Matchings in 1-planar graphs with large minimum degree

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Abstract

In 1979, Nishizeki and Baybars showed that every planar graph with minimum degree 3 has a matching of size $\frac{n}{3}+c$ (where the constant c depends on the connectivity), and even better bounds hold for planar graphs with minimum degree 4 and 5. In this paper, we investigate similar matching-bounds for *1-planar* graphs, i.e., graphs that can be drawn such that every edge has at most one crossing. We show that every 1-planar graph with minimum degree 3 has a matching of size at least $\frac{1}{7}n + \frac{12}{7}$, and this is tight for some graphs. We provide similar bounds for 1-planar graphs with minimum degree 4 and 5, while the case of minimum degree 6 and 7 remains open.

1 Introduction

Matchings are one of the oldest and best-studied problems in graph theory, see for example the extensive reviews of matching theory in [9, 2]. We focus here on matchings in graph classes that are restricted to have special drawings. In particular, a graph is called planar if it can be drawn without crossing in the plane (detailed definitions are below). Nishizeki and Baybars [10] argued that every simple planar graph with $n \geq X$ vertices has a matching of size at least $Yn + Z$, where X, Y, Z depend on the minimum degree δ and the connectivity κ of the graph (they explore all possibilities of δ and κ). Their bounds are tight in the sense that some planar graph that satisfies the restrictions has no bigger matching.

The goal of this paper is to develop similar results for simple 1-planar graphs, i.e., graphs that can be drawn in the plane with at most one crossing per edge. These graphs have been of high interest to the graph theory community ever since Ringel introduced them in 1965 [11]; we refer the reader to an extensive annotated bibliography [8] for many results. To our knowledge, no previous matching-bounds were known for 1-planar graphs of given minimum degree. We prove here the following:

Theorem 1. *Any n-vertex simple 1-planar graph with minimum degree δ has a matching M of the following size:*

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1. $|M| \geq \frac{n+12}{7}$ *if* $\delta = 3$ *and* $n \geq 7$.
2. $|M| \geq \frac{n+4}{3}$ *if* $\delta = 4$ *and* $n \geq 20$.
3. $|M| \geq \frac{2n+3}{5}$ *if* $\delta = 5$ *and* $n \geq 21$.

All of our bounds are tight in the sense that there are arbitrarily large simple 1-planar graphs of the required minimum degree for which no matching can be larger. We also provide some simple 1-planar graphs with minimum degree 6 and 7 with upper bounds on the size of their matchings, though proving that these can always be achieved remains open. No simple 1-planar graph can have minimum degree 8 or higher.

Our proofs follow similar ideas as the proofs in Nishizeki and Baybars, but need some new results that do not immediately transfer from planar to 1-planar graphs. In particular, at the heart of the proofs in [10] lies the idea that a planar bipartite graph has at most $2n-4$ edges. It is known that every 1-planar bipartite graph has at most $3n-6$ edges, but inserting this into the proof from [10] would give no non-trivial matching-bounds for $\delta \leq 5$. We therefore need to develop different techniques to analyze how big an independent set in a 1-planar graph can be, given bounds on the minimum degree; this result may be interesting in its own right.

Preliminaries Let $G = (V, E)$ be a graph with n vertices and m edges; to avoid trivialities assume $n \geq 4$ throughout. We also assume familiarity with basic terms in graph theory; see e.g. [5] for details. A *matching* of G is a set of edges for which the endpoints are all distinct. An *independent set* of G is a set of vertices without edges between them. We assume that the input graph G is *simple*, i.e., has neither a loop nor a multiple edge. It will sometimes be convenient to add multiple edges, but only under restrictions specified below. G is *connected* if any two vertices are connected via a path. The *connectivity* of G is the maximum number κ such that removing any $\kappa - 1$ vertices leaves a connected graph. A *component* of G is a maximal connected subgraph; we call it a *singleton* if it has only one vertex. A *bipartite* graph is a graph $G = (V, E)$ where the vertices can be partitioned into *sides* $V = S \cup T$ such that each side is an independent set.

Nearly all papers that give lower bounds on matching-sizes (see e.g. [10, 6]) use the Tutte-Berge-Formula [1].

Theorem 2 (Tutte-Berge). *The size of a maximum matching M equals the minimum, over all vertex-sets S, of $\frac{1}{2}(n - (odd(G \setminus S) - |S|))$. Here, $odd(G \setminus S)$ denotes the number of components of odd cardinality in the graph $G \setminus S$.*

To prove a lower bound on a matching, one uses the following reformulation of the non-trivial direction of Theorem 2.

