# Induced path factors of regular graphs 

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#### Abstract

An induced path factor of a graph $G$ is a set of induced paths in $G$ with the property that every vertex of $G$ is in exactly one of the paths. The induced path number $\rho(G)$ of $G$ is the minimum number of paths in an induced path factor of $G$. We show that if $G$ is a connected cubic graph on $n>6$ vertices, then $\rho(G) \leqslant(n-1) / 3$.

Fix an integer $k \geqslant 3$. For each $n$, define $\mathcal{M}_{n}$ to be the maximum value of $\rho(G)$ over all connected $k$-regular graphs $G$ on $n$ vertices. As $n \rightarrow \infty$ with $n k$ even, we show that $c_{k}=\lim \left(\mathcal{M}_{n} / n\right)$ exists. We prove that $5 / 18 \leqslant c_{3} \leqslant 1 / 3$ and $3 / 7 \leqslant c_{4} \leqslant 1 / 2$ and that $c_{k}=\frac{1}{2}-O\left(k^{-1}\right)$ for $k \rightarrow \infty$.


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## 1 Introduction

We denote the path of order $n$ by $P_{n}$. A subgraph $H$ of a graph $G$ is said to be induced if, for any two vertices $x$ and $y$ of $H, x$ and $y$ are adjacent in $H$ if and only if they are adjacent in $G$. An induced path factor (IPF) of a graph $G$ is a set of induced paths in $G$ with the property that every vertex of $G$ is in exactly one of the paths. We allow paths of any length in an IPF, including the trivial path $P_{1}$. The induced path number $\rho(G)$ of $G$ is defined as the minimum number of paths in an IPF of $G$. The main aim of this paper is to show:

Theorem 1. Suppose that $G$ is a connected cubic graph on $n$ vertices. If $n \leqslant 6$ then $\rho(G)=2$ and if $n>6$ then $\rho(G) \leqslant(n-1) / 3$.

Of course, for disconnected cubic graphs the smallest IPF consists of a minimal IPF of each component. In particular, Theorem 1 immediately implies:

Corollary 2. A cubic graph on $n$ vertices has an IPF with at most $n / 2$ paths. Equality holds if and only if every component is isomorphic to the complete graph $K_{4}$.

Theorem 1 does not generalise to cubic multigraphs. If $n$ is even, then by adding a parallel edge to every second edge of an $n$-cycle we get a connected cubic multigraph with no IPF with fewer than $n / 2$ paths. Theorem 1 also does not generalise to subcubic graphs. To see this, start with an ( $n / 4$ )-cycle and for every vertex $v$ add a triangle which is connected to $v$ by one edge, as in Figure 1. This graph has $n$ vertices but cannot be covered with fewer than $3 n / 8$ paths.

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Figure 1: Graph showing $3 n / 8$ paths may be required for a subcubic graph.

It is not clear whether the $n / 3-O(1)$ bound in Theorem 1 can be improved. However, in $\S 5$ we construct a family of connected cubic graphs $G$ for which $\rho(G) \geqslant 5 n / 18+O(1)$. In the same section we find asymptotic bounds for the maximum value of $\rho(G) / n$ among connected $k$-regular graphs with $n$ vertices, for general $k$.

The concept of induced path number was introduced by Chartrand et al. [3], who gave the induced path numbers of complete bipartite graphs, complete binary trees, 2-dimensional meshes, butterflies and general trees. Broere et al. [2] determined the induced path numbers for complete multipartite graphs. In [2], it was shown that if $G$ is a graph of order $n$, then $\sqrt{n} \leqslant \rho(G)+\rho(\bar{G}) \leqslant\left\lceil\frac{3 n}{2}\right\rceil$, where $\bar{G}$ denotes the complement of $G$. In [6], the best possible upper and lower bounds for $\rho(G) \rho(\bar{G})$ were given for two variants: (i) when both $G$ and $\bar{G}$ are connected and (ii) when neither $G$ nor $\bar{G}$ has isolated vertices. Pan and Chang [11] presented an $O(|V|+|E|)$-time algorithm for finding a minimal IPF on graphs whose blocks are complete graphs. Le et al. [9] proved for general graphs that it is NP-complete to decide if there is an IPF with a given number of paths.

Several variants of induced path numbers have been investigated in the literature. An IPF in which all paths have order at least two is called an induced nontrivial path factor (INPF). In [1] the following was proved:

Theorem 3. If $k$ is a positive integer and $G$ is a connected $k$-regular graph which is not a complete graph of odd order, then $G$ has an INPF.

In addition, [1] showed that every hamiltonian graph which is not a complete graph of odd order admits an INPF. Also, if $G$ is a cubic bipartite graph of order $n \geqslant 6$, then $G$ has an INPF with size at most $n / 3$.

The path cover number $\mu(G)$ of $G$ is defined to be the minimum number of vertex disjoint paths required to cover the vertices of $G$. Reed [12] proved that $\mu(G) \leqslant\left\lceil\frac{n}{9}\right\rceil$ for any cubic graph of order $n$. Also, Reed [12] conjectured that if $G$ is a 2-connected cubic graph, then $\mu(G) \leqslant\left\lceil\frac{n}{10}\right\rceil$. This conjecture was recently proved by Yu [13].

Magnant and Martin [10] investigated the path cover number of regular graphs. They proposed the following interesting conjecture:

Conjecture 4. Let $G$ be a $k$-regular graph of order $n$. Then $\mu(G) \leqslant \frac{n}{k+1}$.
They proved their conjecture for $k \leqslant 5$. Kawarabayashi et al. [8], proved that every 2connected cubic graph of order at least 6 has a path factor in which the order of each path is at least 6 , and hence it has a path cover using only copies of $P_{3}$ and $P_{4}$. A subgraph $H$ of a graph $G$ is spanning if $H$ has the same vertex set as $G$. The minimum leaf number $m l(G)$ of
a connected graph $G$ is the minimum number of leaves among the spanning trees of $G$. In [5] it was shown that $\mu(G)+1 \leqslant m l(G) \leqslant 2 \mu(G)$. It was conjectured that if $G$ is a 2 -connected cubic graph of order $n$, then $m l(G) \leqslant\left\lceil\frac{n}{10}\right\rceil$.

The structure of this paper is as follows. In the next section we define terms and notation and prove some basic lemmas about the effect of simple graph operations on the induced path number. In $\S 3$ we study IPFs in a certain class of subcubic graphs that arise when we use induction to find IPFs for cubic graphs. In $\S 4$ we prove our main result, Theorem 1, drawing on the results in earlier sections. Finally, in $\S 5$ we study asymptotics for $\rho(G)$ where $G$ is a $k$-regular graph of order $n$, with $k$ fixed and $n \rightarrow \infty$.

## 2 Preliminaries

Throughout our paper the following notation and terminology will be used. When we need to specify the vertices in $P_{n}$ we will write it as $\left[v_{1}, v_{2}, \ldots, v_{n}\right]$, meaning that the edges in the path are $v_{1} v_{2}, v_{2} v_{3}, \ldots, v_{n-1} v_{n}$. Similarly, we use $\left(v_{1}, v_{2}, \ldots, v_{n}\right)$ for a cycle of length $n$, with edges $v_{1} v_{2}, v_{2} v_{3}, \ldots, v_{n-1} v_{n}, v_{n} v_{1}$. We denote the vertex set and the edge set of a graph $G$ by $V(G)$ and $E(G)$, respectively. For a graph $G$ and sets $E \subseteq E(G)$ and $V \subseteq V(G)$, we denote by $G-E$ the graph obtained from $G$ by deleting the edges in $E$ and denote by $G-V$ the graph obtained from $G$ by deleting the vertices in $V$ and all the edges incident on them. The degree of a vertex $v$ in $G$ will be denoted $\operatorname{deg}_{G}(v)$. The set of neighbours of $v$ in $G$ will be denoted $N_{G}(v)$. A connected graph $G$ is said to be $k$-connected if it remains connected whenever fewer than $k$ vertices are removed. Similarly, $G$ is $k$-edge-connected if it remains connected whenever fewer than $k$ edges are removed. A bridge in a connected graph is an edge whose removal disconnects the graph. A graph is called $k$-regular if each vertex has degree $k$. A cubic graph is a 3 -regular graph and a subcubic graph is a graph with maximum degree at most 3 . A $k$-factor of a graph is a spanning $k$-regular subgraph of $G$. So a 2-factor of $G$ is a disjoint union of cycles of $G$ which covers all vertices of $G$. A graph is hamiltonian if it has a 2 -factor consisting of a single cycle. For distinct positive integers $a$ and $b$, an $\{a, b\}$-graph is a graph in which the degree of each vertex is $a$ or $b$. The $\{2,3\}$-graphs will play a major role in our proof of Theorem 1. In particular, we will need $K_{4}^{-}$, the graph obtained by removing one edge from the complete graph $K_{4}$. A block of a graph is a maximal 2-connected subgraph. Throughout the paper when we refer to a block we mean a block of order at least 3 . Note that because we will only be concerned with subcubic graphs, their blocks will be vertex disjoint.

While an IPF is formally defined to be a set of paths, an IPF of a graph $G$ can also be completely specified by giving the set of edges of $G$ that are in its paths (vertices incident with no edges in the set are trivial paths). Throughout the paper we will use set operations to build IPFs from IPFs of subgraphs, as well as to remove or add edges. Whenever we do so, the IPFs should be considered to be sets of edges rather than sets of paths. For any IPF $\mathcal{P}$, we use $\#(\mathcal{P})$ to mean the number of paths in $\mathcal{P}$. When calculating $\#(\mathcal{P})$, it is useful to bear in mind that the number of paths in an IPF $\mathcal{P}$ of a graph $G$ is always equal to the order of $G$ minus the number of edges in $\mathcal{P}$. In particular, there is some dependence on $G$, which will often be implicit.

We start with a lemma showing the effect of two basic operations on graphs.

## Lemma 5.

(i) If $G^{\prime}$ is obtained by subdividing an edge of $G$ then $\rho\left(G^{\prime}\right) \geqslant \rho(G)$.
(ii) If $G$ is obtained from disjoint graphs $A$ and $B$ by identifying a vertex of $A$ with a vertex of $B$, then $\rho(G) \geqslant \rho(A)+\rho(B)-1$.

Proof. To show (i), suppose that edge $u v$ of $G$ is subdivided by a new vertex $w$, thereby forming $G^{\prime}$. Let $\mathcal{P}^{\prime}$ be an IPF for $G^{\prime}$. If a path in $\mathcal{P}^{\prime}$ includes both edges $u w$ and $v w$ then replacing those edges with the edge $u v$ gives an $\operatorname{IPF} \mathcal{P}$ for $G$ with $\#(\mathcal{P})=\#\left(\mathcal{P}^{\prime}\right)$. If a path in $\mathcal{P}^{\prime}$ includes exactly one of the edges $u w$ and $v w$ then $w$ is the end of the path, so removing the edge incident with $w$ gives an IPF $\mathcal{P}$ for $G$ with $\#(\mathcal{P})=\#\left(\mathcal{P}^{\prime}\right)$. Lastly, if $\mathcal{P}^{\prime}$ includes neither of the edges $u w$ and $v w$ then $[w]$ is a trivial path in $\mathcal{P}^{\prime}$. Remove $[w]$ from $\mathcal{P}^{\prime}$ to get a set of paths in $G$. If each of these $\#\left(\mathcal{P}^{\prime}\right)-1$ paths is induced, we are done. The only way one of the paths can be not induced is if it includes both $u$ and $v$. In that case, deleting one of the edges on the path between $u$ and $v$ creates an IPF of $G$ with $\#\left(\mathcal{P}^{\prime}\right)$ paths in it. In all cases, we have succeeded in finding an IPF of $G$ that has at most \#( $\left.\mathcal{P}^{\prime}\right)$ paths.

