Kings in Multipartite Hypertournaments

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Abstract

In his paper "Kings in Bipartite Hypertournaments" (Graphs & Combinatorics 35, 2019), Petrovic stated two conjectures on 4-kings in multipartite hypertournaments. We prove one of these conjectures and give counterexamples for the other.

1 Introduction

Given two integers n and k, $n \geq k > 1$, a *k-hypertournament* T on n vertices is a pair (V, A), where V is a set of vertices, $|V| = n$ and A is a set of k-tuples of vertices, called arcs, so that for any k-subset S of V, A contains exactly one of the $k!$ tuples whose entries belong to S. For an arc $x_1x_2 \dots x_k$, we say that x_i *precedes* x_j if $i < j$. A 2-hypertournament is merely an (ordinary) tournament. Hypertournaments have been studied in a large number of papers, see e.g. [1, 2, 3, 4, 5, 8, 9, 11, 12].

Recently, Petrovic [10] introduced multipartite hypertournaments in a similar way. Let n and k be integers such that $n > k \geq 2$. Let V be a set of n vertices and $V = V_1 \uplus V_2 \uplus \cdots \uplus V_p$ be a partition of V into $p \geq 2$ non-empty subsets. A *p-partite k-hypertournament* (or, *multipartite hypertournament*) H can be obtained from a k-hypertournament T on vertex set V by deleting all arcs $x_1x_2 \dots x_k$ such that $\{x_1, x_2, \dots, x_k\} \subseteq V_i$ for some $i \in [p]$. We call V_i's *partite sets* of H. The set of arcs of $H = (V, A)$ will be denoted by $A(H)$, i.e., $A(H) = A$. A p-partite 2-hypertournament is a *p-partite tournament.*

For $u \in V_i, w \in V_j$ with $i \neq j$, $A_H(u, w)$ is the set of arcs of H which contain u and w and where u precedes w. We will write xey if $e \in A_H(x, y)$. We let $A_H(x, y) = \emptyset$ if either x and y belong to the same partite set of H. A *path* in H is an alternating sequence $P = x_1a_1x_2a_2 \dots x_{q-1}a_{q-1}x_q$ of distinct vertices x_i

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and distinct arcs a_j such that $x_j a_j x_{j+1}$ for every $j \in [q-1]$. We will call P an (x_1, x_q)-*path* of *length* $q-1$.

Let $q \geq 1$ be a natural number. A vertex x of H is a q-*king* if for every $y \in V$, H has an (x, y)-path of length at most q. Generalizing a well-known theorem of Landau that every tournament has a 2-king (see e.g. [6]), Brcanov et al. [4] showed that every hypertournament has a 2-king. A vertex v of H is a *transmitter* if for every vertex u from a different partite set than v, $A_H(u, v) = \emptyset$.

Note that for every $u \in V_i, w \in V_j$ $(i \neq j)$, we have $|A_H(u,w)| + |A_H(w,u)| = \binom{n-2}{k-2}$. A *majority multipartite tournament* $M(H)$ of H has the same partite sets as H and for every $u \in V_i$ and $w \in V_j$ with $i \neq j$, $uw \in M(H)$ if $|A_H(u,w)| > \frac{1}{2}\binom{n-2}{k-2}$. If $|A_H(u,w)| = \frac{1}{2}\binom{n-2}{k-2}$ then we can choose either uw or wu for $M(H)$.

For a graph $G = (V, E)$ and $U \subseteq V$, let $N_G(U) = \{v \in V \setminus U : \ uv \in E, u \in U\}$.

Gutin [7] and independently Petrovic and Thomassen [11] proved the following:

Theorem 1. *[7, 11] Every multipartite tournament with at most one transmitter contains a 4-king.*

Petrovic [10] proved that the same result holds for bipartite k-hypertournaments:

Theorem 2. *[10] Every bipartite k-hypertournaments ($k \geq 2$) with at most one transmitter contains a 4-king.*

In the same paper he conjectured the following:

Conjecture 3. *[10] Every multipartite k-hypertournament ($k \geq 2$) with at most one transmitter contains a 4-king.*

In this short paper, we will solve this conjecture in the affirmative.

The next conjecture of Petrovic [10] is motivated by the fact that Petrovic and Thomassen [11] proved that the assertion of the conjecture holds for bipartite tournaments.

