# On the edge-biclique graph and the iterated edge-biclique operator 

Leandro Monterd* ${ }^{*}$<br>KLaIM team, L@bisen, AIDE Lab., Yncrea Ouest<br>33 Q, Chemin du Champ de Manœuvres<br>44470 Carquefou, France<br>lpmontero@gmail.com<br>Sylvain Legay<br>France


#### Abstract

A biclique of a graph $G$ is a maximal induced complete bipartite subgraph of $G$. The edge-biclique graph of $G, K B_{e}(G)$, is the edge-intersection graph of the bicliques of $G$. A graph $G$ diverges (resp. converges or is periodic) under an operator $H$ whenever $\lim _{k \rightarrow \infty}\left|V\left(H^{k}(G)\right)\right|=\infty\left(\right.$ resp. $\lim _{k \rightarrow \infty} H^{k}(G)=H^{m}(G)$ for some $m$ or $H^{k}(G)=H^{k+s}(G)$ for some $k$ and $s \geq 2$ ). The iterated edge-biclique graph of $G, K B_{e}^{k}(G)$, is the graph obtained by applying the edge-biclique operator $k$ successive times to $G$. In this paper, we first study the connectivity relation between $G$ and $K B_{e}(G)$. Next, we study the iterated edge-biclique operator $K B_{e}$. In particular, we give sufficient conditions for a graph to be convergent or divergent under the operator $K B_{e}$, we characterize the behavior of burgeon graphs and we propose some general conjectures on the subject.


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*Corresponding author.

## 1 Introduction

Intersection graphs of certain special subgraphs of a general graph have been studied extensively. We can mention line graphs (intersection graphs of the edges of a graph), interval graphs (intersection graphs of a family of subpaths of a path), and in particular, clique graphs (intersection graphs of the family of all maximal cliques of a graph) [4, 5, 8, 12, 13, 31, 33].

The clique graph of $G$ is denoted by $K(G)$. Clique graphs were introduced by Hamelink in [21] and characterized in [39]. It was proved in [1] that the clique graph recognition problem is NP-Complete.

The clique graph can be thought as an operator from Graphs into Graphs. The iterated clique graph $K^{k}(G)$ is the graph obtained by applying the clique operator $k$ successive times. It was introduced by Hedetniemi and Slater in [22]. Much work has been done in the field of the iterated clique operator, looking at the possible different behaviors. The goal is to decide whether a given graph converges, diverges, or is periodic under the clique operator when $k$ grows to infinity. This question remains open for the general case, moreover, it is not known if it is computable. However, partial characterizations have been given for convergent, divergent and periodic graphs, restricted to some classes of graphs. Some of them lead to polynomial time algorithms to solve the problem.

For the clique-Helly graph class, graphs which are convergent to the trivial graph have been characterized in [3]. Cographs, $P_{4}$-tidy graphs, and circulararc graphs are examples of classes where the different behaviors were also characterized [7, 25]. On the other hand, divergent graphs were considered. For example, in [36], families of divergent graphs are given. Periodic graphs were studied in [8, 29]. It has been proved that for every integer $i$, there are graphs with period $i$ and graphs which converge in $i$ steps. More results about iterated clique graphs can be found in [11, 26, 27, 28, 30, 37].

A biclique is a maximal induced complete bipartite subgraph. Bicliques have applications in various fields, for example biology: protein-protein interaction networks [6], social networks: web community discovery [24], genetics [2], medicine [35], information theory [20], etc. More applications (including some of these) can be found in [32]. The biclique graph of a graph $G$, denoted by $K B(G)$, is the intersection graph of the family of all bicliques of $G$. It was defined and characterized in [18]. However no polynomial time algorithm is known for recognizing biclique graphs. As for clique graphs, the biclique graph construction can be viewed as an operator between the class
of graphs.
The iterated biclique graph $K B^{k}(G)$ is the graph obtained by applying to $G$ the biclique operator $k$ times iteratively. It was introduced in [16] and all possible behaviors were characterized. It was proven that a graph is either divergent or convergent, but never periodic (with period bigger than 1). Also, general characterizations for convergent and divergent graphs were given. These results were based on the fact that if a graph $G$ contains a clique of size at least 5 , then $K B(G)$ or $K B^{2}(G)$ contains a clique of larger size. Therefore, in that case $G$ diverges. Similarly if $G$ contains the gem or the rocket graphs as an induced subgraph, then $K B(G)$ contains a clique of size 5 , and again $G$ diverges. Otherwise it was shown that after removing false-twin vertices of $K B(G)$, the resulting graph is a clique on at most 4 vertices, in which case $G$ converges. Moreover, it was proved that if a graph $G$ converges, it converges to the graphs $K_{1}$ or $K_{3}$, and it does so in at most 3 steps. These characterizations led to an $O\left(n^{4}\right)$ time algorithm (later improved to $O(n+m)$ time [14]) for recognizing convergent or divergent graphs under the biclique operator.

The edge-biclique graph of a graph $G$, denoted by $K B_{e}(G)$, is the edgeintersection graph of the family of all bicliques of $G$. We recall that edgeintersection means that $K B_{e}(G)$ has a vertex for each biclique of $G$ and two vertices are adjacent in $K B_{e}(G)$ if their corresponding bicliques in $G$ share an edge (and not just a vertex as in $K B(G)$ ). The edge-biclique graph $K B_{e}(G)$ was defined in [19] and studied in [15], however there is no characterization so far to recognize edge-biclique graphs.

In this work we study edge-biclique graphs not only because of their mathematical interest but also because in real-life problems, bicliques often represent the relation between two types of entities (each partition of the biclique) therefore it would make sense to study when two objects (bicliques) share a common relationship (an edge) more than just an entity (a vertex).

We first study the relation between $G$ and $K B_{e}(G)$ in terms of connectivity and we present a polynomial time algorithm to decide if $K B_{e}(G)$ is connected or not. In the rest of the paper, we define and focus on the iterated edge-biclique graph, denoted by $K B_{e}^{k}(G)$, that is, the graph obtained by applying to $G$ the edge-biclique operator $k$ times iteratively. We give some non-trivial sufficient conditions for a graph to be convergent or divergent under the $K B_{e}$ operator that are based on induced substructures. Later, we study burgeon graphs and its relation with line graphs and edge-biclique
graph ${ }^{7}$. We also characterize its behavior under the $K B_{e}$ operator. To finish, we propose some conjectures that would help to fully characterize the behavior of any graph under the $K B_{e}$ operator.

This work is organized as follows. In Section 2 the necessary notation is given. In Section 3 we give connectivity results of $K B_{e}(G)$. In Section 4 and Section 5 we present some results about convergent and divergent graphs, respectively. In Section 6, we study burgeon graphs. Finally, in Section 7 we state some general conjectures on the subject.

## 2 Preliminaries

Along the paper we restrict to undirected simple graphs. Let $G=(V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$, and let $n=|V(G)|$ and $m=|E(G)|$. A subgraph $G^{\prime}$ of $G$ is a graph $G^{\prime}=\left(V^{\prime}, E^{\prime}\right)$, where $V^{\prime} \subseteq V$ and $E^{\prime} \subseteq E$ such that all endpoints of the edges of $E^{\prime}$ are in $V^{\prime}$. When $E^{\prime}$ has all the edges of $E$ whose endpoints belong to the vertex subset $V^{\prime}$, we say that $V^{\prime}$ induces the subgraph $G^{\prime}=\left(V^{\prime}, E^{\prime}\right)$, that is, $G^{\prime}$ is an induced subgraph of $G$. Also, let $G\left[V^{\prime}\right]$ denote the induced subgraph of $G$ by the set $V^{\prime}$. A graph $G=(V, E)$ is bipartite when there exist sets $U$ and $W$ such that $V=U \cup W, U \cap W=\emptyset, U \neq \emptyset, W \neq \emptyset$ and $E \subseteq U \times W$. Say that $G$ is a complete graph when every possible edge belongs to $E$. A complete graph on $n$ vertices is denoted $K_{n}$. A bipartite graph is complete bipartite when every vertex of the first set is connected to every vertex of the second set. A complete bipartite graph on $p$ vertices in one set and $q$ vertices in the other is denoted $K_{p, q}$. A clique of $G$ is a maximal complete induced subgraph, while a biclique is a maximal induced complete bipartite subgraph of $G$. The open neighborhood of a vertex $v \in V(G)$, denoted $N(v)$, is the set of vertices adjacent to $v$. The closed neighborhood of a vertex $v \in V(G)$, denoted $N[v]$, is the set $N(v) \cup\{v\}$. Given a vertex $v \in V(G)$ and set of vertices $S \subseteq V(G)$, we denote by $N_{S}(v)$, to the neigborhood of the vertex $v$ restricted to the set $S$. Given a set of vertices $S \subseteq V(G), \bar{S}$ denotes the set $V(G)-S$. The degree of a vertex $v$, denoted by $d(v)$, is defined as $d(v)=|N(v)|$. A path (cycle) on $k$ vertices $(k \geq 3)$, denoted by $P_{k}\left(C_{k}\right)$, is a sequence of vertices $v_{1}, v_{2}, \ldots, v_{k} \in G$ such that $v_{i} \neq v_{j}$ for all $1 \leq i \neq j \leq k$ and $v_{i}$ is adjacent to $v_{i+1}$ for all $1 \leq i \leq k-1$ (and $v_{k}$ is adjacent to $v_{1}$ ). A graph is connected

[^0]if there exists a path between each pair of vertices. The girth of $G$ is the length of a shortest induced cycle in the graph. Unless stated otherwise, we assume that all graphs of this paper are connected.