We next turn to (ii). Suppose $u \in V(A)$ and $v \in V(B)$ and that $G$ is formed by identifying $u$ with $v$ (for clarity, we will call the merged vertex $w$ ). Let $\mathcal{P}$ be an IPF for $G$ with $\#(\mathcal{P})=\rho(G)$. Then $\mathcal{P}$ induces IPFs $\mathcal{P}_{A}$ and $\mathcal{P}_{B}$ for $A$ and $B$ respectively. The path in $\mathcal{P}$ that contains $w$ contributes one path to $\mathcal{P}_{A}$ and one path to $\mathcal{P}_{B}$. However, every other path in $\mathcal{P}$ is wholly within $A$ or within $B$. It follows that $\#(\mathcal{P})=\#\left(\mathcal{P}_{A}\right)+\#\left(\mathcal{P}_{B}\right)-1 \geqslant \rho(A)+\rho(B)-1$, and we are done.

We remark that in both parts of Lemma 5 equality often holds but strict inequality is possible. For (i), take edges $e_{1}, e_{2}, e_{3}$ that form a 1-factor in $K_{6}$. Let $G=K_{6}-\left\{e_{1}, e_{2}\right\}$ and form $G^{\prime}$ by subdividing $e_{3}$. Then $\rho(G)=2$ but $\rho\left(G^{\prime}\right)=3$. For (ii), take $A=\left[a_{1}, a_{2}, a_{3}\right]$ and $B=\left[b_{1}, b_{2}, b_{3}\right]$ and merge $a_{2}$ with $b_{2}$ to form $G$. In this case, $\rho(A)=\rho(B)=1$ but $\rho(G)=3$.

We now introduce the notion of a well-behaved IPF. This definition is designed for a subsequent application where we will need to ensure that an IPF of a subcubic graph $H$ is also an IPF of a cubic graph $G$ that is formed by adding certain edges to $H$.

Definition 6. Let $G$ be a subcubic graph, let $S=\left\{x \in V(G): \operatorname{deg}_{G}(x) \leqslant 2\right\}$, and let $\mathcal{P}$ be an IPF of $G$. We say that $\mathcal{P}$ is well-behaved (in $G$ ) if, for each path $P$ of $\mathcal{P}$, we have that either
(i) $V(P) \cap S$ is a subset of the vertices of a single block of $G$; or
(ii) $P$ contains a subpath $\left[x, x^{\prime}, y^{\prime}, y\right]$, where $V(P) \cap S=\{x, y\}$ and $x^{\prime} y^{\prime}$ is a bridge of $G$.

If the above definition holds with $S$ replaced by $\left\{x \in V(G): \operatorname{deg}_{G}(x) \leqslant 2\right\} \backslash R$ for some set of vertices $R$, then we say that $\mathcal{P}$ is well-behaved except on $R$.

When we say that an IPF is well behaved except on some set $R$, this does not imply anything about whether the IPF is or is not well-behaved in the graph overall. The remainder of this section will be devoted to proving the following lemma, which describes several surgeries that we will perform on IPFs.

Lemma 7. Let $G$ and $G^{\prime}$ be subcubic graphs.
(i) Suppose $G^{\prime}$ is obtained from $G$ by taking a triangle in $G$ on vertex set $\{a, b, c\}$ such that $\operatorname{deg}_{G}(a)=\operatorname{deg}_{G}(b)=3$ and $\operatorname{deg}_{G}(c)=2$, subdividing ab with a new vertex d, and adding the edge cd. If $G^{\prime}$ has an IPF $\mathcal{P}^{\prime}$, then there is an IPF $\mathcal{P}$ of $G$ such that $\#(\mathcal{P}) \leqslant \#\left(\mathcal{P}^{\prime}\right)$ and a path of $\mathcal{P}$ ends at $c$. Furthermore, $\mathcal{P} \subseteq\left(\mathcal{P}^{\prime} \backslash\{b c\}\right) \cup\{a c\}$ and, if $\mathcal{P}^{\prime}$ is well behaved, then $\mathcal{P}$ is well-behaved except on $\{c\}$.
(ii) Suppose $G^{\prime}$ is obtained from $G$ by deleting an edge ab in $G$ such that $\operatorname{deg}_{G}(a)=\operatorname{deg}_{G}(b)=$ 2 , and then adding new vertices $\{c, d\}$ and edges $\{a c, a d, b c, b d, c d\}$. If $G^{\prime}$ has an IPF $\mathcal{P}^{\prime}$, then there is an IPF $\mathcal{P}$ of $G$ such that $\#(\mathcal{P}) \leqslant \#\left(\mathcal{P}^{\prime}\right)$ and two distinct paths of $\mathcal{P}$ end at a and b. Furthermore, $\mathcal{P} \subseteq \mathcal{P}^{\prime}$ and, if $\mathcal{P}^{\prime}$ is well behaved, then $\mathcal{P}$ is well-behaved except on $\{a, b\}$.
(iii) Suppose $G^{\prime}$ is obtained from $G$ by deleting a degree 2 vertex $c$ in $G$ such that $N_{G}(c)=\{a, b\}$ and $a b \notin E(G)$, and then adding the edge ab. If $G^{\prime}$ has an IPF $\mathcal{P}^{\prime}$, then there is an IPF $\mathcal{P}$ of $G$ such that $\#(\mathcal{P}) \leqslant \#\left(\mathcal{P}^{\prime}\right)+1$ and a path of $\mathcal{P}$ ends at $c$. Furthermore, $\mathcal{P} \subseteq \mathcal{P}^{\prime} \cup\{a c\}$ and, if $\mathcal{P}^{\prime}$ is well behaved, then $\mathcal{P}$ is well-behaved except on $\{c\}$.

If hypothesis (i) holds in Lemma 7, then we say that $G^{\prime}$ is obtained from $G$ by augmenting the triangle on vertex set $\{a, b, c\}$. If hypothesis (ii) holds, then we say that $G^{\prime}$ is obtained from $G$ by pasting a $K_{4}^{-}$over $a b$. If hypothesis (iii) holds, then we say that $G^{\prime}$ is obtained from $G$ by suppressing the vertex $c$.

In order to prove Lemma 7, we will require a definition and a further lemma. Both of these are used only in the proof of Lemma 7.

Definition 8. Let $G$ be a subcubic graph, let $\mathcal{P}$ be an $\operatorname{IPF}$ of $G$, and let $(a, b, c, d)$ be a quadruple of vertices of $G$ that induce a $K_{4}^{-}$subgraph that does not contain the edge $a b$. We say that $\mathcal{P}$ is standardised for $(a, b, c, d)$ if $c$ is an endpoint of a path in $\mathcal{P}$ that includes the edge $a c$, and $d$ is an endpoint of a path in $\mathcal{P}$ that includes the edge $b d$ (note that the two paths must be distinct).

Lemma 9. Let $G$ be a subcubic graph and let $(a, b, c, d)$ be vertices of $G$ that induce a $K_{4}^{-}$ subgraph that does not contain the edge ab. If there is an IPF $\mathcal{P}$ of $G$ then there is an IPF $\mathcal{P}^{*}$ of $G$ such that $\#\left(\mathcal{P}^{*}\right) \leqslant \#(\mathcal{P})$ and $\mathcal{P}^{*}$ is standardised for $(a, b, c, d)$. Moreover, $\mathcal{P}^{*} \subseteq \mathcal{P} \cup\{a c, b d\}$ and if $\mathcal{P}$ is well-behaved then so is $\mathcal{P}^{*}$.

Proof. Suppose that $\mathcal{P}$ is not standardised for $(a, b, c, d)$. Let $H$ be the subgraph induced by $\{a, b, c, d\}$. First suppose that either $a$ and $b$ are in distinct paths in $\mathcal{P}$ or they are both in a path that also includes $c$ or $d$. Either way, $\mathcal{P}^{*}=(\mathcal{P} \backslash E(H)) \cup\{a c, b d\}$ is an IPF of $G$ with $\#\left(\mathcal{P}^{*}\right) \leqslant \#(\mathcal{P})$.

Otherwise $a$ and $b$ are both in a path that includes neither $c$ nor $d$ and hence there is a vertex $e$ in $G-V(H)$ that is adjacent to $b$ in $G$. Then $\mathcal{P}^{*}=(\mathcal{P} \backslash(E(H) \cup\{b e\})) \cup\{a c, b d\}$ is an IPF of $G$ with $\#\left(\mathcal{P}^{*}\right) \leqslant \#(\mathcal{P})$.

In each case, note that $\mathcal{P}^{*}$ is standardised for $(a, b, c, d)$ and $\mathcal{P}^{*} \subseteq \mathcal{P} \cup\{a c, b d\}$. It remains to justify the claim that $\mathcal{P}^{*}$ inherits the well-behaved property from $\mathcal{P}$. This follows from the observation that every path $P$ in $\mathcal{P}^{*}$ has a subpath $P^{\prime}$ that includes all vertices of $P$ that have degree 2 in $G$, and is such that $P^{\prime}$ is itself a subpath of a path in $\mathcal{P}$.

We are now ready to prove Lemma 7. It will be useful to note that by Menger's theorem and the definition of a block, two distinct vertices are in the same block in a graph $G$ if and only if there is a cycle in $G$ containing both of them.

Proof of Lemma 7. Let $G$ and $G^{\prime}$ be graphs such that the hypothesis of (i), (ii) or (iii) holds. We will say we are in case (i), (ii) or (iii) accordingly. Let $\mathcal{P}^{\prime}$ be an IPF of $G^{\prime}$. In cases (i) and (ii), let $\mathcal{P}^{*}$ be an IPF of $G^{\prime}$ such that $\#\left(\mathcal{P}^{*}\right) \leqslant \#\left(\mathcal{P}^{\prime}\right), \mathcal{P}^{*}$ is standardised for $(a, b, c, d)$, $\mathcal{P}^{*} \subseteq \mathcal{P}^{\prime} \cup\{a c, b d\}$, and $\mathcal{P}^{*}$ is well-behaved if $\mathcal{P}^{\prime}$ is. Such a $\mathcal{P}^{*}$ exists by Lemma 9 . In case (iii), let $\mathcal{P}^{*}=\mathcal{P}^{\prime}$.

In case (i), let $\mathcal{P}=\mathcal{P}^{*} \backslash\{b d\}$. Because $\mathcal{P}^{*}$ is standardised for $(a, b, c, d)$, it contains $a c$ and $b d$ but not $a d, b c$ or $c d$. Thus, $\#(\mathcal{P})=\#\left(\mathcal{P}^{*}\right)$, a path of $\mathcal{P}$ ends at $c$, and $\mathcal{P} \subseteq\left(\mathcal{P}^{\prime} \backslash\{b c\}\right) \cup\{a c\}$. In case (ii), let $\mathcal{P}=\mathcal{P}^{*} \backslash\{a c, b d\}$. Similarly, because $\mathcal{P}^{*}$ is standardised for $(a, b, c, d), \mathcal{P}^{*}$ contains $a c$ and $b d$ but not $a d, b c$ or $c d$. Thus, $\#(\mathcal{P})=\#\left(\mathcal{P}^{*}\right)$, two distinct paths of $\mathcal{P}$ end at $a$ and $b$, and $\mathcal{P} \subseteq \mathcal{P}^{\prime}$. In case (iii), let

$$
\mathcal{P}= \begin{cases}\mathcal{P}^{*} & \text { if } a b \notin \mathcal{P}^{*} \\ \left(\mathcal{P}^{*} \backslash\{a b\}\right) \cup\{a c\} & \text { if } a b \in \mathcal{P}^{*}\end{cases}
$$

So a (possibly trivial) path of $\mathcal{P}$ ends at $c$, and $\mathcal{P} \subseteq \mathcal{P}^{\prime} \cup\{a c\}$. Also, $\mathcal{P}$ and $\mathcal{P}^{*}$ have the same number of edges, but $G$ has one more vertex than $G^{\prime}$, so it follows that $\#(\mathcal{P})=\#\left(\mathcal{P}^{*}\right)+1$.