Theorem 4. *[11] Every bipartite tournament B without transmitters has at least two 4-kings in each partite set of B.*

Conjecture 5. *[10] Every bipartite k-hypertournament B ($k \geq 2$) without transmitters has at least two 4-kings in each partite set of B.*

In this paper, we will first show a couterexample to Conjecture 5 and then exhibit a wide family of bipartite hypertournaments for which the conclusion of the conjecture holds.

The paper is organized as follows. In the next section, we prove a lemma (Lemma 7) which we call *the Majority Lemma*, and which is used to show the positive above-mentioned results. In Section 3, we provide the counterexample and positive results. The terminology not introduced in this paper can be found in [6].

2 The Majority Lemma

The Majority Lemma, Lemma 7, is the main technical result of this paper. To prove Lemma 7, we will use the following simple lemma.

Lemma 6. *Let G be a bipartite graph with partite sets U and W and let every vertex in U have degree at least $p \geq 1$ and every vertex in W have degree at most p, except for one vertex which has degree at most $2p-1$. Then G has a matching saturating U.*

Proof. By Hall's theorem, if for every $S \subseteq U$, $|S| \leq |N_G(S)|$ then G has a matching saturating U. Suppose that there is a subset S of U such that $|S| \geq |N_G(S)| + 1$. Let e be the number of edges in the subgraph of G induced by $S \cup N_G(S)$ and observe that

$$p|S| \leq e \leq (|N(S)| - 1)p + (2p-1) \leq (|S| - 2)p + (2p-1) = |S|p - 1,$$

a contradiction. $\square$

Proposition 14 proved in the next section shows that Lemma 7 cannot be extended to $n = 4$ and $p = 2$.

Lemma 7. *Let H be a p-partite k-hypertournament with $p \geq 2$. Let $n \geq 5$ and $n > k \geq 3$. If a majority p-partite tournament $M(H)$ has an (x, y)-path P of length at most 4, then H has such a path of length at most 4.*

Proof. It suffices to prove this lemma for the case when P is of length 4 as the other cases are simpler and similar. Thus, assume that $P = x_1x_2x_3x_4x_5$. By definition of a path, for every $i \in [4]$, x_i and x_{i+1} belong to different partite sets of H. Now consider the following cases covering all possibilities.

Case 1: $n \geq 9$ and $3 \leq k < n$ or $n \geq 7$ and $4 \leq k < n-1$. Observe that if for every $i \in \{1, 2, 3, 4\}$,

$$|A_H(x_i, x_{i+1})| > 3 \tag{1}$$

then we can choose distinct arcs $a_i \in A_H(x_i, x_{i+1})$ such that $x_1a_1x_2a_2x_3a_3x_4a_4x_5$ is the required path in H. In particular, inequalities (1) will hold if $\frac{1}{2}\binom{n-2}{k-2} > 3$.

If $n \geq 9$ and $3 \leq k < n$, we have

$$\frac{1}{2}\binom{n-2}{k-2} \geq \frac{n-2}{2} > 3$$

and hence inequalities (1) hold. If $n \geq 7$ and $4 \leq k < n-1$, we have

$$\frac{1}{2}\binom{n-2}{k-2} \geq \frac{(n-2)(n-3)}{4} > 3.$$

Case 2: $k = 3$ and $5 \leq n \leq 8$. Then

$$|A_H(x_i, x_{i+1})| \geq \frac{1}{2}\binom{n-2}{k-2} \geq \frac{1}{2}\binom{3}{1} = \frac{3}{2} \tag{2}$$

for $i = 1, 2, 3, 4$. Consider a bipartite graph G with partite sets $Z = \{z_1, z_2, z_3, z_4\}$ and $A(H)$. We have an edge $z_i a_j$ if $a_j \in A_H(x_i, x_{i+1})$. By (2), each vertex in Z has degree at least two. Since $k = 3$, vertices z_i and z_j in G have no common neighbor unless $|i - j| = 1$. Thus, every vertex of G in $A(H)$ has degree at most 2. Thus, by Lemma 6, G has a matching saturating Z. In other words, there are distinct $a_1, a_2, a_3, a_4 \in A(H)$ such that $x_1 a_1 x_2 a_2 x_3 a_3 x_4 a_4 x_5$ is a path in H.