Corollary 1. *If $G = (V, E)$ is a graph such that $odd(G \setminus S) - |S| \leq cn - d$ for all vertex-sets $S \subset V$ and some constants c, d, then G has a matching of size at least $\frac{1-c}{2}n + \frac{d}{2}$.*

Planar and 1-planar graphs A *drawing* Γ of a graph G assigns points in $\mathbb{R}^2$ to vertices and curves in $\mathbb{R}^2$ to edges. In what follows, we usually identify the element of G (i.e., vertex or edge) with the geometric element in Γ (i.e., point or curve) that corresponds to it. All drawings are assumed to be *good* (see e.g. [12] for a detailed discussion), which means among others that no two vertices coincide, no edge intersects itself or a non-incident vertex, and any two edges intersect in at most one point and where they either end or fully cross. (For graphs that are not simple, multi-edges are permitted to meet twice, once at each end.) Whenever a drawing is fixed, we think of the edges incident to a vertex v as cyclically ordered according to the order in which they end at v.

A drawing is called *k-planar* if every edge has at most k crossings; in this paper all drawings are 1-planar and sometimes we restrict the attention to 0-planar ("planar") drawings. A graph is called *planar [1-planar]* if it has a planar [1-planar] drawing.

For the following definitions fix a planar drawing Γ. The *faces* of Γ are the connected pieces of $\mathbb{R}^2 \backslash \Gamma$, and described by giving the collection of circuits that form its boundary. A *bigon* is a face whose boundary is a single cycle consisting of two copies of the same edge. Our input graph is assumed to be simple, but we will sometimes add edges for counting arguments, and then allow multi-edges, but never bigons. We never allow loops. Any graph with a planar bigon-free drawing has at most $3n-6$ edges, and at most $2n-4$ edges if it is bipartite.

For the following definitions fix a 1-planar drawing Γ. We call an edge *crossed* if it contains a crossing and *uncrossed* otherwise. The *planarization* Γ_P of Γ is the planar drawing obtained by replacing every crossing with a dummy-vertex of degree 4. The *regions* of Γ are the faces of its planarization Γ_P, the *corners* of a region of Γ are the vertices of the corresponding face of Γ_P; corners are vertices or crossings of Γ.

A *bigon* of Γ is a bigon of Γ_P. Any bigon-free loop-free 1-planar graph has at most $4n-8$ edges, and at most $3n-6$ edges if it is bipartite. We need a slightly more detailed bound.

Observation 1. *Let G be a bipartite graph with a 1-planar bigon-free drawing that has $m_\times$ crossed and m_- uncrossed edges. Then $\frac{1}{2}m_\times + m_- \leq 2n-4$.*

Proof. The crossed edges come in pairs, and if we remove one edge from each pair then we obtain a planar bipartite bigon-free drawing. This has at most $2n-4$ edges, and so $\frac{1}{2}m_\times + m_- \leq 2n-4$. □

2 1-planar graphs without large matchings

In this section, we create some 1-planar graphs that have large minimum degree and for which the maximum matching is small.

Lemma 1. *For any N, there exists a simple 1-planar graph with minimum degree 3 and $n \geq N$ vertices for which any matching has size at most $\frac{n+12}{7}$.*

Proof. Consider the graph in Fig. 1(a), which has been built as follows. Start with an arbitrary planar graph H on s vertices, where $s \geq \max\{3, \frac{N+12}{7}\}$, such that all faces of H are

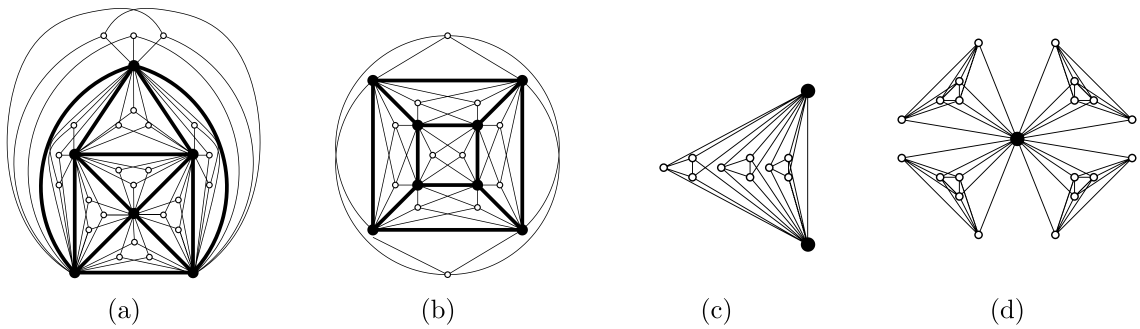

Figure 1: Graphs that do not have large matchings. Graph H is bold, vertices in S are black. (a) Minimum degree 3. (b and c) Minimum degree 4. (d) Minimum degree 5.

triangles. Into each face $\{u, v, w\}$ of H, insert three more vertices that are all adjacent to all of u, v, w. Obviously the resulting graph G has minimum degree 3, and the figure shows that G is 1-planar. Also, H has $2s-4$ faces, hence G has $n = s + 3(2s-4) = 7s - 12 \geq N$ vertices. Setting S to be the s vertices of H, we observe that every vertex in $G \setminus S$ becomes a singleton component. So

$$odd(G \setminus S) - |S| = 3(2s-4) - s = 5s - 12 = \frac{5n-24}{7}.$$

By Theorem 2 therefore any matching has size at most $\frac{n+12}{7}$. □

Lemma 2. *For any N, there exists a simple 1-planar graph with minimum degree 4 and $n \geq N$ vertices for which any matching has size at most $\frac{n+4}{3}$.*

