Antimagic orientation of graphs with minimum degree at least 33

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Abstract

An antimagic labeling of a directed graph D with n vertices and m arcs is a bijection from the set of arcs of D to the integers $\{1, \cdots, m\}$ such that all n oriented vertex sums are pairwise distinct, where an oriented vertex sum is the sum of labels of all arcs entering that vertex minus the sum of labels of all arcs leaving it. A graph G has an antimagic orientation if it has an orientation which admits an antimagic labeling. Hefetz, Mütze, and Schwartz conjectured that every connected graph admits an antimagic orientation. In this paper, we show that every bipartite graph without both isolated and degree 2 vertices admits an antimagic orientation and every graph G with $\delta(G) \geq 33$ admits an antimagic orientation. Our proof relies on a newly developed structural property of bipartite graphs, which might be of independent interest.

Keywords: Labeling; Antimagic labeling; Antimagic orientation; Matching

1 Introduction

All graphs considered are simple and finite unless otherwise stated. For two integers p, q, $[p, q] := \{p, p+1 \ldots, q\}$ if $q \geq p$, and $[p, q] := \emptyset$ if $q < p$. A *labeling* of a graph G with m edges is a bijection from $E(G)$ to a set S of m integers, and the *vertex sum* at a vertex $v \in V(G)$ is the sum of labels on the edges incident to v. A labeling is *antimagic* if $S = [1, m]$ and all the vertex sums are distinct. A graph is *antimagic* if it has an antimagic labeling.

Hartsfield and Ringel [7] introduced antimagic labelings in 1990 and conjectured that every connected graph other than K_2 is antimagic. There have been some significant progress towards this conjecture. Let G be a graph with n vertices other than K_2. In 2004, Alon, Kaplan, Lev, Roditty, and Yuster [1] showed that there exists a constant c such that if G has minimum degree at least $c \cdot \log n$, then G is antimagic. They also proved that G is antimagic when the maximum degree of G is at least $n-2$, and they proved that all complete multipartite graphs (other than K_2) are antimagic. The latter result of Alon et al. was improved by Yilma [18] in 2013.

Apart from the results above on dense graphs, the antimagic labeling conjecture has been also verified for regular graphs. Started with Cranston [4] showing that every bipartite regular graph is antimagic, regular graphs of odd degree [5], and finally all regular graphs [2, 3] were shown to be antimatic sequentially. For more results on the antimagic labeling conjecture for other classes of graphs, see [6, 8, 10, 11].

Hefetz, Mütze, and Schwartz [9] introduced the variation of antimagic labelings, i.e., antimagic labelings on directed graphs. An *antimagic* labeling of a directed graph with m arcs is a bijection from the set of arcs to the integers $\{1, ..., m\}$ such that any two oriented vertex sums are pairwise distinct, where an *oriented vertex sum* is the sum of labels of all arcs entering that vertex minus the sum of labels of all arcs leaving it. A digraph is called *antimagic* if it admits an antimagic labeling. For an undirected graph G, if it has an orientation such that the orientation is antimagic, then we say G admits an *antimagic orientation*. Hefetz, Mütze, and Schwartz in the same paper posted the following problems.

Question 1 ([9])**.** Is every connected directed graph with at least 4 vertices antimagic?

Conjecture 2 ([9])**.** Every connected graph admits an antimagic orientation.

Hefetz, Mütze, and Schwartz [9] showed that every orientation of a dense graph is antimagic and almost all regular graphs have an antimagic orientation. Particulary, they showed that every orientation of stars (other than $K_{1,2}$), wheels, and complete graphs (other than K_3) is antimagic. Conjecture 2 has been also verified for regular graphs [9, 12, 14, 16], biregular bipartite graphs with minimum degree at least two [13], Halin graphs [19], graphs with large maximum degree [17], and graphs with large independence number [15]. In this paper, by supporting Conjecture 2, we obtain the results below.

Theorem 3. *Every bipartite graph with no vertex of degree 0 or 2 admits an antimagic orientation.*

Theorem 4. *Every graph G with $\delta(G) \geq 33$ admits an antimagic orientation.*

The remainder of this paper is organized as follows. We introduce several preliminary results in Section 2. In Section 3, we prove Theorem 3, and in Section 4, we prove Theorem 4.

2 Notation and Preliminary Lemmas

Let G be a graph. We use $e(G)$ for $|E(G)|$. For $S \subseteq V(G)$, $G[S]$ is the subgraph of G induced by S. For two disjoint subsets $S, T \subseteq V(G)$, we denote by $E_G(S,T)$ the set of edges in G with one endvertex in S and the other in T, and let $e_G(S,T) = |E_G(S,T)|$. If G is bipartite with partite sets X and Y, we denote G by $G[X,Y]$ to emphasis the bipartitions. Given an orientation D of G, a labeling σ on $A(D)$ that is the set of arcs of D, and a vertex $v \in V(D)$, we use $s_{[D,\sigma]}(v)$ to denote the *oriented sum at v* in D, which is the sum of labels of all arcs entering v minus the sum of labels of all arcs leaving it in the digraph D.

For a matching M of G, we use $V(M)$ to denote the set of vertices saturated by M. For a vertex $x \in V(M)$, $M(x)$ is the vertex that is matched to x in M. For each subset $X \subseteq V(M)$, if X is an independent set in M, then $M(X)$ is the set of of vertices that are matched to vertices from X in M. By this definition, $|X| = |M(X)|$ and X and $M(X)$ are disjoint. An *M-augmenting path* is a path whose edges are alternating between edges in M and edges not in M and with both endpoints being not saturated by M.