Now further suppose that $\mathcal{P}^{\prime}$ is well-behaved and let $R=\{c\}$ in cases (i) and (iii) and $R=\{a, b\}$ in case (ii). It remains to show that $\mathcal{P}$ is well-behaved except on $R$. Recall that $\mathcal{P}^{*}$ is well-behaved because $\mathcal{P}^{\prime}$ is. Let $S=\left\{x \in V(G): \operatorname{deg}_{G}(x) \leqslant 2\right\} \backslash R$ and note that $S$ is a subset of $S^{\prime}=\left\{x \in V\left(G^{\prime}\right): \operatorname{deg}_{G^{\prime}}(x) \leqslant 2\right\}$. Let $P$ be a path in $\mathcal{P}$ and let $x$ and $y$ be distinct vertices in $V(P) \cap S$. We will complete the proof by showing that either $x$ and $y$ are in the same block in $G$ or $P$ has a subpath $\left[x, x^{\prime}, y^{\prime}, y\right]$ where $V(P) \cap S=\{x, y\}$ and $x^{\prime} y^{\prime}$ is a bridge in $G$. Given that $x, y \in V(P) \cap S$, it follows in each case from our definition of $\mathcal{P}$ that $x$ and $y$ were in the same path $P^{*}$ in $\mathcal{P}^{*}$. Thus, because $\mathcal{P}^{*}$ was well-behaved and $x, y \in S^{\prime}$, either $x$ and $y$ were in the same block in $G^{\prime}$ or $P^{*}$ has a subpath $\left[x, x^{\prime}, y^{\prime}, y\right]$ where $V\left(P^{*}\right) \cap S=\{x, y\}$ and $x^{\prime} y^{\prime}$ is a bridge in $G^{\prime}$. If the former holds, then $x$ and $y$ are in the same block in $G$ (in each case the existence of a cycle in $G^{\prime}$ containing $x$ and $y$ implies the existence of a cycle in $G$ containing $x$ and $y$ ). So suppose the latter holds. In cases (i) and (ii), by our definition of $\mathcal{P}$, either $P=P^{*}$ or $P$ is obtained from $P^{*}$ by removing an edge in $\{a c, b d\}$. Furthermore, [ $x, x^{\prime}, y^{\prime}, y$ ] must be a subpath of $P$ because $x, y \in S$ and $x^{\prime} y^{\prime}$ is a bridge in $G^{\prime}$. In case (iii), either $P$ is a subpath of $P^{*}$ (possibly with $P=P^{*}$ ) or $P$ is obtained from a subpath of $P^{*}$ by adding the edge $a c$. Furthermore, $\left[x, x^{\prime}, y^{\prime}, y\right]$ must be a subpath of $P$ because if $a b$ were an edge in $\left[x, x^{\prime}, y^{\prime}, y\right]$ we would have the contradiction that $x$ and $y$ were in different paths in $\mathcal{P}$. In each case, the fact that $x^{\prime} y^{\prime}$ is a bridge in $G^{\prime}$ implies it is a bridge in $G$. This establishes that $\mathcal{P}$ is well-behaved except on $R$.

## 3 Induced path factors of $\{2,3\}$-graphs

A triangle ring is a graph formed by taking an $n$-cycle $\left(x_{1}, \ldots, x_{n}\right)$ and adding the chords $\left\{x_{n} x_{2}, x_{3} x_{5}, x_{6} x_{8}, \ldots, x_{n-3} x_{n-1}\right\}$ for some integer $n \geqslant 6$ such that $n \equiv 0(\bmod 3)$.

Further we say a graph is bad if it can be obtained from a triangle ring by choosing some (possibly empty) set $S$ of its edges such that no edge in $S$ is in a triangle, and for each edge $e \in S$ proceeding as follows: subdivide $e$ with a vertex $x_{e}$, add a vertex disjoint copy of any hamiltonian $\{2,3\}$-graph $H_{e}$ of order 5 , and add an edge between $x_{e}$ and a degree 2 vertex of $H_{e}$.

Note that every bad graph has order divisible by 3. We refer to the largest block of a bad graph as its hub. The hub of a bad graph has order at least 6 and each of its other blocks has order 5. A fact that will prove useful throughout this section is that a graph cannot be bad if it contains a vertex of degree 2 that is in a block of order at least 6 but is not in a triangle.

The main result of this section is the following.

Theorem 10. Let $G$ be a connected $\{2,3\}$-graph of order $n \geqslant 7$ containing a 2 -factor whose cycles each have length at least 5 . Then $\rho(G) \leqslant n / 3$ if $G$ is a bad graph and $\rho(G) \leqslant(n-1) / 3$ otherwise.

The example in Figure 1 shows that the condition about the existence of the 2 -factor cannot be dropped from Theorem 10. Also, note that no cubic graph is bad and hence Theorem 10 establishes that any cubic graph $G$ on at least 7 vertices with an appropriate 2 -factor has $\rho(G) \leqslant(n-1) / 3$.

Our strategy for building an IPF of a cubic graph $G$ will be to identify a 2-factor $F$ in $G$, and to discard some (but not all) of the edges that join distinct cycles in $F$ so that each cycle in $F$ induces a block. We then stitch together IPFs of these blocks. To be efficient we need to allow some paths to include vertices from more than one block. When this happens the edges that we initially discarded could potentially cause our paths to not be induced in $G$. However, by demanding that the constituent IPFs are well-behaved, we will be able to circumvent this concern.

Theorem 10 will follow with only a little work from Lemma 11 below. Most of our effort in this section will be devoted to proving Lemma 11. The proof proceeds by induction on the number of blocks in $G$, but we will first require a number of preliminary results.

Lemma 11. Let $G$ be a connected $\{2,3\}$-graph of order $n \geqslant 6$ such that each block of $G$ is a hamiltonian graph of order at least 5 and the vertex sets of these blocks partition $V(G)$. Then $G$ has a well-behaved IPF with at most $(n-1) / 3$ paths if $n \geqslant 7$ and $G$ is not a bad graph, and $G$ has a well-behaved IPF with at most $n / 3$ paths otherwise.

We begin with three lemmas on IPFs of small hamiltonian $\{2,3\}$-graphs.
Lemma 12. Let $C$ be a hamiltonian $\{2,3\}$-graph of order 5 . For any vertex $x$ of degree 2 in $C$, there is an IPF of $C$ with two paths such that one path ends at $x$ and every other vertex of this path has degree 3 in $C$.

Proof. Let $C^{\prime}$ be a hamilton cycle in $C$. For our first path, we take a shortest path from $x$ around $C^{\prime}$ that includes one vertex of each chord of $C^{\prime}$. The second path also follows $C^{\prime}$, and joins the vertices not appearing in the first path.

Lemma 13. Let $C$ be a hamiltonian $\{2,3\}$-graph of order 6 . Then $C$ has an IPF with two paths. Furthermore, for any vertex $x$ of degree 2 in $C$, there is an IPF of $C$ with two paths such that one path ends at $x$ and any other vertices on this path have degree 3 in $C$ with the possible exception of the vertex adjacent to $x$ in the path.

Proof. If $C$ is cubic, then it is easy to find an IPF with two paths in each of the two possible cases for $C$. If $C$ has a vertex of degree 2 , then we use exactly the same strategy articulated in the proof of Lemma 12.

Lemma 14. Let $C$ be a hamiltonian $\{2,3\}$-graph of order 7 . Let $p=3$ if $C$ is obtained from a triangle ring of order 6 by subdividing an edge that is not in a triangle, and let $p=2$ otherwise. For any vertex $x$ of degree 2 in $C$, there is an IPF of $C$ with $p$ paths such that one path ends at $x$.

Proof. Let $\left(x, x_{1}, x_{2}, \ldots, x_{6}\right)$ be a hamilton cycle in $C$. If $\left\{x_{1} x_{3}, x_{4} x_{6}, x_{2} x_{5}\right\} \subseteq E(C)$, then we may use $\left\{\left[x, x_{1}, x_{2}, x_{5}\right],\left[x_{3}, x_{4}, x_{6}\right]\right\}$ as our IPF. Otherwise, if $\left\{x_{1} x_{3}, x_{4} x_{6}\right\} \subseteq E(C)$, then $p=3$ and we may use $\left\{\left[x, x_{1}\right],\left[x_{2}, x_{3}, x_{4}\right],\left[x_{5}, x_{6}\right]\right\}$ as our IPF. If only one of the edges $x_{1} x_{3}$ and $x_{4} x_{6}$ is in $E(C)$, then by symmetry we may assume it is $x_{1} x_{3}$. In that case, we may take $\left\{\left[x, x_{1}, x_{2}\right],\left[x_{3}, x_{4}, x_{5}, x_{6}\right]\right\}$ as our IPF. Finally, if $\left\{x_{1} x_{3}, x_{4} x_{6}\right\} \cap E(C)=\varnothing$, then we may use $\left\{\left[x, x_{1}, x_{2}, x_{3}\right],\left[x_{4}, x_{5}, x_{6}\right]\right\}$ as our IPF.

From Lemmas 12, 13 and 14, we can easily prove the following result concerning small $\{2,3\}$-graphs with two blocks.

Lemma 15. Let $G$ be a $\{2,3\}$-graph of order $n \leqslant 12$ consisting of two hamiltonian blocks, each of order at least 5 , with a bridge between them. Then either $G$ has a well-behaved IPF with three paths or $G$ is a bad graph of order 12 with a well-behaved IPF consisting of 4 paths.

Proof. Let the two hamiltonian blocks of $G$ be $G_{1}$ and $G_{2}$. Let $n_{i}=\left|V\left(G_{i}\right)\right|$ for $i \in\{1,2\}$, and assume $n_{1} \geqslant n_{2}$ without loss of generality. Then $\left(n_{1}, n_{2}\right) \in\{(5,5),(6,5),(7,5),(6,6)\}$. Let $x_{1} x_{2}$ be the bridge in $G$ where $x_{i} \in V\left(G_{i}\right)$ for $i \in\{1,2\}$. For $i \in\{1,2\}$ use Lemma 12,13 or 14 as appropriate to create an IPF $\mathcal{P}_{i}$ of $G_{i}$ with one path ending at $x_{i}$. Then $\mathcal{P}=\mathcal{P}_{1} \cup \mathcal{P}_{2} \cup\left\{x_{1} x_{2}\right\}$ is an IPF of $G$. If $\#\left(\mathcal{P}_{1}\right)=3$ then $G$ is a bad graph, $\left(n_{1}, n_{2}\right)=(7,5)$ and $\#(\mathcal{P})=4$. In all other cases, $\#(\mathcal{P})=3$. If $n_{2}=5$, then Lemma 12 ensures that each path in $\mathcal{P}$ obeys (i) in the definition of well-behaved. If $\left(n_{1}, n_{2}\right)=(6,6)$, then Lemma 13 ensures that each path in $\mathcal{P}$ obeys either (i) or (ii) in the definition of well-behaved.

We now prove a more general result for hamiltonian $\{2,3\}$-graphs. Note that triangle rings are the only bad hamiltonian graphs.

Lemma 16. A hamiltonian $\{2,3\}$-graph $G$ of order $n \geqslant 6$ has $\rho(G) \leqslant(n-1) / 3$ if $n \geqslant 7$ and $G$ is not a bad graph, and has $\rho(G) \leqslant n / 3$ otherwise.