Proof. Consider the graph in Fig. 1(b), which has been built as follows. Start with a planar graph H on s vertices, where $s \geq \max\{4, \frac{N+4}{3}\}$, such that all faces of H are simple cycles of length 4. Into each face $\{u, v, w, x\}$ of H, insert two more vertices that are all adjacent to all of u, v, w, x. Obviously the resulting graph G has minimum degree 4 and the figure shows that G is 1-planar. Also, H has $s-2$ faces, hence G has $n = s + 2(s-2) = 3s - 4 \geq N$ vertices. Setting S to be the s vertices of H, we observe that every vertex in $G \setminus S$ becomes a singleton component. So

$$odd(G \setminus S) - |S| = 2(s-2) - s = s - 4 = \frac{n-8}{3}.$$

By Theorem 2 therefore any matching has size at most $\frac{n+4}{3}$. □

We note here that the same lower bound can be achieved with a much simpler construction that combines $(n-2)/3$ copies of the complete graph K_5 at an edge (see Fig. 1(c)), but the connectivity of the resulting graph is not as high.

Lemma 3. *For any N, there exists a simple 1-planar graph with minimum degree 5 and $n \geq N$ vertices for which any matching has size at most $\frac{2n+3}{5}$.*

Proof. Consider the graph in Fig. 1(d), which has been built as follows. Let $n \geq N$ be such that $n \equiv 1 \bmod 5$. Create one vertex v_s, and split the remaining vertices into $(n-1)/5$ groups of five vertices each. For each group $\{v_1, \ldots, v_5\}$, inserted edges to turn $\{s, v_1, \ldots, v_5\}$ into a complete graph K_6. Obviously the resulting graph G has minimum degree 5, and the figure shows that G is 1-planar.

Setting S to be the single vertex v_s, we observe that each of the $(n-1)/5$ groups become an odd component of $G \setminus S$. Hence

$$odd(G \setminus S) - |S| = (n-1)/5 - 1 = \frac{n-6}{5}.$$

By Theorem 2 therefore any matching has size at most $\frac{2n+3}{5}$. $\square$

3 Lower bounds on the matching-size

In this section, we prove Theorem 1 by proving bounds on $odd(G \setminus S) - |S|$ for a 1-planar graph G and an arbitrary vertex-set S under assumptions on the minimum degree.

We first briefly review the technique by Nishizeki and Baybars [10] to prove matchings bounds in a planar graph G of minimum degree 3. Let a vertex-set S be given. By Corollary 1 it suffices to show that $odd(G \setminus S) - |S| \leq \frac{n+8}{3}$. To this end, delete any edge that connects two vertices in S, and delete any component of $G \setminus S$ that has even cardinality; note that both do not increase $odd(G \setminus S)$ and can only decrease n. Also delete any odd component of $G \setminus S$ that contains at least 3 vertices; this decreases $odd(G \setminus S)$ by one and n by at least 3 and so does not make the bound worse. We end with a planar bipartite graph G' where one side is S and the other side T has one vertex for each singleton component of $G \setminus S$. Furthermore, no edges incident to a vertex in T was deleted, so $\deg(t) \geq 3$ for all $t \in T$. Since G' has $n(G') = |S| + |T|$ vertices, it has at most $2(|S|+|T|) - 4$ edges and at least $3|T|$ edges, so $|T| \leq 2|S| - 4$. Therefore $3\,(odd(G' \setminus S) - |S|) = 3|T| - 3|S| \leq 2|T| - 4|S| + n(G') \leq n(G') - 8$ whence the matching-bound follows.

3.1 Independent sets in 1-planar graphs

The crucial ingredient for the proof by Nishizeki and Baybars is the bound $|T| \leq 2|S| - 4$ in a planar bipartite graph $(S \cup T, E)$ where all vertices in T have minimum degree 3. In this section, we aim to show similar bounds for 1-planar graphs. This requires entirely different techniques than the simple edge-counting argument that sufficed for planar graphs. We phrase it as a slightly more general statement about independent sets in 1-planar graphs.

Lemma 4. *Let G be simple 1-planar graph. Let T be a non-empty independent set in G where $\deg(t) \geq 3$ for all $t \in T$. Let T_d be the vertices in T that have degree d. Then*

$$2|T_3| + \sum_{d \geq 4} (3d-6)|T_d| \leq 12|V \setminus T| - 24.^1$$

We will at the same time prove another result that is related (but neither result implies the other); this will be used in a future paper [4]. Define the *crossing-weighted degree* of a vertex $v \in V$ to be the degree plus the number of incident uncrossed edges. Thus, uncrossed edges count doubly.