Case 3: $k = 4$ and $5 \le n \le 6$. Consider the bipartite graph G constructed as in the previous case. Using the computations analogous to those in (2), we see that the minimum degree of a vertex in Z is at least 3 when $n = 6$ and at least 2 when $n = 5$. Since $k = 4$, there is no common neighbor of all vertices in Z. Thus, every vertex of G in $A(H)$ has degree at most 3. Now consider two subcases.

Subcase 1: $n = 6$. Since every vertex of G in $A(H)$ has degree at most 3 and every vertex of G in Z has degree at least 3, by Lemma 6, G has a matching saturating Z and we are done as in Case 2.

Subcase 2: $n = 5$. Recall that the minimum degree of a vertex in Z is at least 2. Suppose that there are two vertices of G in $A(H)$ of degree 3. This means that

$$N_G(z_i) \cap N_G(z_{i+1}) \cap N_G(z_{i+2}) \neq \emptyset \tag{3}$$

for $i = 1$ or 2. Indeed, since $k = 4$, $N_G(z_1) \cap N_G(z_j) \cap N_G(z_4) = \emptyset$ when either $j = 2$ or 3. Without loss of generality, we assume that (3) holds when $i = 1$ and let $e_1 \in N_G(z_1) \cap N_G(z_2) \cap N_G(z_3)$. Thus, $e_1 = x_1 x_2 x_3 x_4$.

If x_1 and x_4 are in different partite sets of H, then $x_1 e_1 x_4$. Since e_1 does not contain x_5, we can choose an arc e_2 of H which is different from e_1 such that $x_4 e_2 x_5$. Then $x_1 e_1 x_4 e_2 x_5$ is a path in H. Now we assume that x_1 and x_4 are in the same partite set of H. Then there is an arc e_1 of H such that $x_1 e_1 x_3$. Since the degree of z_3 in G is at least 2, we can choose an arc e_2 of H which is different from e_1 such that $x_3 e_2 x_4$. We can also choose an arc e_3 of H which is different from e_1 and e_2 such that $x_4 e_3 x_5$. Indeed, $e_3 \neq e_1$ since e_1 does not contain x_5 and $e_3 \neq e_2$ since the degree of z_4 in G is at least 2. Then $x_1 e_1 x_3 e_2 x_4 e_3 x_5$ is a path in H. Thus, we may assume that every vertex of G in $A(H)$ has degree at most 2, except for one vertex which has degree at most 3. Then we can use Lemma 6 and thus we are done as above.

Case 4: $k \in \{5, 6, 7\}$ and $n = k+1$. Consider the bipartite graph G constructed as in Case 2.

Subcase 1: $k \in \{6, 7\}$. Using the computations analogous to those in (2), we see that the minimum degree of a vertex in Z is at least 3. If there is a vertex with degree 4 in $A(H)$, then it means $\{x_1, x_2, x_3, x_4, x_5\}$ is a subset of a vertex set of an arc e_1 and the relative order is x_1, x_2, x_3, x_4, x_5. If x_1 and x_5 are in different partite sets, then $x_1 e_1 x_5$ is a path in H. Otherwise x_1 and x_4 are in different partite sets, so $x_1 e_1 x_4$. There is an arc e_2 different from e_1 such that $x_4 e_2 x_5$ (since the degree of z_4 is at least 3). Now $x_1 e_1 x_4 e_2 x_5$ is a path in H. Thus, we assume each vertex in $A(H)$ has degree at most 3, and we are done by Lemma 6.

Subcase 2: $k = 5$**.** Suppose that the lemma does not hold in this case. Using the computations analogous to those in (2), we see that the minimum degree of a vertex in Z is at least 2. To obtain a contradiction, it suffices to show that G has at most one vertex of degree at least 3 in $A(H)$. Suppose that G has at least two vertices of degree at least 3 in $A(H)$. This means that (3) holds for $i = 1$ or 2. Since H can have only one arc with vertex set $\{x_1, x_2, x_3, x_4, x_5\}$, we have

$$\sum_{j=2}^{3} |N_G(z_1) \cap N_G(z_j) \cap N_G(z_4)| \leq 1 \tag{4}$$

Without loss of generality, we assume that (3) holds when $i = 1$ and let $e_1 \in N_G(z_1) \cap N_G(z_2) \cap N_G(z_3)$. If we restrict e_1 to the vertices $\{x_1, x_2, x_3, x_4\}$, we obtain $e_3' = x_1x_2x_3x_4$.