A *trail* is an alternating sequence of vertices and edges $v_0e_1v_1 \dots e_tv_t$ such that v_{i-1} and v_i are the endvertices of e_i, for each $i \in [1,t]$, and the edges are all distinct (but there might be repetitions among the vertices). A trail is *closed* if $v_0 = v_t$, and is *open* otherwise. An *Euler tour* of G is a closed trail in G that contains all the edges of G. We will need the following classic result of Euler in proving a lemma later on.

Theorem 5 (Euler, 1736)**.** *A mutigraph G has an Euler tour if and only if G has at most one nontrivial component and every vertex of G has an even degree.*

Lemma 6 ([11])**.** *Let t, n be integers with $t \geq 1$ and $n \geq 2$, and let $n = r_1 + \ldots + r_t$ be a partition of n, where r_i is an integer that is at least 2 for each $i \in [1, t]$. Then the set $\{1, \ldots, n\}$ can be partitioned into pairwise disjoint subsets $R_1, \ldots, R_t$ such that for each $i \in [1, t]$, $|R_i| = r_i$ and $\sum_{r \in R_i} r \equiv 0 \pmod{n+1}$ if n is even, and $\sum_{r \in R_i} r \equiv 0 \pmod{n}$ if n is odd.*

The following result was proved in [15] without the furthermore part. However, the furthermore part is easy to obtain by following the same proof of Lemma 2.2 in [15] by just letting vertices in T to be not the endvertices of the edge-disjoint trails that decompose $E(G)$, which can be definitely guaranteed by the conditions imposed on T. So we omit the proof.

Lemma 7. *Let p, m be integers with with $p \geq 0, m \geq 1$, and let G be a graph with m edges. Then there exist an orientation D of G and a bijections $\sigma : A(D) \to \{p+1, \ldots, p+m\}$ such that for each $v \in V(G)$,*

$$-(p+m) + \lfloor \frac{d_G(v) - 1}{2} \rfloor \leq \quad s_{[D,\sigma]}(v) \quad \leq \lfloor \frac{d_G(v) - 1}{2} \rfloor + (p+m).$$

Furthermore, for $T \subseteq V(G)$ and each $v \in T$, if $d_G(v)$ is even and $N_G(v) \cap (V(G) \setminus T) \neq \emptyset$, then we can choose σ so that $s_{[D,\sigma]}(v) = \frac{d_G(v)}{2}$.

Lemma 8. *Let p, m be integers with $p \geq 0$ and $m \geq 1$, and let $G[S, T]$ be a bipartite graph with m edges such that every vertex from T has an even degree in G (so m is even). If $m \equiv 0 \pmod 4$, let $\delta_m = p + m$; and if $m \equiv 2 \pmod 4$, let $\delta_m = p + m + 1$. Then there exist an orientation D of G and a bijection $\sigma : A(D) \to \{p+1, \ldots, p+m-1\} \cup \{\delta_m\}$ such that*

$$\begin{aligned} s_{[D,\sigma]}(v) \quad &= -d_G(v) \qquad\qquad \text{for each } v \in T, \text{ and} \\ \lfloor \frac{d_G(v) - 1}{2} \rfloor - \delta_m \leq \quad s_{[D,\sigma]}(v) \quad &\leq \lfloor \frac{d_G(v) - 1}{2} \rfloor + \delta_m \quad \text{for each } v \in S. \end{aligned}$$

Proof. Suppose G has in total 2ℓ vertices of odd degree for some integer $\ell \geq 0$. We obtain a new graph G^* by pairing up these vertices into ℓ pairs, and for each pair, adding an edge joining the two vertices. Note that $G^* = G$ if $\ell = 0$.

Each component of G^* has an Euler tour by Theorem 5. By deleting all the edges in $E(G^*) \setminus E(G)$, we partition all edges of G into ℓ trails $T_1, T_2, \ldots, T_\ell$ (each T_i is either open or closed). For each $i \in [1, \ell]$, let

$$T_i = x_{t_{i-1}+1} e_{t_{i-1}+1} y_{t_{i-1}+1} f_{t_{i-1}+1} x_{t_{i-1}+2} \ldots y_{t_i - 1} f_{t_i - 1} x_{t_i},$$

where $t_0 = 0$. Note that $t_m - 1 = m/2$ and $|E(T_i)| = 2(t_i - 1 - t_{i-1})$. Since for every $v \in T$, $d_G(v)$ is even, we can further assume that in each T_i, for each $j \in [t_{i-1}+1, t_i]$,

$$x_j \in S \quad \text{and} \quad y_j \in T.$$

Also by the construction of T_i's, each vertex from S is the endvertex of at most one open trail.

For each $j \in [1, m/2-1]$, we direct each edge e_j from x_j to y_j, and direct each edge f_j from y_j to x_{j+1}. Denote by D this orientation of G.

If $m \equiv 0 \pmod 4$, for each $i \in [1, m/4]$, let

$$\begin{aligned} \sigma(e_{2i-1}) &= 4i-3, & \sigma(f_{2i-1}) &= 4i-1; \\ \sigma(e_{2i}) &= 4i-2, & \sigma(f_{2i}) &= 4i. \end{aligned}$$

If $m \equiv 2 \pmod 4$, let

$$\begin{aligned} \sigma(e_{2i-1}) &= 4i-3, & \sigma(f_{2i-1}) &= 4i-1 \quad \text{for each } i \in [1, \frac{m+2}{4}]; \\ \sigma(e_{2i}) &= 4i-2, & \sigma(f_{2i}) &= 4i \quad \text{for each } i \in [1, \frac{m-2}{4}]. \end{aligned}$$

By the definition of σ above, for each $j \in [1, m/2]$, e_j, f_j contributes -2 to the vertex sum at y_j that is shared by e_j and f_j. Since for each vertex $y \in T$, the edges incident to y in G are partitioned into $\frac{d_G(y)}{2}$ pairs of edges in the form of e_j, f_j, it holds $s_{[D,\sigma]}(y) = -d_G(y)$.