Lemma 5. *Let G be a simple graph with a 1-planar drawing Γ. Let T be a non-empty independent set in G where $\deg(t) \geq 3$ for all $t \in T$. Let W_d be the vertices in T that have crossing-weighted degree d. Then*

$$2|W_3| + 2|W_4| + \sum_{d \geq 5} (3d-12)|W_d| \leq 12|V \setminus T| - 24.$$

Proof. (of both Lemma 4 and 5) Fix a 1-planar drawing Γ of G if not given yet. We use a charging scheme, where we assign some *charges* (units of weight) to edges in G (as well as some additions that we make to G), redistribute those to the vertices in T, and then count the number of charges in two ways to obtain the bound.

Step 0: Make G bipartite. Let S be the vertices that are not in T, and note that $|S| \geq 3$ since T is non-empty and vertices in it have degree 3 or more. Delete all edges within S so that G becomes bipartite; this does not affect degrees in T, can only increase crossing-weighted degrees in T by making some edges uncrossed, and so it suffices to prove the bound for the resulting graph.

Step 1: Add more edges. Now add any edge to Γ that can be added without crossing while remaining bipartite. This can only increase degrees of vertices in T and so improve the bound. We are allowed to add multiple edges, as long as they do not form a bigon, see edge $(\mathtt{A}, \mathtt{d})$ in Fig. 2.

We claim that in Γ' no vertex $t \in T$ has three consecutive crossed edges e_1, e_2, e_3 (in the cyclic order of edges defined by Γ'). Assume there were three such consecutive edges in Γ, and let (t', s') (with $t' \in T$ and $s \in S$) be the edge that crosses e_2, say at c. (For an illustration, consider vertex $t = \mathtt{d}$ in Fig. 2, which has three consecutive crossed edges to $\mathtt{E}, \mathtt{A}$ and $\mathtt{C}$ in Γ; hence $(t', s') = (\mathtt{c}, \mathtt{B})$.) We can add an uncrossed edge $e' = (s', t)$ by tracing along c; this edge would end before or after e_2 in the clockwise order at t. Adding e' does not create a bigon, since on one side of e' the region contains c and on the other side the region contains the crossing in e_1 or e_3. So we added this edge when creating Γ' from Γ, and no three consecutive crossed edges in Γ remain consecutive in Γ'.

[1] This bound is not tight, and in fact $2|T_3| + \sum_{d \geq 4}(3d + 3\lceil d/3 \rceil - 12)|T_d| \leq 12|V \setminus T| - 24$ could be shown with much the same proof.

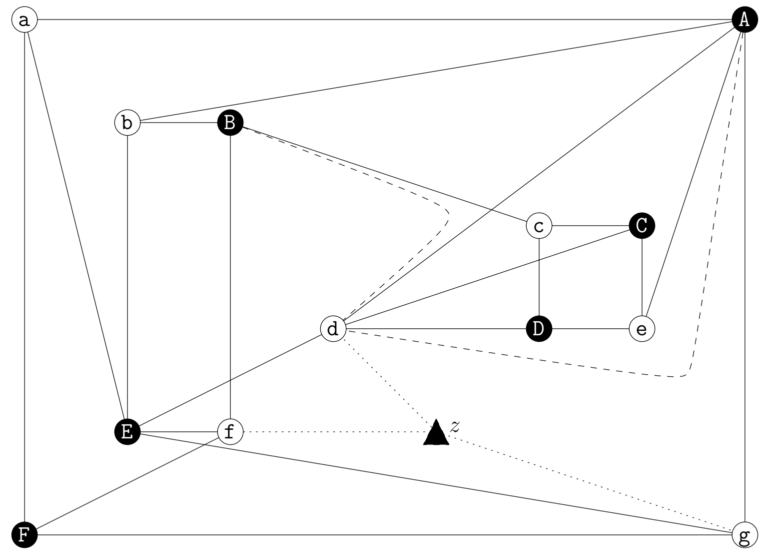

Figure 2: A drawing Γ of a bipartite 1-planar graph with minimum degree 3; vertices of T are white. Edges in $\Gamma' \setminus \Gamma$ are dashed, the unique vertex in S_Δ is a triangle and edges in E_Δ are dotted.

Step 2: Add vertices and edges. As a next step, we possibly augment Γ' further with vertices S_Δ. Assume there exists a region R in Γ' incident to at least three vertices in T. Add a new vertex z to S_Δ and connect it to exactly three vertices of T on R via uncrossed edges within R. This splits R into three regions, each with fewer vertices of T, so we can repeat until no such regions remain. See the new vertex incident to $\mathtt{d}, \mathtt{f}, \mathtt{g}$ in Fig. 2. Let E_Δ be the added edges; we have $|E_\Delta| = 3|S_\Delta|$. Also observe that edges in E_Δ are uncrossed and that the resulting drawing Γ_Δ is again bipartite.

