash K_n$ and $(K_1 \boxtimes K_1)_A^{n+1}$ contains an induced subgraph isomorphic to $K_n \boxslash K_n$. By Lemma 2.2, there is a vertex-minor isomorphic to P_{2n-2} in both cases. □

Lemma 4.12. *Let $n \geqslant 2$ be an integer. If $A = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, then all of $(K_{n+2} \boxminus K_{n+2})_A^{n+1}$, $(K_{n+2} \boxminus \overline{K_{n+2}})_A^{n+1}$, $(K_{n+2} \boxtimes K_{n+2})_A^{n+1}$, and $(K_{n+2} \boxtimes \overline{K_{n+2}})_A^{n+1}$ have vertex-minors isomorphic to $nT_{2,n}$.*

Proof. Let H be the one of $K_{n+2} \boxminus K_{n+2}$, $K_{n+2} \boxminus \overline{K_{n+2}}$, $K_{n+2} \boxtimes K_{n+2}$, or $K_{n+2} \boxtimes \overline{K_{n+2}}$ and let $G = H_A^{n+1}$. Let X_1, Y_1 be the sets of vertices of the first copy of H in G where X_1 denotes the set of vertices in K_{n+2} and Y_1 denotes the other vertices. Let w be a vertex in Y_1.

Then $G*w$ contains an induced subgraph isomorphic to $(K_{n+2} \boxminus K_{n+2})_B^n$, $(K_{n+2} \boxminus \overline{K_{n+2}})_B^n$, $(K_{n+2} \boxtimes K_{n+2})_B^n$, or $(K_{n+2} \boxtimes \overline{K_{n+2}})_B^n$ for $B = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. By Lemma 4.10, it has a vertex-minor isomorphic to $nT_{2,n}$. □

Lemma 4.13. *Let $n \geqslant 2$ be an integer. If $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & d \end{smallmatrix}\right)$ for some $d \in \{0, 1\}$, then all of $(K_{n+2} \boxminus K_{n+2})_A^{n+2}$, $(K_{n+2} \boxminus \overline{K_{n+2}})_A^{n+2}$, $(K_{n+2} \boxtimes K_{n+2})_A^{n+2}$, and $(K_{n+2} \boxtimes \overline{K_{n+2}})_A^{n+2}$ have vertex-minors isomorphic to $nT_{2,n}$.*

Proof. Let H be the one of $K_{n+2} \boxminus K_{n+2}$, $K_{n+2} \boxminus \overline{K_{n+2}}$, $K_{n+2} \boxtimes K_{n+2}$, or $K_{n+2} \boxtimes \overline{K_{n+2}}$ and let $G = H_A^{n+2}$. There exists an induced subgraph G' of G and an edge xy of G' such that $G' - x - y$ is isomorphic to H_A^{n+1}, x is complete to the bottom part of copies of H, and anti-complete to the top part of copies of H, and y is complete to the top part of copies of H, and is either complete or anti-complete to the bottom part of copies of H, because we can choose a vertex x from the top part of H and a neighbor y is chosen from the bottom part in the same copy of H. Now, it is easy to see that $G' * x * y * x - x - y$ is isomorphic to one of $(K_{n+2} \boxminus K_{n+2})_B^{n+1}$, $(K_{n+2} \boxminus \overline{K_{n+2}})_B^{n+1}$, $(K_{n+2} \boxtimes K_{n+2})_B^{n+1}$, and $(K_{n+2} \boxtimes \overline{K_{n+2}})_B^{n+1}$ for a matrix $B = \left(\begin{smallmatrix} 0 & 0 \\ 0 & d \end{smallmatrix}\right)$. By Lemmas 4.10 and 4.12, G has a vertex-minor isomorphic to $nT_{2,n}$. □

5 Main proof

We are ready to prove our main theorem, restated below.

Theorem 1.3. *A vertex-minor ideal $\mathcal{C}$ has bounded depth-2 rank-brittleness if and only if*

$$\{P_1, P_2, P_3, P_4, \ldots\} \nsubseteq \mathcal{C},$$

and

$$\{T_{2,1}, 2T_{2,2}, 3T_{2,3}, 4T_{2,4}, \ldots\} \nsubseteq \mathcal{C}.$$

Before the proof, let us discuss why the two conditions in Theorem 1.3, $\{P_1, P_2, P_3, \ldots\} \nsubseteq \mathcal{C}$ and $\{T_{2,1}, 2T_{2,2}, 3T_{2,3}, \ldots\} \nsubseteq \mathcal{C}$, are incomparable. First we sketch the proof showing that no path contains $T_{2,3}$ as a vertex-minor. The tree $T_{2,3}$ is a tree having a vertex v such that $T_{2,3} - v$ contains three components having linear rank-width 1. (The definition of linear rank-width will be discussed in Section 6.) It implies that it has linear rank-width at least 2, by a characterization of linear rank-width on trees, see [1, 2]. However, paths have linear rank-width 1 and therefore no path contains $T_{2,3}$ as a vertex-minor. Thus, if $\mathcal{C}$ is the set of all vertex-minors of P_n for all n, then $\mathcal{C}$ does not satisfy the first condition but satisfies the second condition. Secondly we claim that no $nT_{2,n}$ contains a long path as a vertex-minor. It is not difficult to see that $nT_{2,n}$ has depth-3 rank-brittleness at most 3. However, $\{P_1, P_2, P_3, \ldots\}$ has unbounded rank-depth [6], and thus unbounded depth-3 rank-brittleness. So, if $\mathcal{C}$ is the set of all vertex-minors of $nT_{2,n}$ for all n, then $\mathcal{C}$ does not satisfy the second condition but satisfies the first condition.