For each $j \in [1, m/2-1]$, f_j, e_{j+1} contributes 1 to the vertex sum at x_{j+1} that is shared by f_j and e_{j+1}. For each vertex $x \in S$, the edges incident to x in G are partitioned into at least $\lfloor \frac{d_G(x)-1}{2} \rfloor$ pairs of edges in the form of f_j, e_{j+1}. If $d_G(x)$ is odd, then x is the endvertex of exactly one open trails in $\{T_1, T_2, \ldots, T_\ell\}$. Thus, the edge incident to x not counted in the pairs f_j, e_{j+1} has a label in $[-\delta_m, \delta_m]$. If $d_G(x)$ is even, then x can be the endvertices of at most one closed trails in $\{T_1, T_2, \ldots, T_\ell\}$. Thus, the two edges incident to x not counted in the pairs f_j, e_{j+1} have a label in $[-\delta_m, \delta_m]$: one is negative and the other is positive, which add up to a value in $[-\delta_m, \delta_m]$. Hence, for each $x \in S$, it holds $\lfloor \frac{d_G(x)-1}{2} \rfloor - \delta_m \le s_{[D,\sigma]}(x) \le \lfloor \frac{d_G(x)-1}{2} \rfloor + \delta_m$. This finishes the proof of Lemma 8. $\square$

The following result on bipartite graphs is heavily used in our proofs, which might be of independent interest to other applications also.

Lemma 9. *If G is a bipartite graph, then $V(G)$ has a partition $S \cup T$ that satisfies the following conditions:*

(a) G has a matching M with $M \subseteq E_G(S,T)$ and M saturates S;

(b) T is an independent set in G.

Proof. It suffices to prove the statement only for every component of G. Thus we may assume that G is connected. Let $[X, Y]$ be a bipartition of G. Assume, without loss of generality, that $|X| \leq |Y|$. Let M be a matching of G that saturates the largest number of vertices from X. We will find a desired partition $S \cup T$ of $V(G)$ based on X and Y.

If $X \setminus V(M) = \emptyset$, then we are done by letting $S = X$ and $T = Y$. Thus, $X \setminus V(M) \neq \emptyset$. Let

$$X_1 = X \cap V(M), \qquad X_0 = X \setminus X_1,$$
$$Y_1 = Y \cap V(M), \qquad Y_0 = Y \setminus Y_1.$$

Since $|X \cap V(M)| = |Y \cap V(M)|$ and $|Y| \geq |X|$, $X_0 \neq \emptyset$ implies $Y_0 \neq \emptyset$. By the maximality of M, it holds

$$E_G(X_0, Y_0) = \emptyset. \tag{1}$$

Let

$$B_0 = N_G(X_0), \qquad A_0 = M(B_0), \qquad C_0 = N_G(Y_0), \qquad D_0 = M(C_0).$$

Clearly, $B_0, C_0 \neq \emptyset$ as G is connected. For each integer i with $i \geq 1$, define

$$B_i = N_G(A_{i-1}) \setminus \bigcup_{j=0}^{i-1} B_j, \quad A_i = M(B_i).$$

Let

$$B = \cup_{i=0}^{\infty} B_i, \quad A = \cup_{i=0}^{\infty} A_i.$$

By the definition, $B_i \cap B_j = \emptyset$ and $A_i \cap A_j = \emptyset$ for every pair of i, j with $i, j \geq 0$ and $i \neq j$. Since $|A_i| = |B_i|$ for each i with $i \geq 0$, it holds

$$|A| = |B|. \tag{2}$$

Let

$$X_r = X_1 \setminus A, \quad Y_r = Y_1 \setminus B.$$

By the definition of A,

$$E_G(A, Y_r) = \emptyset. \tag{3}$$

Let

$$S = B \cup X_r, \quad T = A \cup Y_r \cup X_0 \cup Y_0.$$

It is left to show that $S \cup T$ is a desired partition of $V(G)$. Since $|A| = |B|$ by (2), $|S| = |X_1|$. Furthermore, by the definitions of A and B, $A = M(B)$, and consequently $X_r = M(Y_r)$. Thus, M is still a matching in G that saturates S and has size $|S|$, and $M \subseteq E_G(S, T)$. We only show that T is an independent in G. As each of A, Y_r, X_0 and Y_0 is an independent set in G, (1) and (3), respectively, implies that $A \cup Y_r$ and $X_0 \cup Y_0$ are independent sets in G. Since $E_G(X_0, A) = \emptyset$ and $E_G(X_0, Y_r) = \emptyset$ by $N_G(X_0) = B_0 \subseteq B$, $A \cup Y_r \cup X_0$ is an independent set in G. Since $E_G(Y_0, Y_r) = \emptyset$ and $E_G(Y_0, X_0) = \emptyset$ by (1), we are only left to show that $E_G(Y_0, A) = \emptyset$.

It suffices to only show that $D_0 \subseteq Y_r$. Since $D_0 \subseteq Y_r$ implies that $C_0 \subseteq X_r$ by the definitions of the sets A and B, and $C_0 \subseteq X_r$ implies that $C_0 \cap A = \emptyset$, which yields $E_G(Y_0, A) = \emptyset$.

To show $D_0 \subseteq Y_r$, we just show that for each i with $i \geq 0$, $E_G(A_i, D_0) = \emptyset$. Assume to the contrary and let k be the smallest index such that $E_G(A_k, D_0) \neq \emptyset$. Let $d_0 \in D_0$ and $a_k \in A_k$ such that $d_0 a_k \in E(G)$, $b_k = M(a_k)$, $a_{k-1} \in A_{k-1}$ such that $a_{k-1} b_k \in E(G)$. In general, for each $i = k-1, k-2, \ldots, 1$, let

$$b_i = M(a_i), \quad a_{i-1} \in A_{i-1} \quad \text{such that } a_{i-1} b_i \in E(G).$$

Furthermore, let $b_0 = M(a_0)$ and $x_0 \in X_0$ such that $b_0 x_0 \in E(G)$, and $c_0 = M(d_0)$ and $y_0 \in Y_0$ such that $c_0 y_0 \in E(G)$.