Proof. If $n=6$, the result follows from Lemma 13, so assume $n \geqslant 7$. Let $F$ be a hamilton cycle in $G$. If $G=F$ the result follows easily, so assume $F$ is a proper subgraph of $G$. We can label $F$ as $\left(x_{1}, \ldots, x_{n}\right)$ such that $x_{n} x_{k}$ is a shortest chord of $F$ in $G$ where $k \in\{2, \ldots,\lfloor n / 2\rfloor\}$. If $k=2$, we can further assume that $x_{1}$ is not adjacent in $G$ to any vertex in $\left\{x_{3}, \ldots, x_{\lfloor(n+1) / 2\rfloor}\right\}$ (if this is not satisfied, reassign the labels $x_{2}, \ldots, x_{n}$ in the opposite orientation around $F$ ). If $k=3$, we can further assume that $x_{1} x_{4} \notin E(G)$. (If $x_{1} x_{4} \in E(G)$ but $x_{2} x_{5} \notin E(G)$, then rotate the labels by one position around $F$. If $\left\{x_{1} x_{4}, x_{2} x_{5}\right\} \subseteq E(G)$ then, noting $x_{3} x_{6} \notin E(G)$, rotate the labels by two positions.)

For $k \geqslant 2$, we now construct an IPF $\mathcal{P}$ of $G$ using a greedy algorithm. We add paths one at a time, at each stage taking a path $\left[x_{i}, x_{i+1}, \ldots, x_{j}\right]$ such that $i$ is the smallest element of $\{1, \ldots, n\}$ for which $x_{i}$ is not already in a path, and $j$ is the largest element of $\{i, \ldots, n\}$ such that $\left[x_{i}, x_{i+1}, \ldots, x_{j}\right]$ is induced in $G$. We will establish the following:
(i) the first path added to $\mathcal{P}$ has at least 4 vertices and it has exactly 4 if and only if $x_{a} x_{5} \in E(G)$ for some $a \in\{1,2,3\} ;$
(ii) if $\#(\mathcal{P}) \geqslant 3$, then the final path added to $\mathcal{P}$ has at least 2 vertices;
(iii) other than the first and last paths added, each path $\left[x_{i}, x_{i+1} \ldots, x_{j}\right]$ in $\mathcal{P}$ has at least 3 vertices and has exactly 3 if and only if $x_{i+1} x_{i+3} \in E(G)$.
The properties of our labelling $\left(x_{1}, \ldots, x_{n}\right)$ ensure that (i) holds (recall in particular that $x_{n} x_{k}$ is a shortest chord of $F$ ). That (iii) is satisfied follows from the fact that, by our greedy algorithm,
for each path $\left[x_{i}, \ldots, x_{j}\right]$ in $\mathcal{P}$, there is a chord from $x_{i}$ to a vertex in the path added just prior to $\left[x_{i}, \ldots, x_{j}\right]$. Similarly, because $x_{n} x_{k} \in E(G)$ and $x_{k}$ is in the first path added to $\mathcal{P}$, there is not a path $\left[x_{i}, x_{i+1}, \ldots, x_{n-1}\right]$ in $\mathcal{P}$ for any $i \in\left\{2, \ldots, x_{n-1}\right\}$ and (ii) follows.

From (i), (ii) and (iii) we see immediately that $\#(\mathcal{P}) \leqslant n / 3$. If $\#(\mathcal{P}) \leqslant(n-1) / 3$ or if $G$ is a triangle ring, then the proof is complete, so assume that $\#(\mathcal{P})=n / 3$ and $G$ is not a triangle ring. In this remaining case we give an alternative construction for an IPF $\mathcal{P}^{\prime \prime}$ of $G$ that satisfies the conditions of the lemma. Because $\#(\mathcal{P})=n / 3$ and $n \geqslant 7$, it must be that $\#(\mathcal{P}) \geqslant 3$ and that the first path added to $\mathcal{P}$ has exactly 4 vertices, the final path has exactly 2 vertices and each other path has exactly 3 vertices. So, by (iii), $\left\{x_{6} x_{8}, x_{9} x_{11} \ldots, x_{n-3} x_{n-1}\right\} \subseteq E(G)$. Thus $k=2$ because $x_{n} x_{k}$ is a shortest chord of $F$ in $G$. Then, from (i) and the properties of our labelling $\left(x_{1}, \ldots, x_{n}\right)$, we have that $x_{n} x_{2}, x_{3} x_{5} \in E(G)$. This establishes that a triangle ring is a subgraph of $G$ (note that the labelling given in the definition of triangle ring matches our labelling of $G$ ). By assumption $G$ is not a triangle ring and so there must be an edge $x_{a} x_{b}$ in $E(G)$ where $a, b \in\{1,4,7, \ldots, n-2\}$.

If $n=9$ then without loss of generality $a=1, b=4$ and we may take $\mathcal{P}^{\prime \prime}=\left\{\left[x_{8}, x_{9}, x_{1}, x_{4}\right]\right.$, $\left.\left[x_{2}, x_{3}, x_{5}, x_{6}, x_{7}\right]\right\}$ as our IPF. Henceforth we may assume that $n \geqslant 12$. Note that $\mathcal{P}^{\prime}=$ $E(F) \backslash\left\{x_{i} x_{i+1}: i \in\{1,4,7, \ldots, n-2\}\right\}$ is an IPF of $G$ with $\#\left(\mathcal{P}^{\prime}\right)=n / 3$. Let $\mathcal{P}^{\prime \prime}$ be obtained from $\mathcal{P}^{\prime}$ by removing the edges $\left\{x_{a-1} x_{a}, x_{b-1} x_{b}\right\}$ and adding the edges $\left\{x_{a-1} x_{a+1}, x_{b-1} x_{b+1}, x_{a} x_{b}\right\}$ where we consider subscripts modulo $n$. As $n>9$, it can be seen that $\mathcal{P}^{\prime \prime}$ is an IPF of $G$ with $\#\left(\mathcal{P}^{\prime \prime}\right)=\#\left(\mathcal{P}^{\prime}\right)-1=(n-3) / 3$. This completes the proof.

We require two more lemmas before we can complete our proof of Lemma 11. Both concern the structure of a putative minimal counterexample. Note that by Lemma 16 we know such a counterexample has at least two blocks, and hence contains a bridge.

Lemma 17. Let $G$ be a counterexample to Lemma 11 with the minimum number of blocks. Let $x_{1} x_{2}$ be a bridge in $G$ and let $G_{1}$ and $G_{2}$ be the components of $G-\left\{x_{1} x_{2}\right\}$. Then either
(i) $\left|V\left(G_{1}\right)\right|=5$ or $\left|V\left(G_{2}\right)\right|=5$; or
(ii) for each $i \in\{1,2\}$, either $\left|V\left(G_{i}\right)\right|=6$ or $G_{i}$ is a bad graph.

Proof. Let $n_{i}=\left|V\left(G_{i}\right)\right|$ for $i \in\{1,2\}$, and suppose for a contradiction that neither (i) nor (ii) holds. Then, without loss of generality, $n_{1} \geqslant 7, G_{1}$ is not a bad graph, and $n_{2} \geqslant 6$. By induction there is a well-behaved IPF $\mathcal{P}_{1}$ of $G_{1}$ with $\#\left(\mathcal{P}_{1}\right) \leqslant\left(n_{1}-1\right) / 3$ and a well-behaved IPF $\mathcal{P}_{2}$ of $G_{2}$ with $\#\left(\mathcal{P}_{2}\right) \leqslant n_{2} / 3$. Then $\mathcal{P}=\mathcal{P}_{1} \cup \mathcal{P}_{2}$ is a well-behaved IPF of $G$ with $\#(\mathcal{P}) \leqslant\left(n_{1}+n_{2}-1\right) / 3$, contradicting our assumption that $G$ is a counterexample to Lemma 11.

Lemma 18. Let $G$ be a counterexample to Lemma 11 with the minimum number of blocks. Let $x_{1} x_{2}$ be a bridge in $G$ and let $G_{1}$ and $G_{2}$ be the components of $G-\left\{x_{1} x_{2}\right\}$. Then either $G_{1}$ or $G_{2}$ is a block of order 5 .

Proof. Let $n_{i}=\left|V\left(G_{i}\right)\right|$ for $i \in\{1,2\}$, and suppose for a contradiction that $n_{1}, n_{2} \geqslant 6$. Say $x_{1} \in G_{1}$ and $x_{2} \in G_{2}$. By Lemma 17, for $i \in\{1,2\}$, either $n_{i}=6$ or $G_{i}$ is a bad graph and $n_{i} \geqslant 9$. Lemma 15 eliminates the possibility that $n_{1}=n_{2}=6$. So we may assume without loss of generality that $n_{1} \neq 6$ and hence $G_{1}$ is a bad graph and $n_{1} \geqslant 9$.

Suppose that one of $x_{1}$ or $x_{2}$ is in a block $C$ of order 5 . If $n_{2}=6$, then $G_{2}$ is a block of order 6 and hence it must be $x_{1}$ that is in $C$. Also, we may suppose without loss of generality that it is $x_{1}$ that is in $C$ if $G_{2}$ is a bad graph and $n_{2} \geqslant 9$. Let $y z$ be the bridge of $G$ such that $y$ is in $C$ and $z$ is in the hub $H_{1}$ of $G_{1}$. Let $G_{1}^{\prime}$ and $G_{2}^{\prime}$ be the components of $G-\{y z\}$ where
$V\left(H_{1}\right) \subseteq V\left(G_{1}^{\prime}\right)$. Observe that $H_{1}$ is a block of $G_{1},\left|V\left(H_{1}\right)\right| \geqslant 7, \operatorname{deg}_{G_{1}^{\prime}}(z)=2$ and $z$ is not in a triangle in $H_{1}$. Thus, $\left|V\left(G_{1}^{\prime}\right)\right| \geqslant 7$ and $G_{1}^{\prime}$ is not bad. Clearly $\left|V\left(G_{2}^{\prime}\right)\right| \geqslant n_{2}+5 \geqslant 11$. Thus $y z$ violates Lemma 17.

From the argument above we may assume that $x_{1}$ is in the hub $H_{1}$ of $G_{1}$ and hence $x_{1}$ is in a triangle in $H_{1}$. Furthermore, if $n_{2} \neq 6$ then $x_{2}$ is in a triangle in the hub of $G_{2}$. Because $x_{1}$ is in a triangle in $H_{1}, H_{1}-\left\{x_{1}\right\}$ is hamiltonian, so by induction there is a well-behaved IPF $\mathcal{P}_{1}$ of $G_{1}-\left\{x_{1}\right\}$ with $\#\left(\mathcal{P}_{1}\right) \leqslant\left(n_{1}-3\right) / 3\left(\right.$ recall $\left.n_{1} \equiv 0(\bmod 3)\right)$. If $n_{2} \neq 6$, there is a well-behaved IPF $\mathcal{P}_{2}$ of $G_{2}-\left\{x_{2}\right\}$ with $\#\left(\mathcal{P}_{2}\right) \leqslant\left(n_{2}-3\right) / 3$ by a similar argument. If $n_{2}=6$, use Lemma 13 to take an IPF $\mathcal{P}_{2}$ of $G_{2}$ with two paths, one of which ends at $x_{2}$. In either case, $\mathcal{P}=\mathcal{P}_{1} \cup \mathcal{P}_{2} \cup\left\{x_{1} x_{2}\right\}$ is a well-behaved IPF of $G$. If $n_{2} \neq 6, \#(\mathcal{P})=\#\left(\mathcal{P}_{1}\right)+\#\left(\mathcal{P}_{2}\right)+1 \leqslant$ $\left(n_{1}+n_{2}-3\right) / 3$. If $n_{2}=6, \#(\mathcal{P})=\#\left(\mathcal{P}_{1}\right)+2 \leqslant\left(n_{1}+n_{2}-3\right) / 3$. This contradicts our assumption that $G$ is a counterexample to Lemma 11.