Step 3: Assigning charges. We assign charges as follows: Let E_- be the uncrossed edges of Γ'; each of those receives 6 charges. Let $E_\times$ be the crossed edges of Γ'; each of those receives 3 charges. Finally the (uncrossed) edges E_Δ of $\Gamma_\Delta - \Gamma'$ receive 2 charges each. Using Observation 1 hence

$$\begin{aligned}
\#\text{charges} &= 6|E_-| + 3|E_\times| + 2|E_\Delta| = 6(|E_-| + |E_\Delta|) + 3|E_\times| - 4|E_\Delta| \\
&\leq 12|S| + 12|S_\Delta| + 12|T| - 24 - 12|S_\Delta| \\
&= 12|S| + 12|T| - 24. \qquad (1)
\end{aligned}$$

Step 4: Charges at a vertex. For $t \in T$, let $c(t)$ be the total charges of incident edges of t. We lower-bound $c(t)$ as follows:

- Assume first that t has at least two incident uncrossed edges in Γ'. It obtains 12 charges from these two edges, and at least $3(\deg(t) - 2)$ further charges from the remaining edges that it had in G. Hence $c(t) \geq 12 + 3(\deg(t) - 2) = 3\deg(t) + 6 \geq 15$.
- Now assume that t has at most one uncrossed incident edge in Γ'. We aim to show that $c(t) \geq 14$. By $\deg(t) \geq 3$, and since no three crossed edges are consecutive, vertex

t has at least one uncrossed edge, so it has exactly one, call it e_1. This implies that t has only three incident edges in Γ', else the edges other than e_1 would contain three consecutive crossed edges. So this case can occur only if $t \in T_3$ and it has exactly one uncrossed edge $e_1 =: (t, s_1)$ and two crossed edges (t, s_2) and (t, s_3). See vertex $t = \texttt{f}$ in Fig. 2, where $\{s_2, s_3\} = \{\texttt{B}, \texttt{F}\}$.

Let (s', t') be the edge that crosses (t, s_2) in Γ, with $s' \in S$ and $t' \in T$. Then (as above) an edge (s', t) could be drawn without crossing and could have been added to Γ'. Since the only uncrossed edge at t is e_1 therefore $s' = s_1$. Similarly one argues that (t, s_3) is crossed by edge (s_1, t'') for some $t'' \in T$. (In Fig. 2 we have $\{t', t''\} = \{\texttt{d}, \texttt{g}\}$.) If we had $t' = t''$ then there would be two copies of edge (s_1, t'), and both would be crossed. Since G is simple and no crossed edges were added for Γ', this is impossible and $t' \neq t''$.

Observe that hence t, t', t'' all belong to the same region R between the two crossings in (t, s_2) and (t, s_3). Therefore we added a vertex z of S_Δ inside R and made it adjacent to t. Edge (t, z) has two charges, and in total we have $c(t) \geq 6 + 3 + 3 + 2 = 14$.

- (The following is relevant only for Lemma 5.) Assume $t \in W^d$. Then t receives 6 charges for every uncrossed edge that was incident in Γ, and 3 charges for every crossed edge that was incident in Γ (and possibly some more from edges added in later steps). Since uncrossed edges count twice for the crossing-weighted degree, hence $c(v) \geq 3d$.

To prove Lemma 4, use $c(t) \geq 14$ for all t, $c(t) \geq 18$ for $t \in T_4$ and $c(t) \geq 21$ for $t \in T_d$ with $d \geq 5$ since a vertex in T_d has degree at least d in Γ'. Therefore

$$\#\text{charges} \;=\; \sum_{t \in T} c(t) \geq 14|T_3| + 18|T_4| + 21 \sum_{d \geq 5} |T_d|. \tag{2}$$

To prove Lemma 5, observe that again $c(t) \geq 14$ for all t and $c(t) \geq 3d$ for $t \in W_d$. Therefore

$$\#\text{charges} \;=\; \sum_{t \in T} c(t) \geq 14|W_3| + 14|W_4| + \sum_{d \geq 5} 3d\,|W_d|. \tag{3}$$

Combining this with (1) and subtracting $12|T| = 12 \sum_{d \geq 3} |T_d| = 12 \sum_{d \geq 3} |W_d|$ from both sides gives the results. □

3.2 Matching-bounds

Now we use Lemma 4 to obtain the desired matching-bounds. For minimum degree 3 and 4 we proceed almost exactly as done by Nishizeki and Baybars [10]: preprocess the graph to remove some edges and components that can do no harm, and then use the upper bound on the resulting independent set in a 1-planar graph.

Lemma 6. *Let G be a simple 1-planar graph with minimum degree $\delta \geq 3$. Then for any vertex set S with $|S| \geq 2$, we have*

- $odd(G \setminus S) - |S| \le \frac{5}{7}n - \frac{24}{7}$ *if* $\delta \ge 3$*, and*

- $odd(G \setminus S) - |S| \le \frac{1}{3}n - \frac{8}{3}$ *if* $\delta \ge 4$*.*

Proof. Set $c_3 = \frac{5}{7}$, $d_3 = \frac{24}{7}$, $c_4 = \frac{1}{3}$ and $d_4 = \frac{8}{3}$; the goal is to show $odd(G \setminus S) - |S| \le c_\delta n - d_\delta$.

We first preprocess G by removing any component of $G \setminus S$ that has even size. This does not affect $|S|$ or $odd(G \setminus S)$, or degrees of vertices in $V \setminus S$ that remain, and it can only decrease n. So it suffices to prove the bound for the remaining graph.