Note that for $i, j \in [0, k]$ with $i \neq j$, $a_i \neq a_j$ and $b_i \neq b_j$, as $A_i \cap A_j = \emptyset$ and $B_i \cap B_j = \emptyset$. Furthermore, by the minimality of k, $a_i \neq d_0$ and $b_i \neq d_0$. Thus

$$P := y_0 c_0 d_0 a_k b_k a_{k-1} b_{k-1} \ldots b_1 a_0 b_0 x_0$$

is an M-augmenting path, and $M' := (M \setminus E(P)) \cup (E(P) \setminus M)$ is a matching in G such that $|V(M') \cap X| > |V(M) \cap X|$, showing a contradiction to the choice of M. Therefore, $E_G(A_i, D_0) = \emptyset$ for each i with $i \geq 0$. This completes the proof. $\square$

3 Proof of Theorem 3

Let $S \cup T$ be a partition of $V(G)$ satisfying the requirements in Lemma 9. Let

$$n_1 = |S|, \quad n_2 = |T|, \quad S = \{x_1, x_2, \ldots, x_{n_1}\}, \qquad T = \{y_1, y_2, \ldots, y_{n_2}\}.$$

Assume, without loss of generality, that

$$M = \{x_1 y_1, x_2 y_2, \ldots, x_{n_1} y_{n_1}\}.$$

. For each $i \in [n_1 + 1, n_2]$, let e_i be an edge incident to y_i in G, and let

$$M^* = M \cup \{e_{n_1+1}, \ldots, e_{n_2}\}.$$

In other words, each vertex from T is incident to one and exactly one edge from M^*. Furthermore, let

$$H = G - E(G[S]) - M^*, \quad m_1 = |E(G[S])|, \quad m_2 = e(H).$$

Clearly, $m_1 + m_2 + |M^*| = m_1 + m_2 + n_2 = m := e(G)$. Let $T_1 = \{y \in Y : d_G(y) = 1\}$ and $t_1 = |T_1|$. Assume, without loss of generality, that $T_1 = \{y_{n_2 - t_1 + 1}, y_{n_2 - t_1 + 2}, \ldots, y_{n_2}\}$. Clearly, $d_H(y_i) = 0$ for each $i \in [n_2 - t + 1, n_2]$. We consider two cases below regarding how large n_2 is.

Case 1: $n_2 \leq m_2$.

This case basically follows the same idea as in the Proof of Theorem 1.5 in [15], but we repeat the process for self-completeness.

We give an orientation D of G and a labeling σ of $A(D)$ through four parts below.

(1) Orient and label H: direct each edge from T to S. For each $i \in [1, n_2 - t_1]$, let A_i be the set of all edges incident to y_i in H. Clearly, $|A_1| + |A_2| + \ldots + |A_{n_2-t_1}| = m_2$. Since G has no vertex of degree 2 or isolated vertex, $|A_i| \geq 2$. By applying Lemma 6 to m_2 with $t = n_2 - t_1$ and $r_i = |A_i|$ for each $i \in [1, t]$, the set $\{1, 2, \ldots, m_2\}$ can be partitioned into $R_1, R_2, \ldots, R_{n_2-t_1}$ such that for each $i \in [1, n_2 - t_1]$, $|R_i| = |A_i|$ and $\sum_{r \in R_i} r \equiv 0 \pmod{m_2}$ if m_2 is even, and $\sum_{r \in R_i} r \equiv 0 \pmod{m_2}$ if m_2 is odd. Label edges in A_i by integers in R_i in an arbitrary way as long as distinct edges receive distinct labels.

(2) Orient and label $G[S]$: applying Lemma 7 to get the orientation and labeling with $p = m_2$ and $m = m_1$;

(3) Orient and label $M^* \setminus M = \{e_{n_1+1}, \ldots, e_{n_2}\}$: direct each edge from T to S, and for each $i \in [n_1 + 1, n_2]$, assign $m_1 + m_2 + (i - n_1)$ to e_i.

Let D^* be the union of the digraphs obtained through the three parts above, and σ^* be the labeling on $A(D^*)$ consists of the three labelings above. Assume that the sums at vertices from $S = \{x_1, \ldots, x_{n_1}\}$ satisfy

$$s_{[D^*,\sigma^*]}(x_1) \leq s_{[D^*,\sigma^*]}(x_2) \leq \ldots \leq s_{[D^*,\sigma^*]}(x_{n_1}).$$

(4) Orient and label M: direct each edge from T to S, and for each $i \in [1, n_1]$, assign $m_1 + m_2 + n_2 - n_1 + i$ to $x_i y_i$.

Let D and σ be the resulting orientation and labeling, respectively. It is clear that σ is injective. We show that σ is an antimagic labeling of D. By Step 4, we have

$$s_{[D,\sigma]}(x_1) < s_{[D,\sigma]}(x_2) < \ldots < s_{[D,\sigma]}(x_{n_1}).$$

Furthermore, for each $i \in [1, n_1]$, by Step 2, $s_{[D^*,\sigma^*]}(x_i) \geq \lfloor \frac{d_{G[S]}(x_i)-1}{2} \rfloor - m_1 - m_2$, by Steps 3 and 4, we know $s_{[D,\sigma]}(x_i) \geq s_{[D^*,\sigma^*]}(x_i) + m_1 + m_2 + n_2 - n_1 + i > 0$. For each vertex $y_i \in T$, $i \in [1, n_2]$, all the edges incident to y_i are oriented towards S. Thus, $s_{[D,\sigma]}(y_i) < 0$.