If x_1 and x_4 are in the different partite sets, then $x_1e_1x_4$. Since the degree of z_4 in G is at least 2, we can choose an arc e_2 of H which is different from e_1 such that $x_4e_2x_5$. Then $x_1e_1x_4e_2x_5$ is a path in H, a contradiction. Now we assume x_1 and x_4 are in the same partite set. Then $x_1e_1x_3$. Since the degree of z_3 in G is at least 2, we can choose an arc e_2 of H which is different from e_1 such that $x_3e_2x_4$. Since the degree of z_4 in G is at least 2, we can choose an arc e_3 of H such that $x_4e_3x_5$ and $e_3 \neq e_2$. Suppose $e_3 = e_1$. Then $e_1 = x_1x_2x_3x_4x_5$ and $x_1e_1x_5$, a contradiction. Thus, $e_3 \neq e_1$ and $x_1e_1x_3e_2x_4e_3x_5$ is a path in H, a contradiction. □

3 Main Results

In Section 3.1, using the Majority Lemma and other results, we solve Conjecture 3 in affirmative. In Section 3.2, we describe a family of couterexamples to Conjecture 5 and prove a sufficient condition of when the statement of Conjecture 5 holds.

3.1 Results on Conjecture 3

Lemma 8. *Let $H = (V, A)$ be a multipartite k-hypertournament with at most one transmitter and let $M(H)$ be a majority multipartite tournament of H. Let $n \geq 5$ and $n > k \geq 3$. If $M(H)$ has at least one transmitter, then H has a 2-king.*

Proof. Let V_1 be the partite vertex set containing all transmitters of $M(H)$. Let v be the transmitter of H, if H has a transmitter, and an arbitrary transmitter of $M(H)$, otherwise. Clearly, $v \in V_1$. Observe that for every $u \in V \setminus V_1$, there is an arc $a \in A_H(v, u)$ implying that vau. Note that for every $w \in V_1 \setminus \{v\}$, there are a vertex $u \in V \setminus V_1$ and an arc e of H such that uew. As in Lemma 7, it is easy to see that $|A_H(v, u)| \geq 2$. Thus, there is an arc $a \in A_H(v, u)$ distinct from e implying that $vauew$ is a path. □

Lemma 9. *Let $H = (V, A)$ be a multipartite k-hypertournament and let $n \geq 5$ and $n > k \geq 3$. If H has at most one transmitter then H has a 4-king.*

Proof. Let $M(H)$ be a majority multipartite tournament of H. If $M(H)$ has no transmitters, then by Theorem 1, $M(H)$ has a 4-king x. By Lemma 7, x is a 4-king of H. If $M(H)$ has transmitters, then we apply Lemma 8. □

Lemma 10. *Let $H = (V, A)$ be a p-partite k-hypertournament with $k = 3$, $n = 4$ and $p \geq 2$. If H has at most one transmitter then H has a 4-king.*

Proof. By Theorem 2, this lemma holds for $p = 2$ and so we may assume that $p \geq 3$. It is well known that every k-hypertournament with more than k vertices has a Hamilton path [8]. Observe that for $p = 4$ the first vertex of a Hamilton path in H is a 3-king. Now we may assume that $p = 3$. Let $V = V_1 \cup V_2 \cup V_3$ be a partition of vertices of H. Without loss of generality, we may assume that $V_1 = \{x_1, x_2\}$, $V_2 = \{x_3\}$ and $V_3 = \{x_4\}$.

First assume that H has the unique transmitter v. If $v = x_3$ or $v = x_4$, then v is a 1-king of H. Thus, we assume without loss of generality that $v = x_1$. Since v is a transmitter, va_1x_3 and va_2x_4 for some arcs a_1 and a_2 of H. Since x_2 is not a transmitter, there is an arc e_1 such that ye_1x_2, where $y \in V_2 \cup V_3$. By the definition of a transmitter, v precedes y in every arc containing v and y. Consequently, there is an arc e_2 different from e_1 such that ve_2y. Hence $ve_2ye_1x_2$ is a path from v to x_2. So v is a 2-king.