Next remove all odd components of $G \setminus S$ that have three or more vertices, and let G' be the resulting graph. If k components are removed, then hence $n(G') \le n - 3k$. We will show below that $odd(G' \setminus S) \le c_\delta n(G') - d_\delta$, and therefore (by $c_\delta \ge \frac{1}{3}$)

$$odd(G \setminus S) = odd(G' \setminus S) + k \le c_\delta n(G') - d_\delta + c_\delta 3k \le c_\delta n - d_\delta.$$

It remains to show the claim for G'. Let $T = V(G') \setminus S$, and notice that these are exactly the singleton components of $G \setminus S$ since all other components were removed. In particular they form an independent set in G'. Let T_d be the vertices in T that have degree d in G'.

If $|T| = 0$ then $G' \setminus S$ is empty, so $n(G') = |S| \ge 2$ and

$$odd(G' \setminus S) - |S| = -|S| \le -2 = 2c_\delta - d_\delta \le c_\delta n - d_\delta$$

as desired. If T is non-empty, then apply Lemma 4 to G', and also observe that $n(G') = |S| + |T|$. If $\delta = 3$, then $2|T| = \sum_{d \ge 3} 2|T_d| \le 2|T_3| + 6|T_4| + 9\sum_{d \ge 5} |T_d| \le 12|S| - 24$ by Lemma 4. Therefore

$$7odd(G' \setminus S) - 7|S| = 7|T| - 7|S| = 2|T| - 12|S| + 5n(G') \le 5n(G') - 24.$$

If $\delta = 4$, then T_3 is empty and $6|T| = \sum_{d \ge 4} 6|T_d| \le 12|S| - 24$ by Lemma 4. Therefore

$$9odd(G' \setminus S) - 9|S| = 9|T| - 9|S| = 6|T| - 12|S| + 3n(G') \le 3n(G') - 24.$$

The desired bound follows by dividing suitably. □

With this we can obtain the first two matching bounds.

Proof. (of Theorem 1(a) and (b)) Let G be a 1-planar graph with minimum degree $\delta \in \{3, 4\}$. Fix an arbitrary vertex set S. If $|S| \ge 2$ then Lemma 6 gives $odd(G \setminus S) - |S| \le \frac{5}{7}n - \frac{24}{7}$ and $odd(G \setminus S) - |S| \le \frac{1}{3}n - \frac{8}{3}$, respectively, and Corollary 1 gives the result. So we only have to bound $odd(G \setminus S) - |S|$ for small S. Let X be the smallest odd integer with $X \ge \delta + 1 - |S|$, and note that any odd component of $G \setminus S$ must have at least X vertices by simplicity and the minimum degree requirement. So $odd(G \setminus S) \le \frac{n - |S|}{X}$. Now distinguish cases.

- If $|S| = 0$ then $X = 5$ and $odd(G \setminus S) - |S| \le \frac{n}{5}$. This is at most $\frac{5}{7}n - \frac{24}{7}$ for $n \ge 7$ and at most $\frac{1}{3}n - \frac{8}{3}$ for $n \ge 20$.

- If $|S| = 1$ and $\delta = 3$ then $X = 3$ and $odd(G \setminus S) - |S| \le \frac{n-1}{3} - 1 \le \frac{5}{7}n - \frac{24}{7}$ by $n \ge 6$.

- If $|S| = 1$ and $\delta = 4$ then $X = 5$ and $odd(G \setminus S) - |S| \leq \frac{n-1}{5} - 1 \leq \frac{1}{3}n - \frac{8}{3}$ by $n \geq 11$.

$\square$

For graphs with minimum degree 5 we have to keep more odd components of G.

Lemma 7. *Let G be a 1-planar simple graph with minimum degree 5. Then for any vertex set S with $|S| \geq 1$, we have $odd(G \setminus S) - |S| \leq \frac{1}{5}n - \frac{6}{5}$.*

Proof. As in the proof of Lemma 6 we can remove components of $G \setminus S$ that have even size without affecting the bound. Let G' be the graph obtained from G by removing all odd components of $G \setminus S$ that have five or more vertices. If k components are removed, then hence $n(G') \leq n - 5k$. We will show below that $odd(G' \setminus S) \leq \frac{1}{5}n(G') - \frac{6}{5}$, and therefore

$$odd(G \setminus S) = odd(G' \setminus S) + k \leq \frac{1}{5}n(G') - \frac{6}{5} + \frac{1}{5}5k \leq \frac{1}{5}n - \frac{6}{5}.$$

It remains to show the claim for G'. If $odd(G' \setminus S) = 0$, then by $n(G') = |S| \geq 1$ and $odd(G' \setminus S) - |S| \leq -1 = \frac{1}{5} - \frac{6}{5} \leq \frac{1}{5}n(G') - \frac{6}{5}$ and we are done. So assume $odd(G' \setminus S) > 0$. In contrast to the proof of Lemma 6, $G' \setminus S$ is not necessarily an independent set, because components of size 3 may have edges within them. In contrast to the approach taken by Nishizeki and Baybars [10], we cannot contract such components into a vertex, because 1-planarity may not be preserved under contraction. So we need a different approach.