Proof of Lemma 11. Suppose for a contradiction that $G$ is a counterexample to Lemma 11 with the minimum number of blocks, and let $n=|V(G)|$. Lemma 16 establishes that Lemma 11 holds when $G$ is hamiltonian, so $G$ has at least two blocks. Thus, if $n \leqslant 12, G$ must have exactly two blocks and Lemma 15 establishes that Lemma 11 holds. So we may further assume that $n \geqslant 13$.

It follows from Lemma 18 and the hypotheses of Lemma 11 that $G$ consists of a number $t \geqslant 1$ of hamiltonian blocks $L_{1}, \ldots, L_{t}$ of order 5 , one other hamiltonian block $C$ of order at least 5, and bridges $x_{1} y_{1}, \ldots, x_{t} y_{t}$ where $x_{1}, \ldots, x_{t} \in V(C)$ and $y_{i} \in V\left(L_{i}\right)$ for $i \in\{1, \ldots, t\}$. The proof splits into four cases according to the placement of the vertices $x_{1}, \ldots, x_{t}$ in $C$. In each case we will construct an IPF $\mathcal{P}$ of $G$ that contradicts our assumption that $G$ is a counterexample to Lemma 11.

Case 1. Suppose that there are $i, j \in\{1, \ldots, t\}$ such that $x_{i} x_{j} \in E(C)$. Then $n \geqslant 15$. Without loss of generality, $i=1$ and $j=2$. Let $G_{0}=G-\left(V\left(L_{1}\right) \cup V\left(L_{2}\right)\right)$. Let $G_{0}^{\prime}$ be the $\{2,3\}$-graph of order $n-8 \geqslant 7$ obtained from $G_{0}$ by pasting a $K_{4}^{-}$over $x_{1} x_{2}$, and note that the block $C_{0}^{\prime}$ of $G_{0}^{\prime}$ with $x_{1}, x_{2} \in V\left(C_{0}^{\prime}\right)$ is hamiltonian. So, by induction, there is a well-behaved IPF $\mathcal{P}_{0}^{\prime}$ of $G_{0}^{\prime}$ with $\#\left(\mathcal{P}_{0}^{\prime}\right) \leqslant(n-8) / 3$. By applying Lemma $7($ ii $)$ to $\mathcal{P}_{0}^{\prime}$ we obtain an IPF $\mathcal{P}_{0}$ of $G_{0}$ with $\#\left(\mathcal{P}_{0}\right) \leqslant(n-8) / 3$ that has paths ending at $x_{1}$ and $x_{2}$ and is well-behaved except on $\left\{x_{1}, x_{2}\right\}$. Use Lemma 12 to take IPFs $\mathcal{P}_{1}$ and $\mathcal{P}_{2}$ of $L_{1}$ and $L_{2}$, each with two paths, where one path of $\mathcal{P}_{1}$ ends at $y_{1}$ and one path of $\mathcal{P}_{2}$ ends at $y_{2}$. Then $\mathcal{P}=\mathcal{P}_{0} \cup \mathcal{P}_{1} \cup \mathcal{P}_{2} \cup\left\{x_{1} y_{1}, x_{2} y_{2}\right\}$ is a well-behaved IPF of $G$ with $\#(\mathcal{P})=\#\left(\mathcal{P}_{0}\right)+2 \leqslant(n-2) / 3$.

Case 2. Suppose that we are not in Case 1 and that $|V(C)|=5$. Then $t=2$, because $n \geqslant 13$ implies $t \geqslant 2$ and we would necessarily be in Case 1 if $t \geqslant 3$. So $n=15$. Without loss of generality, let $\left(x_{1}, u, x_{2}, v, w\right)$ be a hamilton cycle in $C$. Use Lemma 12 to take IPFs $\mathcal{P}_{1}$ and $\mathcal{P}_{2}$ of $L_{1}$ and $L_{2}$, each with two paths, where one path of $\mathcal{P}_{1}$ ends at $y_{1}$ and one path of $\mathcal{P}_{2}$ ends at $y_{2}$. Then $\mathcal{P}=\mathcal{P}_{1} \cup \mathcal{P}_{2} \cup\left\{x_{1} y_{1}, x_{2} y_{2}, u x_{1}, u x_{2}, v w\right\}$ is a well-behaved IPF of $G$ with $\#(\mathcal{P})=4$.

Case 3. Suppose that we are not in Case 1 or 2 and that $x_{1}$ is in a triangle of $C$. Because we are not in Case 1 or $2,|V(C)| \geqslant 6$. Let $G_{0}=G-\left(V\left(L_{1}\right) \cup\left\{x_{1}\right\}\right)$, and note $\left|V\left(G_{0}\right)\right|=$ $n-6 \geqslant 7$. Note that the block $C_{0}$ of $G_{0}$ with vertex set $V(C) \backslash\left\{x_{1}\right\}$ has $\left|V\left(C_{0}\right)\right| \geqslant 5$ and is hamiltonian because $x_{1}$ is in a triangle of $C$. Also, $G_{0}$ is not bad because $C_{0}$ contains two degree 2 vertices that are not in triangles. So by induction there is a well-behaved IPF $\mathcal{P}_{0}$ of $G_{0}$ with $\#\left(\mathcal{P}_{0}\right) \leqslant(n-7) / 3$ paths. Use Lemma 12 to take an IPF $\mathcal{P}_{1}$ of $L_{1}$ with two paths such that one path ends at $y_{1}$. Then $\mathcal{P}=\mathcal{P}_{0} \cup \mathcal{P}_{1} \cup\left\{x_{1} y_{1}\right\}$ is a well-behaved IPF of $G$ and
$\#(\mathcal{P})=\#\left(\mathcal{P}_{0}\right)+2 \leqslant(n-1) / 3$.
Case 4. Suppose that we are not in Case 1, 2 or 3 . Then $|V(C)| \geqslant 6$ and $x_{1}$ is not in a triangle in $C$. Let $G_{0}=G-V\left(L_{1}\right)$, and let $G_{0}^{\prime}$ be the graph obtained from $G_{0}$ by suppressing vertex $x_{1}$. Note that $\left|V\left(G_{0}^{\prime}\right)\right|=n-6 \geqslant 7$ and that the block $C_{0}^{\prime}$ of $G_{0}^{\prime}$ with vertex set $V(C) \backslash\left\{x_{1}\right\}$ has $\left|V\left(C_{0}^{\prime}\right)\right| \geqslant 5$ and is hamiltonian (since hamiltonicity is preserved by suppressing a vertex of degree 2). So, by induction, $G_{0}^{\prime}$ has a well-behaved IPF $\mathcal{P}_{0}^{\prime}$ with $\#\left(\mathcal{P}_{0}^{\prime}\right) \leqslant(n-6-\delta) / 3$ paths, where $\delta=0$ if $G_{0}^{\prime}$ is bad and $\delta=1$ otherwise. By applying Lemma 7 (iii) to $\mathcal{P}_{0}^{\prime}$ we obtain an IPF $\mathcal{P}_{0}$ of $G_{0}$ with $\#\left(\mathcal{P}_{0}\right) \leqslant \#\left(\mathcal{P}_{0}^{\prime}\right)+1 \leqslant(n-3-\delta) / 3$ that has a path ending at $x_{1}$ and is wellbehaved except on $\left\{x_{1}\right\}$. Use Lemma 12 to take an IPF $\mathcal{P}_{1}$ of $L_{1}$ with two paths such that one path ends at $y_{1}$. Then $\mathcal{P}=\mathcal{P}_{0} \cup \mathcal{P}_{1} \cup\left\{x_{1} y_{1}\right\}$ is an IPF of $G$ with $\#(\mathcal{P})=\#\left(\mathcal{P}_{0}\right)+1 \leqslant(n-\delta) / 3$. If $G$ is bad or $G_{0}^{\prime}$ is not bad then we are done. So we may assume that $G_{0}^{\prime}$ is bad and $G$ is not bad.

As $G_{0}^{\prime}$ is bad, it must have a hub and that can only be $C_{0}^{\prime}$, since every other block of $G_{0}^{\prime}$ has order 5. So $C_{0}^{\prime}$ is obtained from a triangle ring by subdividing some set of edges not in triangles. Note that $G$ is obtained from $G_{0}^{\prime}$ by subdividing some edge $u v$ with the vertex $x_{1}$ and adding $L_{1}$ and the bridge $x_{1} y_{1}$. So $u v$ is in a triangle in $C_{0}^{\prime}$, since otherwise $G$ is bad or we are in the situation handled by Case 1. Each triangle in $C_{0}^{\prime}$ has two edges in the unique hamilton cycle in $C_{0}^{\prime}$ and one edge not in it. We consider two cases according to which kind of edge $u v$ is.

If $u v$ is not in the hamilton cycle in $C_{0}^{\prime}$, then $C-\left\{x_{1}\right\}$ has order at least 6 and is hamiltonian. Also, $G-\left(V\left(L_{1}\right) \cup\left\{x_{1}\right\}\right)$ has $n-6 \geqslant 7$ vertices and is not bad, so by induction it has a wellbehaved IPF $\mathcal{P}_{2}$ with $\#\left(\mathcal{P}_{2}\right) \leqslant(n-7) / 3$. Now $\mathcal{P}_{2} \cup \mathcal{P}_{1} \cup\left\{x_{1} y_{1}\right\}$ is a well-behaved IPF with at most $2+(n-7) / 3=(n-1) / 3$ paths, as required.

If $u v$ is in the hamilton cycle in $C_{0}^{\prime}$, we can suppose without loss of generality that $\operatorname{deg}_{G_{0}^{\prime}}(u)=$ 3 and $\operatorname{deg}_{G_{0}^{\prime}}(v)=2$. Then $C-\left\{x_{1}, v\right\}$ has order at least 5 and is hamiltonian. Also, $G-\left(V\left(L_{1}\right) \cup\right.$ $\left.\left\{x_{1}, v\right\}\right)$ has $n-7 \geqslant 6$ vertices, so by induction it has a well-behaved IPF $\mathcal{P}_{2}$ with $\#\left(\mathcal{P}_{2}\right) \leqslant$ $(n-7) / 3$. Now $\mathcal{P}_{2} \cup \mathcal{P}_{1} \cup\left\{v x_{1}, x_{1} y_{1}\right\}$ is a well-behaved IPF with at most $2+(n-7) / 3=(n-1) / 3$ paths, as required.

Proof of Theorem 10. If $G$ satisfies the hypotheses of Lemma 11, then we can apply it to complete the proof, so assume otherwise. Of all the 2 -factors of $G$ whose cycles each have length at least 5 , let $F$ be one with the minimum number of cycles. Our first goal will be to obtain a graph $G^{*}$ from $G$ by deleting edges between cycles of $F$ such that $G^{*}$ satisfies the hypotheses of Lemma 11 and is not a bad graph.

Let $S$ be the set of edges of $G$ that are incident with vertices in two distinct cycles of $F$ and let $S^{\prime}$ be a maximal subset of $S$ such that $G-S^{\prime}$ is connected. For each cycle $A$ of $F$ the graph $G-S^{\prime}$ has a hamiltonian block with vertex set $V(A)$. Note that $S^{\prime}$ is nonempty because $G$ does not satisfy the hypotheses of Lemma 11. If $G-S^{\prime}$ is not a bad graph, let $S^{*}=S^{\prime}$ and $G^{*}=G-S^{*}$. Otherwise $G-S^{\prime}$ is bad and we proceed as follows. Choose an arbitrary edge $u v \in S^{\prime}$ and note that without loss of generality $u$ is in a block $L$ of order 5 in $G-S^{\prime}$ and either $v$ is in a different block of order 5 in $G-S^{\prime}$ or $v$ is in a triangle in the hub of $G-S^{\prime}$. Let $S^{*}=\left(S^{\prime} \backslash\{u v\}\right) \cup\{w x\}$ where $w x$ is the unique bridge in $G-S^{\prime}$ with $w$ in the hub of $G-S^{\prime}$ and $x \in V(L)$. Let $G^{*}=G-S^{*}$ and note that $G^{*}$ is not a bad graph because, in $G^{*}, w$ is a vertex of degree 2 that is in a block of order at least 6 but not in a triangle.