Now assume that T has no transmitter. Consider the arc e_1 containing x_1, x_3, and x_4. If x_1 is in the first position of e_1, since x_2 is not a transmitter, there is an arc e_2 different from e_1 such that $x_3e_2x_2$ or $x_4e_2x_2$. Hence $x_1e_1x_3e_2x_2$ or $x_1e_1x_4e_2x_2$ is a path from x_1 to x_2, implying that x_1 is a 2-king. Without loss of generality, we now assume that x_3 is in the first position of e_1. Since x_2 is not a transmitter, there is an arc e_2, where x_3 or x_4 preceds x_2. Hence x_3 is a 2-king. □

Lemmas 9 and 10 imply the following result solving Conjecture 3 in affirmative.

Theorem 11. *Every multipartite hypertournament with at most one transmitter has a 4-king.*

3.2 Results on Conjecture 5

The next result describes a family of counterexamples to Conjecture 5.

Proposition 12. *For every $k \geq 3$, there is a bipartite k-hypertournament B without transmitters which has at most one 4-king in each of its partite sets.*

Proof. Let U and W be partite sets of B. Choose a vertex u in U and a vertex w in W. Let every arc of B with both u and w have both of them in the first and second position such that in at least one such arc u is the first and in at least one such arc w is the first. Let every arc of B containing u but not w have u in the

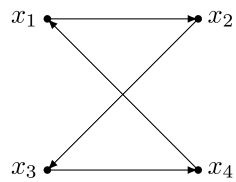

Figure 1: $M(H)$

first position and let every arc of B containing w but not u have w in the first position. Clearly, B has no transmitters, but no vertex v in $(U \cup W) \setminus \{u, w\}$ can be a 4-king as there is no path from v to either u or w. $\square$

The next result is a sufficient condition of when the conclusion of Conjecture 5 holds. It follows directly from Theorem 4 and the Majority Lemma.

Theorem 13. *Let B be a bipartite hypertournament with partite sets U and W and with at least 5 vertices. If a majority bipartite tournament $M(B)$ has no transmitters, then B has at least two 4-kings in each U and W.*

Our final result shows that the Majority Lemma cannot be extended to $n = 4$ and $p = 2$. The proof provides another counterexample to Conjecture 5.

Proposition 14. *For $k = 3$ and $n = 4$, there is a bipartite hypertournament H with partite sets U and W such that (i) $|U| = |W| = 2$, (ii) a majority bipartite tournament $M(H)$ has no transmitters, (iii) $M(H)$ has an (x, y)-path of length 3, but H has no (x, y)-path, (iv) H has only one 4-king in U.*

Proof. Let H be a bipartite hypertournament with partite sets $U = \{x_1, x_3\}$ and $W = \{x_2, x_4\}$, arc set $\{a_1, a_2, a_3, a_4\}$ where

$$a_1 = x_4x_1x_2, a_2 = x_2x_3x_4, a_3 = x_3x_2x_1, a_4 = x_4x_3x_1.$$

Let the arcs of $M(H)$ be $x_4x_1, x_1x_2, x_2x_3, x_3x_4$ (see Fig. 1). Clearly, (i) and (ii) hold and $x_1x_2x_3x_4$ is an (x_1, x_4)-path in $M(H)$.

Now consider H. Suppose that H has an (x_1, x_4)-path P. Since $A_B(x_1, x_4) = \emptyset$, $P = x_1b_1x_2b_2x_3b_3x_4$ for some distinct arcs b_1, b_2, b_3 of H. By inspection of the arcs of H, we conclude that $b_1 = a_1, b_2 = a_2, b_3 = a_2$, which is impossible since b_1, b_2, b_3 must be distinct. So H has no (x_1, x_4)-path and (iii) holds. Observe that x_3 is a 4-king of H since $x_3a_3x_2$, $x_3a_2x_4$ and $x_3a_2x_4a_1x_1$ is an (x_3, x_1)-path of length 2. Moreover, x_1 cannot be a 4-king by the discussion in (iii), so (iv) holds. $\square$

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