For $i = 1, 3$, let $\mathcal{C}_i$ be the components of $G' \setminus S$ that have size i. Use $V(\mathcal{C}_i)$ for the vertices of components in $\mathcal{C}_i$, hence $|V(\mathcal{C}_1)| = |\mathcal{C}_1|$ while $|V(\mathcal{C}_3)| = 3|\mathcal{C}_3|$. Let H be the 1-planar graph obtained by deleting all edges within $V(\mathcal{C}_3)$; this makes $V(\mathcal{C}_3) \cup V(\mathcal{C}_1)$ an independent set in H. Any vertex $v \in V(\mathcal{C}_3)$ has at least five neighbours in G and at most two neighbours in its odd component, so $\deg_H(v) \geq 3$. All vertices in $V(\mathcal{C}_1)$ have degree at least 5 in H. Let T_d for $d \geq 3$ be the vertices of degree d in $V(\mathcal{C}_3) \cup V(\mathcal{C}_1)$, then any vertex in $T_3 \cup T_4$ must belong to $V(\mathcal{C}_3)$. Applying Lemma 4 to H, therefore

$$12|S| - 24 \geq 2|T_3| + 6|T_4| + 9\sum_{d \geq 5} |T_d| \geq 2|V(\mathcal{C}_3)| + 9|V(\mathcal{C}_1)|.$$

Since $n(G') = |V(\mathcal{C}_1)| + |V(\mathcal{C}_3)| + |S|$, this implies

$$\begin{aligned} 21\,(odd(G' \setminus S) - |S|) &= 21|\mathcal{C}_1| + 21|\mathcal{C}_3| - 21|S| \\ &= 21|V(\mathcal{C}_1)| + 7|V(\mathcal{C}_3)| - 21|S| \\ &= 18|V(\mathcal{C}_1)| + 4|V(\mathcal{C}_3)| - 24|S| + 3n(G') \\ &\leq (24|S| - 48) - 24|S| + 3n(G') = 3n(G') - 48 \end{aligned}$$

which gives $odd(G' \setminus S) - |S| \leq \frac{1}{7}n(G') - \frac{16}{7} \leq \frac{1}{5}n(G') - \frac{6}{5}$. $\square$

Proof. (of Theorem 1(c)) Let G be a 1-planar graph with minimum degree 5. Fix an arbitrary vertex set S. If $|S| \geq 1$ then the previous lemma gives $odd(G \setminus S) - |S| \leq \frac{1}{5}n - \frac{6}{5}$. If $|S| = 0$, then every odd component of $G \setminus S = G$ must contain at least 7 vertices since the minimum degree is 5; hence $odd(G \setminus S) - |S| \leq \frac{n}{7} \leq \frac{1}{5}n - \frac{6}{5}$ by $n \geq 21$. Either way the bound follows from Corollary 1. $\square$

4 Other classes of 1-planar graphs

In this section, we construct some other 1-planar graphs that do not have large matchings, and offer some conjectures.

4.1 Non-simple 1-planar graphs

Our matching bounds were proved for simple 1-planar graphs. Obviously no non-trivial matching bounds can exist if we permit bigons, because then $K_{2,n}$ (with edges repeated as needed to achieve any desired minimum degree) has no matching with more than 2 edges.

In fact, even excluding bigons is not enough to ensure a matching of linear size. The 1-planar drawing in Fig. 3(a) has no bigon and the graph has minimum degree 3, yet it has no matching of size exceeding 2 since removing the two black vertices leaves behind $n-2$ singleton components.

The story is different for minimum degree $\delta \geq 4$ or if no two copies of an edge are both crossed. Inspecting the proof of Lemma 4, we see that simplicity is used in two places: we need that no bigons exist to use Observation 1, and we need to exclude the possibility that $t' = t''$ in the second case of Step 4, where we bound $c(t) \geq 14$ if t has only one incident uncrossed edge. This case is never needed if $\deg(t) \geq 4$, and only uses that no two crossed edges (s_1, t') exist otherwise. So Theorem 1 holds for any 1-planar graph with a bigon-free drawing and additionally the minimum degree is at least 4 and/or there are no multiple crossed copies of an edge.

4.2 1-planar graphs of higher minimum degree

For planar graphs, matching-bounds are interesting only for $\delta = 3, 4, 5$, because for $\delta = 2$ there are no linear bounds (consider $K_{2,n}$), and for $\delta \geq 6$ there exists no planar bigon-free graph of minimum degree δ. In contrast to this, there are simple 1-planar graphs of minimum degree 6 or 7, while no simple 1-planar graph can have minimum degree $\delta \geq 8$ since it has at most $4n - 8$ edges [13]. Naturally one wonders what matching bounds can be obtained for $\delta = 6, 7$.