By Lemma 11, there is a well-behaved IPF $\mathcal{P}$ of $G^{*}$ with at most $(n-1) / 3$ paths. We will show that $\mathcal{P}$ is also an IPF of $G$ and so complete the proof. Suppose otherwise that there is
an edge $y z \in S^{*}$ such that $y$ and $z$ are both vertices in the same path of $\mathcal{P}$. Note that, in $G^{*}$, $y$ and $z$ are vertices of degree 2 and are in different blocks. Hence, since $\mathcal{P}$ is well-behaved in $G^{*}$, it must be that $G^{*}$ contains a bridge $y^{\prime} z^{\prime}$ such that $y y^{\prime}, z z^{\prime} \in E\left(G^{*}\right)$. Since $y$ and $z$ are vertices of degree 2 in $G^{*}$, yy $\in E\left(F_{1}\right)$ and $z z^{\prime} \in E\left(F_{2}\right)$ for different cycles $F_{1}$ and $F_{2}$ of $F$. However, then the 2-factor obtained from $F$ by replacing $F_{1}$ and $F_{2}$ with a single cycle with edge set $\left(E\left(F_{1}\right) \cup E\left(F_{2}\right) \cup\left\{y z, y^{\prime} z^{\prime}\right\}\right) \backslash\left\{y y^{\prime}, z z^{\prime}\right\}$ contradicts our choice of $F$.

## 4 Induced path factors of cubic graphs

In the previous section we saw that Theorem 1 holds for any cubic graph containing a 2 -factor whose cycles all have length at least 5. Jackson and Yoshimoto [7] showed that any 3-connected cubic graph on at least 6 vertices has such a 2 -factor. In this section we establish Theorem 1 via contradiction by showing that a minimal counterexample to it must be 3 -connected. Recall that for any subcubic graph the connectivity and edge-connectivity are equal.

Lemma 19. A counterexample to Theorem 1 of minimum order is 2 -connected.
Proof. Aiming for a contradiction, suppose that $G$ is a counterexample to Theorem 1 of minimum order and that $x_{1} x_{2}$ is a bridge in $G$. For $i \in\{1,2\}$, let $G_{i}$ be the component of $G-\left\{x_{1} x_{2}\right\}$ containing $x_{i}$ and let $n_{i}=\left|V\left(G_{i}\right)\right|$. Then $|V(G)|=n_{1}+n_{2}$ and, because $G$ is cubic, $n_{i}$ is odd and at least 5 for $i \in\{1,2\}$.

Let $i \in\{1,2\}$. We claim that $G_{i}$ has an IPF $\mathcal{P}_{i}$ such that $\#\left(\mathcal{P}_{i}\right) \leqslant\left(n_{i}+1\right) / 3$ and one path ends at $x_{i}$. If $n_{i} \in\{5,7\}$, then it is not hard to see that $G_{i}$ is hamiltonian (note that $G_{i}$ can be obtained from a cubic graph of order $n_{i}-1$ by subdividing an edge) and so our claim follows by Lemma 12 or Lemma 14. So we may assume that $n_{i} \geqslant 9$. Let $G_{i}^{\prime}$ be the cubic graph obtained from $G_{i}$ by suppressing $x_{i}$ if it is not in a triangle in $G_{i}$ and augmenting the triangle of $G_{i}$ containing $x_{i}$ otherwise. Let $t=1$ if $x_{i}$ is in a triangle in $G_{i}$ and let $t=0$ otherwise. Then $\left|V\left(G_{i}^{\prime}\right)\right|=n_{i}-1+2 t$ and hence $8 \leqslant\left|V\left(G_{i}^{\prime}\right)\right| \leqslant n_{1}+n_{2}-4$. So, by induction, there is an IPF $\mathcal{P}_{i}^{\prime}$ of $G_{i}^{\prime}$ with $\#\left(\mathcal{P}_{i}^{\prime}\right) \leqslant\left(n_{i}-2+2 t\right) / 3$. Thus our claim holds by applying Lemma $7(\mathrm{i})$ to $\mathcal{P}_{i}^{\prime}$ if $t=1$ and Lemma 7 (iii) to $\mathcal{P}_{i}^{\prime}$ if $t=0$.

Then $\mathcal{P}=\mathcal{P}_{1} \cup \mathcal{P}_{2} \cup\left\{x_{1} x_{2}\right\}$ is an IPF of $G$ and $\#(\mathcal{P})=\#\left(\mathcal{P}_{1}\right)+\#\left(\mathcal{P}_{2}\right)-1 \leqslant\left(n_{1}+n_{2}-1\right) / 3$. This contradicts our assumption that $G$ is a counterexample to Theorem 1.

Next we dispose of a particular configuration that would otherwise cause us problems later.
Lemma 20. A counterexample to Theorem 1 of minimum order does not contain a copy $G_{1}$ of $K_{4}^{-}$such that the two vertices of degree 2 in $G-V\left(G_{1}\right)$ are nonadjacent in $G$.

Proof. Aiming for a contradiction, suppose that $G$ is a counterexample to Theorem 1 of minimum order that contains a copy $G_{1}$ of $K_{4}^{-}$such that the two vertices of degree 2 in $G-V\left(G_{1}\right)$ are nonadjacent in $G$. Let $n=|V(G)|$. By Lemmas 16 and $19, G$ is nonhamiltonian and bridgeless (note that a cubic graph cannot be bad). So $n \geqslant 14$, since the only bridgeless nonhamiltonian cubic graphs with 12 or fewer vertices are the Petersen graph and the Tietze graph (see Figure 2) and neither of these contains a copy of $K_{4}^{-}$.

Let $G_{0}=G-V\left(G_{1}\right)$. Let $x_{0} x_{1}$ and $y_{0} y_{1}$ be the two edges of $G$ such that $x_{0}, y_{0} \in V\left(G_{0}\right)$ and $x_{1}, y_{1} \in V\left(G_{1}\right)$, and note that $x_{0} y_{0} \notin E(G)$ by assumption. Let $G_{0}^{\prime}$ be the graph obtained from $G_{0}$ by suppressing $x_{0}$ if it is not in a triangle in $G_{0}$ and augmenting the triangle of $G_{0}$ containing $x_{0}$ otherwise. In turn, let $G_{0}^{\prime \prime}$ be the cubic graph obtained from $G_{0}^{\prime}$ by suppressing


Figure 2: The Petersen graph and the Tietze graph
$y_{0}$ if it is not in a triangle in $G_{0}^{\prime}$ and augmenting the triangle of $G_{0}^{\prime}$ containing $y_{0}$ otherwise. Let $t^{\prime}$ (respectively $t^{\prime \prime}$ ) be 1 if $x_{0}$ (respectively $y_{0}$ ) is in a triangle in $G_{0}$ and 0 otherwise. Let $t=t^{\prime}+t^{\prime \prime}$ and note that $\left|V\left(G_{0}^{\prime \prime}\right)\right|=n-6+2 t$ and hence $8 \leqslant\left|V\left(G_{0}^{\prime \prime}\right)\right| \leqslant n-2$. So, by induction, there is an IPF $\mathcal{P}_{0}^{\prime \prime}$ of $G_{0}^{\prime \prime}$ with $\#\left(\mathcal{P}_{0}^{\prime \prime}\right) \leqslant(n-7+2 t) / 3$. By applying Lemma 7 to $\mathcal{P}_{0}^{\prime \prime}$ (part (i) if $t^{\prime \prime}=1$ and part (iii) if $t^{\prime \prime}=0$ ) with $c$ chosen to be $y_{0}$, we can obtain an IPF $\mathcal{P}_{0}^{\prime}$ of $G_{0}^{\prime}$ such that a path, $P^{\prime}$ say, of $\mathcal{P}_{0}^{\prime}$ ends at $y_{0}$ and $\#\left(\mathcal{P}_{0}^{\prime}\right) \leqslant \#\left(\mathcal{P}_{0}^{\prime \prime}\right)+1-t^{\prime \prime}$. Let $u$ and $v$ be the neighbours of $x_{0}$ in $G_{0}^{\prime}$ where, without loss of generality, either $u$ is not in $P^{\prime}$ or both $u$ and $v$ are in $P^{\prime}$ and the subpath of $P^{\prime}$ from $y_{0}$ to $v$ does not include $u$. Next we apply Lemma 7 to $\mathcal{P}_{0}^{\prime}$ (part (i) if $t^{\prime}=1$ and part (iii) if $t^{\prime}=0$ ) with $c$ chosen to be $x_{0}$ and $a$ chosen to be $u$. This produces an IPF $\mathcal{P}_{0}$ of $G_{0}$ such that a path of $\mathcal{P}_{0}$ ends at $x_{0}, \mathcal{P}_{0} \subseteq\left(\mathcal{P}_{0}^{\prime} \backslash\left\{x_{0} v\right\}\right) \cup\left\{x_{0} u\right\}$ (note that $x_{0} v \notin E\left(G_{0}^{\prime}\right)$ if $\left.t^{\prime}=0\right)$, and

$$
\#\left(\mathcal{P}_{0}\right) \leqslant \#\left(\mathcal{P}_{0}^{\prime}\right)+1-t^{\prime} \leqslant \#\left(\mathcal{P}_{0}^{\prime \prime}\right)+2-t \leqslant(n-1-t) / 3 \leqslant(n-1) / 3
$$

Furthermore, the fact that $\mathcal{P}_{0} \subseteq\left(\mathcal{P}_{0}^{\prime} \backslash\left\{x_{0} v\right\}\right) \cup\left\{x_{0} u\right\}$ implies there is a path $P$ of $\mathcal{P}_{0}$ such that $P$ ends at $y_{0}, E(P) \subseteq E\left(P^{\prime}\right) \cup\left\{x_{0} u\right\}$ and $P$ does not contain the edge $x_{0} v$. Hence $\left\{x_{0}, v\right\} \nsubseteq V(P)$ and, given our choice of $u$ and $v$, we have $x_{0} \notin V(P)$. So distinct paths of $\mathcal{P}_{0}$ end at $x_{0}$ and $y_{0}$.

Let $\mathcal{P}_{1}$ be an IPF of $G_{1}$ with two paths such that one ends at $x_{1}$ and the other ends at $y_{1}$. Then $\mathcal{P}=\mathcal{P}_{0} \cup \mathcal{P}_{1} \cup\left\{x_{0} x_{1}, y_{0} y_{1}\right\}$ is an IPF of $G$ with $\#(\mathcal{P})=\#\left(\mathcal{P}_{0}\right) \leqslant(n-1) / 3$, contradicting our assumption that $G$ is a counterexample to Theorem 1.