Now let us start the proof of Theorem 1.3. Our first lemma is to prove the forward implication. It is already known that $\{P_1, P_2, P_3, \ldots\}$ has unbounded rank-depth [6] and therefore it has unbounded depth-2 rank-brittleness. Thus, to prove the forward implication, it is enough to show that $\{T_{2,1}, 2T_{2,2}, 3T_{2,3}, 4T_{2,4}, \ldots\}$ has unbounded depth-2 rank-brittleness.

Lemma 5.1. *The class $\{T_{2,1}, 2T_{2,2}, 3T_{2,3}, 4T_{2,4}, \ldots\}$ has unbounded depth-2 rank-brittleness.*

Proof. We claim that $nT_{2,n}$ has depth-2 rank-brittleness at least $n/2$. Suppose that $nT_{2,n}$ admits a $(m, 2)$-decomposition (T, σ) with $m < n/2$. Then T has a root r from which every leaf is within distance at most 2, and we may assume that r is not a leaf. By subdividing an edge if necessary, we may further assume that no leaf is adjacent to r.

Let r_1, r_2, $\ldots$, r_m be the neighbors of r. We color each vertex v of $nT_{2,n}$ by $i \in \{1, 2, \ldots, m\}$ if the component of $T - r$ containing $\sigma(v)$ has r_i. An edge of $nT_{2,n}$ is *colorful* if its ends have distinct colors. Let C_1, C_2, $\ldots$, C_n be the components of $nT_{2,n}$.

Suppose that a component C_i is fully contained in X_j for some j. Then, since C_i contains an induced matching of size n, the width of r_j has to be at least n. This contradicts the assumption that (T, σ) has width less than $n/2$. Thus, we may assume that no component C_i is fully contained in some X_j. So every component C_i has a colorful edge and therefore $nT_{2,n}$ has a set F of n colorful edges in distinct components.

Let X be a subset of $\{1, 2, \dots, m\}$ chosen uniformly at random. A colorful edge of $nT_{2,n}$ is *X-colorful* if one end has a color in X and the other end has a color not in X. Then by the linearity of expectation, the expected number of X-colorful edges in F is $n/2$. This means that there exists X such that there are at least $n/2$ X-colorful edges in distinct components of $nT_{2,n}$ and so the width of r is at least $n/2$, contradicting the assumption on (T, σ). $\square$

The following proposition proves the backward implication of Theorem 1.3.

Proposition 5.2. *For every integer $n \geqslant 2$, there exists an integer $N := d(n)$ such that every graph G of depth-2 rank-brittleness at least N contains a vertex-minor isomorphic to P_n or $nT_{2,n}$.*

Proof. Let σ be the function defined in Proposition 4.5 and let f, g be the functions defined in Lemma 4.9. Let $m := \max(n, f(n+2, n+2), g(n+2, n+2))$. and let $d(n) := \sigma(m)$. By Proposition 4.5, G has a vertex-minor G' satisfying one of the following:

(i) $V(G') = X_1^* \cup X_2^* \cup \cdots \cup X_m^* \cup Q^*$ for disjoint sets $X_1^*, X_2^*, \dots, X_m^*, Q^*$ of m vertices such that each X_i^* is a clique in G', $G'[X_1^* \cup X_2^* \cup \cdots \cup X_m^*]$ is isomorphic to mK_m, and either $K_m \boxminus \overline{K_m}$ or $K_m \boxtimes \overline{K_m}$ is isomorphic to all $G'[X_i^* \cup Q^*]$.

(ii) $V(G') = X_1^* \cup X_2^* \cup \cdots \cup X_m^* \cup Y_1^* \cup Y_2^* \cup \cdots \cup Y_m^*$ for disjoint sets $X_1^*, X_2^*, \dots, X_m^*, Y_1^*, Y_2^*, \dots, Y_m^*$ of m vertices such that each X_i^* is a clique in G', $G'[X_1^* \cup X_2^* \cup \cdots \cup X_m^*]$ is isomorphic to mK_m, and one of $K_m \boxminus \overline{K_m}$, $K_m \boxminus K_m$, $K_m \boxtimes \overline{K_m}$, and $K_m \boxtimes K_m$ is isomorphic to all $G'[X_i^* \cup Y_i^*]$.