Lemma 8. *For any N, there exists a simple 1-planar graph with minimum degree 6 and $n \geq N$ vertices for which any matching has size at most $\frac{3}{7}n + \frac{4}{7}$.*

Proof. Consider the graph in Fig. 3(b), which has been built as follows. Let $n \geq N$ be such that $n \equiv 1 \bmod 7$. Create one vertex v_s, and split the remaining vertices into $(n-1)/7$ groups of seven vertices each. For each group $\{v_1, \ldots, v_7\}$, inserted edges to turn $\{v_s, v_1, \ldots, v_7\}$ into a cube plus a crossing within each face of the cube. Obviously the resulting graph G has minimum degree 6, and the figure shows that G is 1-planar.

Setting S to be the single vertex v_s, we observe that each of the $(n-1)/7$ groups become an odd component of $G \setminus S$. Hence

$$odd(G \setminus S) - |S| = (n-1)/7 - 1 = \frac{n-8}{7}.$$

Therefore any matching has size at most $\frac{3n+4}{7}$. $\square$

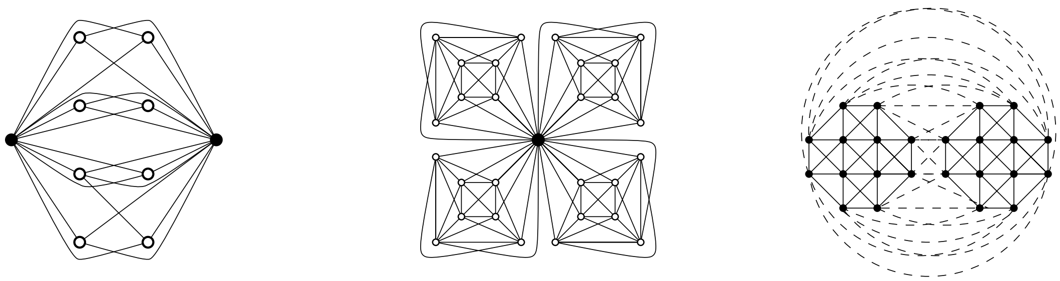

Figure 3: Other 1-planar graphs without large matchings. (a) A bigon-free 1-planar graph with minimum degree 3 and maximum matching size 2. (b) A 1-planar graph with minimum degree 6. (c) A 1-planar graph with minimum degree 7.

We suspect that this is tight.

Conjecture 1. *Any 1-planar graph with minimum degree 6 and $n \geq N$ vertices has a matching of size at least $\frac{3}{7}n + O(1)$.*

One may wonder why the proof of Lemma 7 cannot be generalized to minimum degree 6. The problem are components of size 5 in $G \setminus S$. If we remove these to obtain G', then we remove $5k$ vertices for k odd components and cannot hope for an upper bound better than $\frac{n}{5} + O(1)$ for $odd(G \setminus S) - |S|$. If we keep components of size 5 in G', then each vertex t of a component C of size 5 could have four neighbours in C, hence only two neighbours in S, and Lemma 4 cannot be used.

For minimum degree 7, we similarly have a graph without a perfect matching, but only conjectures as to whether this is tight.

Lemma 9. *For any N, there exists a simple 1-planar graph with minimum degree 7 and $n \geq N$ vertices for which any matching has size at most $\frac{11}{23}n + \frac{12}{23}$.*

Proof. Let $n \geq N$ be such that $n \equiv 1 \bmod 23$. Create one vertex v_s, and split the remaining vertices into $(n-1)/23$ groups of 23 vertices each. For each group, insert edges to turn these 23 vertices, plus vertex v_s, into one a simple 1-planar graph of minimum degree 7, see Fig. 3(c). Setting S to be the single vertex v_s, we observe that each of the $(n-1)/23$ groups become an odd component of $G \setminus S$. Hence

$$odd(G \setminus S) - |S| = \frac{n-1}{23} - 1 = \frac{n-24}{23}.$$

Therefore any matching has size at most $\frac{11n+12}{23}$. $\square$

The above example is best in the sense that any 1-planar simple graph with minimum degree 7 has at least 24 vertices [3]. However, exploiting this to show that the bound in Lemma 9 is tight remains an open problem.

5 Open problems

We leave some gaps in our analysis; in particular one wonders what matching-bounds are tight for simple 1-planar graphs of minimum degree 6 or 7. Also, there are many other related graph classes that could be studied. What, for example, is the size of matchings in 2-planar graphs of minimum degree δ?

1-planar graphs of higher connectivity are also worth exploring. For $\delta = 3, 4$, our constructed graphs have connectivity δ, which is the best one can hope for. But for $\delta \geq 5$ our constructed graphs have a cutvertex. Are there better matching-bounds for 5-connected 1-planar graphs with minimum degree 5? In contrast to planar graphs, 5-connected 1-planar graphs do not necessarily have a Hamiltonian path, and therefore not necessarily a *near-perfect matching* of size $\lceil (n-1)/2 \rceil$ [7]. Do all 6-connected 1-planar graphs have a near-perfect matching? How about all 7-connected ones?

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