Given a family of sets $\mathcal{H}$, the intersection graph of $\mathcal{H}$ is a graph that has the members of $\mathcal{H}$ as vertices, and there is an edge between two sets $E, F \in \mathcal{H}$ when $E$ and $F$ have non-empty intersection.

A graph $G$ is an intersection graph if there exists a family of sets $\mathcal{H}$ such that $G$ is the intersection graph of $\mathcal{H}$. We remark that any graph is an intersection graph [40].

Let $H$ be any graph operator and let $G$ be a graph. The iterated graph under the operator $H$ is defined iteratively as follows: $H^{0}(G)=G$ and for $k \geq 1, H^{k}(G)=H^{k-1}(H(G))$. We say that $G$ diverges (resp. converges or is periodic) under the operator $H$ whenever $\lim _{k \rightarrow \infty}\left|V\left(H^{k}(G)\right)\right|=\infty$ (resp. $\lim _{k \rightarrow \infty} H^{k}(G)=H^{m}(G)$ for some $m$ or $H^{k}(G)=H^{k+s}(G)$ for some $k$ and $s \geq 2$ ). The study of the behavior of a graph $G$ under the operator $H$ consists of deciding if $G$ converges, diverges or is periodic under $H$.

We assume that the empty graph is convergent under the operator $K B_{e}$, as it is obtained by appyling the edge-biclique operator to a graph that does not contain any bicliques.

## 3 Connectivity

In this section we will study the connectivity relation between $G$ and $K B_{e}(G)$. In comparison to the biclique graph $K B(G)$, it was shown in 34, 17] that $G$ is connected if and only if $K B(G)$ is connected. This result is no longer true for edge-biclique graphs. For example, just observe that $K B_{e}\left(K_{n}\right)$ consists of $\frac{n(n-1)}{2}$ isolated vertices, i.e., it is disconnected.

The main result of this section is the following Theorem that characterizes when $K B_{e}(G)$ is connected.

Theorem 3.1. Let $G$ be a connected graph. $K B_{e}(G)$ is connected if and only if there is no subset of vertices $S \subsetneq V(G)$ such that for every $v, w \in S$ we have $N_{\bar{S}}(v)=N_{\bar{S}}(w)$, and $|E(G[S])| \geq 1$, that is, the subgraph induced by $S$ has at least one edge.

Proof. $\Rightarrow)$ Suppose that there exists a subset of vertices $S \subsetneq V(G)$ verifying Theorem's hypothesis. Now, as for every pair of vertices $v, w \in S, N_{\bar{S}}(v)=$ $N_{\bar{S}}(w)$, and $|E(G[S])| \geq 1$, we have that every edge in $E(G[S])$ will not be
part of any biclique containing an edge outside $E(G[S])$. This implies that $K B_{e}(G)$ is disconnected and proves the "only if" part of the Theorem.
$\Leftarrow)$ Suppose that $K B_{e}(G)$ is not connected. We will show how to find a set $S$ of vertices verifying the hypothesis of the Theorem. Let $b$ be a vertex in $K B_{e}(G)$ and let $B$ be its corresponding biclique in $G$. Let $S_{E(B)}=\{e \in$ $E(G)$ : e belongs to a biclique $B^{\prime}$ such that its corresponding vertex $b^{\prime} \in$ $K B_{e}(G)$ is in the same connected component as $\left.b\right\}$. Clearly $S_{E(B)} \neq E(G)$ and $S_{E(B)} \neq \emptyset$ as $E(B) \subseteq S_{E(B)}$. Let $S_{V(B)}=\left\{v \in V(G): \exists e \in S_{E(B)}\right.$ such that $e$ is incident to $v\}$. Clearly $S_{V(B)} \neq \emptyset$ as $V(B) \subseteq S_{V(B)}$. We have the following two cases now.

- Case A) $E\left(G\left[S_{V(B)}\right]\right) \subseteq S_{E(B)}$.

We will show that $S_{V(B)}$ is the desired set. Observe that $G\left[S_{V(B)}\right]$ has at least one edge and it is clearly connected. Now let $u w$ be an edge such that $u \in S_{V(B)}$ and $w \in \overline{S_{V(B)}}$. Let $u^{\prime} \in S_{V(B)}$ be a vertex different to $u$. If $u^{\prime}$ is adjacent to $w$ there is nothing to show. Suppose then that $u^{\prime}$ is not adjacent to $w$. Now, since $G\left[S_{V(B)}\right]$ is connected, there is an induced path $u^{\prime}=u_{1} u_{2} \ldots u_{k}=u$ between $u^{\prime}$ and $u$. Let $u_{i}, i \in\{2, \ldots, k\}$, be the vertex of minimum index of the path that is adjacent to $w$. Clearly $u_{i}$ exists as $u_{k}=u$ is adjacent to $w$. Since $u_{i-1}$ is not adjacent to $w$, the set $\left\{u_{i}, u_{i-1}, w\right\}$ is contained in a biclique that has the edge $u_{i-1} u_{i}$. As the edge $u_{i-1} u_{i} \in S_{E(B)}$, we have that the edge $u_{i} w \in S_{E(B)}$ as well, thus $w \in S_{V(B)}$ which is a contradiction. We conclude that $u^{\prime}$ should be adjacent to $w$, obtaining that for every pair of vertices $u, u^{\prime} \in S_{V(B)}$, we have that $N_{\overline{S_{V(B)}}}(u)=N_{\overline{S_{V(B)}}}\left(u^{\prime}\right)$ as desired.

- Case B) $\exists e \in E\left(G\left[S_{V(B)}\right]\right)-S_{E(B)}$.

There exists a biclique $B_{e}$ such that $e \in E\left(B_{e}\right)$ and $B_{e}$ does not have any edge in common with $S_{E(B)}$. Consider now the sets $S_{E\left(B_{e}\right)}$ and $S_{V\left(B_{e}\right)}$ defined likewise $S_{E(B)}$ and $S_{V(B)}$. It is clear that $S_{E\left(B_{e}\right)} \neq \emptyset$, $S_{E(B)} \cap S_{E\left(B_{e}\right)}=\emptyset$ and $S_{V\left(B_{e}\right)} \neq \emptyset$. Moreover, $S_{V\left(B_{e}\right)} \subseteq S_{V(B)}$. For this, observe that the endpoints of the edge $e$, say $v, w \in B_{e}$, belong to $S_{V(B)}$, as $e \in E\left(G\left[S_{V(B)}\right]\right)$. Furthermore, they should have a common neighbor, say $z \in S_{V(B)}$, with $v z, w z \in S_{E(B)}$. If $B_{e}$ is just the edge $e$, we obtain directly that $S_{V\left(B_{e}\right)} \subseteq S_{V(B)}$. If $B_{e}$ is a larger biclique, there exists a vertex $u \in B_{e}$, without loss of generality, adjacent to $v$ and not adjacent to $w$. Clearly, $u$ belongs to $S_{V\left(B_{e}\right)}$. Now if $u$ is not
adjacent to $z$, then there is a biclique containing $\{u, v, z\}$ and since $v z \in S_{E(B)}$, then $u v \in S_{E(B)}$ and therefore, $e=v w \in S_{E(B)}$ which is a contradiction. Finally, $u$ is adjacent to $z$, thus $\{u, z, w\}$ is contained in a biclique containing the edge $z w \in S_{E(B)}$, therefore $u z \in S_{E(B)}$ implying that $u \in S_{V(B)}$. Since $G\left[S_{V\left(B_{e}\right)}\right]$ is connected, similar arguments can be applied for every other vertex $x \in S_{V\left(B_{e}\right)}$. As there is an induced path from $x$ to vertices $u$, $w$, we will obtain that $x z \in S_{E(B)}$ and thus $x \in S_{V(B)}$, otherwise we would get a contradiction the same way as before.