(iii) G' is isomorphic to P_m.

If (i) holds, then by Lemmas 4.6 and 4.7, G' has a vertex-minor isomorphic to P_{3m-1} or P_{4m-5}. So if (i) or (iii) holds, then G has a vertex-minor isomorphic to P_n. If (ii) holds, then by Lemma 4.9, G' has an induced subgraph isomorphic to one of

$$(K_{n+2} \boxminus K_{n+2})_A^{n+2}, (K_{n+2} \boxminus \overline{K_{n+2}})_A^{n+2}, (K_{n+2} \boxtimes K_{n+2})_A^{n+2}, (K_{n+2} \boxtimes \overline{K_{n+2}})_A^{n+2}$$

for some 0-1 matrix $A = \left(\begin{smallmatrix} 0 & b \\ c & d \end{smallmatrix}\right)$. By Lemmas 4.10, 4.11, 4.12, and 4.13, G' has a vertex-minor isomorphic to P_n to $nT_{2,n}$. $\square$

6 Rank-depth and linear rank-width

By Theorem 1.3, for a fixed positive integer n, nP_5-vertex-minor free graphs have bounded depth-2 rank-brittleness, and thus have bounded rank-depth. We will show that they have bounded linear rank-width. Indeed, we will show that graphs of bounded rank-depth have bounded linear rank-width. This was also proved by Ganian, Hliněný, Nešetřil, Obdržálek, and Ossona de Mendez [11, Proposition 3.4] in terms of shrub-depth and linear clique-width, but our proof provides an explicit bound.

First let us review the definition of linear rank-width [10, 15, 23]. For a graph G, an ordering $(x_1, \ldots, x_n)$ of the vertex set $V(G)$ is called a *linear layout* of G. If $|V(G)| \geqslant 2$, then the *width* of a linear layout $(x_1, \ldots, x_n)$ of G is defined as $\max\limits_{1 \leqslant i \leqslant n-1} \rho_G(\{x_1, \ldots, x_i\})$, and if $|V(G)| = 1$, then the width is defined to be 0. The *linear rank-width* of G, denoted by $\mathrm{lrw}(G)$, is defined as the minimum width over all linear layouts of G. It is easy to see that if H is a vertex-minor of G, then $\mathrm{lrw}(H) \leqslant \mathrm{lrw}(G)$.

Proposition 6.1. *For a graph G, $\mathrm{lrw}(G) \leqslant \mathrm{rd}(G)^2$.*

Proof. If G has 1 vertex, then $\mathrm{lrw}(G) = \mathrm{rd}(G) = 0$. So, we may assume that G has at least 2 vertices. Let $k = \mathrm{rd}(G)$, and let (T, σ) be a (k, k)-decomposition of G. Let r be a node of T within distance at most k from every node of T.

Let $v_1, v_2, \ldots, v_m$ be a DFS ordering of T. Let $n = |V(G)|$. Let $w_1, w_2, \ldots, w_n$ be an ordering of the vertices of G such that for all $1 \leqslant i < j \leqslant n$, $\sigma(i)$ appears before $\sigma(j)$ in the DFS ordering $v_1, v_2, \ldots, v_m$ of T. We claim that $(w_1, w_2, \ldots, w_n)$ has width at most k^2. Let $i \in \{1, 2, \ldots, m\}$, $A_i := \{v_1, \ldots, v_i\}$, $B_i := V(T) \backslash A_i$, $A_i' = \{v \in v(G) : \sigma(v) \in A_i\}$, and $B_i' = \{v \in V(G) : \sigma(v) \in B_i\}$. By the property of the depth-first search, T has a path P_i from r consisting of nodes in A_i such that for each node w in B_i, the first vertex in A_i in the path from w to r is on P_i.

As T has radius at most k, we can take P_i to have length at most $k-1$.

For $w \in V(P_i)$, let X_w be the set of all vertices x of G mapped to a node $\sigma(x)$ in B_i such that w is the first vertex in A_i in the path from $\sigma(x)$ to r is on P_i.

Since (T, σ) has width at most k, the cut-rank of X_w is at most k. As $B_i' = \bigcup_{w \in V(P_i)} X_w$, we deduce that $\rho_G(B_i') \leqslant \sum_{w \in V(P_i)} \rho_G(X_w) \leqslant k^2$ by the submodularity of the cut-rank function. This implies that the width of the linear layout is at most k^2. ☐

Corollary 1.4. *For every positive integer n, graphs with no vertex-minors isomorphic to nP_5 have bounded depth-2 rank-brittleness, bounded rank-depth, and bounded linear rank-width.*

Proof. Let $\mathcal{C}$ be the class of nP_5-vertex-minor free graphs. Then $P_{6n} \notin \mathcal{C}$ and $nT_{2,n} \notin \mathcal{C}$. Thus, by Theorem 1.3, $\mathcal{C}$ has bounded depth-2 rank-brittleness, and thus bounded rank-depth. By Proposition 6.1, it also has bounded linear rank-width. ☐

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