A necessary condition for generic rigidity of bar-and-joint frameworks in d-space

Hakan Guler*
and
Bill Jackson†

Abstract

A graph $G = (V, E)$ is d-sparse if each subset $X \subseteq V$ with $|X| \geq d$ induces at most $d|X| - \binom{d+1}{2}$ edges in G. Maxwell showed in 1864 that a necessary condition for a generic bar-and-joint framework with at least $d+1$ vertices to be rigid in $\mathbb{R}^d$ is that G should have a d-sparse subgraph with $d|X| - \binom{d+1}{2}$ edges. This necessary condition is also sufficient when $d = 1, 2$ but not when $d \geq 3$. Cheng and Sitharam strengthened Maxwell's condition by showing that *every* maximal d-sparse subgraph of G should have $d|X| - \binom{d+1}{2}$ edges when $d = 3$. We extend their result to all $d \leq 11$.

1 Introduction

A *d-dimensional (bar-and-joint) framework* is a pair (G, p) where $G = (V, E)$ is a graph and $p : V \to \mathbb{R}^d$. It is a long standing open problem to determine when a given bar-and-joint framework is *rigid* i.e. every continuous motion of the points $p(v)$ which preserves the distances $\|p(u) - p(v)\|$ for all $uv \in E$ must also preserve the distances $\|p(u) - p(v)\|$ for all $u, v \in V$. It is not difficult to see that a 1-dimensional framework (G, p) is rigid if and only if the graph G is connected. Abbot [1] showed that the problem of determining rigidity is NP-hard for all $d \geq 2$ but the problem becomes more tractable if we assume that the framework is *generic* i.e. there are no algebraic dependencies between the coordinates of the points $p(v)$, $v \in V$.

Given a graph $G = (V, E)$, we can define a $|E| \times d|V|$ matrix, the *d-dimensional rigidity matrix* $R_d(G)$, whose entries are linear combinations of indeterminates representing the coordinates of the points $p(v)$, in such a way that a generic framework (G, p) with at least $d + 1$ vertices is rigid if and only if the rank $r_d(G)$ of $R_d(G)$ is equal to $d|V| - \binom{d+1}{2}$. This naturally gives rise to a matroid on E, the *d-dimensional rigidity matroid* $\mathcal{R}_d(G)$ in which a set of edges $F \subseteq E$ is *independent* if and only if the corresponding rows of $R_d(G)$ are linearly independent. We refer the reader to [10] for a precise definition of

*Department of Mathematics, Faculty of Arts & Sciences, Kastamonu University, Kastamonu, Turkey. e-mail: hakanguler19@gmail.com.

†School of Mathematical Sciences, Queen Mary University of London, Mile End Road, London E1 4NS, England. e-mail: b.jackson@qmul.ac.uk.

the rigidity matrix, the rigidity matroid, and other information on the topic of combinatorial rigidity.

Pollaczek-Geiringer [9] and subsequently Laman [6] characterized when a 2-dimensional generic framework is rigid (see also Lovász and Yemini [7]). Their characterization is based on the following concept. We say that a graph $G = (V, E)$ is *d-sparse* if each $X \subseteq V$ with $|X| \geq d+1$ induces at most $d|X| - \binom{d+1}{2}$ edges of G. Maxwell [8] showed that being d-sparse is a necessary condition for the rows of $R_d(G)$ to be linearly independent. Pollaczek-Geiringer and Laman showed that that this condition is also sufficient when $d = 2$ and deduced that a 2-dimensional generic framework (G, p) is rigid if and only if it has a 2-sparse subgraph with $2|V|-3$ edges. Since every independent set of edges in $\mathcal{R}_2(G)$ can be extended to a base of $\mathcal{R}_2(G)$, Laman's theorem implies that every maximal 2-sparse subgraph of G has the same number of edges.