Thus, for each $x \in S$ and each $y \in T$, $s_{[D,\sigma]}(x) > s_{[D,\sigma]}(y)$. Therefore, it is left to only show that all vertices from T have distinct sums under σ in D.

By Steps 1, 3 and 4, for each $i \in [1, n_2]$ and for some integr $a_i \geq 0$, we have

$$|s_{[D,\sigma]}(y_i)| = \begin{cases} a_i m_2 + m_1 + m_2 + \sigma_i, & \text{if } m_2 \text{ is odd,} \\ a_i(m_2 + 1) + m_1 + m_2 + \sigma_i, & \text{if } m_2 \text{ is even,} \end{cases}$$

where $\sigma_i \in [1, n_2]$ are all distinct. Since $n_2 \leq m_2$, for any two distinct $i, j \in [1, n_2]$,

$$s_{[D,\sigma]}(y_i) - s_{[D,\sigma]}(y_j) \not\equiv \begin{cases} 0 \pmod{m_2}, & \text{if } m_2 \text{ is odd,} \\ 0 \pmod{m_2 + 1}, & \text{if } m_2 \text{ is even.} \end{cases}$$

Consequently, $s_{[D,\sigma]}(y_i) \neq s_{[D,\sigma]}(y_j)$.

The proof for Case 1 is complete.

Case 2: $n_2 \geq m_2 + 1$.

In this case, we develop a result similar to Lemma 6 but using nonconsecutive integers not necessarily starting at 1.

For each $i \in [1, n_2 - t_1]$, let A_i be the set of all edges incident to y_i in H. Clearly, $|A_1| + |A_2| + \ldots + |A_{n_2 - t_1}| = m_2$. Since G has no vertex of degree 2 or isolated vertex, $|A_i| \geq 2$. Let $m_2 = 3k + 2\ell$, for some integers $k, \ell \geq 0$, where k is the number of sets A_i's with an odd cardinality. We may assume that $k + \ell \geq 1$. Otherwise, we follow the same proof as in Case 1, and the vertex sums at vertices from T will naturally be all distinct since all these vertices have degree 1 in G.

Subcase 2.1: $k = 0$.

In this case, all $|A_i|$'s are even. We give an orientation D of G and a labeling σ of $A(D)$ through four parts below.

(1) Orient and label $G[S]$: applying Lemma 7 to get the orientation and labeling with $p = 0$ and $m = m_1$;

(2) Orient and label H: direct each edge from T to S. For each i, partition all edges in A_i into $|A_i|/2$ many 2-element subsets. Thus, we have in total $m_2/2$ many 2-element subsets $B_1, B_2, \ldots, B_{m_2/2}$ of edges. For each B_i, $i \in [1, m_2/2]$, we assign

$$m_1 + n_2 + i, m - (i - 1)$$

to the two edges from it. By the way above of assigning labels to edges in A_i's, $i \in [1, n_2 - t_1]$, the sum of labels assigned to edges from each A_i is

$$a_i(m + m_1 + n_2 + 1) \quad \text{for some integer } a_i \geq 1. \tag{4}$$

(3) Orient and label $M^* \setminus M = \{e_{n_1+1}, \ldots, e_{n_2}\}$: direct each edge from T to S, and for each $i \in [n_1 + 1, n_2]$, assign $m_1 + (i - n_1)$ to e_i.

Let D^* be the union of the digraphs obtained through the three parts above, and σ^* be the labeling on $A(D^*)$ consists of the three labelings above. Assume that the sums at vertices from $S = \{x_1, \ldots, x_{n_1}\}$ satisfy

$$s_{[D^*,\sigma^*]}(x_1) \leq s_{[D^*,\sigma^*]}(x_2) \leq \ldots \leq s_{[D^*,\sigma^*]}(x_{n_1}).$$

(4) Orient and label M: direct each edge from T to S, and for each $i \in [1, n_1]$, assign $m_1 + n_2 - n_1 + i$ to $x_i y_i$.

Let D and σ be the resulting orientation and labeling, respectively. It is clear that σ is injective. We show that σ is an antimagic labeling of D. By Step 4, we have that

$$s_{[D,\sigma]}(x_1) < s_{[D,\sigma]}(x_2) < \ldots < s_{[D,\sigma]}(x_{n_1}).$$

Furthermore, for each $i \in [1, n_1]$, by Lemma 7 and Step 1, $s_{[D^*,\sigma^*]}(x_i) \geq \lfloor \frac{d_{G[S]}(x_i)-1}{2} \rfloor - m_1$, we know $s_{[D,\sigma]}(x_i) \geq s_{[D^*,\sigma^*]}(x_i) + m_1 + n_2 - n_1 + i \geq 0$. For each vertex $y_i \in T$, $i \in [1, n_2]$, all the edges incident to y_i are oriented towards S. Thus, $s_{[D,\sigma]}(y_i) < 0$.

Thus, for each $x \in S$ and each $y \in T$, $s_{[D,\sigma]}(x) > s_{[D,\sigma]}(y)$. Therefore, it is left to only show that all vertices from T have distinct sums under σ in D.

By Steps 2, 3 and 4, for each $i \in [1, n_2]$, we have

$$|s_{[D,\sigma]}(y_i)| = a_i(m_1 + n_1 + m + 1) + m_1 + \sigma_i \quad \text{for some integer } a_i \geq 1,$$

where $\sigma_i \in [1, n_2]$ are all distinct. Since $n_2 < m_1 + n_1 + m + 1$, for any two distinct $i, j \in [1, n_2]$,

$$s_{[D,\sigma]}(y_i) - s_{[D,\sigma]}(y_j) \not\equiv 0 \pmod{m_1 + n_1 + m + 1}$$

Consequently, $s_{[D,\sigma]}(y_i) \neq s_{[D,\sigma]}(y_j)$.