We are now ready to prove the connectivity result we want.
Lemma 21. A counterexample to Theorem 1 of minimum order is 3-connected.
Proof. Aiming for a contradiction, suppose that $G$ is a counterexample to Theorem 1 of minimum order and that $G$ is not 3 -connected. By Lemma 19, $G$ is bridgeless. However, by assumption, there are two edges $e$ and $f$ whose removal disconnects $G$. Note that $e$ and $f$ are independent since $G$ is cubic and bridgeless. Thus $G$ is the union of graphs $G_{1}, G_{2}$ and $H$ (see Figure 3) where

- $V\left(G_{1}\right) \cap V\left(G_{2}\right)=\varnothing$;
- there are vertices $x_{1}, y_{1}, x_{2}, y_{2}$ such that, for $i \in\{1,2\}, V\left(G_{i}\right) \cap V(H)=\left\{x_{i}, y_{i}\right\}$ and $x_{i} y_{i} \notin E\left(G_{i}\right) ;$
- for some positive integer $s, H$ is the vertex disjoint union of two paths $\left[x_{1}=u_{0}, \ldots, u_{s}=\right.$ $\left.x_{2}\right]$ and $\left[y_{1}=v_{0}, \ldots, v_{s}=y_{2}\right]$ and a (possibly empty) matching with edge set $\left\{u_{i} v_{i}: 1 \leqslant\right.$ $i \leqslant s-1\} ;$
To find this decomposition, we initially take the two paths that define $H$ to be the edges $e$ and $f$, but then extend these paths in both directions until their respective endpoints are not


Figure 3: Structure of a bridgeless graph when the removal of 2 edges disconnects it.
adjacent. For $i \in\{1,2\}$, let $n_{i}=\left|V\left(G_{i}\right)\right|$ and note that because $G$ is cubic $n_{i} \geqslant 4$ and $n_{i}$ is even. Note that $|V(G)|=n_{1}+n_{2}+2 s-2$.

Let $i \in\{1,2\}$. We claim that we can find an IPF $\mathcal{P}_{i}$ of $G_{i}$ such that $\#\left(\mathcal{P}_{i}\right) \leqslant\left(n_{i}+2\right) / 3$ and either $\#\left(\mathcal{P}_{i}\right) \leqslant\left(n_{i}-1\right) / 3$ or two distinct paths of $\mathcal{P}_{i}$ end at $x_{i}$ and $y_{i}$. If $n_{i} \in\{4,6\}$, then $G_{i}$ must be $K_{4}^{-}$or one of the three graphs that can be formed by removing an edge from a cubic graph on 6 vertices. In each case it is easy to find an IPF of $G_{i}$ with two paths where one ends at $x_{i}$ and the other ends at $y_{i}$. If $n_{i} \geqslant 8$, then by induction $G_{i}+\left\{x_{i} y_{i}\right\}$ has an IPF $\mathcal{P}^{\prime}$ with $\#\left(\mathcal{P}^{\prime}\right) \leqslant\left(n_{i}-1\right) / 3$. We can see that $\mathcal{P}=\mathcal{P}^{\prime} \backslash\left\{x_{i} y_{i}\right\}$ is an IPF of $G_{i}$ that satisfies the condition of our claim by considering two cases according to whether $x_{i} y_{i} \in \mathcal{P}^{\prime}$.

If, for $i \in\{1,2\}$, we have that two distinct paths of $\mathcal{P}_{i}$ end at $x_{i}$ and $y_{i}$, then $\mathcal{P}=$ $\mathcal{P}_{1} \cup \mathcal{P}_{2} \cup E\left(\left[u_{0}, \ldots, u_{s}\right]\right) \cup E\left(\left[v_{0}, \ldots, v_{s}\right]\right)$ is an IPF of $G$ with $\#(\mathcal{P})=\#\left(\mathcal{P}_{1}\right)+\#\left(\mathcal{P}_{2}\right)-2 \leqslant$ $\left(n_{1}+n_{2}-2\right) / 3$ and $G$ is not a counterexample to Theorem 1. So we may assume without loss of generality that it is not the case that two distinct paths of $\mathcal{P}_{1}$ end at $x_{1}$ and $y_{1}$ and hence that $\#\left(\mathcal{P}_{1}\right) \leqslant\left(n_{1}-1\right) / 3$.

If $s \in\{1,2\}$ and $\#\left(\mathcal{P}_{2}\right) \leqslant\left(n_{2}-1\right) / 3$, then

$$
\mathcal{P}= \begin{cases}\mathcal{P}_{1} \cup \mathcal{P}_{2} & \text { if } s=1 \\ \mathcal{P}_{1} \cup \mathcal{P}_{2} \cup\left\{u_{1} v_{1}\right\} & \text { if } s=2\end{cases}
$$

is an IPF of $G$ with $\#(\mathcal{P})=\#\left(\mathcal{P}_{1}\right)+\#\left(\mathcal{P}_{2}\right)+s-1 \leqslant\left(n_{1}+n_{2}+3 s-5\right) / 3$ and $G$ is not a counterexample to Theorem 1.

So we may further assume that either two distinct paths of $\mathcal{P}_{2}$ end at $x_{2}$ and $y_{2}$ or $\#\left(\mathcal{P}_{2}\right) \leqslant$ $\left(n_{2}-1\right) / 3$ and $s \geqslant 3$. In the former case, let $\mathcal{P}_{2}^{*}=\mathcal{P}_{2}$. In the latter case let $\mathcal{P}_{2}^{*}$ be obtained from $\mathcal{P}_{2}$ by, for $z \in\left\{x_{2}, y_{2}\right\}$, if two edges of $\mathcal{P}_{2}$ are incident with $z$, deleting one of them. In either case $\mathcal{P}_{2}^{*}$ is an IPF of $G_{2}$ such that two distinct paths of $\mathcal{P}_{2}$ end at $x_{2}$ and $y_{2}$ and it can be checked that $\#\left(\mathcal{P}_{2}^{*}\right) \leqslant\left(n_{2}+2 s\right) / 3$.

Let $G_{1}^{\prime \prime}$ be the cubic graph of order $n_{1}+4$ obtained from $G_{1}$ by adding the vertices $\left\{u_{1}, v_{1}\right\}$ and edges $\left\{u_{0} u_{1}, v_{0} v_{1}, u_{1} v_{1}\right\}$, and then pasting a copy $C$ of $K_{4}^{-}$over $u_{1} v_{1}$. By Lemma 20, we can assume that $s \geqslant 2$ if $n_{2}=4$ and so $G_{1}^{\prime \prime}$ has fewer vertices than $G$. So, by induction $G_{1}^{\prime \prime}$, has an IPF $\mathcal{P}_{1}^{\prime \prime}$ with $\#\left(\mathcal{P}_{1}^{\prime \prime}\right) \leqslant\left(n_{1}+3\right) / 3$ paths. By applying Lemma $7(\mathrm{ii})$ to $\mathcal{P}_{1}^{\prime \prime}$ we can obtain an IPF $\mathcal{P}_{1}^{*}$ of the subgraph of $G$ induced by $V\left(G_{1}\right) \cup\left\{u_{1}, v_{1}\right\}$ such that $\#\left(\mathcal{P}_{1}^{*}\right) \leqslant\left(n_{1}+3\right) / 3$ and two distinct paths of $\mathcal{P}_{1}$ end at $u_{1}$ and $v_{1}$.

Then $\mathcal{P}=\mathcal{P}_{1}^{*} \cup \mathcal{P}_{2}^{*} \cup E\left(\left[u_{1}, \ldots, u_{s}\right]\right) \cup E\left(\left[v_{1}, \ldots, v_{s}\right]\right)$ is an IPF $\mathcal{P}$ of $G$ with

$$
\#(\mathcal{P}) \leqslant \#\left(\mathcal{P}_{1}^{*}\right)+\#\left(\mathcal{P}_{2}^{*}\right)-2=\left(n_{1}+n_{2}+2 s-3\right) / 3
$$

This contradicts our assumption that $G$ is a counterexample to Theorem 1.
Proof of Theorem 1. Let $G$ be a counterexample to Theorem 1 of minimal order $n$. By Lemma 21, $G$ is 3 -connected. So, by the theorem of Jackson and Yoshimoto [7], $G$ has a 2 factor whose cycles all have length at least 5 . Theorem 10 then implies that $\rho(G) \leqslant(n-1) / 3$ because $G$ is cubic and so cannot be bad. Hence $G$ is not a counterexample to Theorem 1 after all, and this completes the proof of the theorem.

## 5 Induced path factors of regular graphs

In this section, for a fixed integer $k \geqslant 2$, we consider asymptotics for $\rho(G)$ for $k$-regular graphs $G$ of order $n \rightarrow \infty$. For each $n$, define $\mathcal{M}_{n}$ to be the maximum, over all connected $k$-regular graphs $G$ on $n$ vertices, of $\rho(G)$. (If no such graphs exist then we do not consider such $n$ in what follows.) Define

$$
\begin{aligned}
& \bar{c}_{k}=\limsup _{n \rightarrow \infty} \frac{\mathcal{M}_{n}}{n} \\
& \underline{c}_{k}=\liminf _{n \rightarrow \infty} \frac{\mathcal{M}_{n}}{n} \\
& c_{k}=\lim _{n \rightarrow \infty} \frac{\mathcal{M}_{n}}{n} .
\end{aligned}
$$

Our aim for this section is to find bounds for $c_{k}$. However, first we must show that it is well defined.

Lemma 22. For each integer $k \geqslant 2, c_{k}$ exists.
Proof. First we consider the case when $k$ is even. The case $k=2$ is trivial since $\underline{c}_{2}=c_{2}=$ $\bar{c}_{2}=0$ because every connected 2-regular graph has an IPF with 2 paths. So we may assume that $k \geqslant 4$.

Fix $\varepsilon>0$. As $k$ is constant, the definition of $\bar{c}_{k}$ implies that for some suitably large $n_{1}$ there exists a $k$-regular graph $G$ of order $n_{1}$ such that $\rho(G) \geqslant\left(\bar{c}_{k}-\varepsilon\right)\left(n_{1}+k+4\right)+1$.

By the Erdős-Gallai Theorem [4] there exist graphs $F_{1}$ and $F_{2}$ of respective orders $k+2$ and $k+3$, with one vertex of degree $k-2$ and all other vertices of degree $k$. By subdividing an edge of $G$ with a new vertex that we then identify with the vertex of degree $k-2$ in $F_{1}$, we create a $k$-regular graph $G_{1}$ with $a=n_{1}+k+2$ vertices. We use $F_{2}$ in a similar fashion to create a $k$-regular graph $G_{2}$ with $a+1$ vertices. By Lemma $5, \rho\left(G_{i}\right) \geqslant \rho(G) \geqslant\left(\bar{c}_{k}-\varepsilon\right)(a+2)+1$ for $i \in\{1,2\}$.

Now, for any sufficiently large $n$, we make a $k$-regular graph $G^{\prime}$ of order $n$ as follows. Let $c$ be the least positive integer satisfying $(k-2) c \equiv(k-2) n+2(\bmod k a-2 a+2)$ and let

$$
b=\frac{(k-2)(n-c)+2}{k a-2 a+2}-c .
$$

Note that $\operatorname{gcd}(k-2, k a-2 a+2)=\operatorname{gcd}(k-2,2)=2$ which divides $(k-2) n+2$ so $c$ exists. Also, our choice of $c$ ensures that $b$ is an integer, and $b>0$ because $n$ is large. We start with $b$ copies of $G_{1}$ and $c$ copies of $G_{2}$ and progressively glue these components together to form $G^{\prime}$. Each gluing step takes $k / 2$ existing components, subdivides one edge in each component with a new vertex and identifies these new vertices. The number of gluing steps is $(b+c-1) /(k / 2-1)$
since each step reduces the number of components by $k / 2-1$. The number of vertices in the resulting graph $G^{\prime}$ is

$$
\begin{equation*}
b a+c(a+1)+\frac{b+c-1}{k / 2-1}=(b+c)(a+2 /(k-2))+c-2 /(k-2)=n . \tag{1}
\end{equation*}
$$

Also, we started with $b+c$ components and glued them together, so by Lemma 5 ,

$$
\rho\left(G^{\prime}\right) \geqslant b \rho\left(G_{1}\right)+c \rho\left(G_{2}\right)-(b+c) \geqslant\left(\bar{c}_{k}-\varepsilon\right)(b+c)(a+2) \geqslant\left(\bar{c}_{k}-\varepsilon\right) n
$$

where the last inequality follows from (1). As $n$ was an arbitrary large integer, it follows that $\underline{c}_{k} \geqslant \bar{c}_{k}-\varepsilon$. But $\varepsilon$ was an arbitrary positive quantity, so we must have $\underline{c}_{k}=\bar{c}_{k}$, from which it follows that the limit $c_{k}$ exists.