It is known that the condition that H is a d-sparse subgraph of G is not sufficient for the edges of H to be independent in $\mathcal{R}_d(G)$ when $d \geq 3$. Indeed it is not even true that all maximal d-sparse subgraphs of G have the same number of edges when $d \geq 3$. On the other hand, Cheng and Sitharam [3] have shown that the number of edges in any maximal d-sparse subgraph of G does at least give an upper bound on $r_d(G)$ when $d = 3$. The purpose of this paper is to prove a result, Theorem 3.3 below, which extends Cheng and Sitharam's theorem to all values of $d \leq 11$.

2 Sparse subgraphs

Let $G = (V, E)$ be a graph and $d \geq 1$ be an integer. For $X \subseteq V$ we use $E_G(X)$ to denote the set, and $i_G(X)$ the number, of edges of G joining pairs of vertices of X. We simplify these to $E(X)$ and $i(X)$ when it is obvious to which graph we are referring. We may rewrite the condition for G to be d-sparse as $i(X) \leq d|X| - \binom{d+1}{2}$ for all $X \subseteq V$ with $|X| \geq d$. (Note that if $|X| \in \{d, d+1\}$ then we have $i(X) \leq \binom{|X|}{2} = d|X| - \binom{d+1}{2}$ and the inequality holds trivially.) We will use the fact that the function $i : 2^V \to \mathbb{Z}$ is *supermodular* i.e. $i(X) + i(Y) \leq i(X \cup Y) + i(X \cap Y)$ for all $X, Y \subseteq V$.

A subgraph $H = (U, F)$ of a d-sparse graph G is *d-critical* if either $|U| = 2$ and $|F| = 1$, or $|U| \geq d$ and $|F| = d|X| - \binom{d+1}{2}$. The assumption that G is d-sparse implies that every d-critical subgraph of G is an induced subgraph. A *d-critical component* of G is a d-critical subgraph which is not properly contained in any other d-critical subgraph of G.

Lemma 2.1 *Let $G = (V, E)$ be a d-sparse graph and $H_1 = (U_1, F_1), H_2 = (U_2, F_2)$ be distinct d-critical components of G. Then $|U_1 \cap U_2| \leq d - 1$ and, if equality holds, then $i_G(U_1 \cap U_2) = \binom{d-1}{2}$.*

Proof: Suppose that $|U_1 \cap U_2| \geq d - 1$. When $|U_1 \cap U_2| \geq d$ we have $i(U_1 \cap U_2) \leq d|U_1 \cap U_2| - \binom{d+1}{2}$ since G is d-sparse. When $|U_1 \cap U_2| = d - 1$, we have $i(U_1 \cap U_2) \leq \binom{d-1}{2} = d|U_1 \cap U_2| - \binom{d+1}{2} + 1$ trivially. The maximality of H_1, H_2 and the definition of a d-critical component imply that $|U_1|, |U_2| \geq d$, and $d(|U_1| + |U_2|) - 2\binom{d+1}{2} = i_G(U_1) + i_G(U_2) \leq i_G(U_1 \cup U_2) + i_G(U_1 \cap U_2) \leq$

$d|U_1\cup U_2|-\binom{d+1}{2}-1+d|U_1\cap U_2|-\binom{d+1}{2}+1=d(|U_1|+|U_2|)-2\binom{d+1}{2}$. Equality must hold throughout. In particular we have $i_G(U_1\cap U_2)=d|U_1\cap U_2|-\binom{d+1}{2}+1$. This implies that $|U_1\cap U_2|=d-1$ and $i_G(U_1\cap U_2)=\binom{d-1}{2}$. ■

Let k,t be non-negative integers, $G=(V,E)$ be a graph and $\mathcal{X}$ be a family of subsets of V. We say that $\mathcal{X}$ is *t-thin* if every pair of sets in $\mathcal{X}$ intersect in at most t vertices. A *k-hinge* of $\mathcal{X}$ is a set of k vertices which lie in the intersection of at least two sets in $\mathcal{X}$. A k-hinge U of $\mathcal{X}$ is *closed in G* if $G[U]$ is a complete graph. We use $\Theta_k(\mathcal{X})$ to denote the set of all k-hinges of $\mathcal{X}$. For $U\in\Theta_k(\mathcal{X})$, let $d_{\mathcal{X}}(U)$ denote the number of sets in $\mathcal{X}$ which contain U. Note that if G is t-thin then $\Theta_k(\mathcal{X})=\emptyset$ for all $k\geq t+1$. Note also that $\Theta_0(\mathcal{X})=\{\emptyset\}$ and $d_{\mathcal{X}}(\emptyset)=|\mathcal{X}|$.