Now, if $\left|S_{V\left(B_{e}\right)}\right|=2$ then it is easy to see that $S_{V\left(B_{e}\right)}$ is the desired set. In what follows we assume that $\left|S_{V\left(B_{e}\right)}\right| \geq 3$.
We will show that $S_{V\left(B_{e}\right)} \subsetneq S_{V(B)}$ therefore, if $S_{V\left(B_{e}\right)}$ does not verify Case A), then we obtain another set $S_{V\left(B_{e^{\prime}}\right)}$ such that $S_{V\left(B_{e^{\prime}}\right)} \subsetneq S_{V\left(B_{e}\right)}$ and we repeat the process. Since the graph is finite, in some point we will obtain a set of vertices verifying Case A) which will conclude the proof (see Fig. 2). We will use the following three claims.

Claim 1. $\forall e=v w \in S_{E\left(B_{e}\right)}$ that belongs to two different bicliques, then
$-\exists v_{1}, w_{1} \in S_{V\left(B_{e}\right)}$ such that the pairs of vertices $v, v_{1}$ and $w, w_{1}$ are adjacent, and $v, w_{1}, w, v_{1}$ and $v_{1}, w_{1}$ are not adjacent; or
$-\exists v_{1}, v_{2} \in S_{V\left(B_{e}\right)}$ such that the pairs of vertices $v, v_{1}, v, v_{2}$ and $v_{1}, v_{2}$ are adjacent, and $w, v_{1}$ and $w, v_{2}$ are not adjacent $t^{\ddagger}$

These two options are shown in Figure 1.


Figure 1: Unique two options for an edge $v w$ belonging to two different bicliques.

[^1]Proof of Claim 1. First observe that since $\left|S_{V\left(B_{e}\right)}\right| \geq 3$, there exists a vertex, say $v_{1} \in S_{V\left(B_{e}\right)}$, adjacent to $v$ and not adjacent to $w$, i.e., the biclique containing the edge $v w$ is bigger than a $K_{1,1}$. This implies that $v v_{1} \in S_{E\left(B_{e}\right)}$. Now, since $v w$ belongs to another biclique than the one containing $\left\{v, w, v_{1}\right\}$, then, one case would be to have a vertex, say $w_{1} \in S_{V\left(B_{e}\right)}$, adjacent to $w$ and not adjacent to $v$. Moreover, $w_{1}$ is not adjacent to $v_{1}$, as otherwise, $\left\{v, w, v_{1}, w_{1}\right\}$ would be in the same biclique. Clearly, the edge $w w_{1} \in S_{E\left(B_{e}\right)}$, as both bicliques intersect in the edge $v w \in S_{E\left(B_{e}\right)}$. This shows the first option of the Claim. Now, if such a vertex $w_{1}$ does not exist, then it should exist a vertex $v_{2} \in S_{V\left(B_{e}\right)}$ such that $v_{2}$ is adjacent to $v$ and not adjacent to $w$. Moreover, since the biclique containing $\left\{v, w, v_{2}\right\}$ should be different to the one having $\left\{v, w, v_{1}\right\}$, this vertex $v_{2}$ is adjacent to $v_{1}$. As before, since these two bicliques have $v w \in S_{E\left(B_{e}\right)}$ in common, then $v v_{2} \in S_{E\left(B_{e}\right)}$ as well. Note that the edge $v_{1} v_{2}$ might or might not belong to $S_{E\left(B_{e}\right)}$.

Claim 2. Let $e=u_{1} u_{2} \in E\left(G\left[S_{V(B)}\right]\right)-S_{E(B)}$ and let $x \in S_{V(B)}$ such that $u_{1}, u_{2} \in N(x)$. Then $\forall v \in S_{V\left(B_{e}\right)}$, $v$ is adjacent to $x$ and the edge $v x \in S_{E(B)}$.
Proof of Claim 2. First note that $u_{1} x$ and $u_{2} x$ belong to $S_{E(B)}$ because $u_{1} u_{2} \in E\left(G\left[S_{V(B)}\right]\right)$. Now, since $\left|S_{V\left(B_{e}\right)}\right| \geq 3$, the biclique $B_{e}$ containing $u_{1} u_{2}$ is bigger than a $K_{1,1}$. Let $u_{3} \in V\left(B_{e}\right) \subseteq S_{V\left(B_{e}\right)}$ be a vertex different from $u_{1}$ and $u_{2}$ such that (without loss of generality) $u_{3}, u_{1}$ are not adjacent and $u_{3}, u_{2}$ are adjacent. Clearly, the edge $u_{3} u_{2} \in S_{E\left(B_{e}\right)}$. If $u_{3}$ and $x$ are not adjacent, then $u_{2} x \in S_{E(B)} \cap S_{E\left(B_{e}\right)}$ (as $\left\{u_{2}, x, u_{3}\right\}$ is contained in a biclique that intersects $B_{e}$ ), which is a contradiction. Therefore, $u_{3}, x$ are adjacent. Now, since the set $\left\{x, u_{3}, u_{1}\right\}$ is contained in a biclique that has the edge $u_{1} x \in S_{E(B)}$, it follows that $u_{3} x \in S_{E(B)}$ as well. This same argument applies for every vertex in $B_{e}$, therefore if $V\left(B_{e}\right)=S_{V\left(B_{e}\right)}$, then the proof is complete. Otherwise, there exists another biclique $B^{\prime}$ in $S_{V\left(B_{e}\right)}$ having an edge in common with $B_{e}$. Suppose without loss of generality that the edge $u_{1} u_{2}$ belongs to both. Now, by Claim 1, there are two options for this situation. Observe that in both options, there exists a vertex, say $u_{4} \in S_{V\left(B_{e}\right)}$, that is adjacent to $u_{1}$ and not to $u_{2}$, or adjacent to $u_{2}$ and not $u_{1}$. That is, there is an induced $P_{3}$ containing $u_{4}$ in one of the extremes. Suppose the first case, i.e., $u_{4}$ is adjacent to $u_{1}$ and not to $u_{2}$ (the other option is similar). As
before, $u_{4}$ must be adjacent to $x$, otherwise $u_{1} x \in S_{E(B)} \cap S_{E\left(B_{e}\right)}$, a contradiction. Since $\left\{u_{4}, x, u_{2}\right\}$ is contained in a biclique that has the edge $u_{2} x \in S_{E(B)}$, then we have that $u_{4} x \in S_{E(B)}$. Observe that this argument can be used for every vertex in $B^{\prime}$. Finally, we apply the same reasoning we used for $B^{\prime}$, to the other bicliques having edges in $S_{E\left(B_{e}\right)}$ that intersect previous analyzed bicliques. This completes the proof.

Claim 3. There exists a vertex $x \in S_{V(B)}$ such that $x \notin S_{V\left(B_{e}\right)}$.
Proof of Claim 3. By Claim 2, we have a vertex $x \in S_{V(B)}$ such that $\forall v \in S_{V\left(B_{e}\right)}, v$ is adjacent to $x$ and the edge $v x \in S_{E(B)}$. Finally, since $S_{E(B)} \cap S_{E\left(B_{e}\right)}=\emptyset$, and by definition of the sets $S_{E\left(B_{e}\right)}$ and $S_{V\left(B_{e}\right)}$, we have that $x \notin S_{V\left(B_{e}\right)}$.
To conclude the proof of Case B), by Claim 3, there is a vertex $x \in$ $S_{V(B)}$ such that $x \notin S_{V\left(B_{e}\right)}$, thus $S_{V\left(B_{e}\right)} \subsetneq S_{V(B)}$ as we wanted to show. Therefore, we can always obtain a set of vertices veryfing Case A) as desired.

As there are no cases left to analyze, the proof is complete.


Figure 2: In this example we can see three set of edges, $S_{E(r e d)}, S_{E(b l u e)}$ and $S_{E(\text { black })}$. Note that $S_{V(\text { black })} \subsetneq S_{V(\text { blue })} \subsetneq S_{V(\text { red })}=V(G)$, then following Case B) of the proof of Theorem [3.1, the set $S_{V(\text { black })}$ is the desired one.

To finish the section, we present an $O(n \times m)$ algorithm that, given a graph $G$, decides if $K B_{e}(G)$ is connected or not. Moreover, if $K B_{e}(G)$ is disconnected, the algorithm gives a partition of the edges of $G$ such that each
set of the partition has the edges belonging to bicliques that are in the same connected component in $K B_{e}(G)$. This algorithm relies mostly in Claim 1 of Theorem 3.1, since otherwise, verifying the condition for all subsets of vertices $S \subsetneq V(G)$ would take exponential time. We also remark that, since the number of bicliques of a graph can be exponential [38], constructing $K B_{e}(G)$ to check later if it is connected can take exponential time as well.