It remains to consider the case when $k$ is odd. It works similarly, but is complicated by the fact that $k$-regular graphs only exist for even orders. For some large even integer $a$, we make $G_{1}$ and $G_{2}$ of orders $a$ and $a+2$ with $\rho\left(G_{i}\right) \geqslant\left(\bar{c}_{k}-\varepsilon\right)(a+4)+1$ for $i \in\{1,2\}$. Our gluing steps each involve $k-1$ components. Two new adjacent vertices are introduced and $(k-1) / 2$ components are glued on each of these two vertices. This reduces the number of components by $k-2$, so we want $b, c$ to be solutions to

$$
b a+c(a+2)+2 \frac{b+c-1}{k-2}=n .
$$

We can take $c$ to be the least positive solution to $2(k-2) c \equiv(k-2) n+2(\bmod k a-2 a+2)$ and let

$$
b=\frac{(k-2)(n-2 c)+2}{k a-2 a+2}-c .
$$

Note that $\operatorname{gcd}(2(k-2), k a-2 a+2)=2$ which divides $(k-2) n+2$ so $c$ exists. The remainder of the argument mimics the case for even $k$.

As mentioned in the proof of Lemma 22, $c_{2}=0$. For larger $k$ it seems to be a difficult problem to find the exact value of $c_{k}$, so instead we look for bounds. Of course, $c_{3} \leqslant 1 / 3$ by Theorem 1. We will show that $1 / 3<c_{k} \leqslant 1 / 2$ for all $k>3$ and that $c_{k} \rightarrow 1 / 2$ as $k \rightarrow \infty$. Note that $c_{k} \leqslant 1 / 2$ for all $k$, by Theorem 3 .

In trees all paths are induced. In several of our subsequent results we use constructions based on perfect $(k-1)$-ary trees. The root of a $(k-1)$-ary tree has degree $k-1$, while all other vertices have degree $k$ or degree 1 (in the latter case the vertex is a leaf). A $(k-1)$-ary tree is perfect if all its leaves are at the same distance from the root. In that case the distance from the root to a leaf is called the height. We refer to the distance of a vertex from the root as its depth. The unique neighbour of a vertex that has smaller depth than it is its parent; its other neighbours have greater depth than it and are its children.

Our constructions will also often create graphs that contain blocks that are copies of a complete graph $K_{m}$ with one edge subdivided. An IPF of such a block has at least $\lceil m / 2\rceil$ paths, by Lemma 5 .

Lemma 23. Let $k \geqslant 3$ and let $T$ be a perfect $(k-1)$-ary tree of height $h$. Then $\rho(T)=$ $\frac{1}{k}\left((k-1)^{h+1}+(-1)^{h}\right)$.
Proof. Consider a minimal IPF for $T$. We first argue that without loss of generality, no path ends at a non-leaf vertex. Suppose this is not true for a particular IPF. Locate the vertex $v$ of
least depth at which a path ends. Since $k \geqslant 3$ there is some child $w$ of $v$ which is not in the path that ends at $v$. We add the edge $v w$. By the minimality of our IPF, it must include edges $w x$ and $w x^{\prime}$ where $x$ and $x^{\prime}$ are children of $w$. Remove the edge $w x$. In this way we create another minimal IPF, and we have reduced the number of paths that end at the depth of $v$. So by repeating this process we can move all ends of paths to the leaves.

Let $a(h)$ be the minimum number of disjoint paths needed to cover a perfect ( $k-1$ )-ary tree of height $h$. By the above, we assume that all paths end at leaves. So by removing the vertices on the path through the root, we obtain the recurrence

$$
a(h)=1+(k-3) a(h-1)+2(k-2) \sum_{i=0}^{h-2} a(i)
$$

with initial condition $a(0)=1$. We now show that $a(h)=\frac{1}{k}\left((k-1)^{h+1}+(-1)^{h}\right)$ by induction on $h$. The formula works for $h=0$. Assuming that it works up to $h-1$, we find that

$$
\begin{aligned}
a(h) & =1+\frac{1}{k}(k-3)\left((k-1)^{h}+(-1)^{h-1}\right)+\frac{2}{k}(k-2) \sum_{i=0}^{h-2}\left((k-1)^{i+1}+(-1)^{i}\right) \\
& =\frac{1}{k}\left(k+(k-3)\left((k-1)^{h}+(-1)^{h-1}\right)+2\left((k-1)^{h}-(k-1)\right)+(2 k-4) \chi_{h}\right) \\
& =\frac{1}{k}\left(k+(k-1)^{h+1}+(-1)^{h-1}(k-3)-2 k+2+(2 k-4) \chi_{h}\right) \\
& =\frac{1}{k}\left((k-1)^{h+1}+(-1)^{h}\right),
\end{aligned}
$$

where $\chi_{h}=0$ if $h$ is odd and $\chi_{h}=1$ if $h$ is even. The result follows.
We are now ready to give lower bounds on $c_{k}$ for general $k$. We treat the cases of odd and even $k$ separately. For each case, we will construct a family of graphs that are $k$-regular except that the root vertex has degree less than $k$. It is a simple matter to obtain a $k$-regular graph by adding a fixed gadget to the root. Doing so changes the induced path number by $O(1)$, which will be insignificant for our asymptotics. Hence, for simplicity, we omit details of the gadgets and pretend that the graphs we build are in fact $k$-regular.

Theorem 24. We have $c_{3} \geqslant 5 / 18$. For odd $k>3$ we have

$$
c_{k} \geqslant \frac{1}{2}-\frac{3 k-4}{k^{2}(k-1)}=\frac{1}{2}-O\left(k^{-2}\right) .
$$

Proof. Start with a perfect $(k-1)$-ary tree $T$ of height $h$. We are primarily interested in the behaviour of our construction as $h$ becomes large. On each leaf vertex $\ell$, glue $(k-1) / 2$ blocks, each of which is a copy of $K_{k+1}$ with one edge subdivided (the vertex on the subdivided edge is merged with $\ell$ ). The resulting graph $G$ is $k$-regular (except the root), with

$$
\begin{equation*}
n=\frac{(k-1)^{h+1}}{k-2}+O(1 / k)+\frac{1}{2}(k-1)^{h}(k-1)(k+1) \tag{2}
\end{equation*}
$$

vertices.
Consider an IPF $\mathcal{P}$ for $G$ and suppose for the moment that $k>3$. If $\mathcal{P}$ includes the edge from a leaf $\ell$ of $T$ to its parent, remove this edge from $\mathcal{P}$ and replace it with another edge as
follows. Choose a block $B$ which is glued onto $\ell$, but which contains no neighbour of $\ell$ in $\mathcal{P}$. The neighbours $u$ and $v$ of $\ell$ in $B$ must both be ends of paths in $\mathcal{P}$. If they are on different paths in $\mathcal{P}$ then we simply add the edge $\ell u$ and we are done. Otherwise, $\mathcal{P}$ includes a path [u,w,v]. But $B-\{\ell, u, v, w\}$ is a $K_{k-2}$, and $k-2$ is odd. Hence $\mathcal{P}$ includes a trivial path, say $[x]$. We replace $[u, v, w]$ and $[x]$ by $[\ell, u, x]$ and $[v, w]$. In this way, we have not changed the total number of paths in our IPF, but have removed the edge from $\ell$ to its parent. We repeat this process for all leaves $\ell$.

Now the blocks glued on $\ell$ need $\frac{1}{4}(k-1)(k+1)-1$ paths to cover them (the -1 is from the path that includes $\ell$ ). Applying Lemma 23 to all layers of $T$ except the last, the number of paths needed to cover $G$ is

$$
(k-1)^{h}\left(\left(k^{2}-1\right) / 4-1+1 / k\right)+O(1 / k) .
$$

Combined with (2), and taking $h \rightarrow \infty$, we find that

$$
c_{k} \geqslant \frac{\frac{1}{4}\left(k^{2}-1\right)-1+1 / k}{\frac{k-1}{k-2}+\frac{1}{2}(k-1)(k+1)}=\frac{1}{2}-\frac{3 k-4}{k^{2}(k-1)} .
$$

Finally, consider the case when $k=3$. Here we apply Lemma 23 to the whole initial tree $T$. Then, Lemma 5 tells us the effect of gluing a subdivided $K_{4}$ onto each of the $(k-1)^{h}$ leaves. The conclusion is that $\rho(G) \geqslant \frac{1}{k}(k-1)^{h+1}+O(1 / k)+(k-1)^{h}$. Combined with (2), and taking $h \rightarrow \infty$, we find that

$$
c_{3} \geqslant \frac{\frac{1}{k}(k-1)+1}{\frac{k-1}{k-2}+\frac{1}{2}(k-1)(k+1)}=\frac{2(2 k-1)(k-2)}{k^{2}(k-1)^{2}}=\frac{5}{18}
$$

as claimed.
Theorem 25. We have $c_{4} \geqslant 3 / 7$. For even $k \geqslant 4$ we have

$$
c_{k} \geqslant \frac{1}{2}-\frac{1}{2 k-2}=\frac{1}{2}-O\left(k^{-1}\right) .
$$

Proof. Fix an even $k \geqslant 4$. Start with an $h$-cycle $C$ and on each vertex glue $(k-2) / 2$ blocks, each of which is a $K_{k+1}$ with one edge subdivided. This produces a $k$-regular graph $G$ with $n=h+h(k+1)(k-2) / 2$ vertices. Since $\rho(C)=2$ and $\rho\left(K_{k+1}\right)=(k+2) / 2$, Lemma 5 implies that $\rho(G) \geqslant 2+h(k-2) k / 4$. Taking $h \rightarrow \infty$ gives

$$
c_{k} \geqslant \frac{(k-2) k / 4}{1+(k+1)(k-2) / 2}=\frac{1}{2}-\frac{1}{2 k-2}
$$

as claimed.
While the above argument works for $k=4$, we now provide a separate construction which gives a stronger result in this case. Take a perfect 2 -tree $T$ of height $h$ and add an edge between each pair of vertices that are children of the same parent. Now, for each vertex $\ell$ that was a leaf of $T$, add a copy of $K_{5}$ with a subdivided edge (the vertex on the subdivided edge is identified with $\ell$ ). The result is a graph with $2^{h+1}-1+2^{h} 5$ vertices that is 4 -regular except for the root.

Suppose we have an IPF for this graph. Consider the two children $v$ and $w$ of a non-leaf vertex $u$ in $T$. If our IPF includes the edge $v w$ then there must be a path that ends at $u$ (since
the neighbours of $u$, other than $v$ and $w$, are adjacent). Thus we can remove the edge $v w$ and add the edge $u v$ to get an IPF with the same number of paths. Repeating this process we obtain an IPF in which no path includes both children of a vertex in $T$ and hence no path includes vertices from two distinct subdivided copies of $K_{5}$. There are $2^{h}$ subdivided copies of $K_{5}$, and each requires 3 paths to cover it. Thus the graph needs at least $2^{h} 3$ paths in any IPF. Taking $h \rightarrow \infty$, it follows that $c_{4} \geqslant 3 /(2+5)=3 / 7$, as claimed.

Despite trying several alternative constructions, we were unable to find one for even $k$ which gave an error term matching the one that we obtained for odd $k$ in Theorem 24. Nevertheless, we do not believe there is a great intrinsic difference between the two cases.

Conjecture 26. We have $c_{k}=1 / 2-O\left(k^{-2}\right)$ as $k \rightarrow \infty$.

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