Lemma 2.2 *Let $H=(V,E)$ be a d-sparse graph, $\mathcal{X}$ be a family of subsets of V such that $H[V_i]$ is d-critical for all $V_i\in\mathcal{X}$, and $W\in\Theta_k(\mathcal{X})$ for some $0\leq k\leq d-1$. Suppose that $|V_i|\geq d$ for all $V_i\in\mathcal{X}$ with $W\subseteq V_i$. Then*

$$(d-k)\sum_{\substack{U\in\Theta_{k+1}(\mathcal{X})\\ W\subset U}}(d_{\mathcal{X}}(U)-1)-\sum_{\substack{U\in\Theta_{k+2}(\mathcal{X})\\ W\subset U}}(d_{\mathcal{X}}(U)-1)\leq\binom{d+1-k}{2}(d_{\mathcal{X}}(W)-1)\,.$$

Proof: Let $d_{\mathcal{X}}(W)=t$ and let $V_1,V_2,\ldots,V_t$ be the sets in $\mathcal{X}$ which contain W. Let $H_i=(V_i,E_i)=H[V_i]$ for $1\leq i\leq t$. Let $H'=\bigcup_{i=1}^t H_i$ and put $H'=(V',E')$. Then

$$|V'|=\sum_{i=1}^t|V_i|-k(t-1)-\sum_{\substack{U\in\Theta_{k+1}(\mathcal{X})\\ W\subset U}}(d_{\mathcal{X}}(U)-1)\tag{1}$$

since, for $v\in V'$, if $v\in W$ then v is counted t times in $\sum_{i=1}^t|V_i|$, if $v\in U\setminus W$ for some $U\in\Theta_{k+1}$ with $W\subset U$ then v is counted $d_{\mathcal{X}}(U)$ times in $\sum_{i=1}^t|V_i|$, and all other vertices of V' are counted exactly once in $\sum_{i=1}^t|V_i|$.

Similarly,

$$|E'|\geq\sum_{i=1}^t|E_i|-\binom{k}{2}(t-1)-k\sum_{\substack{U\in\Theta_{k+1}(\mathcal{X})\\ W\subset U}}(d_{\mathcal{X}}(U)-1)-\sum_{\substack{U\in\Theta_{k+2}(\mathcal{X})\\ W\subset U}}(d_{\mathcal{X}}(U)-1)\tag{2}$$

since, for $e=xy\in E'$: if $x,y\in W$ then e is counted t times in $\sum_{i=1}^t|E_i|$ and there are at most $\binom{k}{2}$ such edges; if $x\in W$ and $y\in U\setminus W$ for some $U\in\Theta_{k+1}$ with $W\subset U$ then e is counted $d_{\mathcal{X}}(U)$ times in $\sum_{i=1}^t|E_i|$ and for each such y there are at most k choices for x; if $x,y\in U\setminus W$ for some $U\in\Theta_{k+2}$ with $W\subset U$ then e is counted $d_{\mathcal{X}}(U)$ times in $\sum_{i=1}^t|E_i|$, and all other edges of E' are counted exactly once in $\sum_{i=1}^t|E_i|$.