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Algorithm 1: Connectivity of \(K B_{e}(G)\)
    Input : A connected graph \(G\).
    Output: A partition of \(E(G)=E_{1} \cup \cdots \cup E_{k}\) such that each \(E_{i}\), for
                \(i=1, \ldots, k\), has the edges belonging to bicliques that are
                in the same connected component in \(K B_{e}(G)\).
    mark all edges as not used; \(k \leftarrow 0 ; S_{E} \leftarrow \emptyset\);
    while there exist unused edges do
        \(k \leftarrow k+1 ;\)
        take an unused edge \(e ; S_{E} \leftarrow S_{E} \cup\{e\}\); mark \(e\) as used;
        while \(S_{E} \neq \emptyset\) do
            remove an edge \(e=v w \in S_{E}\);
            \(E_{k} \leftarrow E_{k} \cup\{e\} ;\)
            for every vertex \(z \in N(v)-N(w)\) and \(z v\) not used do
                \(S_{E} \leftarrow S_{E} \cup\{z v\} ;\) mark \(z v\) as used;
            end for
            for every vertex \(z \in N(w)-N(v)\) and \(z w\) not used do
                \(S_{E} \leftarrow S_{E} \cup\{z w\} ;\) mark \(z w\) as used;
            end for
        end while
    end while
    if \(k=1\) then
        return \(K B_{e}(G)\) is connected;
    else
        return \(K B_{e}(G)\) is disconnected and \(E(G)=E_{1} \cup \cdots \cup E_{k}\);
    end if
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It is clear that Algorithm 1 runs in $O(n \times m)$ since each edge is added once to $S_{E}$ and each time we check all its endpoint's neighbors. It only remains to show that the algorithm is correct.
Proposition 3.2. Algorithm 1 correctly finds a partition of $E(G)=E_{1} \cup$ $\cdots \cup E_{k}$ such that each $E_{i}$, for $i=1, \ldots, k$, has the edges belonging to bicliques
that are in the same connected component in $K B_{e}(G)$. In particular, $K B_{e}(G)$ is connected if and only if $k=1$, that is, $E_{1}=E(G)$.

Proof. Observe that in the first while loop, the algorithm takes an unused edge (while exists) and adds it to a set $S_{E}$ of edges to analyze. The second while loop will add all edges that belong to the partition of that edge. For this, it takes an edge $e=v w$ from $S_{E}$ (while $S_{E} \neq \emptyset$ ) and adds it to the current edge partition. Now, if $e$ is not a biclique itself (in which case $N(v)=$ $N(w)$, thus $e$ is alone in its partition), then it must exist other vertex $z$ verifying $z \in N(v)-N(w)$ or $z \in N(w)-N(v)$, therefore it adds all edges of the form $z v$ or $z w$ to $S_{E}$ and to the current partition, respectively. Now, for each other iteration of the second while loop, the algorithm uses Claim 1 of Theorem 3.1 to see if an already used edge belongs to another biclique, and adds these new edges corresponding to those bicliques. When the while loop ends for an iteration $i$, that is, $S_{E}=\emptyset$, then $E_{i}$ has all edges that belong to bicliques that are in the same connected component of $K B_{e}(G)$ as the biclique containing the initial edge of that iteration.

Finally, if $E_{1}=E(G)$, then $K B_{e}(G)$ is connected since all edges of the graph belong to bicliques to one same connected component in $K B_{e}(G)$. Otherwise, one of the sets $S_{V_{i}}$ formed with incident vertices to the edges in $E_{i}$ (analog definition as $S_{E(B)}$ and $S_{V(B)}$ in Theorem 3.1) verifies that $S_{V_{i}} \subsetneq$ $V(G)$ and therefore Theorem 3.1 holds, that is, $K B_{e}(G)$ is disconnected.

## 4 Convergence

To start this section we have this first easy result.
Lemma 4.1. For $n \geq 2$, the complete graph $K_{n}$ converges to the empty graph under the operator $K B_{e}$ in two steps.

Proof. Clearly each edge of $K_{n}$ is a biclique that does not edge-intersect with another one. Then $K B_{e}\left(K_{n}\right)$ consists of $\frac{n(n-1)}{2}$ isolated vertices (and no bicliques), therefore $K B_{e}^{2}\left(K_{n}\right)$ is the empty graph.

Next we show that graphs without induced cycles of length 3 and 4 are convergent.

Theorem 4.2. If $G$ has girth at least five, then the edge-biclique operator applied to $G$ converges towards the graph induced by the union of all the cycles and paths connecting cycles of $G$.

Proof. If $G$ has girth at least five, then every biclique is a star. Moreover $G$ has no triangles, so $N(v)$ is a stable set and thus, for each $v$ of degree more than one, $N[v]$ is a maximal biclique. Notice also that if $u$ is adjacent to $v, N[u]$ and $N[v]$ contain a common edge, therefore the vertices in $K B_{e}(G)$ corresponding to the bicliques $N[u]$ and $N[v]$ will be adjacent. We can conclude that $K B_{e}(G)$ is exactly the graph induced by all vertices of degree at least two of $G$. For $k$ big enough, the only vertices left in $K B_{e}^{k}(G)$ are those which belong to cycles or to paths connecting cycles, that is, $G$ converges under the operator $K B_{e}$ towards the graph induced by the cycles and paths connecting cycles of $G$.

As an immediate result of Theorem 4.2, we obtain the following corollary.
Corollary 4.3. If $G$ has girth at least five and has no vertices of degree one, then $K B_{e}(G)=G$.

One natural question that arises from Corollary 4.3 is: Given a graph $G$ such that $K B_{e}(G)=G$, does $G$ have girth at least five and no vertices of degree one? The answer is no, for instance, the graph $\overline{C_{7}}$ shown in Figure 3 satisfies that $K B_{e}(G)=G$ but its girth is thre ${ }^{\S}$.


Figure 3: The graph $\overline{C_{7}}$ is the smallest graph satisfying $K B_{e}(G)=G$ with girth less than five.

From Theorem 4.2, we also obtain the following results.
Corollary 4.4. For every $k \geq 1$, there is a graph that converges in $k$ steps under the operator $K B_{e}$.

Proof. Just take any induced cycle $C_{n}, n \geq 5$, and join one of its vertices to the endpoint of a path $P_{k}$. Observe that this graph converges to $C_{n}$ in exactly $k$ steps (see Fig 4).

Corollary 4.5. Trees converge to the empty graph under the operator $K B_{e}$.

[^2]

Figure 4: Graph $G$ that converges in $k$ steps under the operator $K B_{e}$

## 5 Divergence

In this section we study the divergence of the operator $K B_{e}$. We start with the following definition.

Definition 5.1. Let $G$ be a graph and let $C=v_{0} v_{1} \ldots v_{n-1}$ be an induced cycle of length $n \geq 5$. We say that $C$ has good neighbors whenever for all vertices $v \in V(G)-C$, if $\left\{v_{i-1}, v_{i+1}\right\} \subseteq N(v)$ then $v_{i} \in N(v)$, for $i=$ $0, \ldots, n-1$ and all subindices taken $(\bmod n)$. (see Fig 5).


Figure 5: $G$ has a cycle with good neighbors while $G^{\prime}$ has not, since $v$ is adjacent to $v_{i-1}$ and $v_{i+1}$ but not adjacent to $v_{i}$.

Now we present an important proposition that assures that the good neighbors property is invariant through the iterations of the operator $K B_{e}$.

Proposition 5.2. Let $G$ be a graph and let $C=v_{0} v_{1} \ldots v_{n-1}$ be an induced cycle of length $n \geq 5$ with good neighbors. Let $B_{i}, i=0, \ldots, n-1$, be bicliques in $G$ containing the vertices $\left\{v_{i-1}, v_{i}, v_{i+1}\right\}(\bmod n)$, respectively, and let $b_{i}$, $i=0, \ldots, n-1$, be the vertices in $K B_{e}(G)$ corresponding to the bicliques $B_{i} \in G$. Then $V\left(B_{i}\right) \subseteq N\left[v_{i}\right]$ and $C^{\prime}=b_{0} b_{1} \ldots b_{n-1}$ is an induced cycle of $K B_{e}(G)$. Moreover, $C^{\prime}$ has good neighbors.

Proof. As $C$ is an induced cycle in $G$, let $B_{i}, i=0, \ldots, n-1$, be bicliques that contain the vertices $\left\{v_{i-1}, v_{i}, v_{i+1}\right\}(\bmod n)$, respectively. Clearly, each $B_{i}$ intersects $B_{i+1}$ in the edge $v_{i} v_{i+1}$, therefore if we call $b_{i}, i=0, \ldots, n-1$, the corresponding vertices in $K B_{e}(G)$ to the bicliques $B_{i}$, then we have that $b_{0} b_{1} \ldots b_{n-1}$ form a cycle $C^{\prime}$ in $K B_{e}(G)$. Now, let $v \in G$ be a vertex in $B_{i}-\left\{v_{i-1}, v_{i}, v_{i+1}\right\}$. As $B_{i}$ is a biclique of $G$, either $v$ is adjacent to $v_{i-1}$ and $v_{i+1}$ but not adjacent to $v_{i}$, which is not possible because $C$ has good neighbors, or $v$ is adjacent to $v_{i}$. Therefore, for all $i=0, \ldots, n-1, V\left(B_{i}\right) \subseteq$ $N\left[v_{i}\right]$ and $C^{\prime}$ is an induced cycle of $K B_{e}(G)$.