The proof for Subcase 2.1 is complete.

Subcase 2.2: $k \geq 1$.

Recall that $m_2 = 3k + 2\ell$ and $n_2 \geq m_2 + 1$. Thus $m \geq n_2 + m_2 \geq 6k + 4\ell + 1$, and $m - 2k - \ell + 2 = m_2 + n_2 + m_1 - 2k - \ell + 2 > m_1 + 3k + \ell$. We assume, without loss of generality, that $|A_1|, \ldots, |A_k|$ are odd, and $|A_{k+1}|, \ldots, |A_{n_2-t_1}|$ are all even.

We will use the labels from the set $A = [1, k] \cup [m_1 + k + 1, m_1 + 2k] \cup [m_1 + 3k + 1, m_1 + 3k + \ell] \cup [m - 2k - \ell + 2, m - 2k + 1] \cup \{m - 2k + 2, m - 2k + 4, \ldots, m - 2, m\}$ for edges from A_i's. For each $i \in [1, k]$, edges in A_i can be partitioned into one 3-subset, and $(|A_i| - 3)/2$ many 2-subsets. For each $i \in [k+1, n_2 - t_1]$, edges in A_i can be partitioned into $|A_i|/2$ many 2-subsets. Let $B_1, B_2, \ldots, B_k$ be the k 3-sets and $C_1, \ldots, C_\ell$ be the ℓ 2-sets obtained by partition edges from each A_i's. For each $i \in [1, k]$, we assign edges in each B_i the following three numbers:

$$i, \quad m_1 + k + i, \quad m - 2i + 2.$$

For each $i \in [1, \ell]$, we assign edges in each C_i the following two numbers:

$$m_1 + 3k + i, \quad m - 2k + 2 - i.$$

By the way above of assigning labels to edges in A_i's, $i \in [1, n_2 - t_1]$, the sum of labels assigned to edges from each A_i is

$$a_i(m + m_1 + k + 2) \quad \text{for some integer } a_i \geq 1. \tag{5}$$

We give an orientation D of G and a labeling σ of $A(D)$ through four parts below.

(1) Orient and label $G[S]$: applying Lemma 7 to get the orientation and labeling with $p = k$ and $m = m_1$;

(2) Orient and label H: direct each edge from T to S. Assign labels in the set A to the edges in $\bigcup_{i=1}^{n_2-t_1} A_i$ as described previously.

Note that the set of unused labels is

$$B = [m_1+2k+1, m_1+3k] \cup [m_1+3k+\ell+1, m-2k-\ell+1] \cup \{m-2k+3, m-2k+5, \ldots, m-1\},$$

and $|B| = k + m - m_1 - 5k - 2\ell + 1 + k - 1 = m_2 + n_2 - 3k - 2\ell = n_2$.

(3) Orient and label $M^* \setminus M = \{e_{n_1+1}, \ldots, e_{n_2}\}$: direct each edge from T to S, and assign the first $n_2 - n_1$ smallest numbers from B to edges in $M^* \setminus M$ such that distinct edges receive distinct labels.

Let D^* be the union of the digraphs obtained through the three parts above, and σ^* be the labeling on $A(D^*)$ consists of the three labelings above. Assume that the sums at vertices from $S = \{x_1, \ldots, x_{n_1}\}$ satisfy

$$s_{[D^*,\sigma^*]}(x_1) \le s_{[D^*,\sigma^*]}(x_2) \le \ldots \le s_{[D^*,\sigma^*]}(x_{n_1}).$$

(4) Orient and label M: direct each edge from T to S, and assign the remaining $n_2-(n_2-n_1) = n_1$ numbers from B to edges in M such that x_iy_i is assigned with the i-th smallest number.

Let D and σ be the resulting orientation and labeling, respectively. It is clear that σ is injective. We show that σ is an antimagic labeling of D. By Step 4, we have

$$s_{[D,\sigma]}(x_1) < s_{[D,\sigma]}(x_2) < \ldots < s_{[D,\sigma]}(x_{n_1}).$$

Furthermore, for each $i \in [1, n_1]$, by Lemma 7 and Step 1, $s_{[D^*,\sigma^*]}(x_i) \ge \lfloor \frac{d_{G[S]}(x_i)-1}{2} \rfloor - m_1 - k$, we know $s_{[D,\sigma]}(x_i) \ge s_{[D^*,\sigma^*]}(x_i) + m_1 + 2k + 1 \ge 0$. For each vertex $y_i \in T$, $i \in [1, n_2]$, all the edges incident to y_i are oriented towards S. Thus, $s_{[D,\sigma]}(y_i) < 0$.

Thus for each $x \in S$ and each $y \in T$, $s_{[D,\sigma]}(x) > s_{[D,\sigma]}(y)$. Therefore, it is left to only show that all vertices from T have distinct sums under σ in D.

By Steps 2, 3, 4 and (5), for each $i \in [1, n_2]$, we have

$$|s_{[D,\sigma]}(y_i)| \ge a_i(m + m_1 + k + 2) + \sigma_i \quad \text{for some integer } a_i \ge 0,$$

where $\sigma_i \in B$ are all distinct. Since $\sigma_i \le m - 1 < m + m_1 + k + 2$, for any two distinct $i, j \in [1, n_2]$,

$$s_{[D,\sigma]}(y_i) - s_{[D,\sigma]}(y_j) \not\equiv 0 \pmod{m + m_1 + k + 2}$$

Consequently, $s_{[D,\sigma]}(y_i) \ne s_{[D,\sigma]}(y_j)$.

The proof for Subcase 2.2 is complete.