Since $H'\subseteq H$, H' is d-sparse Hence $|E'|\leq d|V'|-\binom{d+1}{2}$. We may substitute equations (1) and (2) into this inequality and use the fact that $|E_i|=d|V_i|-$

$\binom{d+1}{2}$ for all $1 \le i \le t$ to obtain

$$
\begin{aligned}
(d-k)\sum_{\substack{U\in\Theta_{k+1}(\mathcal{X})\\ W\subset U}}(d_{\mathcal{X}}(U)-1) \quad - & \sum_{\substack{U\in\Theta_{k+2}(\mathcal{X})\\ W\subset U}}(d_{\mathcal{X}}(U)-1) \\
\le & \left[\binom{d+1}{2}+\binom{k}{2}-dk\right](t-1) \\
= & \binom{d+1-k}{2}(t-1).
\end{aligned}
$$

■

Lemma 2.3 *Let $H=(V,E)$ be a d-sparse graph, $\mathcal{X}$ be a family of subsets of V such that $H[V_i]$ is d-critical and $|V_i|\ge d$ for all $V_i\in\mathcal{X}$. Put $a_k=\sum_{U\in\Theta_k(\mathcal{X})}(d_{\mathcal{X}}(U)-1)$ for $0\le k\le d$. Then for all $0\le k\le d-2$ we have:*

(a) $(d-k)(k+1)a_{k+1}-\binom{k+2}{2}a_{k+2}\le\binom{d+1-k}{2}a_k$;

(b) $(d-k)a_{k+1}-(k+1)a_{k+2}\le\binom{d+1}{k+2}(|\mathcal{X}|-1)$;

(c) if $\mathcal{X}$ is $(d-1)$-thin, $d(d-k)a_{k+1}\le(k+2)(d-k-1)\binom{d+1}{k+2}(|\mathcal{X}|-1)$.

Proof: Part (a) follows by summing the inequality in Lemma 2.2 over all $W\in\Theta_k$, and using the facts that

$$\sum_{W\in\Theta_k(\mathcal{X})}\sum_{\substack{U\in\Theta_{k+1}(\mathcal{X})\\ W\subset U}}(d_{\mathcal{X}}(U)-1)=(k+1)\sum_{U\in\Theta_{k+1}(\mathcal{X})}(d_{\mathcal{X}}(U)-1)=(k+1)a_{k+1}$$

and

$$\sum_{W\in\Theta_k(\mathcal{X})}\sum_{\substack{U\in\Theta_{k+2}(\mathcal{X})\\ W\subset U}}(d_{\mathcal{X}}(U)-1)=\binom{k+2}{2}\sum_{U\in\Theta_{k+2}(\mathcal{X})}(d_{\mathcal{X}}(U)-1)=\binom{k+2}{2}a_{k+2}\,.$$

We prove (b) by induction on k. When $k=0$, (b) follows by putting $k=0$ in (a), and using the fact that $a_0=|\mathcal{X}|-1$. Hence suppose that $k\ge1$. Then (a) gives

$$2(d-k)a_{k+1}-2(k+1)a_{k+2}\le\frac{(d-k+1)(d-k)}{k+1}a_k-ka_{k+2}\,. \tag{3}$$

We may also use (a) to obtain

$$ka_{k+2}\ge\frac{k(d-k)}{k+2}\left(2a_{k+1}-\frac{d-k+1}{k+1}a_k\right)\,. \tag{4}$$

Substituting (4) into (3) and using induction we obtain

$$
\begin{aligned}
(d-k)a_{k+1}-(k+1)a_{k+2} &\le \tfrac{d-k}{k+2}\left[(d-k+1)a_k-ka_{k+1}\right] \\
&\le \tfrac{d-k}{k+2}\tbinom{d+1}{k+1}(|\mathcal{X}|-1) \\
&= \tbinom{d+1}{k+2}(|\mathcal{X}|-1)\,.
\end{aligned}
$$