Now, let $b \in V\left(K B_{e}(G)\right)-C^{\prime}$ be a vertex such that $\left\{b_{i-1}, b_{i+1}\right\} \subseteq N(b)$ for some $i$. If $B$ is the biclique of $G$ corresponding to the vertex $b \in K B_{e}(G)$, then $B$ contains $v_{i-1}$ and $v_{i+1}$, since $V\left(B_{i-1}\right) \subseteq N\left[v_{i-1}\right]$ and $V\left(B_{i+1}\right) \subseteq N\left[v_{i+1}\right]$. As $v_{i-1}$ and $v_{i+1}$ are not adjacent in $G$, there exists a vertex $v \in B \cap B_{i-1} \cap B_{i+1}$ such that $v$ is adjacent to both $v_{i-1}$ and $v_{i+1}$. If $v \neq v_{i}$, since $C$ has good neighbors, $v$ must also be adjacent to $v_{i}$, contradicting the fact that $v \in B_{i-1}$ (or $B_{i+1}$ ). Therefore, $v=v_{i}$ and $B$ and $B_{i}$ have an edge in common, that is, $b$ is adjacent to $b_{i}$ in $K B_{e}(G)$ and thus $C^{\prime}$ has good neighbors.

Before the main theorem, we define the following family of graphs.
Definition 5.3. For $n \geq 3$ and $m \geq 1$, the ( $n, m$ ) - necklace graph on $n+m$ vertices consists of an induced cycle $C_{n}$ and a complete graph $K_{m}$, such that for an edge $e \in C_{n}$, every vertex of the $K_{m}$ is only adjacent to both endpoints of e (see Fig (6).


Figure 6: $(5,1)$ - necklace and $(6,3)$ - necklace graphs.
Now we present the main theorem of this section.

Theorem 5.4. Let $G$ be a graph that contains an induced ( $n, m$ ) - necklace, $n \geq 5, m \geq 1$, such that its cycle has good neighbors. Then, either $K B_{e}^{2}(G)$ or $K B_{e}^{3}(G)$ contains an induced ( $\left.n, m^{\prime}\right)$ - necklace such that its cycle has good neighbors, and $m^{\prime}>m$.

Proof. Let $C_{n}=v_{0} v_{1} \ldots v_{n-1}$ be the induced cycle and $K_{m}=\left\{w_{1}, \ldots, w_{m}\right\}$ be the complete graph of the $(n, m)$ - necklace, respectively. Let $v_{i} v_{i+1}$, for some $i \in\{0, \ldots, n-1\}(\bmod n)$, be the edge of the $C_{n}$ such that $w_{j}$ is adjacent to $v_{i}$ and $v_{i+1}$ for all $j=1, \ldots, m$. Let $B_{t}, t=0, \ldots, n-1$, be bicliques that contain the vertices $\left\{v_{t-1}, v_{t}, v_{t+1}\right\}(\bmod n)$, respectively, and let $b_{t}, t=0, \ldots, n-1$, be the corresponding vertices in $K B_{e}(G)$ to the bicliques $B_{t}$. By Proposition 5.2, $C_{n}^{\prime}=b_{0} b_{1} \ldots b_{n-1}$ is an induced cycle in $K B_{e}(G)$ with good neighbors.

Consider these two families of bicliques $B^{1}=\left\{B_{j}^{1}:\left\{w_{j}, v_{i}, v_{i-1}\right\} \subseteq\right.$ $\left.B_{j}^{1}, j=1, \ldots, m\right\}$ and $B^{2}=\left\{B_{j}^{2}:\left\{w_{j}, v_{i+1}, v_{i+2}\right\} \subseteq B_{j}^{2}, j=1, \ldots, m\right\}$. Clearly, all these $2 m$ bicliques are different and moreover, they are different to the bicliques $B_{t}$ for $t=0, \ldots, n-1$ as $C_{n}$ has good neighbors. Now we can see that $\left(\bigcap_{j=1}^{m} B_{j}^{1}\right) \cap B_{i-1} \cap B_{i}=\left\{v_{i-1}, v_{i}\right\}$ and $\left(\bigcap_{j=1}^{m} B_{j}^{2}\right) \cap B_{i+1} \cap B_{i+2}=$ $\left\{v_{i+1}, v_{i+2}\right\}$. Therefore if $b_{j}^{1}$ and $b_{j}^{2}, j=1, \ldots, m$, are the corresponding vertices in $K B_{e}(G)$ to the bicliques $B_{j}^{1}$ and $B_{j}^{2}$, we have that in $K B_{e}(G)$, $K_{m}^{1}=\left\{b_{1}^{1}, \ldots, b_{m}^{1}\right\}$ and $K_{m}^{2}=\left\{b_{1}^{2}, \ldots, b_{m}^{2}\right\}$ are two complete graphs such that $b_{j}^{1}$ is adjacent to $b_{i-1}$ and $b_{i}$, and $b_{j}^{2}$ is adjacent to $b_{i+1}$ and $b_{i+2}$, for all $j=1, \ldots, m$. Notice that as $C_{n}$ has good neighbors in $G$, then in $K B_{e}(G)$ we have $N\left(b_{j}^{1}\right) \cap C_{n}^{\prime}=\left\{b_{i-1}, b_{i}\right\}$ and $N\left(b_{j}^{2}\right) \cap C_{n}^{\prime}=\left\{b_{i+1}, b_{i+2}\right\}$, for all $j=1, \ldots, m$ (see Fig 7).

Now, let $\widetilde{B}_{t}, t=0, \ldots, n-1$, be the bicliques of $K B_{e}(G)$ that contain the vertices $\left\{b_{t-1}, b_{t}, b_{t+1}\right\}(\bmod n)$, respectively, and $\widetilde{b}_{t}, t=0, \ldots, n-1$, be the corresponding vertices in $K B_{e}^{2}(G)$ to the bicliques $\widetilde{B}_{t}$. Again, by Proposition 5.2, $C_{n}^{\prime \prime}=\widetilde{b}_{0} \widetilde{b}_{1} \ldots \widetilde{b}_{n-1}$ is an induced cycle in $K B_{e}^{2}(G)$ with good neighbors.

Now for each $b_{j}^{1}, j=1, \ldots, m$, we have that $\left\{b_{j}^{1}, b_{i}, b_{i+1}\right\}$ is contained in a biclique $\widetilde{B}_{j}^{1}$. Similarly, for each $b_{j}^{2}, j=1, \ldots, m,\left\{b_{j}^{2}, b_{i}, b_{i+1}\right\}$ is contained in a biclique $\widetilde{B}_{j}^{2}$. In the worst case (to minimize the number of bicliques), if there is exactly a perfect matching between $K_{m}^{1}$ and $K_{m}^{2}$, say $b_{j}^{1}$ is adjacent to $b_{j}^{2}$, for each $j=1, \ldots, m$, then $\widetilde{B}_{j}^{1}=\widetilde{B}_{j}^{2}$. We have the following two cases:

Case A: There is at least one vertex $b_{1}^{1} \in K_{m}^{1}$ not adjacent to any vertex of $K_{m}^{2}$. Clearly, the biclique $\widetilde{B}_{1}^{1}$ is different to the $m$ bicliques $\widetilde{B}_{j}^{2}$, for all


Figure 7: First iteration of the operator $K B_{e}$ applied to $G$ containing an induced $(n, m)$ - necklace with good neighbors.
$j=1, \ldots, m$, and furthermore, these $m+1$ bicliques are different to the bicliques $\widetilde{B}_{t}$ for $t=0, \ldots, n-1$. Observe that $\left(\bigcap_{j=1}^{m} \widetilde{B}_{j}^{2}\right) \cap B_{1}^{1}=\left\{b_{i}, b_{i+1}\right\}$ and moreover, $\widetilde{B}_{i} \cap \widetilde{B}_{i+1}=\left\{b_{i}, b_{i+1}\right\}$. Therefore, if $\widetilde{b}_{1}^{1}$ and $\widetilde{b}_{j}^{2}, j=1, \ldots, m$, are the corresponding vertices in $K B_{e}^{2}(G)$ to the bicliques $\widetilde{B}_{1}^{1}$ and $\widetilde{B}_{j}^{2}$, respectively, we have that in $K B_{e}^{2}(G),\left\{\widetilde{b}_{1}^{1}, \widetilde{b}_{1}^{2}, \ldots, \widetilde{b}_{m}^{2}\right\}$ is a complete graph on $m+1$ vertices such that, as $C_{n}^{\prime}$ has good neighbors, every vertex of this $K_{m+1}$ is only adjacent to $\widetilde{b}_{i}$ and to $\widetilde{b}_{i+1}$ on the cycle $C_{n}^{\prime \prime}$. That is, $K B_{e}^{2}(G)$ contains an induced $(n, m+1)$ - necklace such that its cycle $C_{n}^{\prime \prime}$ has good neighbors. See Fig 8 .