4 Proof of Theorem 4

Let L be a spanning bipartite subgraph of G with the maximum number of edges. Since $|E(L)|$ is maximum among all spanning bipartite subgraphs of G,

$$d_L(v) \ge \frac{d_G(v)}{2} \quad \text{for every } v \in V(G).$$

By Lemma 9, we let $S \cup T$ be a partition of $V(L) = V(G)$, $M \subseteq E_L(S,T)$ be a matching that saturates S and has size $|S|$, and let $L^* = L - E(L[S])$ be the spanning bipartite graph of L between S and T.

Let

$$n_2 = |S|, \quad n_1 = |T|, \quad S = \{x_1, x_2, \ldots, x_{n_2}\}, \qquad T = \{y_1, y_2, \ldots, y_{n_1}\}.$$

Assume, without loss of generality, that

$$M = \{x_1y_1, x_2y_2, \ldots, x_{n_1}y_{n_2}\}.$$

For each $i \in [n_2+1, n_1]$, let e_i be an edge incident to y_i in L^*, and let

$$M^* = M \cup \{e_{n_2+1}, \ldots, e_{n_1}\}.$$

In other words, each vertex from T is incident to one and exactly one edge in M^*. Furthermore, let

$$H = L^* - M^*, \quad G_1 = G - E(H) - M^*.$$

Note that for every vertex $y \in T$,

$$d_H(v) = d_L(v) - 1 \geq \frac{d_G(v)}{2} - 1, \tag{6}$$

and $E(G_1) = E(G) \setminus (E(H) \cup M^*) = E(G[S]) \cup E(G[T]) \cup \big(E_G(S,T) \setminus E_{L^*}(S,T)\big)$. We now modify G_1 to get a new graph by adding some edges from H such that in the new graph the degree of every vertex from T is divisible by 4 and that every vertex from T has a neighbor from S. Specifically, for each $v \in T$, if $d_{G_1}(v) \equiv c \pmod 4$, where $c = 0, 1, 2, 3$, we take exactly $4 - c$ edges incident to v in H and add these $4 - c$ edges into G_1. Call G_2 the resulting in graph from G_1, and H' the resulting in graph from H. From the construction, for each $v \in T$,

$$d_{G_2}(v) \equiv 0 \pmod 4, \quad d_{H'}(v) \geq d_H(v) + c - 4, \tag{7}$$

where $c \in \{0, 1, 2, 3\}$ satisfies $d_{G_1}(v) \equiv c \pmod 4$.

We then split the bipartite graph H' into two spanning subgraphs H_1 and H_2 of H'. For each $v \in T$, we let $A(v)$ be a set of $\frac{d_{G_2}(v)}{2}$ edges incident to v in H'. Now let

$$V(H_2) = V(H'), \quad E(H_2) = \bigcup_{v \in T} A(v), \quad H_1 = H' - E(H_2).$$

From the construction and (7), for each $v \in T$,

$$d_{H_2}(v) = \frac{d_{G_2}(v)}{2} \equiv 0 \pmod 2, \quad d_{H_1}(v) \geq d_H(v) + c - 4 - \frac{d_{G_2}(v)}{2}, \tag{8}$$

where $c \in \{0, 1, 2, 3\}$ satisfies $d_{G_1}(v) \equiv c \pmod 4$. By (6), we have

$$\begin{aligned}
d_{H_1}(v) &\geq d_H(v) + c - 4 - \frac{d_{G_2}(v)}{2} \\
&\geq \lceil \frac{d_G(v)}{2} \rceil - 1 + c - 4 - \frac{d_{G_2}(v)}{2} \\
&\geq \lceil \frac{d_G(v)}{2} \rceil - 1 + c - 4 - \frac{1}{2}\big(\lfloor \frac{d_G(v)}{2} \rfloor + 4 - c\big) \\
&\geq \frac{d_G(v)}{4} - 7,
\end{aligned} \tag{9}$$

which is at least 2, since $\delta(G) \geq 33$.

Let

$$m_1 = e(H_1), \quad m_2 = e(G_2), \quad m_3 = e(H_2).$$

Note that $m_1 + m_2 + m_3 + |M^*| = m := e(G)$. We will now give an orientation D of G and a labeling σ of $A(D)$ through five parts below.

(1) Orient and label H_1: direct each edge from T to S. For each $i \in [1, n_1]$, let A_i be the set of all edges incident to y_i in H_1. Clearly, $|A_1| + |A_2| + \ldots + |A_{n_1}| = m_1$. By (9), $|A_i| \geq 2$. By Lemma 6 applied to m_1 with $t = n_1$ and $r_i = |A_i|$ for each $i \in [1, t]$, the set $\{1, 2, \ldots, m_1\}$ can be partitioned into $R_1, R_2, \ldots, R_{n_1}$ such that for each $i \in [1, n_1]$, $|R_i| = |A_i|$ and $\sum_{r \in R_i} r \equiv 0 \pmod{m_1 + 1}$ if m_1 is even, and $\sum_{r \in R_i} r \equiv 0 \pmod{m_1}$ if m_1 is odd. Label edges in A_i by integers in R_i in an arbitrary way as long as distinct edges receive distinct labels.