We prove (c) by induction on $d-k$. When $d-k=2$, (c) follows by putting $k=d-2$ in (b) and using the fact that $a_d=0$ since $\mathcal{X}$ is $(d-1)$-thin. Hence suppose that $d-k\geq 3$. Then (b) gives

$$d(d-k)a_{k+1} \leq d\tbinom{d+1}{k+2}\,(|\mathcal{X}|-1) + d(k+1)a_{k+2}\,.$$

We may now apply induction to a_{k+2} to obtain

$$\begin{aligned} d(d-k)a_{k+1} &\leq [d\tbinom{d+1}{k+2} + \tfrac{(k+1)(k+3)(d-k-2)}{d-k-1}\tbinom{d+1}{k+3}]\,(|\mathcal{X}|-1) \\ &= (k+2)(d-k-1)\tbinom{d+1}{k+2}\,(|\mathcal{X}|-1)\,. \end{aligned}$$

■

Theorem 2.4 *Let $H=(V,E)$ be a d-sparse graph, $\mathcal{X}$ be a $(d-1)$-thin family of subsets of V such that $H[X]$ is d-critical and $|X|\geq d$ for all $X\in\mathcal{X}$. For each $X\in\mathcal{X}$ let $\theta_k(X)$ be the number of k-hinges of $\mathcal{X}$ contained in X. Then:*
(a) $\theta_1(X)\leq 2d-1$ for some $X\in\mathcal{X}$;
(b) $\theta_2(X)\leq (d-2)(d+1)-1$ for some $X\in\mathcal{X}$;
(c) $\theta_{d-1}(X)\leq d$ for some $X\in\mathcal{X}$.

Proof:
We first prove (a). Putting $k=0$ in Lemma 2.3(c) we obtain

$$d\sum_{U\in\Theta_1(\mathcal{X})}(d_{\mathcal{X}}(U)-1)\leq (d-1)(d+1)(|\mathcal{X}|-1)\,. \tag{5}$$

Since $d_{\mathcal{X}}(U)\geq 2$ for all $U\in\Theta_1(\mathcal{X})$ we have $d_{\mathcal{X}}(U)-1\geq d_{\mathcal{X}}(U)/2$ and hence (5) gives

$$\sum_{U\in\Theta_1(\mathcal{X})} d_{\mathcal{X}}(U) < 2d\,|\mathcal{X}|\,.$$

This tells us that the average number of 1-hinges in a set in $\mathcal{X}$ is strictly less than $2d$.

We next prove (b). Putting $k=1$ in Lemma 2.3(c) we obtain

$$\sum_{U\in\Theta_2(\mathcal{X})}(d_{\mathcal{X}}(U)-1)\leq (d-2)(d+1)(|\mathcal{X}|-1)/2\,. \tag{6}$$

We can now proceed as in (a).

Finally we prove (c). Putting $k=d-2$ in Lemma 2.3(c) gives

$$2\sum_{U\in\Theta_{d-1}(\mathcal{X})}(d_{\mathcal{X}}(U)-1)\leq (d+1)(|\mathcal{X}|-1)\,. \tag{7}$$

We can now proceed as in (a). ■

The bounds given in Theorem 2.4 (a), (b) are close to being best possible. To see this consider the graph $H=H_1\cup H_2\cup\ldots\cup H_m$ where $H_i=(V_i,E_i)$ is

d-critical, $H_i \cap H_j = K_{d-1}$ for $i - j \equiv \pm 1 \mod m$ and otherwise $H_i \cap H_j = \emptyset$. Then H is d-sparse when m is sufficiently large, $\mathcal{X} = \{V_1, V_2, \ldots, V_m\}$ is $(d-1)$-thin and we have $\theta_1(V_i) = 2d-2$ and $\theta_2(V_i) = (d-1)(d-2)$ for all $V_i \in \mathcal{X}$. We do not know whether (c) is close to best possible for large d. It is conceivable that there always exists a set $X \in \mathcal{X}$ with $\theta_{d-1}(X) \le 2$.

3 Main result

In order to prove our main theorem we will need the following result from [4].