Case B: Every vertex of $K_{m}^{1}$ is adjacent to at least one vertex of $K_{m}^{2}$ (and by symmetry every vertex of $K_{m}^{2}$ is adjacent to at least one vertex of $K_{m}^{1}$ ). As explained above, the worst case is when there is a perfect matching between $K_{m}^{1}$ and $K_{m}^{2}$. Without loss of generality, suppose that $b_{j}^{1}$ is adjacent to $b_{j}^{2}$ for each $j=1, \ldots, m$, otherwise we would obtain at least $m+1$ bicliques having the edge $b_{i} b_{i+1}$ in common and therefore $K B_{e}^{2}(G)$ will contain an induced $(n, m+1)$ - necklace such that its cycle $C_{n}^{\prime \prime}$ has good neighbors. As there is a matching between $K_{m}^{1}$ and $K_{m}^{2}$, let $\widetilde{B}_{j}^{\prime}$ be the bicliques that contain the


Figure 8: Case A: Second iteration of the operator $K B_{e}$.
set $\left\{b_{j}^{1}, b_{i}, b_{i+1}, b_{j}^{2}\right\}$ for each $j=1, \ldots, m$. These bicliques contain the edge $b_{i} b_{i+1}$ and they are different to the bicliques $\widetilde{B}_{t}$ for $t=0, \ldots, n-1$. Then, if $\widetilde{b}_{\tilde{B}}^{\prime}, j=1, \ldots, m$, are the corresponding vertices in $K B_{e}^{2}(G)$ to the bicliques $\widetilde{B}_{j}^{\prime}$, we have that in $K B_{e}^{2}(G),\left\{\widetilde{b}_{1}^{\prime}, \ldots, \widetilde{b}_{m}^{\prime}\right\}$ is a complete graph on $m$ vertices such that, as $C_{n}^{\prime}$ has good neighbors, every vertex of this $K_{m}$ is only adjacent to $\widetilde{b}_{i}$ and to $\widetilde{b}_{i+1}$ on the cycle $C_{n}^{\prime \prime}$.

Now for each $b_{j}^{1}, j=1, \ldots, m$, we have that $\left\{b_{j}^{1}, b_{i-1}, b_{i-2}\right\}$ is contained in a biclique $\widetilde{B}_{j}^{1}$. All these $m$ bicliques have the edge $b_{i-1} b_{i-2}$ in common. In addition, they are clearly different to the bicliques $\widetilde{B}_{t}, t=0, \ldots, n-1$ and $\widetilde{B}_{j}^{\prime}, j=1, \ldots, m$. Suppose now that there is an edge in common between the bicliques, say $\widetilde{B}_{1}^{1}$ and $\widetilde{B}_{1}^{\prime}$. Then, there must exist a vertex $b \in K B_{e}(G)$ adjacent to $b_{i-2}, b_{1}^{1}$ and $b_{i+1}$. This implies that in $G$, there must exist a biclique $B$ (corresponding to the vertex $b \in K B_{e}(G)$ ) that has edges in common with the bicliques $B_{i-2}, B_{1}^{1}$ and $B_{i+1}$. Therefore, as $C_{n}$ has good neighbors, there must exist a vertex $v \in B$ adjacent to the vertices $v_{i-2}$ and $v_{i+1}$. Finally, as $B$ has an edge in common with the biclique $B_{1}^{1}, v$ must be adjacent to either to $v_{i}$, or to $v_{i-1}$ and $w_{1}$. In both cases we obtain a
contradiction as $B$ would contain either the $K_{3}=\left\{v, v_{i}, v_{i+1}\right\}$ or the $K_{3}=$ $\left\{v, v_{i-2}, v_{i-1}\right\}$ which is not possible if $B$ is a biclique. We can conclude then that there are no edges in common between the bicliques $\widetilde{B}_{j}^{1}$ and $\widetilde{B}_{j}^{\prime}$, for all $j=1, \ldots, m$. Now let $\widetilde{b}_{j}^{1}$ be the vertices in $K B_{e}^{2}(G)$ corresponding to the bicliques $\widetilde{B}_{j}^{1}$ of $K B(G)$, for $j=1, \ldots, m$ respectively. Then, these vertices form a $K_{m}$ in $K B_{e}^{2}(G)$ and they are only adjacent to the vertices $\widetilde{b}_{i-2}$ and $\widetilde{b}_{i-1}$ of the cycle $C_{n}^{\prime \prime}$. See Fig 9 .


Figure 9: Case B: Second iteration of the operator $K B_{e}$.
Now, let $\beta_{t}, t=0, \ldots, n-1$, be bicliques of $K B_{e}^{2}(G)$ that contain the vertices $\left\{\widetilde{b}_{t-1}, \widetilde{b}_{t}, \widetilde{b}_{t+1}\right\}(\bmod n)$, respectively, and $\widetilde{\beta}_{t}, t=0, \ldots, n-1$, the corresponding vertices in $K B_{e}^{3}(G)$ to the bicliques $\beta_{t}$. By Proposition 5.2, $C_{n}^{\prime \prime \prime}=$ $\widetilde{\beta}_{0} \widetilde{\beta}_{1} \ldots \widetilde{\beta}_{n-1}$ is an induced cycle in $K B_{e}^{3}(G)$ with good neighbors. To finish, consider the following two families of bicliques: $\beta^{1}=\left\{\beta_{j}^{1}:\left\{\widetilde{b}_{j}^{1}, \widetilde{b}_{i-1}, \widetilde{b}_{i}\right\} \subseteq\right.$ $\left.\beta_{j}^{1}, j=1, \ldots, m\right\}$ and $\beta^{2}=\left\{\beta_{j}^{2}:\left\{\widetilde{b}_{j}^{\prime}, \widetilde{b}_{i-1}, \widetilde{b}_{i}\right\} \subseteq \beta_{j}^{2}, j=1, \ldots, m\right\}$. Clearly, all these $2 m$ bicliques are different as there are no edges in common between the bicliques $\widetilde{B}_{j}^{1}$ and $\widetilde{B}_{j}^{\prime}$, for all $j=1, \ldots, m$, and moreover, they are different to the bicliques $\beta_{t}$ for $t=0, \ldots, n-1$ as $C_{n}^{\prime \prime}$ has good neighbors. Since all these $2 m$ bicliques contain the edge $\widetilde{b}_{i-1} \widetilde{b}_{i}$, then if $\widetilde{\beta}_{j}^{1}$ and $\widetilde{\beta}_{j}^{2}, j=1, \ldots, m$, are the corresponding vertices in $K B_{e}^{3}(G)$ to the bicliques $\beta_{j}^{1}$ and $\beta_{j}^{2}$, respectively, we have that in $K B_{e}^{3}(G),\left\{\widetilde{\beta}_{1}^{1}, \ldots, \widetilde{\beta}_{m}^{1}, \widetilde{\beta}_{1}^{2}, \ldots, \widetilde{\beta}_{m}^{2}\right\}$ is a complete graph on $2 m$ vertices such that, as $C_{n}^{\prime \prime \prime}$ has good neighbors, every vertex of this $K_{2 m}$
is only adjacent to $\widetilde{\beta}_{i}$ and to $\widetilde{\beta}_{i-1}$ on the cycle $C_{n}^{\prime \prime \prime}$. That is, $K B_{e}^{3}(G)$ contains an induced $(n, 2 m)$ - necklace such that its cycle $C_{n}^{\prime \prime \prime}$ has good neighbors. See Fig 10 .


Figure 10: Case B: Third iteration of the operator $K B_{e}$.
As a corollary, we obtain the following divergence theorem.
Theorem 5.5. Let $G$ be a graph that contains an induced ( $n, m$ ) - necklace, $n \geq 5, m \geq 1$, such that its cycle has good neighbors. Then $G$ diverges under the operator $K B_{e}$.
Proof. Applying Theorem 5.4 several times, we obtain that either $K B_{e}^{2}(G)$ or $K B_{e}^{3}(G)$ contains an induced $\left(n, m^{\prime}\right)-$ necklace, and $m^{\prime}>m$. Then $K B_{e}^{4}(G), K B_{e}^{5}(G)$ or $K B_{e}^{6}(G)$ contains an induced ( $n, m^{\prime \prime}$ ) - necklace, and $m^{\prime \prime}>m^{\prime}$, etc., all having its cycles with good neighbors. Therefore, $G$ is divergent under the operator $K B_{e}$ as $\lim _{k \rightarrow \infty}\left|V\left(K B_{e}^{k}(G)\right)\right|=\infty$.