(2) Orient and label G_2: Note that for each $y \in T$, $d_{G_2}(y)$ is even and $N_{G_2}(y) \cap S \neq \emptyset$ by the construction of G_2. Thus, we apply Lemma 7 to get the orientation and labeling of G_2 with $p = m_1$ and $m = m_2$ with the furthermore requirement for vertices in T. Let D_2 be the orientation of G_2 and σ_2 be the labeling. We have

$$\begin{aligned} -(m_1 + m_2) + \lfloor \frac{d_{G_2}(x) - 1}{2} \rfloor \leq \quad s_{[D_2,\sigma_2]}(x) \quad &\leq \lfloor \frac{d_{G_2}(x) - 1}{2} \rfloor + (m_1 + m_2) \quad \text{for } x \in S, \\ s_{[D_2,\sigma_2]}(y) \quad &= \frac{d_{G_2}(y)}{2} \quad \text{for } y \in T. \end{aligned} \tag{10}$$

(3) Orient and label H_2: applying Lemma 8 to get the orientation and labeling of H_2 with $p = m_1 + m_2$ and $m = m_3$. Let D_3 be the orientation of H_2 and σ_3 be the labeling. We have

$$\begin{aligned} \lfloor \frac{d_{H_2}(x) - 1}{2} \rfloor - \delta_m \leq \quad s_{[D_3,\sigma_3]}(x) \quad &\leq \lfloor \frac{d_{H_2}(x) - 1}{2} \rfloor + \delta_m \quad \text{for each } x \in S, \\ s_{[D_3,\sigma_3]}(y) \quad &= -d_{H_2}(y) \qquad\qquad \text{for each } y \in T, \end{aligned} \tag{11}$$

where $\delta_m = m_1 + m_2 + m_3$ if $m_3 \equiv 0 \pmod 4$, and $\delta_m = m_1 + m_2 + m_3 + 1$ if $m_3 \equiv 2 \pmod 4$.

(4) Orient and label $M^* \setminus M = \{e_{n_2+1}, \ldots, e_{n_1}\}$: direct each edge from T to S. If $m_3 \equiv 0 \pmod 4$, for each $i \in [n_2+1, n_1]$, assign $m_1+m_2+m_3+(i-n_2)$ to e_i. If $m_3 \equiv 2 \pmod 4$, assign $m_1+m_2+m_3$ to e_{n_2+1}, and for each $i \in [n_2+2, n_1]$, assign $m_1+m_2+m_3+(i-n_2)$ to e_i.

Let D^* be the union of the digraphs obtained through the four parts above, and σ^* be the labeling on $A(D^*)$ consists of the four labelings above. Assume that the sums at vertices from $S = \{x_1, \ldots, x_{n_2}\}$ satisfy

$$s_{[D^*,\sigma^*]}(x_1) \leq s_{[D^*,\sigma^*]}(x_2) \leq \ldots \leq s_{[D^*,\sigma^*]}(x_{n_2}).$$

(5) Orient and label M: direct each edge from T to S. If $n_1 \geq n_2 + 1$ or $m_3 \equiv 0 \pmod 4$, for each $i \in [1, n_2]$, assign $m_1 + m_2 + m_3 + n_1 - n_2 + i$ to $x_i y_i$. If $n_1 = n_2$ and $m_3 \equiv 2 \pmod 4$, assign $m_1 + m_2 + m_3$ to $x_1 y_1$, and for each $i \in [2, n_2]$, assign $m_1 + m_2 + m_3 + i$ to $x_i y_i$.

Let D and σ be the resulting orientation and labeling, respectively. It is clear that σ is injective. We show that σ is an antimagic labeling of D.

By Step 5, we have

$$s_{[D,\sigma]}(x_1) < s_{[D,\sigma]}(x_2) < \ldots < s_{[D,\sigma]}(x_{n_2}).$$

Furthermore, for each $i \in [1, n_2]$, by (10) and (11), $s_{[D^*,\sigma^*]}(x_i) \geq \lceil \frac{d_{G_2}(x_i)-1}{2} \rceil - m_1 - m_2 + \lceil \frac{d_{H_2}(x_i)-1}{2} \rceil - m_1 - m_2 - m_3 - 1$, we know $s_{[D,\sigma]}(x_i) \geq s_{[D^*,\sigma^*]}(x_i) + m_1 + m_2 + m_3 \geq -m_1 - m_2 - 1$. For each vertex $y_i \in T$, $i \in [1, n_1]$, for all the edges incident to y_i that are contained in $G_2 \cup H_2$, the partial sum at y_i of the labels assigned to these edges is zero by (8), (10) and (11). All other edges incident to y_i that are contained in $H_1 \cup M^*$ are oriented towards S. Thus, $s_{[D,\sigma]}(y_i) < 0$. Furthermore, by Steps 1, 4 and 5, $s_{[D,\sigma]}(y_i) \leq -m_1 - m_2 - m_3 - 3$.

Thus, for each $x \in S$ and each $y \in T$, $s_{[D,\sigma]}(x) > s_{[D,\sigma]}(y)$. Therefore, it is left to only show that all vertices from T have distinct sums under σ in D.

By Steps 1, 4, 5, and (10) and (11), for each $i \in [1, n_1]$ and some integer $a_i \geq 1$, we have

$$|s_{[D,\sigma]}(y_i)| = \begin{cases} \frac{d_{G_2}(y_i)}{2} - d_{H_2}(y_i) + a_i m_1 + m_1 + m_2 + m_3 + \sigma_i, & \text{if } m_1 \text{ is odd,} \\ \frac{d_{G_2}(y_i)}{2} - d_{H_2}(y_i) + a_i(m_1+1) + m_1 + m_2 + m_3 + \sigma_i, & \text{if } m_1 \text{ is even,} \end{cases}$$

where $\sigma_i \in [1, n_1]$ are all distinct, and $\frac{d_{G_2}(y_i)}{2} - d_{H_2}(y_i) = 0$. Since $m_1 \geq 2n_1 > n_1$ by (9), for any two distinct $i, j \in [1, n_1]$,

$$s_{[D,\sigma]}(y_i) - s_{[D,\sigma]}(y_j) \not\equiv \begin{cases} 0 \pmod{m_1}, & \text{if } m_1 \text{ is odd,} \\ 0 \pmod{m_1+1}, & \text{if } m_1 \text{ is even.} \end{cases}$$

Consequently, $s_{[D,\sigma]}(y_i) \neq s_{[D,\sigma]}(y_j)$.

The proof is now complete.

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