Lemma 3.1 *Let $G = (V, E)$ be a graph such that E is a non-rigid circuit in $\mathcal{R}_d(G)$. Then $|E| \ge d(d+9)/2$.* ■

Let $G = (V, E)$ be a graph and $\mathcal{X}$ be a family of subsets of V. We say that $\mathcal{X}$ is a *cover* of G if every set in $\mathcal{X}$ contains at least two vertices, and every edge of G is induced by at least one set in $\mathcal{X}$.

Lemma 3.2 *Let $G = (V, E)$ be a graph, $H = (V, F)$ be a maximal d-sparse subgraph of G, and $H_1, H_2, \ldots, H_m$ be the d-critical components of H. Let X_i be the vertex set of H_i for $1 \le i \le m$. Then $\mathcal{X} = \{X_1, X_2, \ldots, X_m\}$ is a $(d-1)$-thin cover of G and each $(d-1)$-hinge of $\mathcal{X}$ is closed in H.*

Proof: The definition of a d-critical subgraph implies that each H_i has at least two vertices and that every edge of H belongs to at least one H_i. Thus $\mathcal{X}$ is a cover of H. To see that $\mathcal{X}$ also covers G we choose $e = uv \in E \setminus F$. The maximality of H implies that $H + e$ is not d-sparse. Hence $\{u, v\}$ is contained in some d-critical subgraph of H. Thus $\mathcal{X}$ also covers G. The facts that $\mathcal{X}$ is $(d-1)$-thin and that each $(d-1)$-hinge of $\mathcal{X}$ is closed follow from Lemma 2.1. ■

We refer to the $(d-1)$-thin cover of G described in Lemma 3.2 as the *H-critical cover* of G. Note that the definition of a d-critical set implies that each set in the H-critical cover has size two or has size at least d.

Theorem 3.3 *Let $G = (V, E)$ be a graph, $d \le 11$ be an integer and $H = (V, F)$ be a maximal d-sparse subgraph of G. Then $r_d(G) \le |F|$.*

Proof: We proceed by contradiction. Suppose the theorem is false and choose a counterexample (G, H) such that $|E|$ is as small as possible. Let $H_1, H_2, \ldots, H_m$ be the d-critical components of H where $H_i = (V_i, F_i)$ for $1 \le i \le m$. Then $\mathcal{X}_0 = \{V_1, V_2, \ldots, V_m\}$ is the H-critical cover of G.

Choose a cover $\mathcal{X}$ of G such that $\mathcal{X} \subseteq \mathcal{X}_0$ and $|\mathcal{X}|$ is as small as possible. Note that $\mathcal{X}_0$, and hence also $\mathcal{X}$, are $(d-1)$-thin. For each $V_i \in \mathcal{X}$, let F_i^* be the set of all edges $uv \in F_i$ such that $\{u, v\}$ is a 2-hinge of $\mathcal{X}$, and let E_i be the set of edges of G induced by V_i.

Claim 3.4 *If $e = uv \in E$ satisfies $r_d(G) = r_d(G - e)$, then $\{u, v\}$ is a 2-hinge of $\mathcal{X}$.*

Proof: First suppose that $e \in E \setminus F$. Since H is a maximal d-sparse subgraph of $G - e$, the minimality of $|E|$ gives $r_d(G-e) \leq |F|$. Since $r_d(G) = r_d(G-e)$ this gives a contradiction.

Thus we can assume that $e \in F$. Let $d_{\mathcal{X}}(e)$ be the number of $V_i \in \mathcal{X}$ such that $e \in F_i$. Since $H - e$ is a d-sparse subgraph of $G - e$, we may choose a maximal d-sparse subgraph $H' = (V, F')$ of $G - e$ which contains $H - e$. Let $V_i \in \mathcal{X}$. If $e \notin F_i$, then no edge of $E_i \setminus F_i$ can be in F', since H_i is d-critical. On the other hand, if $e \in F_i$, then at most one edge of $E_i \setminus F_i$ can be in F', since $|F_i - e| = d|V_i| - \binom{d+1}{2} - 1$. These observations imply that $|F'| \leq |F| - 1 + d_{\mathcal{X}}(e)$. By the minimality of $|E|$ we have $r_d(G-e) \leq |F'|$, and hence $r_d(G) = r_d(G-e) \leq |F| - 1 + d_{\mathcal{X}}(e)$. Combining this with $r_d(G) > |F|$ gives $d_{\mathcal{X}}(e) \geq 2$. ■