To finish the section, we obtain a second corollary.
Corollary 5.6. Let $G$ be a graph and let $C_{n}$ be an induced cycle of length $n \geq 5$ with good neighbors. If there is a vertex $v \in V(G)-C_{n}$ such that $N(v) \cap C_{n}$ has at least one edge and not all $C_{n}$, then $G$ diverges under the operator $K B_{e}$.
Proof. Just observe that $K B_{e}(G)$ satisfies conditions of Theorem 5.5.

## 6 Burgeon graphs

In this section we will study the iterated edge-biclique graph of burgeon graphs and its relationship with the iterated line graph.

Definition 6.1. Let $G$ be a graph. We define the burgeon graph of $G$, denoted by $B(G)$, as the graph obtained by replacing each vertex $v$ of $G$ by a clique $C_{v}$ of $d(v)$ vertices, such that each vertex of the clique $C_{v}$ is only adjacent (to the outside of $C_{v}$ ) to exactly one vertex of another clique $C_{u}$ if and only if $u$ and $v$ are adjacent in G. See Fig 11.


Figure 11: Graph $G$ and the construction of $B(G)$.
Recall the definition of the line graph of a graph $G$, denoted by $L(G)$, as the intersection graph of the edges of $G$, that is, $L(G)$ has one vertex for each edge of $G$ and two vertices $v, w$ in $L(G)$ are adjacent if their corresponding edges in $G$ have a common endpoint. Next theorem shows the connection between the three operators $K B_{e}, B$ and $L$.

Theorem 6.2. Let $G$ be a graph on $n \geq 2$ vertices. Then $K B_{e}(B(G))=$ $B(L(G))$.

Proof. Observe first that in $B(G)$ we have two types of edges. Edges of type I will be the edges inside the cliques and edges of type II will be the edges joining different cliques (these are in one-to-one correspondence with the edges of $G$ ). Now, as $B(G)$ has no induced $C_{4}$ and there is at most one edge of type II between each pair of cliques, we have that all bicliques of $B(G)$ are isomorphic to $K_{1,2}$. Moreover, each biclique is formed with an edge of type I and an edge of type II sharing a common vertex. Consider now
an edge $e=v w \in G$ and its corresponding edge $e_{B}=v_{B} w_{B} \in B(G)$, with $v_{B} \in C_{v}$ and $w_{B} \in C_{w}$. Note that $e_{B}$ is of type II. The edge $e_{B}$ belongs to exactly $d(v)-1+d(w)-1=d(v)+d(w)-2$ bicliques in $B(G)$. First $d(v)-1$ bicliques are formed with the edge $e_{B}$ and each choice of an edge in $C_{v}$ having $v_{B}$ as an endpoint, while the other $d(w)-1$ bicliques, with the edge $e_{B}$ and each choice of an edge in $C_{w}$ having $w_{B}$ as an endpoint. Since all these bicliques have the edge $e_{B}$ in common, their corresponding vertices in $K B_{e}(B(G))$ form a clique of size $d(v)+d(w)-2$. For each edge $e_{B} \in B(G)$ of type II, call $C_{e_{B}}$ this clique in $K B_{e}(B(G))$. Finally, observe that there is exactly one edge between two cliques $C_{e_{B}}, C_{e_{B}^{\prime}}$ of $K B_{e}(B(G))$ if and only if there are two bicliques in $B(G)$ containing $e_{B}$ and $e_{B}^{\prime}$ respectively, and a common edge of type I. That is, $e$ and $e^{\prime}$ are adjacent in $G$.

Now, in $L(G)$, each vertex, say $e_{L}$ (that corresponds to an edge $e=v w$ of $G)$, is adjacent to $d(v)+d(w)-2$ other vertices in $L(G)$. Thus, each vertex of $L$ will form a clique, say $C_{e_{L}}$, of size $d(v)+d(w)-2$ vertices in $B(L(G))$. Finally, there is exactly one edge between two cliques $C_{e_{L}}, C_{e_{L}^{\prime}}$ of $B(L(G))$ if and only if the vertices $e_{L}$ and $e_{L}^{\prime}$ are adjacent in $L(G)$. That is, $e$ and $e^{\prime}$ are adjacent in $G$.

We conclude therefore that $K B_{e}(B(G))=B(L(G))$ as desired (see Figure 12 .

As a corollary, we can characterize the behavior of burgeon graphs under the $K B_{e}$ operator.

Corollary 6.3. Let $G$ be a graph. $B(G)$ is divergent under the $K B_{e}$ operator if and only if $G$ is not a cycle, a path or a $K_{1,3}$.

Proof. Theorem 6.2 implies that $K B_{e}^{n}(B(G))=B\left(L^{n}(G)\right)$. Also, we know by [41] that $G$ diverges under the $L$ operator if and only if $G$ is not a cycle, a path, or a $K_{1,3}$. Combining both last statements along with the fact that $B(G)$ has at least as many vertices as $G$ (for $|V(G)| \neq 2$ ), the result holds.

Corollary 6.4. Let $G$ be a graph. $B(G)$ is convergent under the $K B_{e}$ operator if and only if $G$ is a cycle, a path or a $K_{1,3}$. Moreover, it converges to itself, to the empty graph or to $C_{6}$, respectively.

Last corollary can be stated only in terms of burgeon graphs applying the $B$ operator as follows.


Figure 12: Example of the relationship of Theorem 6.2.

Corollary 6.5. Let $G=B(H)$ for some graph $H$. $G$ is convergent under the $K B_{e}$ operator if and only if $G$ is a cycle, a path or the net graph (see Fig 13). Moreover, it converges to itself, to the empty graph or to $C_{6}$, respectively.


Figure 13: The net graph.
Note that one can verify in polynomial time if given a graph $G$, there exists some graph $H$ such that $G=B(H)$. Moreover, since checking if $G$ is a cycle, a path or the net graph can also be done in polynomial time, we
can conclude that deciding the behavoir of a burgeon graph under the $K B_{e}$ operator is polynomial as well.

We finish the section with the following result.
Proposition 6.6. Let $G=B(H)$ for some graph $H$. Then $K B_{e}(G)$ is a cycle, a path or it contains an induced $(n, m)-$ necklace, $n \geq 6$ and $m \geq 1$, with good neighbors.

Proof. By previous results, $G$ is either divergent or convergent under the $K B_{e}$ operator, therefore if it is convergent, then $G$ is a cycle, a path or the net graph, thus $K B_{e}(G)$ is a cycle, a (shorter) path or a $C_{6}$, respectively. Now, if it is divergent, then $G$ is not a cycle, a path or the net graph, therefore $H$ is not a cycle, a path or a $K_{1,3}$. This implies that $H$ contains the paw graph, the chair graph (see Fig 14) or a $K_{1,4}$, not necessarily induced. We will show that $B(L(H))$ contains an induced $(n, m)$-necklace, $n \geq 6$ and $m \geq 1$, with good neighbors, then by Theorem 6.2, $B(L(H))=K B_{e}(B(H))=K B_{e}(G)$ contains it as well. We will also use the following remark; given a graph $X$, and $X^{\prime}$ a subgraph of $X$ not necessarily induced, then $L\left(X^{\prime}\right)$ and $B\left(X^{\prime}\right)$ are induced subgraphs of $L(X)$ and $B(X)$, respectively.

Observe now that $L($ paw $)=$ diamond, $L\left(K_{1,4}\right)=K_{4}$ and $L($ chair $)=$ paw, and $B($ diamond $), B\left(K_{4}\right)$ and $B($ paw $)$, contain an induced $(n, m)$ necklace, $n \geq 6$ and $m \geq 1$, therefore following the remark, $B(L(H))=$ $K B_{e}(G)$ also contains an induced $(n, m)$ - necklace, $n \geq 6$ and $m \geq 1$.

Note that the induced ( $n, m$ ) - necklace, $n \geq 6$ and $m \geq 1$, in $K B_{e}(G)$ always have good neighbors, since the operator $B$ applied to any graph, never contains an induced $C_{4}$.


Figure 14: The paw and the chair graphs.

## 7 Open problems

We propose the following conjectures.
Conjecture 7.1. A graph $G$ is either divergent or convergent under the $K B_{e}$ operator but never periodic (with period bigger than 1).

Conjecture 7.2. $G=K B_{e}(G)$ if and only if $G=\overline{C_{7}}, G=G_{9}$ or $G$ has girth at least five and has no vertices of degree one (see Fig. 15).


Figure 15: Graphs $\overline{C_{7}}$ and $G_{9}$ satisfying $K B_{e}(G)=G$ with girth less than five.