We next show that F_i^* is dependent in $\mathcal{R}_d(G)$ for all $V_i \in \mathcal{X}$. Suppose this is not the case. Then E_i is independent in $\mathcal{R}_d(G)$ by Claim 3.4. Thus E_i can have at most $d|V_i| - \binom{d+1}{2}$ edges. Since H_i is d-critical, this gives $E_i = F_i$. The minimality of $\mathcal{X}$ implies that $F_i \neq F_i^*$ and hence we may choose an edge $e \in F_i \setminus F_i^*$. Since $F_i = E_i$, all edges of $G - e$ which are induced by V_i are in $H - e$. Since each $V_j \in \mathcal{X} - V_i$ induce a d-critical subgraph of $H - e$, we conclude that $H - e$ is a maximal d-sparse subgraph of $G - e$. The minimality of $|E|$ now gives $r_d(G-e) \leq |F - e| = |F| - 1$. Since $e \notin F_i^*$, Claim 3.4 gives $r_d(G-e) = r_d(G) - 1$. Hence $r_d(G) = r_d(G-e) + 1 \leq |F|$. This contradicts the choice of G and implies that F_i^* is dependent in $\mathcal{R}_d(G)$ for all $V_i \in \mathcal{X}$.

By Theorem 2.4(b) we may choose $V_i \in \mathcal{X}$ such that $|F_i^*| \leq (d-2)(d+1) - 1$. Since F_i^* is dependent in $\mathcal{R}_d(G)$, it contains a circuit of $\mathcal{R}_d(G)$. This circuit cannot be rigid, since H is d-sparse. Lemma 3.1 now gives $\frac{d^2+9d}{2} \leq |F_i^*| \leq (d-2)(d+1) - 1$ which implies that $d \geq 12$. ■

We have the following immediate corollary.

Corollary 3.5 *Let $d \leq 11$ be an integer and $G = (V, E)$ be a graph with $|V| \geq d+1$. If G is generically rigid in $\mathbb{R}^d$ then every maximal d-sparse subgraph of G has $d|V| - \binom{d+1}{2}$ edges.* ■

4 Closing remarks

1. Given a graph G, let $s_d(G)$ be the minimum number of edges in a maximal d-sparse subgraph of G. Theorem 3.3 tells us that $r_d(G) \leq s_d(G)$ when $d \leq 11$. We can use the following operation to construct graphs for which strict inequality holds. Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with $V_1 \cap V_2 = \{u, v\}$ and $E_1 \cap E_2 = \{uv\}$, we refer to the graph $G = G_1 \cup G_2$ as the *parallel connection of G_1 and G_2 along the edge uv.*

The graph G obtained by taking the parallel connection of two copies of K_5 along an edge uv and then deleting uv, is 3-sparse and is not rigid in $\mathbb{R}^3$. Hence $s_3(G) = |E(G)| = 18 > 17 = r_3(G)$. On the other hand we may improve the upper bound on $r_3(G)$ in this example by considering the graph $H = G + uv$.

A maximal 3-sparse subgraph of H which contains uv has 17 edges. Thus we have $17 = r_3(G) \leq r_3(H) \leq s_3(H) = 17$.

More generally, for any graph G, let $s^*_d(G) = \min\{s_d(H) : G \subseteq H\}$. Then $r_d(G) \leq s^*_d(G)$ for all $d \leq 11$. The following example shows that strict inequality can also hold in this inequality. Let G be obtained from K_5 by taking parallel connections with 10 different K_5's along each of the edges of the original K_5. We have $r_3(G) = 89$. On the other hand, $s_3(G) = 90$ (obtained by taking a maximal 3-sparse subgraph which contains nine of the edges of the original K_5). Furthermore we have $s_3(H) \geq r_3(H) > r_3(G)$ for all graphs H which properly contain G. Thus $s^*_3(G) = 90 > r_3(G)$.

2. For fixed d, we can use network flow algorithms to test whether a graph is d-sparse in polynomial time, see for example [2]. This means we can greedily construct a maximal d-sparse subgraph H of a graph G in polynomial time and hence obtain an upper bound on $r_d(G)$ when $d \leq 11$ via Theorem 3.3. We do not know whether $s_d(G)$ or $s^*_d(G)$ can be determined in polynomial time.

3. We believe that the conclusion of Theorem 3.3 should be valid for all d. However the graph G given in the example at the end of Section 2 shows that our proof technique will not give this: G is d-sparse and we have $\theta_2(V_i) = (d-1)(d-2)$ for all V_i in the G-critical cover of G. On the other hand, the lower bound on the number of edges in a non-rigid circuit in $\mathcal{R}_d(G)$ given by Lemma 3.1 is $\frac{d(d+9)}{2}$, so we cannot use it to deduce that the set of 2-hinges in some $G[V_i]$ is $\mathcal{R}_d$-independent when $d \geq 15$. One way to get round this problem would be to show that the d-critical components in a d-sparse graph form a cover which is 'iteratively independent' i.e. we can order the vertex sets of these components as $V_1, V_2, \ldots V_m$ such that the set of 2-hinges of $\{V_1, V_2, \ldots, V_i\}$ which belong to V_i is $\mathcal{R}_d$-independent for all $2 \leq i \leq m$. We refer the reader to [5] for more information on iteratively independent covers.

Acknowledgement

The authors would like to thank Meera Sitharam for helpful conversations on this topic and the London Mathematical Society for providing partial financial support through a scheme 5 grant. The first author would also like to thank the Ministry of National Education of Turkey for PhD funding through a YLSY grant.

References

[1] T. G. Abbot, Generalizations of Kempe's Universality Theorem. MSc thesis, MIT (2008). http://web.mit.edu/tabbott/www/papers/mthesis.pdf

[2] A. Berg and T. Jordán, Algorithms for graph rigidity and scene analysis, in: *Proceedings of the 11th Annual European Symposium on Algorithms*, Springer Lecture Notes in Computer Science **2832**, 2003, 78–89.

[3] M. J. Cheng and M. Sitharam, Maxwell-independence: a new rank estimate for the 3-dimensional generic rigidity matroid, *Journal of Combinatorial Theory Series B* **105**, (2014), 26-43.

[4] G. Grasegger, H. Guler, B. Jackson and A. Nixon, Flexible circuits, preprint available at `https://arxiv.org/pdf/2003.06648.pdf`.

[5] B. Jackson and T. Jordán, On the Rank Function of the 3-Dimensional Rigidity Matroid, *International Journal of Computational Geometry and Applications* **16** (2006), 415-429.

[6] G. Laman, On graphs and rigidity of plane skeletal structures. *J. Engineering Math.* **4** (1970), 331-340

[7] L. Lovász and Y. Yemini, On generic rigidity in the plane, *SIAM J. Algebraic Discrete Methods* **3** (1982), 91-98.

[8] J. C. Maxwell, On the calculation of the equilibrium and stiffness of frames, *Philosophical Magazine* **27** (1864), 294 - 299.

[9] H. Pollaczek-Geiringer, Über die Gliederung ebener Fachwerke, Zeitschrift für, *Angewandte Mathematik und Mechanik (ZAMM)* **7** (1927), 58–72.

[10] W. Whiteley, Some matroids from discrete applied geometry, in *Matroid Theory, Bonin*, J. E. et al., (eds.), Contemp. Math., **197**, Amer. Math. Soc., Providence, RI, 1996, 171–311.