Note that Corollary 4.3 together with the fact that $K B_{e}\left(\overline{C_{7}}\right)=\overline{C_{7}}$, $K B_{e}\left(G_{9}\right)=G_{9}$ prove the "only if" part of Conjecture 7.2 .

Conjecture 7.3. It is computable to decide if a graph diverges or converges under the operator $K B_{e}$.

Despite that it seems that a small number of graphs contain an induced ( $n, m$ ) - necklace, $n \geq 5, m \geq 1$, we believe that all divergent graphs will contain one in some iteration under the operator $K B_{e}$. We propose therefore the following conjecture.

Conjecture 7.4. A graph $G$ is divergent under the operator $K B_{e}$ if and only if there exists some $k$ such that $K B_{e}^{k}(G)$ contains an induced $(n, m)-$ necklace, $n \geq 5, m \geq 1$, with its cycle having good neighbors.

Clearly Theorem 5.5 proves the "only if" part of Conjecture 7.4 and moreover, the "if" part along with Conjecture 7.1 imply Conjecture 7.3. Note that all these conjectures are true for burgeon graphs.

## References

[1] L. Alcón, L. Faria, C. M. H. de Figueiredo, and M. Gutierrez. The complexity of clique graph recognition. Theoret. Comput. Sci., 410(21-23):2072-2083, 2009.
[2] G. Atluri, J. Bellay, G. Pandey, C. Myers, and V. Kumar. Discovering coherent value bicliques in genetic interaction data. In Proceedings of 9th International Workshop on Data Mining in Bioinformatics (BIOKDD'10), 2000.
[3] H.-J. Bandelt and E. Prisner. Clique graphs and Helly graphs. J. Combin. Theory Ser. B, 51(1):34-45, 1991.
[4] K. Booth and G. Lueker. Testing for the consecutive ones property, interval graphs, and graph planarity using $P Q$-tree algorithms. J. Comput. System Sci., 13(3):335-379, 1976.
[5] A. Brandstädt, V. Le, and J. P. Spinrad. Graph Classes: a Survey. SIAM Monographs on Discrete Mathematics and Applications. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1999.
[6] D. Bu, Y. Zhao, L. Cai, H. Xue, X. Zhu, H. Lu, J. Zhang, S. Sun, L. Ling, N. Zhang, G. Li, and R. Chen. Topological structure analysis of the protein-protein interaction network in budding yeast. Nucleic Acids Research, 31(9):2443-2450, 2003.
[7] C. P. de Mello, A. Morgana, and M. Liverani. The clique operator on graphs with few $P_{4}$ 's. Discrete Appl. Math., 154(3):485-492, 2006.
[8] F. Escalante. Über iterierte Clique-Graphen. Abh. Math. Sem. Univ. Hamburg, 39:59-68, 1973.
[9] O. Favaron. Irredundance in inflated graphs. J. Graph Theory, 28(2):97104, 1998.
[10] O. Favaron. Inflated graphs with equal independence number and upper irredundance number. Discrete Mathematics, 236(1):81-94, 2001.
[11] M. E. Frías-Armenta, V. Neumann-Lara, and M. A. Pizaña. Dismantlings and iterated clique graphs. Discrete Math., 282(1-3):263-265, 2004.
[12] D. R. Fulkerson and O. A. Gross. Incidence matrices and interval graphs. Pacific J. Math., 15:835-855, 1965.
[13] F. Gavril. The intersection graphs of subtrees in trees are exactly the chordal graphs. J. Combinatorial Theory Ser. B, 16:47-56, 1974.
[14] M. Groshaus, A. L. Guedes, and L. Montero. Almost every graph is divergent under the biclique operator. Discrete Appl. Math., 201:130 140, 2016.
[15] M. Groshaus, P. Hell, and J. Stacho. On edge-sets of bicliques in graphs. Discrete Appl. Math., 160(18):2698-2708, 2012.
[16] M. Groshaus and L. Montero. On the iterated biclique operator. J. Graph Theory, 73(2):181-190, 2013.
[17] M. Groshaus and L. Montero. Structural properties of biclique graphs and the distance formula. CoRR, abs/1708.09686v5, 2021.
[18] M. Groshaus and J. L. Szwarcfiter. Biclique graphs and biclique matrices. J. Graph Theory, 63(1):1-16, 2010.
[19] M. E. Groshaus. Bicliques, cliques, neighborhoods y la propiedad de Helly. PhD thesis, Universidad de Buenos Aires, 2006.
[20] W. H. Haemers. Bicliques and eigenvalues. J. Combinatorial Theory Ser. B, 82(1):56-66, 2001.
[21] R. C. Hamelink. A partial characterization of clique graphs. J. Combinatorial Theory, 5:192-197, 1968.
[22] S. T. Hedetniemi and P. J. Slater. Line graphs of triangleless graphs and iterated clique graphs. In Graph theory and applications (Proc. Conf., Western Michigan Univ., Kalamazoo, Mich., 1972; dedicated to the memory of J. W. T. Youngs), pages 139-147. Lecture Notes in Math., Vol. 303. Springer, Berlin, 1972.
[23] M. A. Henning and A. P. Kazemi. Total domination in inflated graphs. Discrete Appl. Math., 160(1-2):164-169, 2012.
[24] R. Kumar, P. Raghavan, S. Rajagopalan, and A. Tomkins. Trawling the web for emerging cyber-communities. In Proceeding of the 8 th international conference on World Wide Web, pages 1481-1493, 1999., 2000.
[25] F. Larrión, C. P. de Mello, A. Morgana, V. Neumann-Lara, and M. A. Pizaña. The clique operator on cographs and serial graphs. Discrete Math., 282(1-3):183-191, 2004.
[26] F. Larrión and V. Neumann-Lara. A family of clique divergent graphs with linear growth. Graphs Combin., 13(3):263-266, 1997.
[27] F. Larrión and V. Neumann-Lara. Clique divergent graphs with unbounded sequence of diameters. Discrete Math., 197/198:491-501, 1999. 16th British Combinatorial Conference (London, 1997).
[28] F. Larrión and V. Neumann-Lara. Locally $C_{6}$ graphs are clique divergent. Discrete Math., 215(1-3):159-170, 2000.
[29] F. Larrión, V. Neumann-Lara, and M. A. Pizaña. Whitney triangulations, local girth and iterated clique graphs. Discrete Math., 258(1-3):123-135, 2002.
[30] F. Larrión, M. A. Pizaña, and R. Villarroel-Flores. Equivariant collapses and the homotopy type of iterated clique graphs. Discrete Math., 308:3199-3207, 2008.
[31] P. G. H. Lehot. An optimal algorithm to detect a line graph and output its root graph. J. ACM, 21(4):569-575, 1974.
[32] G. Liu, K. Sim, and J. Li. Efficient mining of large maximal bicliques. In Proceedings of the 8th International Conference on Data Warehousing and Knowledge Discovery, DaWaK'06, page 437-448, Berlin, Heidelberg, 2006. Springer-Verlag.
[33] T. A. McKee and F. R. McMorris. Topics in Intersection Graph Theory. SIAM Monographs on Discrete Mathematics and Applications. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1999.
[34] L. Montero. Convergencia y divergencia del grafo biclique iterado. Master's thesis, Departamento de Computación, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, 2008.
[35] N. Nagarajan and C. Kingsford. Uncovering genomic reassortments among influenza strains by enumerating maximal bicliques. 2012 IEEE International Conference on Bioinformatics and Biomedicine, 0:223230, 2008.
[36] V. Neumann Lara. Clique divergence in graphs. In Algebraic methods in graph theory, Vol. I, II (Szeged, 1978), volume 25 of Colloq. Math. Soc. János Bolyai, pages 563-569. North-Holland, Amsterdam, 1981.
[37] M. A. Pizaña. The icosahedron is clique divergent. Discrete Math., 262(1-3):229-239, 2003.
[38] E. Prisner. Bicliques in graphs i: Bounds on their number. Combinatorica, 20(1):109-117, 2000.
[39] F. S. Roberts and J. H. Spencer. A characterization of clique graphs. J. Combinatorial Theory Ser. B, 10:102-108, 1971.
[40] E. Szpilrajn-Marczewski. Sur deux propriétés des classes d'ensembles. Fund. Math., 33:303-307, 1945.
[41] A. C. M. van Rooij and H. S. Wilf. The interchange graph of a finite graph. Acta Math. Acad. Sci. Hungar., 16:263-269, 1965.


[^0]:    ${ }^{\dagger}$ Burgeon graphs have been studied under the name of inflated graphs mainly considering the domination problem (9, 10, 23.

[^1]:    ${ }^{\ddagger}$ Note that this Claim is valid for any edge in a graph that belongs to two bicliques.

[^2]:    ${ }^{\S}$ Found using the computer.

